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Asymptotic Estimates for a Class of Summatory Functions II

by

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Abstract. As in two previous papers [2] and [3] we address here the problem of providing a precise asymptotic estimate for the sum $\sum_{n\le x} a(n)$ with generating function $\sum a(n)n^{-s} = \zeta(s)\Pi(s)H(s)$ $(\sigma > \sigma_0)$, where $\Pi(s) := \prod_{i=1}^{k} \zeta^{\alpha_i}(rs+\gamma_i)$ $(r \in N;\ 1-r < \gamma_1 < \cdots < \gamma_k)$ and H is some "innocuous" factor. In the first part of this work [3] we allow the α_i to take any nonzero complex values, as a result zeros of the zeta function may very well correspond to singularities of $\Pi(s)$, and for this reason we are able to exploit only the pole of $\zeta(s)$ at $s = 1$ and the right-most singularity of the first factor in $\Pi(s)$. In this sequel however, we assume all the α_i to be positive integers, whence all the singularities of $\Pi(s)$ are poles, which we are able to use. Our result can be applied for instance to estimating $\sum_{n\le x} \sigma_a^m(n)$ for positive integral values of m, where $\sigma_a(n) = \sum_{d|n} d^a$, with, for negative a small enough in modulus, a significant explicit smallest term of order x^{1+ma}: as an illustration we treat extensively the cases $m = 2$ and $m = 3$. Our method appeals to a simple idea, an optimal use of Hölder's inequality on known bounds for the Riemann zeta function and its mean in the critical strip, but requires extensive computation.

1. Introduction

1.1. Preliminary remark on the symbol σ

The symbol σ, sometimes with an index, is traditionally used in number theory to denote (at least) two types of very different objects which, somehow unfortunately, both constantly appear in this paper. The use of σ has become for both such a matter of course, that we feel the use of another symbol would look incongruous and be misleading. We are however confident that in the context the reader will not mistake the real part σ of a complex number for the sum-of-divisors function σ, nor the abscissa of absolute convergence σ_a of some Dirichlet series for the divisors function σ_a.

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1.2. Introduction and notation

We consider the problem of giving a precise asymptotic estimate for the sum

$$\sum_{n\le x} a(n)\,,$$

where for large enough σ we have the representation

$$\sum_{n\ge1}\frac{a(n)}{n^s}=\zeta(s)\zeta^{\alpha}(rs+\beta)F(s)$$

$$=\zeta(s)\zeta^{\alpha_1}(rs+\beta)\prod_{i=2}^{k}\zeta^{\alpha_i}(rs+\beta+\beta_i)H(s)\,,$$

where r is a positive integer, $-r+1<\beta<1$, and $\beta_1:=0<\beta_2<\cdots<\beta_k$ (see [3, Section 2] for a complete treatment of the easier case $\beta>1$; see [2] for the case $\beta=1$ and $r=1$), and where the product is allowed to be 1 (in which case $k=1$ and $H=F$). We assume that H has a representation

$$H(s)=\sum_{n\ge1}\frac{b(n)}{n^s}\,,$$

with an abscissa of absolute convergence $\sigma_a=\sigma_a(H)<\frac{1-\beta}{r}$, and this implies in particular that for every $\epsilon>0$ we have

$$|H(\sigma+\varepsilon+it)|\le C_\varepsilon:=\bar{H}(\sigma+\varepsilon)\,, \tag{1}$$

where $\bar{H}(s):=\sum_{n\ge1}|b(n)|n^{-s}$. We also assume that the exponents

$$\alpha_i \ (i=1,\cdots,k) \text{ are positive integers}.$$

Theorem 1 below then provides an asymptotic estimate for $\sum_{n\le x}a(n)$ which is more precise than that provided by the first part of this work [3] (in which the exponent α_1 ($=\alpha$ there) is not assumed to be a real positive integer). But without an additional assumption on H Theorem 1 is valid only under the rather strong condition $\sigma_a(H)<0$. However, in all the classical applications of this type of results we can think of, the coefficients $b(n)$ are all zero when n is not the r-th power m^r of an integer (in which case we say that H *has its support* on the r-th powers of integers). In this case we write $H(s)=h(rs+\beta+\beta_k)$ and $F(s)=f(rs+\beta)$, and we obtain, without assuming $\sigma_a(H)<0$, a better estimate.

We need to introduce some additional notation. We put, for $\sigma>(1-\beta)/r$,

$$\zeta^{\alpha}(rs+\beta)F(s)=\sum_{n\ge1}\frac{v(n)}{n^s}\,,$$

and we note that if $F(s)=f(rs+\beta)$ has its support on the r-th powers of integers, then we may write

$$v(m^r)=v_1(m)m^{-\beta}\,,\quad \text{where } \zeta^{\alpha}(z)f(z)=\sum_{n\ge1}\frac{v_1(m)}{m^z}\,.$$

In general (this notation will be used in Lemma 1 below), if

$$F_1(s) = f_1(rs + \beta)$$

has its support on the r-th powers of integers, and if we write

$$\zeta^\alpha(rs + \beta) F_1(s) = \sum_{n \ge 1} \frac{w(n)}{n^s} \quad \text{and} \quad \zeta^\alpha(z) f_1(z) = \sum_{m \ge 1} \frac{w_1(m)}{m^z},$$

then

$$w(m^r) = w_1(m) m^{-\beta}.$$

Moreover we put

$$R_\rho(f_1, b) := \text{the residue of } \frac{\zeta^\alpha(s + \beta - \rho) f_1(s + \beta - \rho)}{s} x^{\frac{s}{r}} \quad \text{at } s = b.$$

We also let

$$\ell_*(\beta) = \begin{cases} \ell(\beta) & \text{if } \beta \text{ is not a singularity of } \ell \\ 0 & \text{otherwise} \end{cases}$$

(this will be used when $\ell = \zeta$, $\ell = f$, $\ell = f_1$). For u a real number we put

$$\psi(u) := u - [u] - \frac{1}{2} = \{u\} - \frac{1}{2},$$

where $[u]$ and $\{u\}$ denote as usual the integral and fractional parts of u.

We shall also consider nonnegative numbers $p(\sigma)$ and $G(A, \sigma)$, where $A \ge 0$ and $1/2 \le \sigma$, satisfying

$$|\zeta(\sigma + it)| \ll |t|^{p(\sigma)+\varepsilon} \tag{2}$$

and

$$\int_1^T |\zeta(\sigma + it)|^A dt \ll T^{1+G(A,\sigma)+\varepsilon}, \tag{3}$$

the associated quantity

$$P(\sigma) = \sum_{i=1}^{k} p(\sigma + \beta_i)\alpha_i, \tag{4}$$

and for real positive numbers q_i $(1 \le i \le k)$ with $\sum_{i=1}^k 1/q_i = 1$, the associated quantities

$$K(q_1, \cdots, q_k; \sigma) = \sum_{i=1}^{k} \frac{G(\alpha_i q_i, \sigma + \beta_i)}{q_i} \tag{5}$$

and

$$K_0 = K_0(\sigma) = \inf_{\{q_i\}} K(q_1, \cdots, q_k; \sigma). \tag{6}$$

In Theorem 1 we obtain an expression for $\sum_{n \le x} a(n)$ involving an error term of the type $O(x^{\mu+\varepsilon})$. For this purpose we choose an abscissa of integration of the form $\sigma + \varepsilon$, where ε is small, and with

$$\sigma = \max\left\{ r\sigma_a(H_1) + \beta, \frac{1}{2} \right\}, \tag{7}$$

(where H_1 will be H or 1). The value of μ, as well as that of an auxiliary quantity δ, depend on a parameter λ with

$$\lambda = \frac{1}{r}\left(\beta - \frac{K_0 + \sigma}{K_0 + 1}\right).$$

We state here the definitions of $\mu = \mu(\lambda)$ and $\delta = \delta(\lambda)$. We put

$$\mu = \frac{1}{r}(1-\beta)(1-\delta)\,, \tag{8}$$

where δ is the solution of

$$\frac{1}{r}(1-\beta)(1-\delta) = \begin{cases} \delta - \lambda(1-\delta) & \text{if } \lambda \geq 0 \\ \delta - \lambda & \text{otherwise}\,. \end{cases}$$

Thus we have

$$\delta = \begin{cases} \dfrac{\dfrac{1-\beta}{r} + \lambda}{\dfrac{1-\beta}{r} + \lambda + 1} & \text{if } \lambda \geq 0 \\[2ex] \dfrac{\dfrac{1-\beta}{r} + \lambda}{\dfrac{1-\beta}{r} + 1} & \text{otherwise}\,. \end{cases}$$

REMARKS 1. It is well known that in (2) we may take $p(\sigma) = 0$ as soon as $\sigma \geq 1$ (see for instance [10, Théorème II.1.17]). For more precise information see Section 3.

2. Since $\delta > 0$, we have $\mu(\lambda) > -\lambda$.

2. Main result

THEOREM 1. *Let the notation be that introduced in Section* 1, *and assume in addition that for the abscissa σ defined in* (7) *we have*

$$P(\sigma) < 1 + K_0\,. \tag{9}$$

Now suppose either that (1)

$$\sigma_a(H) < 0 \quad \textit{and} \quad \lambda < -\sigma_a(H)$$

or that (2)

$H(s) = h(rs + \beta + \beta_k)$ *has its support on the r-th powers of integers, and* $v_1(m) \ll m^{\varepsilon}$.

Then we have

$$\sum_{n\leq x} a(n) = \zeta^{\alpha}(r+\beta) f(r+\beta) x + \sum_{i=1}^{k} x^{\frac{1-\beta-\beta_i}{r}} \sum_{n=0}^{\alpha_i - 1} A_{ni} \log^n x + O(x^{\mu+\varepsilon})\,,$$

where the A_{ni} are some computable constants.

LEMMA 1. *Suppose, with the notation of Section* 1, *that*

$$\zeta^{\alpha}(rs+\beta)F_1(s)=\zeta^{\alpha}(rs+\beta)\prod_{i=2}^{k}\zeta^{\alpha_i}(rs+\beta+\beta_i)H_1(s)\,,$$

where

$$F_1(s)=f_1(rs+\beta)\quad\textit{and}\quad H_1(s)=h_1(rs+\beta+\beta_k)$$

have their support on the r-th powers of integers. Suppose further that

$$\sigma_a(H_1)<\frac{1-\beta}{r}\,,$$

where σ *is as in* (7), *that hypothesis* (9) *is satisfied, and that*

$$w_1(m)\ll m^{\varepsilon}\,.$$

Then

$$\sum_{n\le x}n^{\rho/r}w(n)=\sum_{\substack{i=1\\ \beta_i\neq\beta_0-1}}^{k}R_\rho(f_1,-\beta_0-\beta_i+1)+R_\rho(f_1,0)+O\Big(x^{\frac{K_0+\sigma}{r(K_0+1)}-\frac{\beta_0}{r}+\varepsilon}\Big)\,,$$

where $\beta_0:=\beta-\rho$.

Proof. Since $w(n)$ is zero when n is not of the form m^r, we have

$$\sum_{n=1}^{\infty}\frac{n^{\rho/r}w(n)}{n^s}=\sum_{m=1}^{\infty}\frac{m^{\beta}w(m^r)}{m^{rs+\beta_0}}$$
$$=\zeta^{\alpha}(rs+\beta_0)f_1(rs+\beta_0)\,,$$

whence, as noted in Section 1, $w(m^r)=w_1(m)m^{-\beta}$ where $w_1(m)$ is the coefficient of m^{-z} in the Dirichlet series expansion for $\zeta^{\alpha}(z)f_1(z)$. It follows that

$$\sum_{m\ge1}\frac{|w_1(m)|}{m^{1+\vartheta}}\ll\vartheta^{-\alpha}\quad\text{as }\vartheta\to0\,. \tag{10}$$

An application of Perron's truncated formula (see for instance [10, Théorèmes II.2.1 and II.2.2]) yields

$$\sum_{n\le x}n^{\frac{\rho}{r}}w(n)$$
$$=\frac{1}{2\pi i}\int_{(1/r)(1-\beta_0+\vartheta-i\infty)}^{(1/r)(1-\beta_0+\vartheta+i\infty)}\zeta^{\alpha}(rs+\beta_0)f_1(rs+\beta_0)\frac{x^s}{s}ds+O(x^{(-\beta_0+\varepsilon)/r})$$
$$=\frac{1}{2\pi i}\int_{1-\beta_0+\vartheta-iT}^{1-\beta_0+\vartheta+iT}\zeta^{\alpha}(s+\beta_0)f_1(s+\beta_0)\frac{x^{\frac{s}{r}}}{s}ds$$
$$+O\Big(x^{\frac{1-\beta_0}{r}}\sum_{m=1}^{\infty}\frac{m^{\rho}|w(m^r)|m^{\beta_0-1-\vartheta}}{1+T|\log(x^{\frac{1}{r}}/m)|}\Big) \tag{11}$$
$$=\frac{1}{2\pi i}\int_{1-\beta_0+\vartheta-iT}^{1-\beta_0+\vartheta+iT}\zeta^{\alpha}(s+\beta_0)f_1(s+\beta_0)\frac{x^{\frac{s}{r}}}{s}ds+O\Big(x^{\frac{1-\beta_0}{r}+\varepsilon}/T\Big)\,, \tag{12}$$

where $\vartheta := 1/\log x$. Here the parameter T is still to be defined precisely in terms of the parameter x, but we already fix the condition $T \ll x^{1/(2r)}$, which ensures the correctness of (11) and (12). In order to infer (12) from (11) we estimate in a standard way the sum in the error term of (11) by splitting it in three sums, the first one indexed by the integers $m \geq 1$ with $m \leq x^{\frac{1}{r}}/2$ or $m \geq 2x^{\frac{1}{r}}$, the second one by those remaining with $|m - x^{\frac{1}{r}}| \geq 1$, and the third one by those with $|m - x^{\frac{1}{r}}| < 1$. We also appeal to (10).

Let now σ be as in (7). We take a different path of integration P joining the points $1-\beta_0+\vartheta+iT, \sigma-\beta_0+iT, \sigma-\beta_0-iT, 1-\beta_0+\vartheta-iT$ in that order, by line segments in the s-plane (We make a small indentation at $s = 0$ if necessary, that is if $\sigma = \beta_0$ or if $\sigma = 1 - \beta_\ell$). Hence, if C is the contour formed by P and the line segment joining $1 - \beta_0 + \vartheta - iT$ and $1 - \beta_0 + \vartheta + iT$, then

$$\sum_{n\leq x} n^{\rho/r} w(n)$$

$$= \text{sum of the residues of the integrand within the contour } C$$

$$- \frac{1}{2\pi i}\int_P \prod_{i=1}^{k} \zeta^{\alpha_i}(s+\beta_0+\beta_i) h_1(s+\beta_0+\beta_k)\frac{x^{s/r}}{s} ds + O\Big(x^{\frac{1-\beta_0}{r}+\varepsilon}/T\Big).$$

With the estimates (1) and (3), the notation (5) and (6), and by using Hölder's inequality, we see that if the q_i $(i \leq k)$ are real positive numbers with $\sum_{i=i}^{k} 1/q_i = 1$ we have

$$\int_{\sigma+\varepsilon/2-\beta_0-iT}^{\sigma+\varepsilon/2-\beta_0+iT} \prod_{i=1}^{k} \zeta^{\alpha_i}(s+\beta_0+\beta_i) h_1(s+\beta_0+\beta_k)\frac{x^{s/r}}{s} ds$$

$$\ll x^{\frac{\sigma-\beta_0}{r}+\frac{\varepsilon}{2}} \prod_{i=1}^{k} \left(\int_1^T |\zeta(\sigma+\beta_i+it)|^{\alpha_i q_i}\frac{dt}{t}\right)^{\frac{1}{q_i}}$$

$$\ll x^{\frac{\sigma-\beta_0}{r}+\frac{\varepsilon}{2}} T^{K(q_1,\cdots,q_k;\sigma)+\frac{\varepsilon}{4}}.$$

The last expression is

$$x^{\frac{\sigma-\beta_0}{r}+\frac{\varepsilon}{2}} T^{K_0+\frac{\varepsilon}{2}} \tag{13}$$

if we choose the numbers q_i such that $K(q_1, \cdots, q_k; \sigma) \leq K_0 + \varepsilon/4$. Now we make use of (2) and (4), of Remark 1 at the end of Section 1, and again of Hölder's inequality, and verify that the integral on the horizontal sides of the contour is

$$\ll x^{(\sigma-\beta_0)/r+\varepsilon/2} T^{P(\sigma)-1+\varepsilon} + x^{(1-\beta_0)/r} T^{-1+\varepsilon}.$$

By hypothesis (9) we see that the first term is dominated by (13), whence the integral on the path P is

$$\ll x^{\frac{\sigma-\beta_0}{r}+\frac{\varepsilon}{2}} T^{K_0+\frac{\varepsilon}{2}} + x^{(1-\beta_0)/r} T^{-1+\varepsilon} \ll O\Big(x^{\frac{K_0+\sigma}{r(K_0+1)}-\frac{\beta_0}{r}+\varepsilon}\Big)$$

if we choose $T = x^{\frac{1-\sigma}{r(1+K_0)}}$. Thus the lemma is proved. (Note that in case some of the residues in the statement of the lemma are not within the contour C, they are then absorbed in the error term).

Now if we set $\rho = -r$ and $\rho = 0$ this yields the following.

LEMMA 2. *Let* $\lambda = \frac{1}{r}\left(\beta - \frac{K_0+\sigma}{K_0+1}\right)$. *Then, under the assumptions of Lemma* 1, *we have*

$$x\sum_{n\le x}\frac{w(n)}{n} = \zeta^{\alpha}(r+\beta)f_1(r+\beta)x + \sum_{i=1}^{k} x^{\frac{1-\beta-\beta_i}{r}}\sum_{n=0}^{\alpha_i} d_{ni}\log^n x + O(x^{-\lambda+\varepsilon})$$

and

$$\sum_{n\le x} w(n) = \sum_{i=1}^{k} x^{\frac{1-\beta-\beta_i}{r}}\sum_{n=0}^{\alpha_i} c_{ni}\log^n x + \zeta_*^{\alpha}(\beta)f_{1*}(\beta) + O(x^{-\lambda+\varepsilon}),$$

where the c_{ni} *and* d_{ni} *are some constants.*

If the assumptions (2) of Theorem 1 are satisfied we shall apply Lemma 2 with $f_1 = f$, that is with $w = v$. Otherwise we use Lemma 3 with $f_1(rs+\beta) \equiv \prod_{i=2}^{k}\zeta^{\alpha_i}(rs+\beta+\beta_i)$ and also need the following.

LEMMA 3. *Under the assumptions* (1) *of Theorem* 1 *we have*

$$x\sum_{n\le x}\frac{v(n)}{n} = \zeta^{\alpha}(r+\beta)f(r+\beta)x + \sum_{i=1}^{k} x^{\frac{1-\beta-\beta_i}{r}}\sum_{n=0}^{\alpha_i} g_{ni}\log^n x + O(x^{-\lambda+\varepsilon}) \tag{14}$$

and

$$\sum_{n\le x} v(n) = \sum_{i=1}^{k} x^{\frac{1-\beta-\beta_i}{r}}\sum_{n=0}^{\alpha_i} e_{ni}\log^n x + \zeta_*^{\alpha}(\beta)f_*(\beta) + O(x^{-\lambda+\varepsilon}), \tag{15}$$

where the e_{ni} *and* g_{ni} *are some constants.*

Proof. We can write $v(n)$ as a Dirichlet convolution $(w * b)(n)$, where Lemma 2 holds for w, and thus

$$\begin{aligned}\sum_{n\le x} v(n) &= \sum_{n\le x}(w*b)(n) = \sum_{m\le x} b(k)\sum_{d\le x/m} w(d)\\ &= \sum_{m\le x} b(m)\left(\sum_{i=1}^{k}\left(\frac{x}{m}\right)^{\frac{1-\beta-\beta_i}{r}}\sum_{n=0}^{\alpha_i} c_{ni}\log^n\left(\frac{x}{m}\right) + \zeta^{\alpha}(\beta) + O\left(\frac{x}{m}\right)^{-\lambda+\varepsilon}\right)\\ &= \sum_{i=1}^{k} x^{\frac{1-\beta-\beta_i}{r}}\sum_{m\le x}\frac{b(m)}{m^{(1-\beta-\beta_i)/r}}\sum_{n=0}^{\alpha_i} c_{ni}\log^n\left(\frac{x}{m}\right) + \zeta^{\alpha}(\beta)\sum_{m\le x} b(m)\\ &\quad + O\left(x^{-\lambda+\varepsilon}\sum_{m\le x}\frac{|b(m)|}{m^{-\lambda+\varepsilon}}\right).\end{aligned}$$

Since the last sum remains bounded as $x \to \infty$ if ε is small enough, the error term is a $O(x^{-\lambda+\varepsilon})$. And for the same reason we have

$$\sum_{m\le x} b(m) = f(\beta) - \sum_{m>x} b(m) = f(\beta) + O(x^{-\lambda}).$$

Now we have

$$\begin{aligned}
&\sum_{m\le x}\frac{b(m)}{m^{\frac{1-\beta-\beta_i}{r}}}\sum_{n=0}^{\alpha_i}c_{ni}\log^n\left(\frac{x}{m}\right)\\
&=\sum_{m\le x}\frac{b(m)}{m^{\frac{1-\beta-\beta_i}{r}}}\sum_{n=0}^{\alpha_i}c_{ni}\log^n(x)\left(1-\frac{\log m}{\log x}\right)^n\\
&=\sum_{n=0}^{\alpha_i}c_{ni}\log^n x\sum_{\ell=0}^{n}(-1)^\ell\binom{n}{\ell}\log^{-\ell}x\sum_{m\le x}\frac{b(m)}{m^{\frac{1-\beta-\beta_i}{r}}}\log^\ell m\\
&=\sum_{n=0}^{\alpha_i}c_{ni}\log^n x\sum_{\ell=0}^{n}(-1)^\ell\binom{n}{\ell}\log^{-\ell}x\left(C_{\ell,i}+O(x^{-\lambda-\frac{1-\beta-\beta_i}{r}}\log^\ell x)\right),
\end{aligned}$$

for some constants $C_{\ell,i}$. From these observations we see that (15) holds. We then derive (14) from (15) by noting that

$$\sum_{n\ge 1}\frac{v(n)}{n}=\zeta^\alpha(r+\beta)f(r+\beta)$$

and that

$$\sum_{n>x}\frac{v(n)}{n}=\int_x^\infty\frac{dV(u)}{u}=-\frac{V(x)}{x}+\int_x^\infty\frac{V(u)}{u^2}du\,,$$

where $V(u):=\sum_{n\le u}v(n)$.

LEMMA 4. *Under the assumptions* (1) *or* (2) *of Theorem* 1 *we have*

$$\begin{aligned}
\sum_{n\le x}a(n)&=\zeta^\alpha(r+\beta)f(r+\beta)x+\sum_{i=1}^{k}x^{\frac{1-\beta-\beta_i}{r}}\sum_{n=0}^{\alpha_i}h_{ni}\log^n x\\
&\quad-\frac{1}{2}\zeta_*^\alpha(\beta)f_*(\beta)-\sum_{n\le x}v(n)\psi\left(\frac{x}{n}\right)+O(x^{-\lambda+\varepsilon})\,,
\end{aligned}$$

where the h_{ni} *are some constants.*

Proof. We have $\sum_{n=1}^{\infty}\frac{a(n)}{n^s}=\zeta(s)\sum_{n=1}^{\infty}\frac{v(n)}{n^s}$ and so $a(n)=\sum_{d|n}v(d)$. Hence

$$\sum_{n\le x}a(n)=\sum_{d\le x}v(d)\left[\frac{x}{d}\right]=x\sum_{n\le x}\frac{v(n)}{n}-\frac{1}{2}\sum_{n\le x}v(n)-\sum_{n\le x}v(n)\psi\left(\frac{x}{n}\right)\qquad(16)$$

where $\psi(u)=u-[u]-\frac{1}{2}$, as defined in Section 1. Now the lemma follows from Lemmas 2 and 3.

LEMMA 5. *If we let* $y = x^{1-\delta}$ *where* δ *is some real number with* $0 < \delta < 1$, *then under the assumptions of Theorem* 1 *we have*

$$\sum_{y<n\le x} v(n)\psi\left(\frac{x}{n}\right) = \sum_{i=1}^{k} x^{\frac{1-\beta-\beta_i}{r}} \sum_{n=0}^{\alpha_i} k_{ni} \log^n x + O(x^{\frac{1}{r}(1-\beta)(1-\delta)-\delta} \log^{\alpha-1} x) + \begin{cases} O(x^{\delta-\lambda(1-\delta)+\epsilon}) & \textit{if } \lambda \ge 0 \\ O(x^{\delta-\lambda+\varepsilon}) & \textit{if } \lambda < 0, \end{cases}$$

where the k_{ni} *are some constants.*

Proof. By Lemmas 2 and 3 we have

$$\begin{aligned}\sum_{y<n\le x} v(n)\psi\left(\frac{x}{n}\right) &= \int_y^x \psi\left(\frac{x}{u}\right) dV(u) \\ &= \int_y^x \psi\left(\frac{x}{u}\right) d\left(\sum_{i=1}^{k} u^{\frac{1-\beta-\beta_i}{r}} \sum_{n=0}^{\alpha_i} e_{ni} \log^n u\right) + \int_y^x \psi\left(\frac{x}{u}\right) d(\varepsilon(u)) \\ &:= I + I_\varepsilon ,\end{aligned}$$

where $\varepsilon(u) \ll u^{-\lambda+\varepsilon}$. The last integral is

$$\begin{aligned} I_\varepsilon = -\psi\left(\frac{x}{y}\right)\varepsilon(y) + \psi(1)\varepsilon(x) + \int_1^{\frac{x}{y}} \varepsilon\left(\frac{x}{u}\right) d\psi(u) &\ll (y^{-\lambda+\varepsilon} + x^{-\lambda+\varepsilon})\frac{x}{y} \\ &\ll \begin{cases} x^{\delta-\lambda(1-\delta)+\varepsilon} & \text{if } \lambda \ge 0 \\ x^{\delta-\lambda+\varepsilon} & \text{if } \lambda < 0 . \end{cases}\end{aligned}$$

As for I, it is

$$\sum_{i=1}^{k} \sum_{n=0}^{\alpha_i} \int_y^x u^{\frac{1-\beta-\beta_i}{r}-1} e_{ni} n \log^{n-1} u \psi\left(\frac{x}{u}\right) du ,$$

to which we must add

$$\sum_{\beta_i \ne 1-\beta} \frac{1-\beta-\beta_i}{r} \sum_{n=0}^{\alpha_i-1} \int_y^x u^{\frac{1-\beta-\beta_i}{r}-1} e_{ni} \log^n u \psi\left(\frac{x}{u}\right) du$$

(since if $\beta_i \neq 1-\beta$ then $e_{\alpha_i i} = 0$). So we can write, for some constants e'_{ni}

$$
\begin{aligned}
I &= \sum_{i=1}^{k} \sum_{n=0}^{\alpha_i-1} e'_{ni} \int_y^x u^{\frac{1-\beta-\beta_i}{r}-1} \log^n u \psi\left(\frac{x}{u}\right) du \\
&= \sum_{i=1}^{k} x^{\frac{1-\beta-\beta_i}{r}} \sum_{n=0}^{\alpha_i-1} e'_{ni} \int_1^{x/y} t^{-1+\frac{\beta-\beta_i-1}{r}} \psi(t) \log^n (x/t) dt \\
&= \sum_{i=1}^{k} x^{\frac{1-\beta-\beta_i}{r}} \sum_{n=0}^{\alpha_i-1} e'_{ni} \log^n x \int_1^{x/y} t^{-1+\frac{\beta-\beta_i-1}{r}} \psi(t) \left(1 - \frac{\log t}{\log x}\right)^n dt \\
&= \sum_{i=1}^{k} x^{\frac{1-\beta-\beta_i}{r}} \sum_{n=0}^{\alpha_i-1} e'_{ni} \log^n x \int_1^{x/y} t^{-1+\frac{\beta-\beta_i-1}{r}} \psi(t) \left(\sum_{m=0}^{n} \binom{n}{m} (-1)^m \left(\frac{\log t}{\log x}\right)^m\right) dt . \\
&= \sum_{i=1}^{k} x^{\frac{1-\beta-\beta_i}{r}} \sum_{n=0}^{\alpha_i-1} e'_{ni} \log^n x \left(\sum_{m=0}^{n} \binom{n}{m} (-1)^m \log^{-m} x C_{mi}(1) + O\left(\left(\frac{y}{x}\right)^{d_i}\right)\right),
\end{aligned}
$$

where

$$d_i := 1 + \frac{1-\beta-\beta_i}{r}$$

and

$$C_{mi}(z) := \int_z^\infty t^{-d_i} \psi(t) \log^m t dt \ll z^{-d_i} \log^m z .$$

Thus, for some constants k_{ni}, we have

$$
\begin{aligned}
\sum_{y<n\le x} v(n)\psi\left(\frac{x}{n}\right) - \sum_{i=1}^{k} x^{\frac{1-\beta-\beta_i}{r}} \sum_{n=0}^{\alpha_i} k_{ni} \log^n x &\ll x^{\frac{1}{r}(1-\beta)(1-\delta)-\delta} \log^{\alpha-1} x \\
&\quad + \begin{cases} x^{\delta-\lambda(1-\delta)+\varepsilon} & \text{if } \lambda \ge 0 \\ x^{\delta-\lambda+\varepsilon} & \text{if } \lambda < 0 , \end{cases} \qquad (17)
\end{aligned}
$$

and the lemma is proved.

LEMMA 6. *If we let* $y = x^{1-\delta}$ *where* δ *is some real number with* $0 < \delta < 1$, *then under the assumptions of Theorem* 1 *we have*

$$\sum_{n\le y} |v(n)| = O(x^{\frac{1}{r}(1-\beta)(1-\delta)} \log^{\alpha-1} x) .$$

Proof. Lemmas 1, 2 and 3 apply to $|w(n)|$ and $|v(n)|$, with $\bar{H}(s) = \sum_{n\ge 1} |b(n)|/n^s$ instead of $H(s)$. The result follows immediately.

Proof of Theorem 1. By Lemmas 4, 5 and 6 we have

$$
\begin{aligned}
\sum_{n\le x} a(n) &= \zeta^\alpha(r+\beta) f(r+\beta) x + \sum_{i=1}^{k} x^{\frac{1-\beta-\beta_i}{r}} \sum_{n=0}^{\alpha_i} A_{ni} \log^n x \\
&\quad + O(x^{\frac{1}{r}(1-\beta)(1-\delta)} \log^{\alpha-1} x) + \begin{cases} O(x^{\delta-\lambda(1-\delta)+\varepsilon}) & \text{if } \lambda \ge 0 \\ O(x^{\delta-\lambda+\varepsilon}) & \text{if } \lambda < 0 . \end{cases}
\end{aligned}
$$

And it is easy to check that all the constants $c_{\alpha_i i}$, $d_{\alpha_i i}$, $e_{\alpha_i i}$, $g_{\alpha_i i}$, $h_{\alpha_i i}$ and $k_{\alpha_i i}$, occuring in Lemmas 2 through 5, are all zero when $\beta_i \neq 1 - \beta$: thus this property is also satisfied by the constants $A_{\alpha_i i}$. Theorem 1 now follows from the definition (8) of μ, and from the fact that when $\beta_i = 1 - \beta$, then the terms in the double sum above corresponding to i are negligeable with respect to the estimate of the error term.

3. Bounds for ζ and its mean in the critical strip

In the next section we shall give some applications of Theorem 1. For that purpose we first need explicit estimates of types (2) and (3) for $|\zeta(\sigma+it)|$ and for $\int_2^T |\zeta(\sigma+it)|^A dt$. We borrow the seven lemmas below from the literature.

LEMMA A (Phragmén-Lindelöf). *If $f(s)$ is regular, and $O(e^{\varepsilon t})$ as $t \to \infty$ for every positive ε, in the region $\alpha \le \sigma \le \beta$, $t \ge e$, and if*

$$f(\alpha+it) \ll t^a \quad \text{and} \quad f(\beta+it) \ll t^b, \quad \text{as } t \to \infty,$$

where a and b are nonnegative real constants, then

$$f(\sigma+it) \ll (t^a)^{\frac{\beta-\sigma}{\beta-\alpha}} (t^b)^{\frac{\sigma-\alpha}{\beta-\alpha}}$$

uniformly for $\alpha \le \sigma \le \beta$, $t \ge 1$.

A proof of Lemma A can be found in § 237 of Landau's book [7], and another one in § 5.65 of Titchmarsh's book [11]. Both consist in two parts: first a proof of the case where $a = b = 0$, and then using this a proof of the general case. The first case is clearer in [7, Erster Hilfssatz, p. 849–850], as there is an oversight in the other version. The general case is established in a simpler manner in [11].

LEMMA B (Carlson; Hardy-Ingham-Pólya). *Let f be a complex function, real for real s, regular for $s \ge \alpha$ except possibly for a pole at $s = s_0$, and $O(e^{\varepsilon|t|})$ as $|t| \to \infty$, for every positive ε, in the strip $\alpha \le \sigma \le \beta$, and suppose that*

$$\int_2^T |f(\alpha+it)|^2 dt \ll T^a \quad \text{and} \quad \int_2^T |f(\beta+it)|^2 dt \ll T^b, \quad \text{as } T \to \infty,$$

where a and b are positive constants. Then

$$\int_2^T |f(\sigma+it)|^2 dt \ll (T^a)^{\frac{\beta-\sigma}{\beta-\alpha}} (T^b)^{\frac{\sigma-\alpha}{\beta-\alpha}}$$

uniformly for $\alpha \le \sigma \le \beta$, $T \ge 2$.

A proof of Lemma B can be found in § 7.8 of [12]. (Note however that the argument given by Titchmarsh doesn't work in the case $a = b = 0$, which we don't consider here.)

Now we appeal to known bounds for the Riemann zeta-function and its mean in the critical strip.

LEMMA C. *We have*

$$\zeta\left(\frac{1}{2}+it\right) \ll |t|^{89/570+\varepsilon}, \tag{18}$$

$$\zeta(1+it) \ll \log|t| \tag{19}$$

and

$$\zeta(C+it) \ll 1 \quad \text{if } C>1. \tag{20}$$

Estimate (18) (under a more precise form) is due to M. N. Huxley [5, Chapter 21]. Estimate (19) is well-known (see for instance [7, § 46]; see [9] for a better one). Estimate (20) is trivial.

LEMMA D. *The estimate*

$$\int_1^T |\zeta(\sigma+it)|^2 dt \ll T\log T$$

holds uniformly for $1/2 \le \sigma \le 2$.

For a proof see the more precise [12, Theorem 7.2], due to J. E. Littlewood.

LEMMA E. *We have*

$$\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^A dt \ll T^{1+g(A)+\varepsilon},$$

where

$$g(A) = \begin{cases} 0 & (0 \le A \le 4); \\ \dfrac{A-4}{8} & (4 \le A \le 12); \\ \dfrac{3A-14}{22} & \left(12 \le A \le \dfrac{178}{13}\right); \\ \dfrac{89}{570}A - \dfrac{3361}{3705} & \left(A \ge \dfrac{178}{13}\right). \end{cases}$$

For $A \le 4$ this follows from Lemma D. For $4 \le A \le 12$ and $12 \le A \le 178/13$, this is Theorem 8.3 in Ivić's book [6]. And for $A \ge 178/13$ the estimate given by Ivić in his Theorem 8.3 can be slightly improved with a use of Huxley's result (18) instead of the weaker Corollary 7.1 in [6].

LEMMA F. *For* $1/2 < \sigma < 1$ *let* $m(\sigma)$ *be the supremum of all numbers* m *such that*

$$\int_1^T |\zeta(\sigma+it)|^m dt \ll T^{1+\varepsilon}$$

for any $\varepsilon > 0$. *Then for* $1/2 \le \sigma_1 < \sigma < \sigma_2 < 1$ *we have*

$$m(\sigma) \ge \frac{m(\sigma_1)m(\sigma_2)(\sigma_2-\sigma_1)}{m(\sigma_2)(\sigma_2-\sigma)+m(\sigma_1)(\sigma-\sigma_1)}.$$

This is Theorem 8.1 of [6].

LEMMA G. *With the notation of Lemma F we have*

$$m(\sigma) \geq \begin{cases} \dfrac{4}{3-4\sigma} & \textit{for } \dfrac{1}{2} < \sigma \leq \dfrac{5}{8}; \\ \dfrac{10}{5-6\sigma} & \textit{for } \dfrac{5}{8} \leq \sigma \leq \dfrac{35}{54}; \\ \dfrac{180'351}{65'154-69'606\sigma} & \textit{for } \dfrac{35}{54} \leq \sigma \leq \dfrac{2}{3}; \\ \dfrac{220'429}{80'470-86'330\sigma} & \textit{for } \dfrac{2}{3} \leq \sigma \leq \dfrac{7}{10}; \\ \dfrac{132}{61-70\sigma} & \textit{for } \dfrac{7}{10} \leq \sigma \leq \dfrac{5}{7}; \\ \dfrac{132}{46-49\sigma} & \textit{for } \dfrac{5}{7} \leq \sigma \leq \dfrac{3}{4}; \\ \dfrac{12'408}{4537-4890\sigma} & \textit{for } \dfrac{3}{4} \leq \sigma \leq \dfrac{5}{6}; \\ \dfrac{4324}{1031-1044\sigma} & \textit{for } \dfrac{5}{6} \leq \sigma \leq \dfrac{7}{8}; \\ \dfrac{9016\sqrt{5385}-522'928}{2821\sqrt{5385}-173'817-(2944\sqrt{5385}-182'408)\sigma} & \textit{for } \dfrac{7}{8} < \sigma \leq \dfrac{271+\sqrt{5385}}{376} \\ & \quad = .91591106\ldots; \\ \dfrac{24\sigma-9}{(4\sigma-1)(1-\sigma)} & \textit{for } \dfrac{271+\sqrt{5385}}{376} \\ & \quad \leq \sigma \leq 1-\varepsilon. \end{cases}$$

In the intervals (0, 35/54], [3/4, 7/8] and [.91591106..., 1), this is Theorem 8.4 of [6] for which, as Ivić writes in the notes following his Chapter 8, "no effort has been made (except when $\sigma = 2/3$) to obtain the best possible estimates for $m(\sigma)$ that [his] method allows, as this would involve tedious computations with exponent pairs, and the possible improvements would be rather small". In the intervals [35/54, 3/4] and [7/8, .91591106...] we nevertheless replaced Ivić's printed estimates by some slightly better ones, using in a straightforward manner Lemma F and the bounds $m(2/3) \geq 9.61872$, $m(7/10) \geq 11$ and $m(5/7) \geq 12$ (which Ivić also gives in his Theorem 8.4 of [6]). This has in particular the æsthetic advantage of providing a continuous lower bound for $m(\sigma)$. In our calculation we wrote the number 9.61872 under the form 60'117/62'500, in order to ensure a certain uniformity in the expression of the results. This choice is partly responsible (but only partly) for the occurence of very large integers in the estimates we obtain in the next section.

The following theorem is established by straightforward arguments exploiting the seven lemmas above.

THEOREM 2. *For A a nonnegative real number we have, uniformly in $\frac{1}{2} \leq \sigma$,*

$$|\zeta(\sigma+it)| \ll |t|^{\frac{89}{285}(1-\sigma)+\varepsilon} + 1, \tag{21}$$

and

$$\int_1^T |\zeta(\sigma+it)|^A dt \ll T^{1+G(A,\sigma)+\varepsilon}, \tag{22}$$

where

$$
G(A,\sigma) = \begin{cases}
0 \quad \left(\sigma \geq \frac{1}{2}\right) & (0 \leq A \leq 4)\,; \\[2ex]
\begin{cases} \frac{A}{2}\left(\frac{3A-4}{4A} - \sigma\right) & \left(\frac{1}{2} \leq \sigma \leq \frac{3A-4}{4A}\right) \\ 0 & \left(\sigma \geq \frac{3A-4}{4A}\right) \end{cases} & (4 \leq A \leq 8)\,; \\[2ex]
\begin{cases} \frac{3A(A-4)}{8(A-5)}\left(\frac{5(A-2)}{6A} - \sigma\right) & \left(\frac{1}{2} \leq \sigma \leq \frac{5(A-2)}{6A}\right) \\ 0 & \left(\sigma \geq \frac{5(A-2)}{6A}\right) \end{cases} & (8 \leq A \leq 9)\,; \\[2ex]
\begin{cases} \frac{11601A(A-4)}{4(10'117A-60'117)}\left(\frac{21'718A-60'117}{23'202A} - \sigma\right) & \left(\frac{1}{2} \leq \sigma \leq \frac{21'718A-60'117}{23'202A}\right) \\ 0 & \left(\sigma \geq \frac{21'718A-60'117}{23'202A}\right) \end{cases} & (9 \leq A \leq 9.61872)\,; \\[2ex]
\begin{cases} \frac{43'165A(A-4)}{4(37'305A-220'429)}\left(\frac{80'470A-220'429}{86'330A} - \sigma\right) & \left(\frac{1}{2} \leq \sigma \leq \frac{80'470A-220'429}{86'330A}\right) \\ 0 & \left(\sigma \geq \frac{80'470A-220'429}{86'330A}\right) \end{cases} & (9.61872 \leq A \leq 11)\,; \\[2ex]
\begin{cases} \frac{35A(A-4)}{4(26A-132)}\left(\frac{61A-132}{70A} - \sigma\right) & \left(\frac{1}{2} \leq \sigma \leq \frac{61A-132}{70A}\right) \\ 0 & \left(\sigma \geq \frac{61A-132}{70A}\right) \end{cases} & (11 \leq A \leq 12)\,; \\[2ex]
\begin{cases} \frac{49A(3A-14)}{11(43A-264)}\left(\frac{46A-132}{49A} - \sigma\right) & \left(\frac{1}{2} \leq \sigma \leq \frac{46A-132}{49A}\right) \\ 0 & \left(\sigma \geq \frac{46A-132}{49A}\right) \end{cases} & \left(12 \leq A \leq \frac{178}{13}\right); \\[2ex]
\begin{cases} \frac{49A(1157A-6722)}{3705(43A-264)}\left(\frac{46A-132}{49A} - \sigma\right) & \left(\frac{1}{2} \leq \sigma \leq \frac{46A-132}{49A}\right) \\ 0 & \left(\sigma \geq \frac{46A-132}{49A}\right) \end{cases} & \left(\frac{178}{13} \leq A \leq \frac{528}{37}\right); \\[2ex]
\begin{cases} \frac{163A(1157A-6722)}{247(2092A-12'408)}\left(\frac{4537A-12408}{4890A} - \sigma\right) & \left(\frac{1}{2} \leq \sigma \leq \frac{4537A-12408}{4890A}\right) \\ 0 & \left(\sigma \geq \frac{4537A-12'408}{4890A}\right) \end{cases} & \left(\frac{528}{37} \leq A \leq \frac{564}{21}\right); \\[2ex]
\begin{cases} \frac{174A(1157A-6722)}{1235(509A-4324)}\left(\frac{1031A-4324}{1044A} - \sigma\right) & \left(\frac{1}{2} \leq \sigma \leq \frac{1031A-4324}{1044A}\right) \\ 0 & \left(\sigma \geq \frac{1031A-4324}{1044A}\right) \end{cases} & \left(\frac{564}{21} \leq A \leq \frac{184}{5}\right); \\[2ex]
\begin{cases} \frac{A(1157A-6722)(1472\sqrt{5385}-91'204)}{3705((1349\sqrt{5385}-82'613)A-(9016\sqrt{5385}-522'928))} \\ \quad \times\left(\frac{(2821\sqrt{5385}-173'817)A-(9016\sqrt{5385}-522'928)}{A(2944\sqrt{5385}-182'408)} - \sigma\right) & \left(\frac{1}{2} \leq \sigma \leq \frac{(2821\sqrt{5385}-173'817)A-(9016\sqrt{5385}-522'928)}{A(2944\sqrt{5385}-182'408)}\right) \\ 0 & \left(\sigma \geq \frac{(2821\sqrt{5385}-173'817)A-(9016\sqrt{5385}-522'928)}{A(2944\sqrt{5385}-182'408)}\right) \end{cases} & \left(\frac{184}{5} < A \leq \frac{4606}{373-4\sqrt{5385}}\right); \\[2ex]
\begin{cases} \frac{4A(1157A-6722)}{3705(A-24+\sqrt{9A^2-96A+576})}\left(\frac{5A-24+\sqrt{9A^2-96A+576}}{8A} - \sigma\right) & \left(\frac{1}{2} \leq \sigma \leq \frac{5A-24+\sqrt{9A^2-96A+576}}{8A}\right) \\ 0 & \left(\sigma \geq \frac{5A-24+\sqrt{9A^2-96A+576}}{8A}\right) \end{cases} & \left(\frac{4606}{373-4\sqrt{5385}} \leq A\right).
\end{cases}
$$

4. Applications

Our first application is to $\sum_{n \leq x} \sigma_b^2(n)$, $|b| < 1$, where $\sigma_b(n)$ denotes the divisors function $\sum_{d|n} d^b$, and more generally to $\sum_{n \leq x} P_{b,r}^2(n)$, $|rb| < 1$, $r \in N$, where $P_{b,r}(n)$ (capital "rho") denotes the Gegenbauer rho-function $\sum_{d|n,\, d^{1/r} \in N} d^b$ (hence $\sigma_b = P_{b,1}$). From Crum's generalization of the Ramanujan's formula (which is the case $r = 1$)

$$\sum_{n=1}^{\infty} \frac{P_{-a,r}(n)P_{-b,r}(n)}{n^s} = \frac{\zeta(s)\zeta(rs+ra)\zeta(rs+rb)\zeta(rs+ra+rb)}{\zeta(2rs+ra+rb)}$$

(see for instance [8]), we have

$$\sum_{n=1}^{\infty} \frac{P^2_{-a,r}(n)}{n^s} = \frac{\zeta(s)\zeta^2(rs+ra)\zeta(rs+2ra)}{\zeta(2rs+2ra)}. \tag{23}$$

If $0 < a < 1$ the hypotheses of Theorem 1 are satisfied with $k = 2$, $\alpha = \alpha_1 = 2$, $\alpha_2 = 1$, $\beta = ra$, $\beta_1 = 0$, $\beta_2 = ra$, and $H(s) = (\zeta(2rs+2ra))^{-1}$. Indeed $\sigma_a(H) = \frac{1}{2r} - a$, whence the abscissa σ of (7) is $\frac{1}{2}$. Thus from estimate (21) in Theorem 2 we see that $P(1/2) < 1/2$ and that hypothesis (9) is (amply) satisfied.

Now we have

$$K\left(q_1, q_2; \frac{1}{2}\right) = \frac{1}{q_1} G\left(2q_1, \frac{1}{2}\right) + \frac{1}{q_2} G\left(q_2, \frac{1}{2} + ra\right),$$

and by choosing $q_1 = q_2 = 2$ we see that $K_0 = 0$. Thus $\lambda = (ra - 1/2)/r$,

$$\delta = \begin{cases} \dfrac{1}{2r+1} & \text{if } ra \geq \dfrac{1}{2} \\ \dfrac{1}{2r+2-2ra} & \text{if } ra < \dfrac{1}{2}, \end{cases} \quad \text{and} \quad \mu = \begin{cases} \dfrac{2r}{2r+1}\left(\dfrac{1}{r} - a\right) & \text{if } ra \geq \dfrac{1}{2} \\ \dfrac{2r+1-2ra}{2r+2-2ra}\left(\dfrac{1}{r} - a\right) & \text{if } ra < \dfrac{1}{2}. \end{cases}$$

Hence we established the following.

THEOREM 3. *If $0 < a < 1/r$, we have*

$$\sum_{n \leq x} (P_{-a,r}(n))^2 = Ax + A_{11} x^{\frac{1}{r}-a} \log x + A_{01} x^{\frac{1}{r}-a} + A_{02} x^{\frac{1}{r}-2a} + E_a(x),$$

where the error term $E_a(x) = E_{a,r}(x)$ satisfies

$$E_a(x) \ll \begin{cases} x^{\frac{2r}{2r+1}\left(\frac{1}{r}-a\right)+\varepsilon} & \text{if } \dfrac{1}{2r} \leq a < \dfrac{1}{r}; \\ x^{\frac{2r+1-2ra}{2r+2-2ra}\left(\frac{1}{r}-a\right)+\varepsilon} & \text{if } 0 < a < \dfrac{1}{2r}. \end{cases}$$

Note in comparison that if we apply to this case our earlier (and more general) result [3], the estimate for the error term is only $\ll x^{\frac{1}{r}-a-\varepsilon}$ for some $\varepsilon = \varepsilon(x) \to 0$, so that the term of order $x^{\frac{1}{r}-2a}$ is never significant. Here this term becomes significant as soon as

$$a < \frac{r^2 + 3r/2 - \sqrt{(r^2+3r/2)^2 - 2r^2}}{2r^2},$$

that is for instance when $a < .219\ldots$ for $r = 1$ and when $a < .0746\ldots$ for $r = 2$.

In the special case where $r = 1$, if we gather all the information at our disposition we can state the more precise and complete following result.

THEOREM 4. *If* $0 < |b| \le 1$, *we have*

$$\sum_{n\le x}(\sigma_b(n))^2 = \frac{\zeta^2(1-b)\zeta(1-2b)}{\zeta(2-2b)}x + K_b x^{1+b} + \frac{\zeta(1-b)\zeta(1+b)}{\zeta(2)(1+b)}x^{1+b}\log x - \begin{cases} 0 & \text{if } b \ne -1 \\ \frac{1}{4}\log^2 x & \text{if } b = -1 \end{cases} + \frac{\zeta^2(1+b)\zeta(1+2b)}{\zeta(2+2b)(1+2b)}x^{1+2b} + E_b(x),$$

where the error term $E_b(x)$ *satisfies*

$$E_b(x) \ll x^{b+|b|}(\log x)^{4/3}(\log\log x)^{8/3} \quad \text{if } |b| = 1; \tag{24}$$

$$E_b(x) \ll x^{b+|b|+\frac{2}{3}(1-|b|)+\varepsilon} \quad \text{if } \frac{1}{2} \le |b| < 1; \tag{25}$$

$$E_b(x) \ll x^{b+|b|+\frac{1}{2+2|b|}+\varepsilon} \quad \text{if } 0 < |b| < \frac{1}{2}. \tag{26}$$

This requires some comments. We didn't compute the coefficients of the five terms forming the main part of the sum: for a justification see the second (corrected) part of [4] (where an explicit expression of K_b is also given). Estimate (24) is a result of our previous paper [2]. Estimate (25) for $b < 0$ is provided by our Theorem 2 and improves on the current best estimates, which are due to Ishibashi and Kanemitsu [4] when $1/2 < |b| < \sqrt{2}/2$, and to ourselves [3] when $\sqrt{2}/2 \le |b| < 1$. On the other hand estimate (26) is due to Ishibashi and Kanemitsu [4], and supersedes what is given by Theorem 2.

Proof of the theorem. There remain to treat the case where $1/2 \le b < 1$. This follows easily from the case where $-1 < b < -1/2$ by putting $B = 2b$ and $A(n) = \sigma_{-b}^2(n)$ in the following lemma.

LEMMA 7. *If* $S(B,x) := \sum_{n\le x} n^B A(n)$ *where A is a real arithmetical function and B a nonzero real number, then*

$$S(B,x) = x^B S(0,x) - B\int_1^x t^{B-1}S(0,t)dt\,.$$

Proof. We have

$$\frac{1}{B}\sum_{n\le x}(x^B - n^B)A(n) = \sum_{n\le x}A(n)\int_n^x t^{B-1}dt = \int_1^x t^{B-1}S(0,t)dt\,,$$

whence the lemma.

Since $K_0 = 0$ for every value of $\beta = ra$ in this first application, a very limited use of Theorem 2 was sufficient for our purpose. The other applications we discuss now exploit more fully these estimates.

Ramanujan's formula was generalized to sums of the type $\sum_{n=1}^{\infty}\sigma_{-a_1}(n)\sigma_{-a_2}(n)\cdots\sigma_{-a_k}(n)n^{-s}$ $(k \ge 3)$ by Balakrishnan, and from his general result [1, Theorem 1] it is clear that our Theorem 1 can be applied to the arithmetical functions $a(n) = \sigma_{-a_1}(n)\sigma_{-a_2}(n)\cdots\sigma_{-a_k}(n)$. As an illustration we choose to treat the cases $a(n) = \sigma_b^m(n)$, $1/2 \le |b| \le 1, m = 3, 4, 5, \cdots$, where the use of Lemma E, instead of Theorem 2, is sufficient for our purpose, and then more extensively the particular case $a(n) = \sigma_b^3(n)$, $|b| \le 1$.

It is for the latter, when $|b| < 1/2$, that we appeal to Theorem 2. From Balakrishnan's Theorem 1 of [1] we have

$$\sum_{n=1}^{\infty} \frac{\sigma_{-a}^m(n)}{n^s} = \zeta(s)\zeta^m(s+a)\zeta^{\binom{m}{2}}(s+2a)\zeta^{\binom{m}{3}}(s+3a)\cdots\zeta(s+ma)(s)H_*(s)$$
$$= \zeta(s)\zeta^m(s+a)H(s)\,,$$

where $H(s)$ as a Dirichlet series has an abscissa of absolute convergence $\sigma_a(H) = \max\{1-2a, \frac{1}{2}-a\}$. If $1/2 \le a \le 1$ the hypotheses of Theorem 1 are satisfied with $r = 1, k = 1, \alpha = \alpha_1 = m, \beta = a, \beta_1 = 0$, and $\sigma = \frac{1}{2}$. We have

$$K\left(q_1; \frac{1}{2}\right) = K\left(1; \frac{1}{2}\right) = G\left(m, \frac{1}{2}\right),$$

and with Lemma E it is easy to see that

$$K_0 = \begin{cases} 0 & \text{if } 3 \le m \le 4\,; \\ \dfrac{m-4}{8} & \text{if } 5 \le m \le 12\,; \\ \dfrac{3m-14}{22} & \text{if } m = 13\,; \\ \dfrac{1157m-6722}{7410} & \text{if } m \ge 14\,; \end{cases}$$

whence

$$\mu = \mu_m(a) = (1-a) \times \begin{cases} \dfrac{2}{3} & \text{if } 3 \le m \le 4\,; \\ \left(1 - \dfrac{4}{m+8}\right) & \text{if } 5 \le m \le 12\,; \\ \left(1 - \dfrac{11}{3m+19}\right) & \text{if } m = 13\,; \\ \left(1 - \dfrac{3705}{1157m+4393}\right) & \text{if } m \ge 14\,. \end{cases}$$

Thus hypothesis (9) of Theorem 1 is satisfied and we have obtained

THEOREM 5. *If $1/2 \le |b| \le 1$ and $m \ge 3$ is an integer, we have*

$$\sum_{n \le x} (\sigma_b(n))^m = Ax^{1+\frac{m}{2}(b+|b|)} + x^{1-|b|+\frac{m}{2}(b+|b|)} \sum_{n=0}^{m} A_n \log^n x + \mathcal{E}_{m,b}(x)\,,$$

where A and the A_n are computable constants, and where the error term $\mathcal{E}_{m,b}(x)$ satisfies

$$\mathcal{E}_{m,b}(x) \ll x^{\frac{m}{2}(b+|b|)}(\log x)^{2m/3}(\log\log x)^{4m/3} \quad \text{if } |b| = 1\,; \tag{27}$$

$$\mathcal{E}_b(x) \ll x^{\frac{m}{2}(b+|b|)+\mu+\varepsilon} \quad \text{if } \frac{1}{2} \le |b| < 1\,; \tag{28}$$

for $\mu = \mu_m(|b|)$ as defined just above.

Estimate (27) is a result of our previous paper [2]. And, similarly as in Theorem 4, estimate (28) for $1/2 \le b < 1$ is obtained from the case where $-1 < b \le -1/2$ by using Lemma 7. Note that $A_m \neq 0$ only when $b = -1$.

We pass now to the case $a(n) = \sigma_b^3(n)$, $|b| \le 1$. From Balakrishnan's formula on page 148 of [1] we have

$$\sum_{n=1}^{\infty} \frac{\sigma_{-a}^3(n)}{n^s} = \zeta(s)\zeta^3(s+a)\zeta^3(s+2a)\zeta(s+3a)H(s)\,,$$

where $H(s)$ is given explicitly under the form of an Euler product and, as a Dirichlet series, has an abscissa of absolute convergence $\sigma_a(H) = \frac{1}{2} - a$. If $0 < a \le 1$ the hypotheses of Theorem 1 are satisfied with $r = 1$, $k = 3$, $\alpha = \alpha_1 = 3$, $\alpha_2 = 3$, $\alpha_3 = 1$, $\beta = a$, $\beta_1 = 0$, $\beta_2 = a$, $\beta_3 = 2a$ and $\sigma = \frac{1}{2}$. We have

$$K\left(q_1, q_2, q_3; \frac{1}{2}\right) = \frac{1}{q_1} G\left(3q_1, \frac{1}{2}\right) + \frac{1}{q_2} G\left(3q_2, \frac{1}{2} + a\right) + \frac{1}{q_3} G\left(q_3, \frac{1}{2} + 2a\right),$$

and with (tedious but) straightforward calculations we see that

$$K_0 = 0 \qquad \left(a \ge \frac{445 + \sqrt{852'009}}{6112} = .2238\ldots =: a_0\right);$$

$$K_0 = \frac{3}{8} - \begin{cases}
\dfrac{6112a^2 + 2278a + 91}{8448a + 528} & \left(a_0 \ge a \ge \dfrac{3}{14} = .2\overline{142857} =: a_1\right); \\[2ex]
\dfrac{2032a^2 + 493a + 16}{2112a + 132} & \left(a_1 \ge a \ge \dfrac{83 + \sqrt{5385}}{752} = .2079\ldots =: a_2\right); \\[2ex]
\dfrac{25'733\sqrt{5385} - 1'444'433 + (222'548\sqrt{5385} - 13'164'216)a}{198'352\sqrt{5385} - 11'504'416} & \left(a_2 \ge a > \dfrac{1}{5} = .2 =: a_3\right); \\[2ex]
\dfrac{680'994'487\sqrt{5385} - 38'534'185'087}{3'974'775'728\sqrt{5385} - 230'536'992'224} & \\
\quad + \dfrac{(3'632'939'792\sqrt{5385} - 215'849'148'784)a}{3'974'775'728\sqrt{5385} - 230'536'992'224} & \left(a_3 \ge a > \dfrac{3}{16} = .1875 =: a_4\right); \\[2ex]
\dfrac{357'016'175 + 1'580'128'512a}{1'906'269'992} & \left(a_4 \ge a \ge \dfrac{1}{6} = .1\overline{6} =: a_5\right); \\[2ex]
\dfrac{60'892'941 + 362'903'877a}{372'965'868} & \left(a_5 \ge a \ge \dfrac{4}{27} = .\overline{148} = a_6\right); \\[2ex]
\dfrac{14'356 + 160'572a}{124'080} & \left(a_6 \ge a \ge \dfrac{1}{8} = .125 =: a_7\right); \\[2ex]
\dfrac{23 + 988a}{528} & \left(a_7 \ge a \ge \dfrac{3}{28} = .10\overline{714285} =: a_8\right); \\[2ex]
\dfrac{7 + 536a}{264} & \left(a_8 \ge a \ge \dfrac{1}{10} = .1 =: a_9\right); \\[2ex]
\dfrac{71'209 + 3'335'788a}{1'763'432} & \left(a_9 \ge a \ge \dfrac{1}{12} = .08\overline{3} =: a_{10}\right); \\[2ex]
\dfrac{19'649 + 907'020a}{480'936} & \left(a_{10} \ge a \ge \dfrac{2}{27} = .\overline{074} =: a_{11}\right); \\[2ex]
\dfrac{1 + 84a}{40} & \left(a_{11} \ge a \ge \dfrac{1}{16} = .0625 =: a_{12}\right); \\[2ex]
\dfrac{5a}{2} & (a_{12} \ge a > 0)\,.
\end{cases}$$

The last equality is obtained by choosing $q_3 = 4/(1-8a)$, and any q_1, q_2 both in the interval $[4/3, 4]$ (and satisfying $q_1^{-1}+q_2^{-1} = 1-q_3^{-1}$). All the other equalities are obtained by finding the maximal quantity $q_i^{-1} = 1 - q_2^{-1} - q_3^{-1}$ satisfying $K(q_1, q_2, q_3; \frac{1}{2}) = G(3q_1, \frac{1}{2})/q_1$. Now

$$\lambda = a-1+\begin{cases} \dfrac{1}{2} & (a \geq a_0); \\ \dfrac{4224a+264}{635+9338a-6112a^2} & (a_0 \geq a \geq a_1); \\ \dfrac{2112a+132}{331+4822a-4064a^2} & (a_1 \geq a \geq a_2); \\ \dfrac{99'176\sqrt{5385}-5'752'208}{247'001\sqrt{5385}-14'374'139-(222'548\sqrt{5385}-13'164'216)a} & (a_2 \geq a \geq a_3); \\ \dfrac{1'987'387'864\sqrt{5385}-115'268'496'112}{4'784'322'139\sqrt{5385}-278'454'179'221-(3'632'939'792\sqrt{5385}-215'849'148'784)a} & \\ & (a_3 \geq a \geq a_4); \\ \dfrac{953'134'996}{2'264'105'064-1'580'128'512a} & (a_4 \geq a \geq a_5); \\ \dfrac{372'965'868}{903'870'255-725'807'754a} & (a_5 \geq a \geq a_6); \\ \dfrac{31'020}{78'127-80'286a} & (a_6 \geq a \geq a_7); \\ \dfrac{264}{703-988a} & (a_7 \geq a \geq a_8); \\ \dfrac{33}{89-134a} & (a_8 \geq a \geq a_9); \\ \dfrac{440'858}{1'176'755-1'667'894a} & (a_9 \geq a \geq a_{10}); \\ \dfrac{120'234}{320'819-453'510a} & (a_{10} \geq a \geq a_{11}); \\ \dfrac{10}{27-42a} & (a_{11} \geq a \geq a_{12}); \\ \dfrac{4}{11-20a} & (a_{12} \geq a); \end{cases}$$

and with estimate (21) in Theorem 2 we can ensure that hypothesis (9) is satisfied. Finally we verify that $\lambda < 0$ when $a < 1/2$ and compute the parameter δ, whence

$$\frac{\mu}{1-a}=\begin{cases} \frac{2}{3} & \left(a \geq \frac{1}{2}\right); \\ \left(1-\frac{1}{(2-a)2}\right) & \left(\frac{1}{2} \geq a \geq a_0\right); \\ \left(1-\frac{4224a+264}{(2-a)(635+9338a-6112a^2)}\right) & (a_0 \geq a \geq a_1); \\ \left(1-\frac{2112a+132}{(2-a)(331+4822a-4064a^2)}\right) & (a_1 \geq a \geq a_2); \\ \left(1-\frac{99'176\sqrt{5385}-5'752'208}{(2-a)(247'001\sqrt{5385}-14'374'139-(222'548\sqrt{5385}-13'164'216)a)}\right) & (a_2 \geq a \geq a_3); \\ \left(1-\frac{1'987'387'864\sqrt{5385}-115'268'496'112}{(2-a)(4'784'322'139\sqrt{5385}-278'454'179'221-(3'632'939'792\sqrt{5385}-215'849'148'784)a)}\right) & \\ & (a_3 \geq a \geq a_4); \\ \left(1-\frac{953'134'996}{(2-a)(2'264'105'064-1'580'128'512a)}\right) & (a_4 \geq a \geq a_5); \\ \left(1-\frac{372'965'868}{(2-a)(903'870'255-725'807'754a)}\right) & (a_5 \geq a \geq a_6); \\ \left(1-\frac{31'020}{(2-a)(78'127-80'286a)}\right) & (a_6 \geq a \geq a_7); \\ \left(1-\frac{264}{(2-a)(703-988a)}\right) & (a_7 \geq a \geq a_8); \\ \left(1-\frac{33}{(2-a)(89-134a)}\right) & (a_8 \geq a \geq a_9); \\ \left(1-\frac{440'858}{(2-a)(1'176'755-1'667'894a)}\right) & (a_9 \geq a \geq a_{10}); \\ \left(1-\frac{120'234}{(2-a)(320'819-453'510a)}\right) & (a_{10} \geq a \geq a_{11}); \\ \left(1-\frac{10}{(2-a)(27-42a)}\right) & (a_{11} \geq a \geq a_{12}); \\ \left(1-\frac{4}{(2-a)(11-20a)}\right) & (a_{12} \geq a). \end{cases}$$

Hence we proved the following, in the case where $-1 < b < 0$.

THEOREM 6. *If* $0 < |b| \leq 1$, *we have*

$$\sum_{n\leq x}(\sigma_b(n))^3 = Ax + x^{1+b}\sum_{n=0}^{3} A_{n1}\log^n x + x^{1+2b}\sum_{n=0}^{2} A_{n2}\log^n x + A_{03}x^{1+3b} + \mathcal{E}_b(x),$$

where A and the A_{nk} *are computable real constants and where the error term* $\mathcal{E}_b(x)$ *satisfies*

$$\mathcal{E}_b(x) \ll x^{\frac{3}{2}(b+|b|)}(\log x)^2(\log\log x)^4 \quad \textit{if } |b| = 1\,; \tag{29}$$

$$\mathcal{E}_b(x) \ll x^{\frac{3}{2}(b+|b|)+\mu+\varepsilon} \quad \textit{if } 0 < |b| < 1\,; \tag{30}$$

for $\mu = \mu(|b|)$ *as defined just above.*

Estimate (29) is again a result of our previous paper [2]. And again, estimate (30) for $0 < b < 1$ is obtained from the case where $-1 < b < 0$ by using Lemma 7.

Note that by Theorem 1 we have $A = \zeta^3(1-b)\zeta^3(1-2b)\zeta(1-3b)H(1)$ for $b < 0$. And that for $b < 0$, the terms of order $x^{1+2b}\log^n x$ are significant when $0 > b > -.2184$, and the term of order x^{1+3b} when $0 > b > -.1038$. Finally note that the term $A_{31}x^{1+b}\log^3 x$ is non zero and significant in the single case $b = -1$, which is not provided for by Theorem 1 but by [2] (see (29)).

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