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Converse Theorem for Not Necessarily Cuspidal Siegel Modular Forms of Degree 2 and Saito-Kurokawa Lifting

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§0. Introduction

In [Im], Imai studied a converse theorem for Siegel cusp forms of degree 2, which characterizes Siegel cusp forms of degree 2 with the functional equations satisfied by the corresponding Koecher-Maass series with Grössencharacters. Her work was an attempt to generalize the celebrated converse theorem of Hecke for elliptic modular forms to the case of Siegel modular forms. Since then there have been almost no applications of Imai's converse theorem. Only recently Duke-Imamoḡlu [DI] gave a new proof of a substantial part of the Saito-Kurokawa lifting for Siegel modular forms of degree 2 by using a slight modification of Imai's converse theorem as well as the Shimura correspondence for Maass wave forms due to Katok-Sarnak [KS]. However their work is restricted to Siegel cusp forms, since Imai's converse theorem can be applied to only cusp forms.

The aim of the present paper is to obtain a convenient converse theorem applicable to not necessarily cuspidal Siegel modular forms of deree 2 and moreover to apply it to a proof of the Saito-Kurokawa lifting for degree 2 including the case of Eisenstein series along the lines of [DI].

We explain our purpose and results in a little more precise manner. Imai's original formulation of the converse theorem is not suitable to actual applications, since the assumptions of her theorem contain other condition that seems hard to verify than analytic properties of Koecher-Maass series. Therefore Duke-Imamoḡlu [DI] reformulated the converse theorem in a form which serves an actual use (see Ibukiyama [Ib] for a detailed discussion on this point). On the other hand, shortly after the appearence of Imai's paper [Im], Weissauer [We] has already given this kind of improvement of the converse theorem as well as its extention to Siegel cusp forms of arbitrary degree. The method of Imai ([DI], and [Ib]) is based on the full spectral decomposition of $L^2(SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/SO(2))$, while the method of Weissauer is based on an easier part of the theory of Eisenstein series.

In this paper, following the method of Weissauer, we generalize the converse theorem to the one (Theorem 14) which is available to not necessarily cuspidal Siegel modular forms of degree 2. More precisely let P_2 denote the set of half-integral positive semi-definite symmetric matrices of size 2. For a mapping $A : P_2 \to \mathbb{C}$ satisfying a certain growth condition ((A-0) in §4.1), we form a function $f_A(Z)$ on the Siegel upper half space of degree 2 by

$$f_A(Z) = \sum_{T \in P_2} A(T) \exp(2\pi i \sigma(TZ)).$$

We can also associate with the function f_A Koecher-Maass type zeta functions with Grössencharacters. Then, under the assumptions on some analytic properties of the Koecher-Maass type zeta functions ((A-1)–(A-4) in §4.1), our converse theorem asserts that $f_A(Z)$ is a Siegel modular form of degree 2 and weight k with respect to the full Siegel modular group. The assumptions of Imai's theorem involve the analytic continuation of the real analytic Eisenstein series $E(z, u)$ for the group $SL_2(\mathbb{Z})$ to the critical line $\mathrm{Re}\, u = 1/2$, while ours concern only with the Eisenstein series $E(z, u)$ in the domain of absolute convergence, which are easier to handle. This is one of the merits of Weissauer's method.

The final part of this paper is devoted to an application of our converse theorem, where we construct a mapping $A : P_2 \to \mathbb{C}$ starting from a modular form of half-integral weight $k - 1/2$ belonging to the Kohnen space (see (5.21)). Then we can prove that this mapping A satisfies the condition from (A-0) to (A-4), and consequently that f_A becomes a Siegel modular form of degree 2 and weight k. This gives another proof of the Saito-Kurokawa lifting whcih is a generalization of the result in [DI] to not necessarily cuspidal Siegel modular forms of degree 2.

§1. Preliminaries

1.1. Grössencharacters (Maass wave forms) on $SL_2(\mathbb{Z})\backslash\mathfrak{H}$

We put

$$\tilde{\Gamma} = GL_2(\mathbb{Z}), \quad \Gamma = SL_2(\mathbb{Z}).$$

Denote by B the group of upper triangular matrices in GL_2 and put

$$\tilde{\Gamma}_\infty = \tilde{\Gamma} \cap B, \quad \Gamma_\infty = \Gamma \cap B.$$

A real matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$ with positive determinant acts on the upper half plane $\mathfrak{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$ by

$$g \cdot z = \frac{az + b}{cz + d} \quad (z \in \mathfrak{H}).$$

For $u \in \mathbb{C}$, we define the Eisenstein series $E(z, u)$ by

$$E(z, u) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\mathrm{Im}\, \gamma \cdot z)^u.$$

We recall the well-known basic properties of $E(z, u)$.

LEMMA 1. *The series $E(z, u)$, initially defined for $\mathrm{Re}\, u > 1$, has an analytic continuation to a meromorphic function of u and satisfies the functional equation*

$$E(z, u) = C(u) E(z, 1 - u), \quad C(u) = \sqrt{\pi} \cdot \frac{\zeta(2u - 1)\Gamma(u - 1/2)}{\zeta(2u)\Gamma(u)}. \tag{1.1}$$

Moreover $E(z,u)$ *is holomorphic in* $\{u \in \mathbb{C} \mid \frac{1}{2} \leq \mathrm{Re}\, u, u \neq 1\}$ *and*

$$|E(z,u)| \leq C|\mathrm{Im}\, u|^a \quad (|\mathrm{Im}\, u| \to \infty) \quad \textit{uniformly on } \frac{1}{2} \leq \mathrm{Re}\, u$$

for some positive constants a *and* C. $E(z,u)$ *has a simple pole at* $u = 1$ *with residue* $3/\pi$.

For later use we recall the formula for the constant term of the Fourier series expansion of $E(z,u)$:

$$\int_0^1 E(x+iy,u)dx = y^u + C(u)y^{1-u}. \tag{1.2}$$

We denote by P_2 (resp. P_2^+) the set of half-integral positive semi-definite (resp. definite) symmetric matrices of size 2:

$$P_2 = \left\{ T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in M_2(\mathbb{Q}) \,\middle|\, T \geq 0,\ a,b,c \in \mathbb{Z} \right\},$$

$$P_2^+ = \left\{ T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in M_2(\mathbb{Q}) \,\middle|\, T > 0,\ a,b,c \in \mathbb{Z} \right\}.$$

We also put

$$\mathcal{P}_2 = \{Y \in Sym_2(\mathbb{R}) \mid Y > 0\}, \quad \mathcal{SP}_2 = \{W \in \mathcal{P}_2 \mid \det W = 1\},$$

$Sym_2(\mathbb{R})$ denoting the set of real symmetric matrices of size two. The space $\mathcal{SP}_2$ has a parametrization

$$\mathcal{SP}_2 \ni W = \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix}\begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} = \begin{pmatrix} y^{-1} & -xy^{-1} \\ -xy^{-1} & y^{-1}(x^2+y^2) \end{pmatrix}, \tag{1.3}$$

where we have employed Siegel's notation $T[V] = {}^tVTV$, and, via the mapping $W \mapsto z = x+iy$, one can identify $\mathcal{SP}_2$ with $\mathfrak{H}$. The correspondence can also be written as

$$\mathcal{SP}_2 \ni W = I_2[g^{-1}] \leftrightarrow z = g \cdot \sqrt{-1} \in \mathfrak{H} \quad (g \in SL_2(\mathbb{R})).$$

In the following, the variables z and W are always assumed to be related to each other by this correspondence.

From the identity $\mathrm{Im}\, z = 1/W_{11}$ (W_{11} is the $(1,1)$-entry of the matrix W), it follows that

$$E(z,u) = \sum_{U \in \Gamma_\infty \backslash \Gamma} (W[U^{-1}]_{11})^{-u} = \frac{1}{\zeta(2u)} \sum_{(m,n) \in \mathbb{Z}^2 \backslash \{(0,0)\}} \left(W\begin{bmatrix} m \\ n \end{bmatrix} \right)^{-u} \tag{1.4}$$

$$(\mathrm{Re}\, u > 1).$$

It is known that the ring of $SL_2(\mathbb{R})$-invariant differential operators on $\mathfrak{H}$ is generated by the Laplacian

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

For Γ-invariant functions f, g on $\mathfrak{H}$, we define the inner product $\langle f, g\rangle$ by setting

$$\langle f, g\rangle = \int_{\Gamma\backslash\mathfrak{H}} f(z)\overline{g(z)}d\mu(z)$$

where we put $d\mu(z) = \dfrac{dxdy}{y^2}$, a usual invariant measure on $\mathfrak{H}$. Here $\Gamma\backslash\mathfrak{H}$ denotes the fundamental domain of $\mathfrak{H}$ with respect to Γ and we can take

(1.5) $$\mathcal{F} = \left\{ z \in \mathfrak{H} \,\middle|\, |z| \geq 1,\ -\frac{1}{2} \leq \operatorname{Re} z \leq \frac{1}{2} \right\}$$

as a fundamental domain.

Consider the L^2-space

$$L^2(\Gamma\backslash\mathfrak{H}) = \{f : \mathfrak{H} \to \mathbb{C} \,|\, f(\gamma\cdot z) = f(z)\ (\forall\gamma \in \Gamma),\ \langle f, f\rangle < \infty\}.$$

We say that $f \in L^2(\Gamma\backslash\mathfrak{H})$ is **cuspidal** if it satisfies

$$\int_0^1 f(x+iy)dx = 0 \quad \text{(almost everywhere)},$$

and denote by $L_0^2(\Gamma\backslash\mathfrak{H})$ the subspace of cuspidal functions. The space $L_0^2(\Gamma\backslash\mathfrak{H})$ is a closed subspace of the Hilbert space $L^2(\Gamma\backslash\mathfrak{H})$.

For $\psi \in C_0^\infty(\mathbb{R}_+^\times)$, the pseudo-Eisenstein series θ_ψ is defined by

(1.6) $$\theta_\psi(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \psi(\operatorname{Im}(\gamma\cdot z)).$$

Then θ_ψ is in $L^2(\Gamma\backslash\mathfrak{H})$. Denote by Θ the closed subspace of $L^2(\Gamma\backslash\mathfrak{H})$ spanned by $\{\theta_\psi \,|\, \psi \in C_0^\infty(\mathbb{R}_+^\times)\}$. Then it is known that Θ is the orthogonal complement of $L_0^2(\Gamma\backslash\mathfrak{H})$ ([Ku], Theorem 5.1.1):

(1.7) $$L^2(\Gamma\backslash\mathfrak{H}) = \Theta \oplus L_0^2(\Gamma\backslash\mathfrak{H}).$$

Following Maass, we mean by a **Grössencharacter** on $\mathfrak{H}$ any function $\mathcal{U}$ on $\mathfrak{H}$ satisfying the three conditions

(i) $\mathcal{U}(\gamma\cdot z) = \mathcal{U}(z)$ $(\forall\gamma \in SL_2(\mathbb{Z}))$

(ii) $\mathcal{U}$ is a C^∞-function on $\mathfrak{H}$ with respect to x, y which verifies a differential equation $\Delta\mathcal{U} = -\lambda\mathcal{U}$ with some $\lambda \in \mathbb{C}$.

(iii) $\mathcal{U}$ has a **moderate** growth condition; namely there exists a certain $\alpha > 0$ with $\mathcal{U}(x+iy) = O(y^\alpha)$ $(y \to \infty)$.

A Grössencharacter on $\mathfrak{H}$ is also called a **Maass wave form**. It is known that any cuspidal function satisfying the conditions (i), (ii) becomes necessarily bounded and is a Maass wave form. We put

(1.8) $$\nu_0(z) = \sqrt{\frac{3}{\pi}},$$

which is a constant function in $L_0^2(\Gamma\backslash\mathfrak{H})$ normalized so that $\langle\nu_0, \nu_0\rangle = 1$. Obviously, ν_0 is a Maass wave form.

We extend a Grössencharacter $\mathcal{U}(z)$ to a function on $\mathcal{P}_2$ by setting

$$\mathcal{U}(T) = \mathcal{U}(z) \quad (z \leftrightarrow W = (\det T)^{-1/2}T,\ T \in \mathcal{P}_2)\,. \tag{1.9}$$

1.2. Siegel modular forms of degree 2

Let $\mathfrak{H}_2$ be the Siegel upper half space of degree 2:

$$\mathfrak{H}_2 = \{Z = X + iY \in Sym_2(\mathbb{C}) \mid Y > 0\}\,.$$

The real symplectic group $Sp_2(\mathbb{R})$ acts on $\mathfrak{H}_2$ via

$$M\langle Z\rangle = (AZ + B)(CZ + D)^{-1}\,, \quad Z \in \mathfrak{H}_2\,, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_2(\mathbb{R})\,.$$

We put

$$j(M, Z) = \det(CZ + D)\,, \quad Z \in \mathfrak{H}_2\,, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_2(\mathbb{R})\,.$$

A function $f : \mathfrak{H}_2 \to \mathbb{C}$ is called a **Siegel modular form of degree 2 and weight** k, if it satisfies the conditions

i. f is holomorphic on $\mathfrak{H}_2$.
ii. $f(M\langle Z\rangle) = j(M, Z)^k f(Z)$ $(\forall M \in \Gamma_2 := Sp_2(\mathbb{Z}))$.

We denote by $\mathfrak{M}_2^k(\Gamma_2)$ the space of Siegel modular forms of degree 2 and weight k.

A Siegel modular form $f \in \mathfrak{M}_2^k(\Gamma_2)$ has a Fourier series expansion of the form

$$f(Z) = \sum_{T \in P_2} A(T) \exp(2\pi i \sigma(TZ))\,,$$

where $\sigma(Y)$ denotes the trace of a matrix Y.

We define the **Siegel operator** Φ, which maps a Siegel modular form $f \in \mathfrak{M}_2^k(\Gamma_2)$ to an elliptic modular form Φf of the same weight k, by

$$\Phi f(z) = \lim_{\lambda \to +\infty} f\begin{pmatrix} z & 0 \\ 0 & i\lambda \end{pmatrix} \quad (z \in \mathfrak{H})\,.$$

The modular form Φf satisfies

$$\Phi f(\gamma \cdot z) = (cz + d)^k f(z) \quad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\right).$$

In terms of the Fourier expansion, Φf is given by

$$\Phi f(z) = \sum_{n=0}^{\infty} A(T_n) e^{2n\pi i z}\,, \quad T_n = \begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}.$$

For further properties of Siegel modular forms, we refer to [Kl], [Ma2].

§2. The Koecher-Maass type Dirichlet series

2.1. Convergence

Let us consider a function $A : P_2 \to \mathbb{C}$ satisfying the following conditions:

(A-0) (i) $A(T[U]) = A(T)$ ($\forall T \in P_2^+$, $U \in \tilde{\Gamma}$),

(ii) There exists a positive α such that $A(T) = O((\det T)^\alpha)$ (equivalently $= O((ac)^\alpha)$) for $T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in P_2^+$,

(iii) There exists a positive β such that $A(T_n) = O(n^\beta)$ for T_n, $n \geq 0$.

For such an A and a Grössencharacter $\mathcal{U}$, we define a Dirichlet series

$$D_2(A, \mathcal{U}, s) = \sum_{\{T\}} \frac{A(T)}{\varepsilon(T)} \mathcal{U}(T)(\det T)^{-s},$$

where $\{T\}$ is a complete set of representatives of $\tilde{\Gamma}\backslash P_2^+$, and $\varepsilon(T) = \sharp\{U \in \tilde{\Gamma} \mid T[U] = T\}$. We introduce another Dirichlet series obtained from the values of A for matrices T of rank 1:

$$D_1(A, u) = \sum_{n=1}^{\infty} a(n) n^{-u}, \quad a(n) := A(T_n). \tag{2.1}$$

LEMMA 2. *Let α (resp. β) be a positive constant satisfying the condition* (A-0), (ii) (*resp.* (A-0), (iii)).

(i) *When* $\mathrm{Re}\, u > \beta + 1$, *the Dirichlet series $D_1(A, u)$ is absolutely convergent.*

(ii) *When* $\mathrm{Re}\, s > \frac{3}{2} + \alpha$ *and $\mathcal{U}$ is bounded, the Dirichlet series $D_2(A, \mathcal{U}, s)$ are absolutely convergent.*

(iii) *When* $\mathrm{Re}\, s > \max\left\{\frac{\mathrm{Re}\, u}{2} + \alpha + 1, \frac{1-\mathrm{Re}\, u}{2} + \alpha + 1, \alpha + \frac{3}{2}\right\}$ *and $C(u) \neq \infty$, the Dirichlet series $D_2(A, E(\cdot, u), s)$ is absolutely convergent. Moreover the function*

$$u(1-u)\pi^{-u}\Gamma(u)\zeta(2u)D_2(A, E(\cdot, u), s)$$

represents a holomorphic function of (s, u) in the domain $\mathrm{Re}\, s > \max\left\{\frac{\mathrm{Re}\, u}{2} + \alpha + 1, \frac{1-\mathrm{Re}\, u}{2} + \alpha + 1, \alpha + \frac{3}{2}\right\}$ *and is invariant under $(s, u) \mapsto (s, 1-u)$.*

For the proof of the lemma, we need the following.

LEMMA 3 (Shintani [Shn]). (i) *The Dirichlet series*

$$\sum_{\{T\}} \frac{1}{\varepsilon(T)} (\det T)^{-s}$$

is absolutely convergent for $\mathrm{Re}\, s > \frac{3}{2}$.

(ii) *The Dirichlet series*

$$\sum_{T \in P_2^+/\Gamma_\infty} (T_{11})^{-u} (\det T)^{-s}$$

is absolutely convergent for $\mathrm{Re}\, s > 1$ *and* $\mathrm{Re}\, u > 1$.

Proof of Lemma 2. (i) The first assertion is obvious from the condition (A-0), (iii).

(ii) The second assertion is an immediate consequence of (A-0), (ii) and Lemma 3(i).

(iii) By (1.4) and the assumption (A-0), (ii), we have

$$
\begin{aligned}
(2.2)\qquad &\sum_{\{T\}} \left| \frac{A(T)}{\varepsilon(T)} E(z,u)(\det T)^{-s} \right| \\
&\leq C \sum_{\{T\}} \frac{1}{\varepsilon(T)} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (T[\gamma^{-1}]_{11})^{-\mathrm{Re}\, u} (\det T)^{-(\mathrm{Re}\, s - \alpha)} \\
&= \frac{C}{2} \sum_{T \in P_2^+/\Gamma_\infty} (T_{11})^{-\mathrm{Re}\, u} (\det T)^{-(\mathrm{Re}\, s - \frac{\mathrm{Re}\, u}{2} - \alpha)}
\end{aligned}
$$

for some positive constant C. Here for each T, $z \in \mathfrak{H}$ corresponds to $\frac{1}{\sqrt{\det T}} T$. Hence, by Lemma 3(ii), the Dirichlet series $D_2(A, E(\cdot, u), s)$ is absolutely convergent for $\mathrm{Re}\, s > \frac{\mathrm{Re}\, u}{2} + \alpha + 1$ and $\mathrm{Re}\, u > 1$. To prove the stronger statement given in the lemma, we use the following integral representation of $E(z,u)$. For a $T \in \mathcal{P}_2$, we put

$$
\begin{aligned}
Z(T,u) &= \int_0^\infty t^{u-1} \sum_{x \in \mathbb{Z}^2 \backslash \{(0,0)\}} e^{-\pi t (x T\, {}^t x)} dt\,, \\
Z_+(T,u) &= \int_1^\infty t^{u-1} \sum_{x \in \mathbb{Z}^2 \backslash \{(0,0)\}} e^{-\pi t (x T\, {}^t x)} dt\,.
\end{aligned}
$$

Then, $Z(T,u)$ is absolutely convergent for $\mathrm{Re}\, u > 1$ and we have

$$
E(z,u) = \frac{1}{2\pi^{-u}\Gamma(u)\zeta(2u)} \cdot (\det T)^{u/2} Z(T,u) \quad (z \leftrightarrow W = (\det T)^{-1/2} T \in \mathcal{SP}_2)\,.
$$

On the other hand, $Z_+(T,u)$ is absolutely convergent for any $u \in \mathbb{C}$ and represents an entire function of u. Moreover we have

$$
\begin{aligned}
(2.3)\quad E(z,u) &= \frac{1}{2\pi^{-u}\Gamma(u)\zeta(2u)} \\
&\times \left\{ (\det T)^{u/2} Z_+(T,u) + (\det T)^{(1-u)/2} Z_+(T, 1-u) - \frac{1}{u} - \frac{1}{1-u} \right\}.
\end{aligned}
$$

The right hand side gives a meromorphic continuation of $E(z,u)$ on the whole complex plane $\mathbb{C}$.

We see easily from this integral representation that

$$
E(z,0) \equiv 1\,, \qquad E(z,1/2) \equiv 0\,.
$$

Hence, if $u = 0$ or $1/2$, then the convergence of the Dirichlet series is obvius. In the following, we assume always that $u \neq 0, 1/2$. Then the identity (2.3) implies the inequality

$$|E(z,u)| \leq \frac{1}{2\pi^{-\mathrm{Re}\,u}|\Gamma(u)\zeta(2u)|} \times \left\{(\det T)^{\mathrm{Re}\,u/2} Z_+(T, \mathrm{Re}\,u) + (\det T)^{(1-\mathrm{Re}\,u)/2} Z_+(T, 1-\mathrm{Re}\,u) + \frac{1}{|u|} + \frac{1}{|1-u|}\right\}.$$

Therefore, if $u \neq 1$ and $\Gamma(u)\zeta(2u) \neq 0$, then for some positive constant C we have

$$\sum_{\{T\}} \left|\frac{A(T)}{\varepsilon(T)} E(z,u)(\det T)^{-s}\right| \leq \frac{C}{\pi^{-\mathrm{Re}\,u}|\Gamma(u)\zeta(2u)|} \left\{\left(\frac{1}{|u|} + \frac{1}{|1-u|}\right) \sum_{\{T\}} \frac{1}{\varepsilon(T)} (\det T)^{-(\mathrm{Re}\,s-\alpha)} + \sum_{\{T\}} \frac{1}{\varepsilon(T)} Z_+(T, \mathrm{Re}\,u)(\det T)^{-(\mathrm{Re}\,s - \frac{\mathrm{Re}\,u}{2} - \alpha)} + \sum_{\{T\}} \frac{1}{\varepsilon(T)} Z_+(T, 1-\mathrm{Re}\,u)(\det T)^{-(\mathrm{Re}\,s - \frac{1-\mathrm{Re}\,u}{2} - \alpha)}\right\}.$$

By Lemma 3(i), the series $\sum_{\{T\}} \frac{1}{\varepsilon(T)} (\det T)^{-(\mathrm{Re}\,s-\alpha)}$ is convergent for $\mathrm{Re}\,s > \alpha + \frac{3}{2}$. It is easy to see that the integrand of $Z_+(T, \mathrm{Re}\,u)$ is an increasing function of $\mathrm{Re}\,u$ on $\mathbb{R}$. Hence, for any $M > 0$, we have

$$\begin{aligned}
&\sum_{\{T\}} \frac{1}{\varepsilon(T)} Z_+(T, \mathrm{Re}\,u)(\det T)^{-(\mathrm{Re}\,s - \frac{\mathrm{Re}\,u}{2} - \alpha)} \\
&\leq \sum_{\{T\}} \frac{1}{\varepsilon(T)} Z_+(T, \mathrm{Re}\,u + M)(\det T)^{-(\mathrm{Re}\,s - \frac{\mathrm{Re}\,u}{2} - \alpha)} \\
&\leq \sum_{\{T\}} \frac{1}{\varepsilon(T)} Z(T, \mathrm{Re}\,u + M)(\det T)^{-(\mathrm{Re}\,s - \frac{\mathrm{Re}\,u}{2} - \alpha)} \\
&\leq \pi^{-\mathrm{Re}\,u} \Gamma(\mathrm{Re}\,u + M)\zeta(2(\mathrm{Re}\,u + M)) \\
&\quad \times \sum_{\{T\}} \frac{1}{\varepsilon(T)} E(z, \mathrm{Re}\,u + M)(\det T)^{-(\mathrm{Re}\,s + \frac{M}{2} - \alpha)}.
\end{aligned}$$

As we have already seen, the last expression is convergent if $\mathrm{Re}\,u + M > 1$ and $\mathrm{Re}\,s + \frac{M}{2} - \alpha > \frac{\mathrm{Re}\,u + M}{2} + 1$. Since M can be taken arbitrarily large, this implies that

$$\sum_{\{T\}} \frac{1}{\varepsilon(T)} Z_+(T, \mathrm{Re}\,u)(\det T)^{-(\mathrm{Re}\,s - \frac{\mathrm{Re}\,u}{2} - \alpha)}$$

converges for $\operatorname{Re} s > \frac{\operatorname{Re} u}{2} + \alpha + 1$. Similarly

$$\sum_{\{T\}} \frac{1}{\varepsilon(T)} Z_+(T, 1 - \operatorname{Re} u)(\det T)^{-(\operatorname{Re} s - \frac{1-\operatorname{Re} u}{2} - \alpha)}$$

converges for $\operatorname{Re} s > \frac{1-\operatorname{Re} u}{2} + \alpha + 1$. Thus we see that, if $\Gamma(u)\zeta(2u) \neq 0$, $u \neq 1$ (equivalently, $C(u) \neq \infty$) and $\operatorname{Re} s > \max\{\alpha + \frac{3}{2}, \frac{\operatorname{Re} u}{2} + \alpha + 1, \frac{1-\operatorname{Re} u}{2} + \alpha + 1\}$, then the right hand side of (2.2) is convergent. The holomorphy of the function $u(1-u)\pi^{-u}\Gamma(u)\zeta(2u)D_2(A, E(\cdot, u), s)$ follows immediately from the expression

$$\begin{aligned} (2.4)\qquad & u(1-u)\pi^{-u}\Gamma(u)\zeta(2u)D_2(A, E(\cdot, u), s) \\ &= \sum_{\{T\}} \frac{A(T)}{\varepsilon(T)} (\det T)^{-s} \{u(1-u)\big((\det T)^{\frac{u}{2}} Z_+(T, u) \\ &\quad + (\det T)^{\frac{1-u}{2}} Z_+(T, 1-u)\big) - 1\}, \end{aligned}$$

since the convergence is uniform on every compact subset of the domain $\operatorname{Re} s > \max\{\alpha + \frac{3}{2}, \frac{\operatorname{Re} u}{2} + \alpha + 1, \frac{1-\operatorname{Re} u}{2} + \alpha + 1\}$. The invariance of the function under $(s, u) \mapsto (s, 1-u)$ is obvious from the right hand side of the identity above. ∎

LEMMA 4. *Let σ be a sufficiently large positive number. Then, for any ρ with $\frac{1}{2} < \rho < 2(\sigma - \alpha - 1)$, there exist positive constants C and δ (independent of $\operatorname{Im} s$) such that*

$$|D_2(A, E(z, u), s)| < C|\operatorname{Im} u|^{\delta} \quad (|\operatorname{Im} u| \to \infty)$$

uniformly on $\{(s, u) \mid \operatorname{Re} s = \sigma, \frac{1}{2} \leq \operatorname{Re} u \leq \rho\}$.

Proof. It is sufficient to prove the lemma under the assumption that $\rho > 1$. Fix an $s \in \mathbb{C}$ with $\operatorname{Re} s = \sigma$. Then, there exists a positive constant M satisfying

$$(2.5)\qquad |u(u-1)\zeta(2u)D_2(A, E(z, u), s)| < M(\operatorname{Im} u)^2, \quad (|\operatorname{Im} u| \to \infty) \quad \text{on } \operatorname{Re} u = \rho.$$

In fact, as M we may take any positive number greater than $\zeta(2\rho)D_2(|A|, E(z, \rho), \sigma)$. Note that M does not depend on $\operatorname{Im} s$. By the functional equation given in Lemma 2(iii), we have

$$\begin{aligned} & u(1-u)\zeta(2u)D_2(A, E(z, u), s) \\ &= u(1-u)\zeta(2(1-u))D_2(A, E(z, 1-u), s) \times \pi^{1-2u} \frac{\Gamma(1-u)}{\Gamma(u)}. \end{aligned}$$

Hence, by Stirling's estimate of the gamma function and (2.5), we have

$$\begin{aligned} (2.6)\qquad & |u(1-u)\zeta(2u)D_2(A, E(z, u), s)| \\ &< M\pi^{2\rho-1}(\operatorname{Im} u)^2 \left|\frac{\Gamma(1-u)}{\Gamma(u)}\right| \\ &\sim M\pi^{2\rho-1}(\operatorname{Im} u)^2 \cdot (\operatorname{Im} u)^{2\rho-1} \\ &= M\pi^{2\rho-1}(\operatorname{Im} u)^{2\rho+1} \quad (|\operatorname{Im} u| \to \infty) \quad \text{on } \operatorname{Re} u = 1-\rho. \end{aligned}$$

The identity (2.4) implies that the function $u(1-u)\zeta(2u)D_2(A, E(z,u), s)$ is a holomorphic function of u in $1-\rho \leq \mathrm{Re}\, u \leq \rho$ and is bounded from above by

$$\frac{1}{|\Gamma(u)|}\sum_{\{T\}}\frac{|A(T)|}{\varepsilon(T)}(\det T)^{-\sigma}\{2\rho(\rho+1)(\det T)^{\rho/2}Z_+(T,\rho)+1\} = O(e^{c|\mathrm{Im}\, u|}).$$

Hence, by the Phragmén-Lindelöf theorem, we have

$$|u(1-u)\zeta(2u)D_2(A, E(z,u), s)| < C(\mathrm{Im}\, u)^{2\rho+1} \tag{2.7}$$
$$(|\mathrm{Im}\, u| \to \infty) \quad \text{uniformly on } 1-\rho \leq \mathrm{Re}\, u \leq \rho .$$

Since the constants contained in the right hand sides of (2.5) and (2.6) are independent of $\mathrm{Im}\, s$, the inequality (2.7) holds uniformly for any s with $\mathrm{Re}\, s = \sigma$. Since $\frac{1}{|\zeta(2u)|} = O(\log^7 |\mathrm{Im}\, u|)$ in $\frac{1}{2} \leq \mathrm{Re}\, u \leq \rho$ (cf. [Ti, Chapter III, 3.6]), this implies the lemma. ∎

2.2. Integral representations

Let $A : P_2 \to \mathbb{C}$ be a function satisfying the condition (A-0) in §2.1. In order to give an integral representation of $D_2(A, \mathcal{U}, s)$, we consider the Fourier series

$$f_A(Z) = \sum_{T\in P_2} A(T)\exp(2\pi i\sigma(TZ)). \tag{2.8}$$

LEMMA 5. *The series $f_A(Z)$ is absolutely convergent for any $Z \in \mathfrak{H}_2$ and defines a holomorphic function of Z.*

Proof. We decompose the Fourier series $f_A(Z)$ into 3 parts according to the rank of T:

$$f_A(Z) = f_A^{(0)}(Z) + f_A^{(1)}(Z) + f_A^{(2)}(Z), \tag{2.9}$$
$$f_A^{(i)}(Z) = \sum_{\substack{T\in P_2 \\ \mathrm{rank}\, T = i}} A(T)\exp(2\pi i\sigma(TZ)) \quad (i = 0,1,2).$$

Obviously, $f_A^{(0)}(Z) = a(0)$. By [Im, Proposition 2.5], $f_A^{(2)}(Z)$ is absolutely convergent and gives a holomorphic function on $\mathfrak{H}_2$. Let us consider the rank 1 part. Since $\{U \in \tilde{\Gamma} \mid T_n[U] = T_n\} = {}^t(\tilde{\Gamma}_\infty)$, we have

$$\begin{aligned} f_A^{(1)}(Z) &= \sum_{n=1}^{\infty}\sum_{U\in {}^t(\tilde{\Gamma}_\infty)\backslash\tilde{\Gamma}} A(T_n[U])\exp(2\pi i\sigma(T_n[U]\cdot Z)) \\ &= \sum_{n=1}^{\infty} a(n) \sum_{U\in {}^t(\Gamma_\infty)\backslash\Gamma} \exp(2\pi i\sigma(T_n[U]\cdot Z)) \\ &= \sum_{n=1}^{\infty} a(n) \sum_{U\in \Gamma/\Gamma_\infty} \exp(2\pi i\sigma(T_n\cdot Z[U])). \end{aligned}$$

Therefore

$$f_A^{(1)}(Z) = \sum_{n=1}^{\infty} a(n) \sum_{U\in\Gamma/\Gamma_\infty} \exp(2n\pi i Z[U]_{11}). \tag{2.10}$$

Hence, writing $\operatorname{Im} Z = \sqrt{t}\, W$ ($t = \det(\operatorname{Im} Z)$, $W \in \mathcal{SP}_2$), we have $W[U]_{11} = \operatorname{Im}(U^{-1} \cdot z)^{-1}$ and

$$f_A^{(1)}(i\sqrt{t}\, W) = \sum_{n=1}^{\infty} a(n) \sum_{U \in \Gamma/\Gamma_\infty} \exp(-2\pi n \sqrt{t} \operatorname{Im}(U^{-1} \cdot z)^{-1}) . \tag{2.11}$$

This implies that

$$|f_A^{(1)}(Z)| \le C \sum_{n=1}^{\infty} n^{\beta} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \exp(-2\pi n \sqrt{t} \operatorname{Im}(\gamma \cdot z)^{-1}) .$$

Fix a $\sigma > \beta + 1$. Then one can choose a positive constant C' such that

$$\exp(-\pi n \sqrt{t} \operatorname{Im}(\gamma \cdot z)^{-1}) \le C'(n\sqrt{t} \operatorname{Im}(\gamma \cdot z)^{-1})^{-\sigma} .$$

Hence

(2.12)

$$\begin{aligned} |f_A^{(1)}(Z)| &\le C'' t^{-\sigma/2} \sum_{n=1}^{\infty} n^{-(\sigma-\beta)} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \operatorname{Im}(\gamma \cdot z)^{\sigma} \exp(-\pi n \sqrt{t} \operatorname{Im}(\gamma \cdot z)^{-1}) \\ &\le C'' t^{-\sigma/2} \zeta(\sigma - \beta) \Bigg(y^{\sigma} \exp(-\pi \sqrt{t} y^{-1}) + \exp(-\pi y \sqrt{t}) \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma \\ c \neq 0}} \operatorname{Im}(\gamma \cdot z)^{\sigma} \Bigg) . \end{aligned}$$

where C'' is some positive constant and the last summation indicates that γ runs over all representatives of $\Gamma_\infty \backslash \Gamma$ with the left lower component c of γ being not zero. Here we have used the estimate

$$\operatorname{Im} \gamma \cdot z = \frac{y}{(cx+d)^2 + (cy)^2} < y^{-1} \quad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \right) \qquad \text{if } c \neq 0 .$$

By the choice of σ, the right hand of the inequality above is absolutely convergent and the convergence is uniform on every compact subset of $\mathfrak{H}_2$. Hence $f_A^{(1)}(Z)$ defines a holomorphic function on $\mathfrak{H}_2$. ∎

COROLLARY 6. (i) *For sufficiently large σ there exists positive constants c, c' such that*

$$\begin{aligned} |f_A^{(1)}(Z)| < c(\det \operatorname{Im} Z)^{-\sigma/2} \{ & y^{\sigma} \exp(-\pi y^{-1} (\det \operatorname{Im} Z)^{1/2}) \\ & + c' \exp(-\pi y (\det \operatorname{Im} Z)^{1/2}) \} \end{aligned}$$

($\forall Z \in \mathfrak{H}_2$ *such that* $z \in \mathcal{F}$).

(ii) *There exist some constants* $c, \ell, \delta > 0$ *such that*

$$|f_A^{(2)}(Z)| < c(\det \operatorname{Im} Z)^{-\ell} \exp(-\delta \sqrt{\det \operatorname{Im} Z}) \quad (\forall Z \in \mathfrak{H}_2 \text{ such that } z \in \mathcal{F}) .$$

Proof. The first estimate follows immediately from (2.12) and the estimate

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma, c \neq 0} \operatorname{Im}(\gamma \cdot z)^{\sigma} \le C y^{1-\sigma} \quad (y \ge 1/2)$$

with some constant $C > 0$. The second estimate is given in [Im, Proposition 2.5]. ∎

We define $\Phi f_A(z)$ ($z \in \mathfrak{H}$) as in the case of Siegel modular forms:

$$\Phi f_A(z) = \lim_{\lambda \to +\infty} f_A\begin{pmatrix} z & 0 \\ 0 & i\lambda \end{pmatrix} = \sum_{n=0}^{\infty} a(n) e^{2n\pi i z}, \quad a(n) = A(T_n). \tag{2.13}$$

Set

$$\xi_1(A, u) = (2\pi)^{-u} \Gamma(u) D_1(A, u). \tag{2.14}$$

Then the following integral representation of the Dirichlet series $D_1(A, u)$ is classical.

LEMMA 7. *When* $\mathrm{Re}\, u > \beta + 1$, *the integral*

$$\xi_1(\Phi f_A, u) = \int_0^\infty y^{u-1}(\Phi f_A(iy) - a(0))dy$$

is absolutely convergent and we have

$$\xi_1(\Phi f_A, u) = \xi_1(A, u).$$

We normalize the $GL_2(\mathbb{R})$-invariant measure dY on $\mathcal{P}_2$ by setting

$$dY = \frac{dy_1 dy_2 dy_3}{(\det Y)^{3/2}} \quad \left(Y = \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} \right).$$

The following is the integral representation of $D_2(A, \mathcal{U}; s)$ given by Maass ([Ma2]):

LEMMA 8. *Let* $\mathcal{U}$ *be a Grössencharacter corresponding to the eigenvalue* $-\lambda$ *of* Δ (*namely,* $\Delta\mathcal{U} = -\lambda\mathcal{U}$) *and denote by* $\mathcal{D}_{\mathcal{U}}$ *the domain of convergence of* $D_2(A, \mathcal{U}, s)$ *given in Lemma* 2:

$$\mathcal{D}_{\mathcal{U}} = \begin{cases} \{s \in \mathbb{C} \mid \mathrm{Re}\, s > \alpha + \frac{3}{2}\} & \text{if } \mathcal{U} \text{ is bounded}, \\ \{s \in \mathbb{C} \mid \mathrm{Re}\, s > \max\{\alpha + \frac{3}{2}, \frac{\mathrm{Re}\, u}{2} + \alpha + 1, \frac{1-\mathrm{Re}\, u}{2} + \alpha + 1\}\} & \text{if } \mathcal{U} = E(z, u). \end{cases}$$

When $s \in \mathcal{D}_{\mathcal{U}}$ (*and* $C(u) \neq \infty$ *if* $\mathcal{U} = E(z, u)$), *then the integral*

$$\xi_2(f_A, \mathcal{U}, s) = \int_{\Gamma \backslash \mathcal{P}_2} (\det Y)^s \mathcal{U}(Y) f_A^{(2)}(\sqrt{-1}\, Y) dY \tag{2.15}$$

is absolutely convergent and we have

$$\xi_2(f_A, \mathcal{U}, s) = 4\pi^{1/2}(2\pi)^{-2s} \Gamma(\mathcal{U}, s) D_2(A, \mathcal{U}, s),$$

where

$$\Gamma(\mathcal{U}, s) = \Gamma\left(s - \frac{1 + \sqrt{1 - 4\lambda}}{4}\right)\Gamma\left(s - \frac{1 - \sqrt{1 - 4\lambda}}{4}\right).$$

REMARK. If $\mathcal{U} = E(z, u)$, then we have

$$\Gamma(E(z, u), s) = \Gamma\left(s - \frac{u}{2}\right)\Gamma\left(s - \frac{1 - u}{2}\right).$$

§3. Functional equation and analytic continuation of the Koecher-Maass series

In this section we consider the Koecher-Maass Dirichlet series obtained from Siegel modular forms of degree 2. Let k be an even positive integer. Let

$$f(Z) = \sum_{T \in P_2} A(T) \exp(2\pi i \sigma(TZ))$$

be the Fourier series expansion of a *not necessarily cuspidal* Siegel modular form $f \in \mathfrak{M}_2^k(\Gamma_2)$. Then it is known that $A(T) = O((\det T)^k)$ ([Kl, Chapter VI, Lemma 1]) and $a(n) = A(T_n) = O(n^{k-1+\varepsilon})$ for any $\varepsilon > 0$ ([Mi, Chapter 4]). Using the Fourier coefficients $A(T)$ $(T \in P_2)$, we can define the Koecher-Maass Dirichlet series $D_2(f, \mathcal{U}, s) := D_2(A, \mathcal{U}, s)$ and the Hecke L series $D_1(\Phi f, u) = D_1(A, u)$. All the results in §1 apply to these Dirichlet series.

The analytic properties of $D_1(\Phi f, u)$ summarized in the following lemma are well known (cf. [Mi]).

LEMMA 9. *The integral $\xi_1(\Phi f, u)$ is absolutely convergent for* $\operatorname{Re} u > k$ *and the function*

$$\xi_1(\Phi f, u) - a(0)\left(\frac{(-1)^{k/2}}{u-k} - \frac{1}{u}\right)$$

can be extended to an entire function of u in $\mathbb{C}$, which is bounded in every vertical strip

$$\mathcal{B}_{\sigma_1,\sigma_2} = \{u \in \mathbb{C} \,|\, \sigma_1 < \operatorname{Re} u < \sigma_2\}.$$

Moreover it satisfies the functional equation

$$\xi_1(\Phi f, k-u) = (-1)^{k/2} \xi_1(\Phi f, u).$$

Recall that the lemma is an immediate consequence of the formula

$$(3.1)\quad \xi_1(\Phi f, u) = \int_1^\infty (y^u + (-1)^{k/2} y^{k-u})(\Phi f(z) - a(0)) \frac{dy}{y} + a(0)\left(\frac{(-1)^{k/2}}{u-k} - \frac{1}{u}\right).$$

We summarize the formulas originally due to Maass [Ma1] and Roelcke [Ro] which give the explicit determination of the principal parts of the Koecher-Maass series $\xi_2(f, \mathcal{U}, s)$.

THEOREM 10. (i) *If $C(u) \neq \infty$, then the function*

$$\xi_2(f, E(z,u), s) - \left\{\xi_1(\Phi f, 1-u)\left(\frac{1}{s-k+\frac{1-u}{2}} - \frac{1}{s-\frac{1-u}{2}}\right)\right.$$
$$\left. + C(u)\xi_1(\Phi f, u)\left(\frac{1}{s-k+\frac{u}{2}} - \frac{1}{s-\frac{u}{2}}\right)\right\}$$

can be extended to an entire function of s and is bounded in every vertical strip $\mathcal{B}_{\sigma_1,\sigma_2}$.

(ii) *Let us recall that ν_0 is the constant function given by* (1.8). *Then the function*

$$\xi_2(f, \nu_0, s) - \sqrt{\frac{3}{\pi}}\left\{\frac{a(0)\pi}{3}\left(\frac{1}{s-k} - \frac{1}{s}\right) + \xi_1(\Phi f, 1)\left(\frac{1}{s-k+\frac{1}{2}} - \frac{1}{s-\frac{1}{2}}\right)\right\}$$

can be extended to an entire function of s and is bounded in every vertical strip $\mathcal{B}_{\sigma_1,\sigma_2}$.

(iii) *If* $\mathcal{U}$ *is cuspidal, then the function* $\xi_2(f,\mathcal{U},s)$ *can be extended to an entire function of s and is bounded in every vertical strip* $\mathcal{B}_{\sigma_1,\sigma_2}$.

(iv) *They satisfy the functional equation*

$$\xi_2(f,\mathcal{U},k-s)=\xi_2(f,\mathcal{U},s).$$

REMARK. The functional equation and meromorphic continuation are proved in [Ma2]. The determination of the principal parts of $\xi_2(f,\mathcal{U},s)$ has been studied by Maass [Ma1] ($\mathcal{U}=\nu_0$) and by Roelcke [Ro] ($\mathcal{U}=E(z,u)$). The case of ν_0 has been generalized to Siegel modular forms of degree n in [Ar1]. We give here an another easier proof of (i), (ii) than those of [Ma1], [Ro] along the line in [Ar1]. It will be convenient to the reader to treat the formulas all together as a whole.

Theorem 10 together with Lemma 2(iii) implies the following.

THEOREM 11. *The function*

$$u(u-1)\left(s-\frac{u}{2}\right)\left(s-k+\frac{u}{2}\right)\left(s-\frac{1-u}{2}\right)\left(s-k+\frac{1-u}{2}\right)$$
$$\times\Gamma(u)\Gamma\left(s-\frac{u}{2}\right)\Gamma\left(s-\frac{1-u}{2}\right)\zeta(2u)D_2(f,E(z,u),s)$$

is an entire function of (s,u) *in* $\mathbb{C}^2$.

The proof of Theorem 10 is based on the following lemma.

LEMMA 12. *When* $\operatorname{Re} s$ *is sufficiently large, we have*

$$\xi_2(f,\mathcal{U};s)=\int_{\substack{\Gamma\backslash\mathcal{P}_2\\ \det Y\geq 1}}((\det Y)^s+(\det Y)^{k-s})\mathcal{U}(Y)f^{(2)}(\sqrt{-1}Y)dY+I(f,\mathcal{U};s),\tag{3.2}$$

where

$$I(f,\mathcal{U};s)=\int_{\substack{\Gamma\backslash\mathcal{P}_2\\ \det Y\geq 1}}(f^{(2)}(\sqrt{-1}Y^{-1})-f^{(2)}(\sqrt{-1}Y)(\det Y)^k)(\det Y)^{-s}\mathcal{U}(Y)dY.$$

The integral

$$\int_{\substack{\Gamma\backslash\mathcal{P}_2\\ \det Y\geq 1}}((\det Y)^s+(\det Y)^{k-s})\mathcal{U}(Y)f^{(2)}(\sqrt{-1}Y)dY\tag{3.3}$$

is absolutely convergent for any $s\in\mathbb{C}$, *unless* $\mathcal{U}(Y)=E(z,u)$ $(z\leftrightarrow W=(\det Y)^{-1/2}Y)$ *and represents an entire function of s. If* $\mathcal{U}=E(z,s)$, *then the function*

$$u(1-u)\pi^{-u}\Gamma(u)\zeta(2u)\int_{\substack{\Gamma\backslash\mathcal{P}_2\\ \det Y\geq 1}}((\det Y)^s+(\det Y)^{k-s})\mathcal{U}(Y)f^{(2)}(\sqrt{-1}Y)dY\tag{3.4}$$

is a holomorphic function of (s,u) *in* $\mathbb{C}^2$.

Proof. The identity (3.2) is an immediate consequence of the fact $\mathcal{U}(Y) = \mathcal{U}(Y^{-1})$. The convergence and holomorphy of the integral (3.3) follow from Lemma 2 and Lemma 8. ∎

Now we have to calculate $I(f, \mathcal{U}, s)$ explicitly. For this purpose, we introduce the following auxiliary functions ϕ and Q.

For $z \in \mathfrak{H}$, we put

$$\phi(z) = \Phi f(z) - a(0) = \sum_{n=1}^{\infty} a(n) e^{2n\pi i z}, \quad a(n) = A(T_n).$$

We also put

$$Q(y) = \phi(\sqrt{-1}y) - (-1)^{k/2} a(0) y^{-k}.$$

Then, by the automorphic property of Φf, we have

$$Q(y^{-1}) = (-1)^{k/2} y^k Q(y).$$

LEMMA 13. (i) *If* $\operatorname{Re} s < 0$, *then*

$$\int_0^{\infty} v^s (Q(v) + a(0)) \frac{dv}{v} = \xi_1(\Phi f, s).$$

(ii) *If* $0 < \operatorname{Re} s < k$, *then*

$$\int_0^{\infty} v^s Q(v) \frac{dv}{v} = \xi_1(\Phi f, s).$$

Proof. (i) If $\operatorname{Re} s < 0$, then by Lemma 9 we have

$$\begin{aligned}
\int_0^{\infty} v^{s-1}(Q(v) + a(0)) dv &= \int_0^{\infty} v^{s-1}(\Phi f(\sqrt{-1}\, v) - (-1)^{k/2} v^{-k} a(0)) dv \\
&= \int_0^{\infty} v^{s-1}\{(-1)^{k/2} v^{-k} \phi(\sqrt{-1}\, v^{-1})\} dv \\
&= (-1)^{k/2} \int_0^{\infty} v^{s-k-1} \phi(\sqrt{-1} v^{-1}) dv \\
&= (-1)^{k/2} \int_0^{\infty} v^{k-s-1} \phi(\sqrt{-1} v) dv \\
&= (-1)^{k/2} \xi_1(\Phi f, k-s) = \xi_1(\Phi f, s).
\end{aligned}$$

(ii) If $0 < \operatorname{Re} s < k$, then by (3.1) we have

$$\begin{aligned}
\int_0^{\infty} v^s Q(v) \frac{dv}{v} &= \int_1^{\infty} v^s (\phi(\sqrt{-1} v) - (-1)^{k/2} a(0) v^{-k}) \frac{dv}{v} \\
&\quad + \int_1^{\infty} v^{-s} (-1)^{k/2} v^k (\phi(\sqrt{-1} v) - (-1)^{k/2} a(0) v^{-k}) \frac{dv}{v} \\
&= \int_1^{\infty} \phi(\sqrt{-1}\, v)(v^s + (-1)^{k/2} v^{k-s}) \frac{dv}{v} - \frac{a(0)}{s} - \frac{(-1)^{k/2} a(0)}{k-s} \\
&= \xi_1(\Phi f, s).
\end{aligned}$$

This proves the lemma. ∎

Proof of Theorem 10. Since $f(Z)$ satisfies

$$f(-Z^{-1}) = (\det Z)^k f(Z),$$

we have

$$\begin{aligned} &f^{(2)}(\sqrt{-1}\,Y^{-1}) - (\det Y)^k f^{(2)}(\sqrt{-1}Y) \\ &\quad = a(0)\{(\det Y)^k - 1\} + (\det Y)^k f^{(1)}(\sqrt{-1}Y) - f^{(1)}(\sqrt{-1}Y^{-1}). \end{aligned}$$

By (2.10), we have

$$f^{(1)}(\sqrt{-1}Y) = \sum_{U \in \Gamma/\Gamma_\infty} \{\Phi f(\sqrt{-1}Y[U]_{11}) - a(0)\}.$$

Since the mapping $U \mapsto {}^tU^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ yields a bijection of Γ/Γ_∞ onto itself, we have

$$\begin{aligned} f^{(1)}(\sqrt{-1}Y^{-1}) &= \sum_{U \in \Gamma/\Gamma_\infty} \{\Phi f(\sqrt{-1}Y^{-1}[U]_{11}) - a(0)\} \\ &= \sum_{U \in \Gamma/\Gamma_\infty} \{\Phi f(\sqrt{-1}(Y[U]^{-1})_{22}) - a(0)\}. \end{aligned}$$

Therefore we have

$$\begin{aligned} &(\det Y)^k f^{(1)}(\sqrt{-1}Y) - f^{(1)}(\sqrt{-1}Y^{-1}) \\ &\quad = \sum_{U \in \Gamma/\Gamma_\infty} \{(\det Y)^k (Q(Y[U]_{11}) + a(0)(-1)^{k/2}(Y[U]_{11})^{-k}) \\ &\qquad\qquad - (Q((Y[U]^{-1})_{22}) + a(0)(-1)^{k/2}((Y[U]^{-1})_{22})^{-k})\} \\ &\quad = \sum_{U \in \Gamma/\Gamma_\infty} \{(\det Y)^k Q(Y[U]_{11}) - Q((Y[U]^{-1})_{22})\}. \end{aligned}$$

Thus we obtain

(3.5)

$$\begin{aligned} I(f, \mathcal{U}; s) = \int_{\substack{\Gamma\backslash\mathcal{P}_2 \\ \det Y \geq 1}} \Big[&a(0)\{(\det Y)^k - 1\} \\ &+ \sum_{U \in \Gamma/\Gamma_\infty} \{(\det Y)^k Q(Y[U]_{11}) - Q((Y[U]^{-1})_{22})\}\Big] (\det Y)^{-s}\mathcal{U}(Y)dY. \end{aligned}$$

CASE 1. $\mathcal{U}$ is cuspidal. In this case we have

$$\int_{\substack{\Gamma\backslash\mathcal{P}_2 \\ \det Y \geq 1}} a(0)\{(\det Y)^k - 1\}(\det Y)^{-s}\mathcal{U}(Y)dY = a(0)\int_1^\infty t^{-s}(t^k - 1)\frac{dt}{t}\int_{\Gamma\backslash\mathfrak{H}} \mathcal{U}(z)d\mu(z).$$

Since any cusp form is orthogonal to constant functions, we have

$$\int_{\Gamma\backslash\mathfrak{H}} \mathcal{U}(z)d\mu(z) = 0,$$

which implies

$$\int_{\substack{\Gamma\backslash\mathcal{P}_2\\ \det Y\geq 1}} a(0)\{(\det Y)^k-1\}(\det Y)^{-s}\mathcal{U}(Y)dY=0\,.$$

Hence

$$\begin{aligned}I(f,\mathcal{U};s)&=\int_{\substack{\Gamma\backslash\mathcal{P}_2\\ \det Y\geq 1}}\sum_{U\in\Gamma/\Gamma_\infty}\{(\det Y)^kQ(Y[U]_{11})-Q((Y[U]^{-1})_{22})\}(\det Y)^{-s}\mathcal{U}(Y)dY\\ &=\int_{\substack{{}^t(\Gamma_\infty)\backslash\mathcal{P}_2\\ \det Y\geq 1}}\{(\det Y)^kQ(Y[U]_{11})-Q((Y[U]^{-1})_{22})\}(\det Y)^{-s}\mathcal{U}(Y)dY\,.\end{aligned}$$

A fundamental domain ${}^t(\Gamma_\infty)\backslash\mathcal{P}_2$ is given by

$$F=\left\{\begin{pmatrix}v&0\\0&w\end{pmatrix}[n_x]\,\middle|\,v>0,\ w>0,\ |x|\leq\frac{1}{2}\right\},\quad n_x=\begin{pmatrix}1&x\\0&1\end{pmatrix}. \tag{3.6}$$

Therefore the cuspidal condition

$$\int_{-1/2}^{1/2}\mathcal{U}(x+iy)dx=0$$

implies that, if $\operatorname{Re}s>k$, then

(3.7)

$$\begin{aligned}&I(f,\mathcal{U},s)\\ &=\int_{F,vw\geq 1}\{(vw)^kQ(v)-Q(w^{-1})\}(vw)^{-s}\mathcal{U}\left(\begin{pmatrix}v&0\\0&w\end{pmatrix}[n_x]\right)v^{1/2}w^{-1/2}\frac{dv}{v}\frac{dw}{w}dx\\ &=\int_{vw\geq 1}\{(vw)^kQ(v)-Q(w^{-1})\}(vw)^{-s}v^{1/2}w^{-1/2}\frac{dv}{v}\frac{dw}{w}\int_{-1/2}^{1/2}\mathcal{U}(x+i\sqrt{w/v})dx\\ &=0\,,\end{aligned}$$

where we note that the integral on the right hand side of the first equality is absolutely convergent.

CASE 2. $\mathcal{U}=\nu_0\equiv\sqrt{\frac{3}{\pi}}$. In this case, if $\operatorname{Re}s>k$, then we have

$$\begin{aligned}\int_{\substack{\Gamma\backslash\mathcal{P}_2\\ \det Y\geq 1}} a(0)\{(\det Y)^k-1\}(\det Y)^{-s}\mathcal{U}(Y)dY&=a(0)\int_1^\infty t^{-s}(t^k-1)\frac{dt}{t}\int_{\Gamma\backslash\mathfrak{H}}\sqrt{\frac{3}{\pi}}d\mu(z)\\ &=a(0)\sqrt{\frac{\pi}{3}}\left(\frac{1}{s-k}-\frac{1}{s}\right).\end{aligned}$$

The remaining term of $I(f,\nu_0,s)$ is equal to

$$\begin{aligned}&\sqrt{\frac{3}{\pi}}\int_{\substack{\Gamma\backslash\mathcal{P}_2\\ \det Y\geq 1}}(\det Y)^{-s}\sum_{U\in\Gamma/\Gamma_\infty}\{(\det Y)^kQ(Y[U]_{11})-Q((Y[U]^{-1})_{22})\}dY\\ &=\sqrt{\frac{3}{\pi}}\int_{\substack{{}^t(\Gamma_\infty)\backslash\mathcal{P}_2\\ \det Y\geq 1}}(\det Y)^{-s}\{(\det Y)^kQ(Y_{11})-Q((Y^{-1})_{22})\}dY.\end{aligned}$$

If $\operatorname{Re} s > k - \frac{1}{2}$, then by a calculation similar to that in Case 1 and Lemma 13, we can rewrite the integral above as follows:

$$\begin{aligned}
&\sqrt{\frac{3}{\pi}}\int_{vw\geq 1}(vw)^{-s}v^{1/2}w^{-1/2}((vw)^k Q(v) - Q(w^{-1}))\frac{dv}{v}\frac{dw}{w} \\
&= \sqrt{\frac{3}{\pi}}\left\{\int_{vw\geq 1} v^{-s+k+1/2}w^{-s+k-1/2}Q(v)\frac{dv}{v}\frac{dw}{w}\right. \\
&\quad \left. - \int_{vw\geq 1} v^{-s+1/2}w^{-s-1/2}Q(w^{-1})\frac{dv}{v}\frac{dw}{w}\right\} \\
&= \sqrt{\frac{3}{\pi}}\left\{\frac{1}{s-k+1/2}\int_0^\infty vQ(v)\frac{dv}{v} - \frac{1}{s-1/2}\int_0^\infty w^{-1}Q(w^{-1})\frac{dw}{w}\right\} \\
&= \xi_1(\Phi f, 1)\sqrt{\frac{3}{\pi}}\left(\frac{1}{s-k+1/2} - \frac{1}{s-1/2}\right).
\end{aligned}$$

Summing up these calculations, we obtain for $\operatorname{Re} s > k$

$$I(f, \nu_0, s) = a(0)\sqrt{\frac{\pi}{3}}\left(\frac{1}{s-k} - \frac{1}{s}\right) + \xi_1(\Phi f, 1)\sqrt{\frac{3}{\pi}}\left(\frac{1}{s-k+1/2} - \frac{1}{s-1/2}\right). \tag{3.8}$$

CASE 3. $\mathcal{U} = E(z, u)$. In this case we have

$$\begin{aligned}
I(f, \mathcal{U}; s) &= \int_{\substack{\Gamma\backslash\mathcal{P}_2 \\ \det Y\geq 1}} \sum_{U\in\Gamma/\Gamma_\infty} [\{(\det Y)^k Q(Y[U]_{11}) - Q((Y[U]^{-1})_{22})\}(\det Y)^{-s}\mathcal{U}(Y) \\
&\qquad + a(0)\{(\det Y)^k - 1\}(\det Y)^{-s+u/2}(Y[U]_{11})^{-u}]dY \\
&= \int_{\substack{{}^t(\Gamma_\infty)\backslash\mathcal{P}_2 \\ \det Y\geq 1}} [\{(\det Y)^k Q(Y_{11}) - Q((Y^{-1})_{22})\}(\det Y)^{-s}\mathcal{U}(Y) \\
&\qquad + a(0)\{(\det Y)^k - 1\}(\det Y)^{-s+u/2}Y_{11}{}^{-u}]dY \\
&= \int_{F, vw\geq 1}\left[\{(vw)^k Q(v) - Q(w^{-1})\}(vw)^{-s}\mathcal{U}\left(\begin{pmatrix} v & 0 \\ 0 & w\end{pmatrix}[n_x]\right)\right. \\
&\qquad \left. + a(0)\{(vw)^k - 1\}(vw)^{-s+u/2}v^{-u}\right]v^{1/2}w^{-1/2}\frac{dv}{v}\frac{dw}{w}dx \\
&= \int_{vw\geq 1}\left[\{(vw)^k Q(v) - Q(w^{-1})\}(vw)^{-s}\int_{-1/2}^{1/2}\mathcal{U}(x + i\sqrt{w/v})dx\right. \\
&\qquad \left. + a(0)\{(vw)^k - 1\}(vw)^{-s+u/2}v^{-u}\right]v^{1/2}w^{-1/2}\frac{dv}{v}\frac{dw}{w}.
\end{aligned}$$

Since

$$\int_{-1/2}^{1/2}\mathcal{U}(x + i\sqrt{w/v})dx = \left(\frac{w}{v}\right)^{u/2} + C(u)\left(\frac{w}{v}\right)^{(1-u)/2}$$

for $\mathcal{U} = E(z,u)$ (see (1.2)), we have for $\mathrm{Re}\, s > \max\left\{k + \frac{\mathrm{Re}\, u-1}{2}, \frac{1-\mathrm{Re}\, u}{2}\right\}$

$$\begin{aligned}
&I(f, E(z,u), s)\\
&= \int_{vw\geq 1} \{(vw)^k(Q(v)+a(0)) - (Q(w^{-1})+a(0))\}(vw)^{-s}(w/v)^{u/2}v^{1/2}w^{-1/2}\frac{dv}{v}\frac{dw}{w}\\
&\quad + C(u)\int_{vw\geq 1}\{(vw)^kQ(v) - Q(w^{-1})\}(vw)^{-s}(w/v)^{(1-u)/2}v^{1/2}w^{-1/2}\frac{dv}{v}\frac{dw}{w}\\
&= \frac{1}{s-k+\frac{1-u}{2}}\int_0^\infty v^{1-u}(Q(v)+a(0))\frac{dv}{v} - \frac{1}{s+\frac{u-1}{2}}\int_0^\infty w^{u-1}(Q(w^{-1})+a(0))\frac{dw}{w}\\
&\quad + C(u)\left\{\frac{1}{s-k+\frac{u}{2}}\int_0^\infty v^u Q(v)\frac{dv}{v} - \frac{1}{s-\frac{u}{2}}\int_0^\infty w^{-u}Q(w^{-1})\frac{dw}{w}\right\}\\
&= \left(\frac{1}{s-k+\frac{1-u}{2}} - \frac{1}{s+\frac{u-1}{2}}\right)\int_0^\infty v^{1-u}(Q(v)+a(0))\frac{dv}{v}\\
&\quad + C(u)\left(\frac{1}{s-k+\frac{u}{2}} - \frac{1}{s-\frac{u}{2}}\right)\int_0^\infty v^u Q(v)\frac{dv}{v}.
\end{aligned}$$

Hence, if we further assume that $1 < \mathrm{Re}\, u < k$, then by Lemma 13 we obtain

$$\begin{aligned}
I(f, E(z,u), s) &= \xi_1(\Phi f, 1-u)\left(\frac{1}{s-k+\frac{1-u}{2}} - \frac{1}{s+\frac{u-1}{2}}\right)\\
&\quad + C(u)\xi_1(\Phi f, u)\left(\frac{1}{s-k+\frac{u}{2}} - \frac{1}{s-\frac{u}{2}}\right).
\end{aligned} \tag{3.9}$$

Now the theorem follows immediately from (3.7), (3.8), (3.9) and Lemma 12. ∎

§4. Converse theorem

4.1. Statement of the converse theorem

Let $A : P_2 \to \mathbb{C}$ be a mapping satisfying the condition (A-0) in §2.1; namely

(A-0) (i) $A(T[U]) = A(T)$ $(\forall T \in P_2^+,\ U \in \tilde{\Gamma})$,

(ii) $A(T) = O((\det T)^\alpha)$ for some $\alpha > 0$.

(iii) There exists a positive β such that $A(T_n) = O(n^\beta)$ for T_n, $n \geq 0$.

Let $\mathcal{U}$ be a Grössencharacter in the sense of §0.1. Then, by Lemma 2, the Koecher-Maass series

$$D_2(A, \mathcal{U}; s) = \sum_{\{T\}} \frac{A(T)}{\varepsilon(T)}\mathcal{U}(T)(\det T)^{-s}$$

is absolutely convergent for sufficiently large $\mathrm{Re}(s)$. In particular, if $\mathcal{U}(z) = E(z,u)$ and c is a sufficiently large positive real number, then the function

$$u(1-u)\pi^{-u}\Gamma(u)\zeta(2u)D_2(A, E(*,u); s)$$

is absolutely convergent in the domain

$$\mathcal{D} := \left\{ (s,u) \in \mathbb{C}^2 \,\middle|\, \mathrm{Re}(s) > \max\left\{ \frac{\mathrm{Re}(u)}{2}, \frac{1-\mathrm{Re}(u)}{2} \right\} + c \right\} \tag{4.1}$$

and defines a holomorphic function invariant under $(s,u) \mapsto (s, 1-u)$.

Consider the integral representation

$$\begin{aligned} \xi_2(A,\mathcal{U};s) &= \int_{\Gamma\backslash\mathcal{P}_2} (\det Y)^s \mathcal{U}(Y) f_A^{(2)}(\sqrt{-1}Y)dY \\ &= 4\sqrt{\pi}(2\pi)^{-2s}\Gamma(\mathcal{U};s)D_2(A,\mathcal{U};s), \end{aligned}$$

given in Lemma 8.

We assume the following conditions:

(A-1) For any Grössencharacter $\mathcal{U}$, the function $\xi_2(A,\mathcal{U};s)$ has an analytic continuation to a meromorphic function of s in $\mathbb{C}$ and satisfies the functional equation

$$\xi_2(A,\mathcal{U};k-s) = \xi_2(A,\mathcal{U};s). \tag{4.2}$$

(A-2) If $\mathcal{U}$ is a cusp form, then $\xi_2(A,\mathcal{U};s)$ is an entire function.

(A-3) If $\mathcal{U} = E(z,u)$, there exists a sufficiently large number c such that

$$\left(s - \frac{u}{2}\right)\left(s - \frac{1-u}{2}\right)\left(s-k+\frac{u}{2}\right)\left(s-k+\frac{1-u}{2}\right)\xi_2(A,\mathcal{U};s)$$

is a holomorphic function of finite order in

$$\mathcal{D}^* = \{(s,u) \in \mathbb{C}^2 \,|\, \mathrm{Re}(u) > c\}.$$

The assumptions (A-0) and (A-3) imply that the function

$$\left(s-\frac{u}{2}\right)\left(s-\frac{1-u}{2}\right)\left(s-k+\frac{u}{2}\right)\left(s-k+\frac{1-u}{2}\right)u(1-u)\pi^{-u}\Gamma(u)\zeta(2u)\xi_2(A,E(z,u);s)$$

can be extended to a holomorphic function in $\mathbb{C}^2$.

We define a Dirichlet series of rank one part by (2.1):

$$D_1(A,u) = \sum_{n=1}^{\infty} a(n)n^{-u}, \quad a(n) := A(T_n).$$

Moreover attaching the gamma factor we define a function $\xi_1(A,u)$ by (2.14).

Our final assumption is the following:

(A-4) The function $\xi_1(A,u)$ can be continued to a meromorphic function of u in the whole u plane and

$$\xi_1(A,u) - a(0)\left(\frac{(-1)^{k/2}}{u-k} - \frac{1}{u}\right)$$

is an entire function of u in $\mathbb{C}$ of finite order. In view of (A-3) we moreover assume that for $\mathcal{U} = E(z,u)$

$$\xi_1(A,1-u) = \lim_{s\to k-\frac{1-u}{2}} \left(s-k+\frac{1-u}{2}\right)\xi_2(A,E(z,u);s). \tag{4.3}$$

By (4.2) and the functional equation for $\xi_2(A, E(z,u); s)$ under $(s,u) \mapsto (s, 1-u)$, we see that

$$\xi_2(A, E(*,u); s) - \left\{ \xi_1(A, 1-u) \left(\frac{1}{s-k+\frac{1-u}{2}} - \frac{1}{s-\frac{1-u}{2}} \right) + C(u)\xi_1(A,u) \left(\frac{1}{s-k+\frac{u}{2}} - \frac{1}{s-\frac{u}{2}} \right) \right\} \tag{4.4}$$

is holomorphic in $\mathcal{D}^* = \{(s,u) \in \mathbb{C}^2 \mid \mathrm{Re}(u) > c\}$.

THEOREM 14. *Let $A : P_2 \to \mathbb{C}$ be a mapping satisfying* (A-0)–(A-4). *Then the series*

$$f_A(Z) = \sum_{T \in P_2} A(T) \exp(2\pi i \sigma(TZ)) \quad (Z \in \mathfrak{H}_2)$$

is a Siegel modular form of weight k of degree 2 (*namely $f_A \in \mathfrak{M}_2^k(\Gamma_2)$*).

4.2. Strategy to prove the converse theorem and preparatory lemmas

For simplicity we write $f(Z)$ instead of $f_A(Z)$ and put $f_t(W) = f(\sqrt{-t}W)$. By the principle of analytic continuation, it is enough to prove that

$$f_t(W) = t^{-k} f_{t^{-1}}(W). \tag{4.5}$$

By Corollary 6 the function $f_t(W)$ is of moderate growth as a function of W on $\Gamma \backslash \mathcal{P}S_2$, namely on $SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})/SO(2)$. Therefore, by (1.7), the identity (4.5) is a consequence of the following:

(C-1) If $\mathcal{U}$ is a cuspform, then $\langle f_t, \mathcal{U} \rangle = \langle t^{-k} f_{t^{-1}}, \mathcal{U} \rangle$.

(C-2) For $\psi \in C_0^\infty(\mathbb{R}_+^\times)$, let θ_ψ be the pseudo-Eisenstein series defined by (1.6). Then $\langle f_t, \theta_\psi \rangle = \langle t^{-k} f_{t^{-1}}, \theta_\psi \rangle$.

To be more precise, the function $f_t - t^{-k} f_{t^{-1}}$ is of moderate growth on $\Gamma \backslash \mathcal{P}S_2$, and moreover by the condition (C-2), cuspidal. Then the condition (C-1) implies (4.5).

REMARK. This kind of procedure of proving (4.5) was first emplyed by Weissauer in [We]. He generalized Imai's converse theorem to the one for arbitrary degree, though it was restricted to only cusp forms.

Accordingly we have only to prove (C-1), (C-2).

First note the following estimates, which are consequences of the assumptions (A-0)–(A-4) in Theorem 14, Stirling's estimate of the gamma function and the Phragmén-Lindelöf theorem.

LEMMA 15. (i) *For any $\gamma > 0$ and for any α_1, α_2 with $\alpha_1 < \alpha_2$, there exists a constant $c = c_{\gamma,\alpha_1,\alpha_2,\mathcal{U}}$ such that*

$$|\xi_2(A, \mathcal{U}, s)| < c|\mathrm{Im}\, s|^{-\gamma} \quad (|\mathrm{Im}\, s| \to \infty) \quad \textit{uniformly on } \alpha_1 \le \mathrm{Re}\, s \le \alpha_2 .$$

(ii) *For any $\gamma > 0$ and for any α_1, α_2 with $\alpha_1 < \alpha_2$, there exists a constant $c = c_{\gamma,\alpha_1,\alpha_2}$ such that*

$$|\xi_1(A, s)| < c|\mathrm{Im}\, s|^{-\gamma} \quad (|\mathrm{Im}\, s| \to \infty) \quad \textit{uniformly on } \alpha_1 \le \mathrm{Re}\, s \le \alpha_2 .$$

Let $f^{(i)}(Z)$ $(0 \le i \le 2)$ be the same as in (2.9) for $f = f_A$. For $F = f^{(i)}$ $(0 \le i \le 2)$ we put

$$F_t(W) = F(\sqrt{-t}W) \quad (W \in \mathcal{SP}_2,\ t > 0)$$

and for $i = 1, 2$,

$$\tilde{F}_s(W) = \int_0^\infty F_t(W) t^{s-1} dt\,.$$

By Corollary 6, the integral above $(i = 1, 2)$ is absolutely convergent when $\mathrm{Re}\, s$ is sufficiently large. Since $F_t(W[U]) = F_t(W)$ $(U \in \tilde{\Gamma})$, it is obvious that

$$\tilde{F}_s(W[U]) = \tilde{F}_s(W) \quad (\forall U \in \tilde{\Gamma})\,.$$

Hence, if we consider $\tilde{F}_s$ as a function on $\mathfrak{H}$ through the correspondence (1.3), then

$$\tilde{F}_s(\gamma \cdot z) = \tilde{F}_s(z) \quad (\forall \gamma \in \Gamma)\,.$$

LEMMA 16. *We have*

$$\widetilde{f^{(1)}}_s(z) = 2\xi_1(A, 2s)E(z, 2s) \quad \left(\mathrm{Re}\, s > \frac{\beta + 1}{2}\right).$$

Proof. By (2.11), we have

$$\begin{aligned}
\widetilde{f^{(1)}}_s(W) &= \int_0^\infty t^{s-1} f_t^{(1)}(W) dt \\
&= 2\sum_{n=1}^\infty a(n)(2\pi n)^{-2s} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma \cdot z)^{2s} \int_0^\infty t^{2s-1} e^{-t} dt \\
&= 2(2\pi)^{-2s}\Gamma(2s)D_1(A, 2s)E(z, 2s) \\
&= 2\xi_1(A, 2s)E(z, 2s)\,.
\end{aligned}$$

This proves the lemma. ∎

4.3. Proof of the converse theorem

First we prove (C-1). Since cusp forms are orthogonal to constant functions (cf. [Ku]), we have

$$\langle f_t^{(0)}, \mathcal{U} \rangle = \langle a(0), \mathcal{U} \rangle = 0\,.$$

Recall the identity (2.11)

$$f_t^{(1)}(W) = \sum_{n=1}^\infty a(n) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \exp(-2\pi n \sqrt{t}\, \mathrm{Im}(\gamma \cdot z)^{-1})\,.$$

Since $a(n) = O(n^\beta)$,

$$\exp(-2\pi n \sqrt{t} y^{-1}) = O((n\sqrt{t} y^{-1})^{-\sigma})$$

for sufficiently large $\sigma > 0$, and $\mathcal{U}$ is rapidly decreasing, the Fubini theorem can apply to the calculation of $\langle f_t^{(1)}, \mathcal{U}\rangle$ and we obtain

$$
\begin{aligned}
(4.7) \qquad \langle f_t^{(1)}, \mathcal{U}\rangle &= \sum_{n=1}^{\infty} a(n) \Big\langle \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \exp(-2\pi n \sqrt{t}\, \mathrm{Im}(\gamma \cdot z)^{-1}), \mathcal{U} \Big\rangle \\
&= \sum_{n=1}^{\infty} a(n) \int_{\Gamma_\infty \backslash \mathcal{H}} \exp(-2\pi n \sqrt{t} y^{-1}) \bar{\mathcal{U}}(x+iy) \frac{dxdy}{y^2} \\
&= \sum_{n=1}^{\infty} a(n) \int_0^\infty \frac{\exp(-2\pi n \sqrt{t} y^{-1})}{y^2} dy \int_0^1 \bar{\mathcal{U}}(x+iy) dx \\
&= 0 .
\end{aligned}
$$

For the calculation of $\langle f_t^{(2)}, \mathcal{U}\rangle$, we use the expression

$$
f_t^{(2)}(W) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=s_0} t^{-s} \tilde{f}_s^{(2)}(W) ds , \quad \tilde{f}_s^{(2)}(W) := \int_0^\infty t^{s-1} f_t^{(2)}(W) dt ,
$$

where s_0 is a sufficiently large real number. By the integration by part, we can prove easily that, for any $\sigma > 0$, one can choose $\beta > 0$ such that

$$
\tilde{f}_s^{(2)}(W) = O(y^\beta |\mathrm{Im}(s)|^{-\sigma}) \quad (z \in \mathcal{F}) .
$$

Therefore the function $t^{-s} \tilde{f}_s^{(2)}(W) \bar{\mathcal{U}}(z)$ is integrable as a function of $(\mathrm{Im}(s), z)$ on $(s_0 + i\mathbb{R}) \times (\Gamma \backslash \mathcal{H})$ and we have

$$
\begin{aligned}
(4.8) \qquad \langle f_t^{(2)}, \mathcal{U}\rangle &= \int_{\Gamma \backslash \mathcal{H}} \Big\{ \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=s_0} t^{-s} \tilde{f}_s^{(2)}(W) ds \Big\} \bar{\mathcal{U}}(z) dz \\
&= \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=s_0} t^{-s} \Big\{ \int_{\Gamma \backslash \mathcal{H}} \tilde{f}_s^{(2)}(W) \bar{\mathcal{U}}(z) dz \Big\} ds \\
&= \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=s_0} t^{-s} \xi_2(A, \bar{\mathcal{U}}; s) ds .
\end{aligned}
$$

By (A-1), (A-2) and Lemma 15, we further obtain

$$
\begin{aligned}
(4.9) \qquad \langle f_t^{(2)}, \mathcal{U}\rangle &= \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=s_0} t^{-s} \xi_2(A, \bar{\mathcal{U}}; k-s) ds \\
&= \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=k-s_0} t^{s-k} \xi_2(A, \bar{\mathcal{U}}; s) ds \\
&= \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=s_0} t^{s-k} \xi_2(A, \bar{\mathcal{U}}; s) ds \\
&= \langle t^{-k} f_{t^{-1}}^{(2)}, \mathcal{U}\rangle .
\end{aligned}
$$

Summing up (4.6), (4.7) and (4.9), we obtain

$$
\langle f_t, \mathcal{U}\rangle = \langle t^{-k} f_{t^{-1}}, \mathcal{U}\rangle .
$$

Let us prove (C-2). For $\psi \in C_0^\infty(\mathbb{R}_+^\times)$, we put

$$L_\psi(u) = \int_0^\infty y^{-u}\psi(u)\frac{dy}{y}.$$

The function $L_\psi(u)$ is an entire function of u in $\mathbb{C}$. Since $\bar{\theta_\psi} = \theta_{\bar\psi}$, we have

$$\text{(4.10)} \qquad \langle f_t^{(0)}, \theta_\psi\rangle = a(0)\int_{\Gamma\backslash\mathcal{H}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \bar\psi(y(\gamma\cdot z))d\mu(z)$$
$$= a(0)\int_{\Gamma_\infty\backslash\mathcal{H}} \bar\psi(y)\frac{dxdy}{y^2} = a(0)L_{\bar\psi}(1).$$

We have

$$\langle f_t^{(1)}, \theta_\psi\rangle = \int_{\Gamma\backslash\mathcal{H}}\left\{\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=s_0} t^{-s}\tilde f_s^{(1)}(W)ds\right\}\theta_{\bar\psi}(z)d\mu(z)$$
$$= \int_{\Gamma\backslash\mathcal{H}}\left\{\frac{2}{2\pi i}\int_{\mathrm{Re}(s)=s_0} t^{-s}\xi_1(A, 2s)E(z, 2s)ds\right\}\theta_{\bar\psi}(z)d\mu(z)$$
$$= \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=s_0/2} t^{-s/2}\xi_1(A, s)\left\{\int_{\Gamma\backslash\mathcal{H}} E(z, s)\theta_{\bar\psi}(z)dz\right\}ds.$$

Now we have

$$\int_{\Gamma\backslash\mathcal{H}} E(z, s)\theta_{\bar\psi}(z)d\mu(z) = \int_{\Gamma_\infty\backslash\mathcal{H}} E(z, s)\bar\psi(y)d\mu(z)$$
$$= \int_0^\infty \bar\psi(y)\frac{dy}{y^2}\int_0^1 E(x+iy, s)dx$$
$$= \int_0^\infty (y^s + C(s)y^{1-s})\bar\psi(y)\frac{dy}{y^2} = L_{\bar\psi}(1-s) + C(s)L_{\bar\psi}(s).$$

Hence

$$\langle f_t^{(1)}, \theta_\psi\rangle = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=s_0/2} t^{-s/2}\xi_1(A, s)\{L_{\bar\psi}(1-s) + C(s)L_{\bar\psi}(s)\}ds$$
$$= \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=s_0/2} t^{-s/2}\xi_1(A, s)C(s)L_{\bar\psi}(s)ds$$
$$+ \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=s_0/2} t^{-s/2}\xi_1(A, s)L_{\bar\psi}(1-s)ds.$$

By the assumption (A-4), the second term in the final expression is equal to

$$\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=1-s_0/2} t^{(s-1)/2}\xi_1(A, 1-s)L_{\bar\psi}(s)ds$$
$$= \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=s_0/2} t^{(s-1)/2}\xi_1(A, 1-s)L_{\bar\psi}(s)ds - a(0)L_{\bar\psi}(1)$$
$$+ t^{-k/2}a(0)(-1)^{k/2}L_{\bar\psi}(1-k).$$

Therefore we obtain

$$(4.11)\qquad \begin{aligned}&\langle f_t^{(1)} + f_t^{(0)}, \theta_\psi\rangle\\ &= \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=s_0/2} L_{\bar\psi}(s)\{t^{-s/2}\xi_1(A,s)C(s) + t^{(s-1)/2}\xi_1(A,1-s)\}ds\\ &\quad + t^{-k/2}a(0)(-1)^{k/2}L_{\bar\psi}(1-k)\,.\end{aligned}$$

For the calculation of $\langle f_t^{(2)}, \theta_\psi\rangle$, we need the following expression of θ_ψ (see [Ku], §3.1):

$$\theta_\psi(z) = \frac{1}{2\pi i}\int_{\mathrm{Re}(u)=u_0} L_\psi(u)E(z,u)du\,.$$

Since $\tilde f_s^{(2)}(W)$ is bounded with respect to W and rapidly decreasing with respect to $\mathrm{Im}(s)$, we have

$$\begin{aligned}&\langle f_t^{(2)}, \theta_\psi\rangle\\ &= \int_{\Gamma\backslash\mathcal{H}}\left\{\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=s_0} t^{-s}\tilde f_s^{(2)}(W)ds\right\}\theta_{\bar\psi}(z)d\mu(z)\\ &= \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=s_0} t^{-s}\left\{\int_{\Gamma\backslash\mathcal{H}}\tilde f_s^{(2)}(W)\theta_{\bar\psi}(z)d\mu(z)\right\}ds\\ &= \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=s_0} t^{-s}\left\{\int_{\Gamma\backslash\mathcal{H}}\tilde f_s^{(2)}(W)\left\{\frac{1}{2\pi i}\int_{\mathrm{Re}(u)=u_0} L_{\bar\psi}(u)E(z,u)du\right\}d\mu(z)\right\}ds\,.\end{aligned}$$

Since $L_{\bar\psi}(u)$ is a rapidly decreasing function of $\mathrm{Im}(u)$ on $\mathrm{Re}(u) = u_0$, we have for sufficiently large s_0 and u_0

$$\begin{aligned}&\langle f_t^{(2)}, \theta_\psi\rangle\\ &= \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=s_0} t^{-s}\left\{\frac{1}{2\pi i}\int_{\mathrm{Re}(u)=u_0} L_{\bar\psi}(u)\left\{\int_{\Gamma\backslash\mathcal{H}}\tilde f_s^{(2)}(W)E(z,u)d\mu(z)\right\}du\right\}ds\\ &= \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=s_0} t^{-s}\left\{\frac{1}{2\pi i}\int_{\mathrm{Re}(u)=u_0} L_{\bar\psi}(u)\xi_2(A, E(*,u); s)du\right\}ds\\ &= \frac{1}{2\pi i}\int_{\mathrm{Re}(u)=u_0} L_{\bar\psi}(u)\left\{\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=s_0} t^{-s}\xi_2(A, E(*,u); s)ds\right\}du\,.\end{aligned}$$

By (A-1), (A-3) and the holomorphy of (4.4), we have

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=s_0} t^{-s}\xi_2(A, E(*,u); s)ds\\ &\quad= \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=s_0} t^{-s}\xi_2(A, E(*,u); k-s)ds\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=k-s_0} t^{s-k} \xi_2(A, E(*, u); s) ds \\
&= \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=s_0} t^{s-k} \xi_2(A, E(*, u); s) ds \\
&\quad - \{\xi_1(A, 1-u)(t^{-(1-u)/2} - t^{-k+(1-u)/2}) + C(u)\xi_1(A, u)(t^{-u/2} - t^{-k+u/2})\},
\end{aligned}
$$

where we used the residue calculation to get the final equality. Hence the identity just obtained together with (4.11) implies that

$$
\begin{aligned}
\langle f_t^{(2)}, \theta_\psi \rangle &= \langle t^{-k} f_{t^{-1}}^{(2)}, \theta_\psi \rangle \\
&\quad - \frac{1}{2\pi i} \int_{\mathrm{Re}(u)=u_0} L_{\bar{\psi}}(u) \{\xi_1(A, 1-u)(t^{-(1-u)/2} - t^{-k+(1-u)/2}) \\
&\quad + C(u)\xi_1(A, u)(t^{-u/2} - t^{-k+u/2})\} du \\
&= \langle t^{-k} f_{t^{-1}}^{(2)}, \theta_\psi \rangle - \{\langle f_t^{(1)} + f_t^{(0)}, \theta_\psi \rangle - t^{-k} \langle f_{t^{-1}}^{(1)} + f_{t^{-1}}^{(0)}, \theta_\psi \rangle\}.
\end{aligned}
$$

This proves the identity $\langle f_t, \theta_\psi \rangle = \langle t^{-k} f_{t^{-1}}, \theta_\psi \rangle$, which completes the proof of Theorem 14.

§5. Saito-Kurokawa lifting (Improvement of Duke-Imamoḡlu's method)

In this section we improve the method of [DI] by using our converse theorem (Theorem 14) so that it can be applied to not necessarily cuspidal Siegel modular forms of degree two. In [DI] they used three real analytic Eisenstein series on $\Gamma_0(4)$ with level 4 to deal with modular forms on the full modular groups $SL_2(\mathbb{Z})$ and $Sp_2(\mathbb{Z})$. Here we shall prove the modularity of lifted forms by using the unique real analytic Eisenstein series for $SL_2(\mathbb{Z})$. This will enable us to treat not necessarily cuspidal Siegel modular forms together with the help of Zagier's trick in [Za].

5.1. Shimura correspondence for Maass wave forms

We use the symbol $e(w)$ ($w \in \mathbb{C}$) as an abbreviation for $\exp(2\pi i w)$. Throughout the rest of this paper we assume that k is a non-negative even integer. Let $J_{k,1}$ denote the space of Jacobi forms of weight k and index 1 on the full modular group $SL_2(\mathbb{Z})$, for the precise definition of which we refer to Eichler-Zagier [EZ]. Each $\phi \in J_{k,1}$ has an expression as a linear combination of two theta series:

$$
\phi(\tau, z) = h_0(\tau)\theta_0(\tau, z) + h_1(\tau)\theta_1(\tau, z), \tag{5.1}
$$

where $\theta_i(\tau, z) = \sum_{n\in\mathbb{Z}} e((n+i/2)^2\tau + (2n+i)z)$ ($i = 0, 1$).

For $w \in \mathbb{C} - \{0\}$ the function $w^{1/2} := \exp((1/2)\log w)$ denotes a holomorphic function of w with the branch $-\pi < \arg w \leq \pi$. As usual let $\Gamma_0(4)$ be the congruence subgroup of $\Gamma = SL_2(\mathbb{Z})$ consisting elements ($\in \Gamma$) whose lower left components are congruent to one mod 4. Set $\theta(z) = \theta_0(\tau, 0) = \sum_{n\in\mathbb{Z}} e(n^2\tau)$. As is well-known this theta function is non-zero on $\mathfrak{H}$ and verifies the the transformation formula

$$\theta(M\cdot z)/\theta(z) = j(M,z) \quad \text{for any } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4),$$

where $j(M,z)$ is characterized by

$$j(M,z) = \left(\frac{c}{d}\right)\varepsilon_d^{-1}(cz+d)^{1/2}. \tag{5.2}$$

Here $\left(\frac{c}{d}\right)$ is Shimura's residue symbol on whose precise definition we refer the reader to [Shm1] and $\varepsilon_d = 1$ (resp. $\varepsilon_d = i$) according to $d \equiv 1 \bmod 4$ (resp. $d \equiv 3 \bmod 4$). With the help of this factor of automorphy the space $M_{k-1/2}(\Gamma_0(4))$ of modular forms of half-integral weight $k-1/2$ on $\Gamma_0(4)$ has been defined. Namely, $M_{k-1/2}(\Gamma_0(4))$ consists of holomorphic functions f on $\mathfrak{H}$ satisfying the conditions

(i) $f(M\cdot\tau) = j(M,\tau)^{2k-1}f(\tau)$ for all $M \in \Gamma_0(4)$.

(ii) f is holomorphic at all cusps of $\Gamma_0(4)$.

By (i), (ii), each f has a Fourier expansion of the form

$$f(\tau) = \sum_{n=0}^{\infty} c(n)e(n\tau).$$

Moreover $M_{k-1/2}^{+}$ denotes the subspace of $M_{k-1/2}(\Gamma_0(4))$ so called the plus space consisting of $f \in M_{k-1/2}(\Gamma_0(4))$ whose Fourier coefficients $c(n)$ satisfy the condition

$$c(n) = 0 \quad \text{if } (-1)^k n \equiv 1, 2 \bmod 4. \tag{5.3}$$

Let $f(z) = \sum_{n=0}^{\infty} c(n)e(nz) \in M_{k+1/2}^{+}$, not necessarily cuspidal. Then f corresponds to a Jacobi form $\phi \in J_{k,1}$ in such a manner that $f(\tau) = h_0(4\tau) + h_1(4\tau)$, where $h_0(\tau)$ and $h_1(\tau)$ are given by (5.1). Furthermore h_0 and h_1 have the Fourier expansions:

$$h_0(\tau) = \sum_{n=0}^{\infty} c(4n)e(n\tau), \quad h_1(\tau) = \sum_{\substack{n\in\mathbb{Z}_{>0} \\ n\equiv 3 \bmod 4}} c(n)e\left(\frac{n\tau}{4}\right) \tag{5.4}$$

and that

$$f\left(-\frac{1}{4\tau}\right) = \sqrt{2}\left(\frac{\tau}{i}\right)^{k-1/2}(-1)^{k/2}h_0(\tau), \tag{5.5}$$

$$\begin{pmatrix} h_0(-1/\tau) \\ h_1(-1/\tau) \end{pmatrix} = \frac{1}{\sqrt{2}}(\tau/i)^{k-1/2}(-1)^{k/2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} h_0(\tau) \\ h_1(\tau) \end{pmatrix}. \tag{5.6}$$

This correspondence $f \mapsto \phi$ gives an isomorphism from $M_{k-1/2}^{+}(\Gamma_0(4))$ onto $J_{k,1}$ (see [EZ]).

We now recall the **Shimura correspondence** for Maass wave forms. For the definition of Maass wave forms (Größencharacters) we refer to 1.1.

A Maass wave form $\mathcal{U}$ on $\mathfrak{H}$ (or on $\mathcal{P}_2$) is called **even**, if it satisfies

$$\mathcal{U}(-\bar{z}) = \mathcal{U}(z) \quad (\text{or } \mathcal{U}(\tilde{Y}) = \mathcal{U}(Y)),$$

where $\tilde{Y} = {}^tI_0 Y I_0$ with $I_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. On the contrary a Maass wave form $\mathcal{U}$ is called **odd**, if $\mathcal{U}(-\bar{z}) = -\mathcal{U}(z)$.

Moreover we have to introduce Maass wave forms of weight 1/2. For $r \in \mathbb{C}$ let T_r^+ denote the $\mathbb{C}$-linear space consisting of functions $g : \mathfrak{H} \to \mathbb{C}$ satisfying the following three conditions:

(i) Each g is a C^∞-function of x and y verifying the transformation formula

$$g(M \cdot z) = g(z) j(M, z) |cz + d|^{-1/2}$$

for all $M \in \Gamma_0(4)$ and it has a moderate growth condition at any cusps of $\Gamma_0(4)$; namely there exists $\alpha > 0$ such that for all $M \in SL_2(\mathbb{Z})$

$$|g(M \cdot z)| = O(y^\alpha) \quad (y \to \infty)\,.$$

(ii) g has a Fourier expansion of the form

$$g(z) = \sum_{n \in \mathbb{Z}} B(n, y) e(nx)\,, \tag{5.7}$$

where the Fourier coefficients $B(n, y)$ for $n \neq 0$ are given by

$$B(n, y) = b(n) W_{\mathrm{sign} n/4, ir/2}(4\pi y|n|)\,. \tag{5.8}$$

Here $W_{\alpha,\beta}$ is the usual Whittaker function.

(iii) If $n \equiv 2, 3 \bmod 4$, then necessarily $B(n, y) = 0$.

The Shimura correspondence from the space of Maass wave forms to the space of Maass wave forms of weight 1/2 has been obtained by Katok-Sarnak [KS]. They treated the case of cusp forms, while Duke-Imamoḡlu [DI] improved it to cover the case of Eisenstein series. Kojima in [Ko] also discussed such a correspondence in the case with levels.

For $Y \in \mathcal{P}_2$ we denote by z_Y the point in $\mathfrak{H}$ corresponding to $\frac{1}{\sqrt{\det Y}} \cdot Y$ via the correspondence (1.3).

THEOREM 17 (Katok-Sarnak, Duke-Imamoḡlu). *Let $\mathcal{U}$ be an even Maass wave form and assume that $\Delta\mathcal{U} = -(\frac{1}{4} + r^2)\mathcal{U}$ with some $r \in \mathbb{C}$. Then there exists $g = \sum_{n \in \mathbb{Z}} B(n, y) e(nx) \in T_r^+$ which satisfies the relation*

$$b(-n) = n^{-3/4} \sum_{T \in P_2^+/SL_2(\mathbb{Z}), \det 2T = n} \mathcal{U}(z_T) |Aut T|^{-1} \quad (n \in \mathbb{Z}_{>0})\,,$$

where $b(n)$'s are given by (5.8). *Here T runs through all the $SL_2(\mathbb{Z})$-equivalence classes of elements of P_2^+ with* $\det 2T = n$ *and $|Aut T|$ denotes the order of the unit group $Aut T := \{U \in SL_2(\mathbb{Z}) \mid {}^tUTU = T\}$ of T.*

Let $g = \sum_{n \in \mathbb{Z}} B(n, y) e(nx) \in T_r^+$. Set

$$g_0(z) = \frac{1}{\sqrt{2}} \left(\frac{z}{i}\right)^{-1/2} g\left(-\frac{1}{4z}\right) |z|^{-1/2}$$

and

$$g_1(z) = g\left(\frac{z}{4}\right) - g_0(z)\,.$$

Then it is immediate to see that

$$\begin{pmatrix} g_0(-1/\tau) \\ g_1(-1/\tau) \end{pmatrix} = \frac{1}{\sqrt{2}}(\tau/i)^{1/2}|z|^{-1/2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} g_0(\tau) \\ g_1(\tau) \end{pmatrix}. \tag{5.9}$$

It is known by [DI] and [Ib] that

$$g_0(z) = \sum_{n\in\mathbb{Z}} B\left(4n, \frac{y}{4}\right) e(nx), \tag{5.10}$$

$$g_1(z) = \sum_{\substack{n\in\mathbb{Z} \\ n\equiv 1 \bmod 4}} B\left(n, \frac{y}{4}\right) e\left(\frac{nx}{4}\right).$$

To avoid the use of real analytic Eisenstein series on $\Gamma_0(4)$ we consider the following function:

$$H(z) = h_0(z)g_0(z) + h_1(z)g_1(z). \tag{5.11}$$

LEMMA 18. *The function $H(z)$ satifies the transformation formula*

$$H(M\cdot z) = (cz+d)^k|cz+d|^{-1/2}H(z)$$

for any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

Proof. We have only to check it for $M = J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which is easily seen from (5.4), (5.6), (5.9), and (5.10). ∎

A real analytic Eisenstein series $E_\infty(z, s)$ of weight k with respect to $\Gamma = SL_2(\mathbb{Z})$ is given by

$$E_\infty(z, s) = \sum_{M\in\Gamma_\infty\backslash\Gamma}\left(\frac{cz+d}{|cz+d|}\right)^k \operatorname{Im}(Mz)^s,$$

which is absolutely convergent for $\operatorname{Re}(s) > 1$. If $k = 0$, then, $E_\infty(z, s) = E(z, s)$. It is well-known that this function can be analytically continued to a meromorphic function in the whole s-plane and moreover that it satisfies the functional equation

$$\tilde{E}_\infty(z, s) = \tilde{E}_\infty(z, 1-s),$$

where we put

$$\tilde{E}_\infty(z, s) = \gamma(s)E_\infty(z, s) \quad \text{with } \gamma(s) = \pi^{-s}\Gamma\left(s + \frac{k}{2}\right)\zeta(2s).$$

It is known that $\tilde{E}_\infty(z, s)$ has a Fourier expansion of the form

$$\tilde{E}_\infty(z, s) = e(y, s) + \sum_{n\in\mathbb{Z}, n\neq 0} e_n(y, s)e(nx) \tag{5.12}$$

$$\text{with } e(y, s) = \gamma(s)y^s + \gamma(1-s)y^{1-s}$$

and that the function $\sum_{n\in\mathbb{Z},n\neq 0} e_n(y, s)e(nx)$ is of rapid decay for any $s \in \mathbb{C}$.

5.2. Rankin-Selberg convolution

To include the case of not necessarily cuspidal modular forms we employ the method of Zagier [Za] which enables us to deal with the Rankin-Selberg convolution of not cuspidal automorphic forms.

Denote by $\mathcal{F}$ the usual fundamental domain of $SL_2(\mathbb{Z})$ in $\mathfrak{H}$ given by (1.5). Choose $T > 0$ sufficiently large and set

$$\mathcal{F}_T = \{z = x + iy \in \mathcal{F} \mid y \leq T\}.$$

Take $f \in M^+_{k-1/2}(\Gamma_0(4))$ and let $H(z)$ be the same as in (5.11). Let us consider the integral

$$I_T(s) = \int_{\mathcal{F}_T} y^{\frac{k}{2}-\frac{1}{4}} H(z) E_\infty(z, s) d\mu(z). \tag{5.13}$$

Here note that the integrand $y^{\frac{k}{2}-\frac{1}{4}} H(z) E_\infty(z, s)$ is invariant under the action of $SL_2(\mathbb{Z})$.

Unfolding the integral (5.13) faithfully as in [Za], (22), we see that

$$I_T(s) = \int_0^T \int_0^1 y^{\frac{k}{2}-\frac{1}{4}} H(z) y^s d\mu(z) - \sum_{c=1}^{\infty} \sum_{\substack{a \bmod c \\ (a,c)=1}} \int_{S_{a/c}} y^{\frac{k}{2}-\frac{1}{4}} H(z) y^s d\mu(z),$$

where $S_{a/c}$ is the disc in $\mathfrak{H}$ of radius $1/(2c^2T)$ tangent to the real axis at a/c. If we choose some $M_0 \in SL_2(\mathbb{Z})$ of the form $M_0 = \begin{pmatrix} a & \cdot \\ c & \cdot \end{pmatrix}$, then, $M_0^{-1} \cdot S_{a/c} = \{z \in \mathfrak{H} \mid y \geq T\}$. With the use of this expression of $S_{a/c}$ and replacing z with $M_0 \cdot z$ we have

$$I_T(s) = \int_0^T \int_0^1 y^{\frac{k}{2}-\frac{1}{4}} H(z) y^s d\mu(z) - \int_{\mathcal{F}-\mathcal{F}_T} y^{\frac{k}{2}-\frac{1}{4}} H(z)(E_\infty(z, s) - y^s) d\mu(z). \tag{5.14}$$

Let

$$A_0(y) = \int_0^1 H(x + iy) dx$$

denote the constant term of the Fourier expansion of $H(z)$. We immediately have

$$\begin{aligned} A_0(y) = c(0) B\left(0, \frac{y}{4}\right) &+ \sum_{n=1}^{\infty} c(4n) b(-4n) e^{-2\pi n y} W_{-\frac{1}{4}, \frac{ir}{2}}(4\pi n y) \\ &+ \sum_{\substack{n \equiv 3 \bmod 4 \\ n > 0}} c(n) b(-n) e^{-\frac{\pi n y}{2}} W_{-\frac{1}{4}, \frac{ir}{2}}(\pi n y). \end{aligned} \tag{5.15}$$

We attach the gamma factor $\gamma(s)$ to $I_T(s)$. Set

$$\tilde{I}_T(s) = \int_{\mathcal{F}_T} y^{\frac{k}{2}-\frac{1}{4}} H(z) \tilde{E}_\infty(z, s) d\mu(z).$$

Immediately from (5.14),

$$\tilde{I}_T(s) = \gamma(s) \int_0^T y^{s+\frac{k}{2}-\frac{1}{4}-2} A_0(y) dy - \int_{\mathcal{F}-\mathcal{F}_T} y^{\frac{k}{2}-\frac{1}{4}} H(z)(\tilde{E}_\infty(z, s) - \gamma(s) y^s) d\mu(z).$$

Then with the use of $e(y, s)$ (see (5.12)),

$$\begin{aligned}\tilde{I}_T(s) = \gamma(s) \int_0^T y^{s+\frac{k}{2}-\frac{1}{4}-2} A_0(y)dy \\ - \int_{\mathcal{F}-\mathcal{F}_T} y^{\frac{k}{2}-\frac{1}{4}} H(z)(\tilde{E}_\infty(z, s) - e(y, s))d\mu(z) \\ - \gamma(1-s) \int_{\mathcal{F}-\mathcal{F}_T} y^{1-s+\frac{k}{2}-\frac{1}{4}} H(z)d\mu(z),\end{aligned}$$

or rearranging,

$$\begin{aligned}(5.16) \qquad \gamma(s) \int_0^T y^{s+\kappa-2} A_0(y)dy - \gamma(1-s) \int_T^\infty y^{1-s+\kappa-2} A_0(y)dy \\ = \int_{\mathcal{F}_T} y^\kappa H(z)\tilde{E}_\infty(z, s)d\mu(z) \\ + \int_{\mathcal{F}-\mathcal{F}_T} y^\kappa H(z)(\tilde{E}_\infty(z, s) - e(y, s))d\mu(z),\end{aligned}$$

where we put $\kappa = (k - 1/2)/2$. This identity has been proved only for $\mathrm{Re}(s)$ sufficiently large.

Write

$$A_0(y) = \alpha_0(y) + A_0^*(y)$$

with $\alpha_0(y) = c(0)B(0, y/4)$. Here the function $A_0^*(y)$ is rapidly decreasing with respect to y. We note that if either f or $\mathcal{U}$ is a cusp form, then $\alpha_0(y) = 0$.

Set

$$\Lambda_\infty(f, g, s) = \gamma(s) \int_0^\infty y^{s+\kappa-2} A_0^*(y)dy,$$

Then the congruence condition for $c(n)$, $b(n)$ (i.e., $c(n) = b(-n) = 0$ if $n \equiv 1, 2 \bmod 4$) in the expression (5.15) being taken into account, this function $\Lambda_\infty(f, g, s)$ can be written as

$$\begin{aligned}&\Lambda_\infty(f, g, s) \\ &= \pi^{-2s-\kappa+1}\zeta(2s)\Gamma\left(s+\kappa+\frac{ir}{2}-\frac{1}{2}\right)\Gamma\left(s+\kappa-\frac{ir}{2}-\frac{1}{2}\right)\sum_{n=1}^\infty \frac{c(n)b(-n)}{n^{s+\kappa-1}},\end{aligned}$$

where we have used the integral formula

$$\int_0^\infty y^{s+\kappa-1} e^{-\frac{y}{2}} W_{-\frac{1}{4},\frac{ir}{2}}(y)\frac{dy}{y} = \frac{\Gamma\left(s+\kappa+\frac{ir}{2}-\frac{1}{2}\right)\Gamma\left(s+\kappa-\frac{ir}{2}-\frac{1}{2}\right)}{\Gamma\left(s+\frac{k}{2}\right)}.$$

Set

$$h_T(s) = c(0) \int_0^T y^{s+\kappa-2} B\left(0, \frac{y}{4}\right) dy.$$

In the case of $\mathcal{U}$ being Eisenstein series we have to compute $h_T(s)$. Now let

$$\mathcal{U}(z) = E(z, u) = \sum_{M \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(Mz)^u$$

with $\mathrm{Re}(u) > 1$. We write $\zeta^*(s)$ for the function $\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$. It is known by [DI], Lemma 5 and [Ib], p. 159 that the automorphic form $g(z,u)$ which corresponds to $\mathcal{U} = E(z,u)$ by the Katok-Sarnak correspondence (Theorem 17) is given by

$$g(z,u) = \frac{1}{2}(4\pi)^{-1/4}y^{1/4}2^u\zeta^*(u)F(-1,u,z)\,,$$

where

$$F(-1,u,z) = E(-1,u,z) + 2^{-u-\frac{1}{2}}(e(1/8)+e(-1/8))E^*(-1,u,z)$$

and for each $l \in \mathbb{Z}$ the Eisenstein series $E(l,u,z)$ and $E^*(l,u,z)$ are given by

$$\begin{aligned} E(l,u,z) &= \sum_{M\in\Gamma_\infty\backslash\Gamma_0(4)} j(M,z)^l \mathrm{Im}(Mz)^{u/2} \\ &= y^{u/2}\sum_{\substack{d=1\\ d:\mathrm{odd}}}\sum_{c\in\mathbb{Z}}\left(\frac{4c}{d}\right)\varepsilon_d^{-l}(4cz+d)^{l/2}|4cz+d|^{-u}\,, \\ E^*(l,u,z) &= E\left(l,u,-\frac{1}{4z}\right)(-2iz)^{l/2}\,. \end{aligned}$$

Moreover it is known that $F(-1,u,z)$ has a Fourier expansion of the form

$$\text{(5.17)}\qquad \begin{aligned} F(-1,u,z) &= y^{\frac{u}{2}} + 2^{1-2u}\frac{\zeta^*(2u-1)}{\zeta^*(2u)}y^{\frac{1-u}{2}} \\ &\quad + \sum_{\substack{d\in\mathbb{Z}\\ d\neq 0}} C(d,u)e(dx)y^{-1/4}W_{\mathrm{sign}(d)/4,u/2-1/4}(4\pi|d|y) \end{aligned}$$

(see for instance [Ib], p. 158 and also [Shm2]) and moreover that via this expression it can be analytically continued to a meromorphic function verifying the functional equation

$$2^u\zeta^*(2u)F(-1,u,z) = 2^{1-u}\zeta^*(2(1-u))F(-1,1-u,z)\,.$$

In this case ($g = g(z,u)$) by the above (5.17),

$$\text{(5.18)}\qquad B\left(0,\frac{y}{4}\right) = \frac{\pi^{-1/4}}{4}\zeta^*(u)\left(y^{\frac{u+1/2}{2}} + \frac{\zeta^*(2u-1)}{\zeta^*(2u)}y^{\frac{1-u+1/2}{2}}\right).$$

By integrating the function on the right hand side of (5.18) from 0 to T we have

(5.19)
$$h_T(s) = \frac{c(0)\pi^{-1/4}}{4}\zeta^*(u)\left(\frac{T^{s+\frac{k}{2}+\frac{u}{2}-1}}{s+\frac{k}{2}+\frac{u}{2}-1} + \frac{\zeta^*(2u-1)}{\zeta^*(2u)}\cdot\frac{T^{s+\frac{k}{2}+\frac{1-u}{2}-1}}{s+\frac{k}{2}+\frac{1-u}{2}-1}\right).$$

Therefore in this case if $\mathrm{Re}(s)$ is sufficiently large,

$$\gamma(s)\int_0^T y^{s+\kappa-2}A_0(y)dy = \Lambda_\infty(f,g,s) - \gamma(s)\int_T^\infty y^{s+\kappa-2}A_0^*(y)dy + \gamma(s)h_T(s)$$

and moreover

$$\gamma(1-s)\int_T^{\infty} y^{1-s+\kappa-2}A_0(y)dy = \gamma(1-s)\int_T^{\infty} y^{1-s+\kappa-2}A_0^*(y)dy - \gamma(1-s)h_T(1-s)\,. \tag{5.20}$$

To derive (5.20) we used the identity (5.18) and the integration of $y^{1-s+\kappa-2}B(0, y/4)$ from T to ∞.

By (5.19), (5.20) and (5.16) we obtain the following key identity.

THEOREM 19. *Let $f \in M_{k-1/2}^{+}$ and $\mathcal{U}$ an even Maass wave form, to which g corresponds via the Katok-Sarnak correspondence. Let T be a sufficiently large positive number. We have*

$$\begin{aligned} &\Lambda_\infty(f, g, s) + \gamma(s)h_T(s) + \gamma(1-s)h_T(1-s) \\ &= \gamma(s)\int_T^{\infty} y^{s+\kappa-2}A_0^*(y)dy + \gamma(1-s)\int_T^{\infty} y^{1-s+\kappa-2}A_0^*(y)dy \\ &\quad + \int_{\mathcal{F}_T} y^{\kappa}H(z)\tilde{E}_\infty(z,s)d\mu(z) + \int_{\mathcal{F}-\mathcal{F}_T} y^{\kappa}H(z)(\tilde{E}_\infty(z,s) - e(y,s))d\mu(z)\,. \end{aligned}$$

The function on the right hand side is an entire function of s and bounded in any vertical strip of s. Moreover this expression gives meromorphic continuation of $\Lambda_\infty(f, g, s)$, and the functional equation

$$\Lambda_\infty(f, g, s) = \Lambda_\infty(f, g, 1-s)\,.$$

follows.

5.3. Application of the converse theorem

We continue the assumption that k is a positive even integer. Let $f(z) = \sum_{n=0}^{\infty} c(n)e(nz) \in M_{k-1/2}^{+}$. We define a mapping $A : P_2 \to \mathbb{C}$ from f as follows: Set, for $T = \begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix} \in P_2$, $T \neq 0$,

$$A(T) = \sum_{0<d \mid (m,r,n)} d^{k-1}c\left(\frac{\det 2T}{d^2}\right), \tag{5.21}$$

(m, r, n) denoting the greatest common integer of m, r, n, and for $T = 0$,

$$A(0) = \frac{\zeta(k)\Gamma(k)}{(2\pi i)^k}c(0) = \frac{1}{2}\zeta(1-k)c(0)\,.$$

Since $c(n) = O(n^k)$, it is easy to see that this mapping A satisfeis the condition (A-0) in §2.1.

We now define a holomorphic function $l(f)(Z)$ on $\mathfrak{H}_2$ by putting

$$l(f)(Z) = \sum_{T \in P_2} A(T)e(\sigma(TZ))\,. \tag{5.22}$$

We write simply $F(Z)$ instead of $l(f)(Z)$ ($F(Z) = f_A(Z)$ with the notation of (2.8)). Let $F^{(2)}(Z)$ denote the rank two part of $F(Z)$ given in the same manner as in (2.9). By the action of the Φ operator given in (2.13) on F, we easily have, from (5.21),

$$(5.23)\qquad \begin{aligned}(\Phi F)(z) &= \sum_{n=0}^{\infty} A\begin{pmatrix} n & 0 \\ 0 & 0\end{pmatrix} e(nz) \\ &= \left\{\frac{1}{2}\zeta(1-k) + \sum_{n=1}\Big(\sum_{0<d\,|\,n} d^{k-1}\Big)e(nz)\right\}c(0) \\ &= \left\{\frac{1}{2}\zeta(1-k)\sum_{M\in\Gamma_\infty\backslash\Gamma}(cz+d)^{-k}\right\}c(0)\,.\end{aligned}$$

We consider the Mellin transform of $F^{(2)}(iY)$ on a fundamental domain of $\Gamma = SL_2(\mathbb{Z})$ in $\mathcal{P}_2$:

$$\xi_2(F, \mathcal{U}, s) := \int_{\Gamma\backslash\mathcal{P}_2} (\det Y)^s \mathcal{U}(Y) F^{(2)}(\sqrt{-1}Y)dY$$

where $\mathcal{U}$ is a Grössencharacter on $\mathcal{P}_2$ (see (2.15)). This integral is absolutely convergent for Re(s) sufficiently large. We note that if $\mathcal{U}$ is an odd Maass wave form, then trivially, $\xi_2(F, \mathcal{U}, s) = 0$. Therefore we may assume that $\mathcal{U}$ is even. Let $g \in T_r^+$ denote a Maass wave form of weight 1/2 corresponding to $\mathcal{U}$ via Theorem 17.

A key identity connecting Koecher-Maass series $\xi_2(F, \mathcal{U}, s)$ with some integral of Rankin-Selberg type is the following:

$$(5.24)\qquad \xi_2(F, \mathcal{U}, s) = c_k \Lambda_\infty\left(f, g, s - \frac{k-1}{2}\right)$$

with $c_k = 2\pi^{-\kappa}$. A proof of this identity which uses Theorem 17 of Katok-Sarnak is given in [DI], [Ib] if f is a cusp form. That proof is available also in the non-cuspidal case (see also [Ar2]).

To apply our converse theorem (Theorem 14) we have to compute the residues of the function at poles. First we note that if either f or $\mathcal{U}$ is a cusp form, then (5.24) and Theorem 19 implies that $\xi_2(F, \mathcal{U}, s)$ is an entire function of s.

So let $\mathcal{U}(z) = E(z, u)$. Here we assume that Re$(u) > k$. Denote by $\xi_1(\Phi F, u)$ the Mellin transform of ΦF:

$$\xi_1(\Phi F, u) = \int_0^\infty (\Phi F)(iy) y^{u-1} dy\,.$$

Then, by (5.23)

$$(5.25)\qquad \begin{aligned}\xi_1(\Phi F, u) &= (2\pi)^{-u}\Gamma(u)\zeta(u)\zeta(u-k+1)\cdot c(0) \\ &= 2^{-1}\pi^{-(u+1)/2}\Gamma\left(\frac{u+1}{2}\right)\zeta^*(u)\zeta(u-k+1)\cdot c(0)\,.\end{aligned}$$

We see from Theorem 19, (5.19) and (5.24) that the function $\xi_2(F, \mathcal{U}, s)$ can be written as in the form

$$\xi_2(F,\mathcal{U},s) = -c_k\left\{\gamma\left(s-\frac{k-1}{2}\right)h_T\left(s-\frac{k-1}{2}\right) + \gamma\left(k-s-\frac{k-1}{2}\right)h_T\left(k-s-\frac{k-1}{2}\right)\right\} + (\text{some entire function of } s)\,, \tag{5.26}$$

where

$$h_T\left(s-\frac{k-1}{2}\right) = \frac{c(0)\pi^{-1/4}}{4}\zeta^*(u)\left(\frac{T^{s-\frac{1-u}{2}}}{s-\frac{1-u}{2}} + \frac{\zeta^*(2u-1)}{\zeta^*(2u)}\cdot\frac{T^{s-\frac{u}{2}}}{s-\frac{u}{2}}\right).$$

Let $C(u)$ be the same function as in (1.1). Then we have

$$C(u) = \frac{\zeta^*(2u-1)}{\zeta^*(2u)}.$$

If we write

$$\gamma\left(s-\frac{k-1}{2}\right)h_T\left(s-\frac{k-1}{2}\right) = \frac{c(0)\pi^{-1/4}}{4}\zeta^*(u)\left(\frac{\gamma\left(\frac{1-u}{2}-\frac{k-1}{2}\right)}{s-\frac{1-u}{2}} + C(u)\frac{\gamma\left(\frac{u}{2}-\frac{k-1}{2}\right)}{s-\frac{u}{2}}\right) + H_T(s)$$

with some function $H_T(s)$ of s, then we observe that $H_T(s)$ is holomorphic except for $s = k/2$, since $\frac{T^s}{s} - \frac{1}{s}$ is an entire function of s. The principal part of $H_T(s)$ at $s = k/2$ is given by

$$H_T(s) = \frac{2^{-1}\pi^{-1/2}\Gamma\left(\frac{k+1}{2}\right)h_T\left(\frac{1}{2}\right)}{s-\frac{k}{2}} + \cdots. \tag{5.27}$$

To continue the computation furthermore we need

$$c(0)\zeta^*(u)\gamma\left(\frac{1-u}{2}-\frac{k-1}{2}\right) = \pi^{\frac{k}{2}}\pi^{\frac{u}{2}-1}\zeta^*(1-u)\Gamma\left(1-\frac{u}{2}\right)\zeta(2-u-k)c(0) = 2\pi^{\frac{k}{2}}\xi_1(\Phi F, 1-u)$$

and

$$c(0)\zeta^*(u)\gamma\left(\frac{u}{2}-\frac{k-1}{2}\right) = \pi^{\frac{k-1}{2}-\frac{u}{2}}\zeta^*(u)\Gamma\left(\frac{u+1}{2}\right)\zeta(u-k+1)c(0) = 2\pi^{\frac{k}{2}}\xi_1(\Phi F, u)\,,$$

where we have used (2.5). Therefore

$$\gamma\left(s-\frac{k-1}{2}\right)h_T\left(s-\frac{k-1}{2}\right) = \frac{1}{2}\pi^{\frac{k-1/2}{2}}\left\{\frac{\xi_1(\Phi F, 1-u)}{s-\frac{1-u}{2}} + C(u)\frac{\xi_1(\Phi F, u)}{s-\frac{u}{2}}\right\} + H_T(s)\,. \tag{5.28}$$

Though $H_T(s)$ itself has a simple pole only at $s = k/2$ as in (5.27), the function

$$H_T(s) + H_T(k-s)$$

is entire and bounded in any vertical strip of s.

Thus we obtain the following theorem which enables us to apply our converse theorem to the function F.

THEOREM 20. *Let the notation be the same as before. The Koecher-Maass series* $\xi_2(F, \mathcal{U}, s)$ *can be continued analytically to a meromorphic function of* s *verifying the functional equation*

$$\xi_2(F, \mathcal{U}, s) = \xi_2(F, \mathcal{U}, k-s)\,.$$

If either f *or* $\mathcal{U}$ *is a cusp form, then* $\xi_2(F, \mathcal{U}, s)$ *is an entire function and bounded in any vertical strip. If* $\mathcal{U}(z) = E(z, u)$ *with* $\mathrm{Re}(u) > k$, *then* $\xi_2(F, E(z, u), s)$ *has the following expression*

$$\begin{aligned}\xi_2(F, E(z,u), s) &= \xi_1(\Phi F, 1-u)\left(\frac{1}{s-k+\frac{1-u}{2}} - \frac{1}{s-\frac{1-u}{2}}\right)\\ &+ C(u)\xi_1(\Phi F, u)\left(\frac{1}{s-k+\frac{u}{2}} - \frac{1}{s-\frac{u}{2}}\right) + \eta_2(F, E(z,u), s)\,,\end{aligned} \tag{5.29}$$

where $\eta_2(F, E(z, u), s)$ *is an entire function of* s. *Moreover* $\eta_2(F, E(z, u), s)$ *is bounded in any vertical strip of* s.

REMARK. If either f or $\mathcal{U}$ is a cusp form, then the assertion has been proved by Duke-Imamoḡlu [DI].

Proof. We have only discuss the case of $\mathcal{U}(z) = E(z, u)$ with $\mathrm{Re}(u) > k$. In this case the assertion is easily seen from (5.26) and (5.28). ∎

Let $\mathfrak{Ma}_k(\Gamma_2)$ denote the subspace of $\mathfrak{M}_2^k(\Gamma_2)$ consisting of F whose Fourier coefficients $a(T)$ satisfy the **Maass relation**:

$$a\begin{pmatrix} m & r \\ r & n\end{pmatrix} = \sum_{0<d\,|\,(m,r,n)} d^{k-1} a\begin{pmatrix} 1 & r/d \\ r/d & mn/d^2\end{pmatrix} \quad ((m, r, n) \neq (0,0,0))\,.$$

Finally we apply Theorem 15 to prove the modularity of $F = l(f)$.

THEOREM 21. *Let* $f \in M^+_{k-1/2}$ *and* $F = l(f)$ *be the function on* $\mathfrak{H}_2$ *defined by* (5.22). *Then the function* F *is a Siegel modular form of* $\mathfrak{M}_2^k(\Gamma_2)$ *and moreover it belongs to the Maass space* $\mathfrak{Ma}_k(\Gamma_2)$.

Proof. Let $A : P_2 \to \mathbb{C}$ be the same as in (5.21). If A is restricted to P_2^+, then the condition (A-0) is obvious. The conditions (A-1), (A-2) are immediately checked from Theorem 20. We see easily from (5.29) that the conditions (A-3), (A-4) hold true. Therefore Theorem 14 implies that $F \in \mathfrak{M}_2^k(\Gamma_2)$. The latter assertion is derived from the definition (5.22) and (5.21). ∎

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