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COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 52, No. 2 2003

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171–8501 JAPAN

Discrete Lax Pair for Discrete Toda Equation

by

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(Received June 27, 2003)
(Revised September 26, 2003)

1. Introduction

H. Flaschka [2, 3], developed the inverse scattering method for the solution of the Toda equation. On the other hand, R. Hirora [5, 6] and R. Hirota, S. Tsujimoto and T. Imai [4] proposed a time-discrete Toda equation of the form

$$\begin{aligned} \frac{q_{n-1}(t+\delta)-q_{n-1}(t)}{\delta} &= q_{n-1}(t+\delta)r_{n-1}(t+\delta)-q_{n-1}(t)r_n(t) \\ \frac{r_n(t+\delta)-r_n(t)}{\delta} &= q_{n-1}(t+\delta)-q_n(t) \end{aligned} \tag{1.1}$$

($\delta > 0, t \in Z_\delta = \{k\delta; k \in Z\}$), and constructed a family of pairs of infinite matrices $\{L(t), A(t)\}$, $t \in Z_\delta$, called a discrete Lax pair (see [1], [2], [7]), which satisfies a discrete Lax equation

$$L(t+\delta)A(t) = A(t)L(t)\,, \qquad t \in Z_\delta \tag{1.2}$$

equivalent to (1.1). However, the discrete Lax pair constructed by Hirota-Tsujimoto-Imai does not have the property that $L(t)$ is symmetric for each $t \in Z_\delta$. This seems to make it difficult to develop the inverse scattering theory for discrete Toda equation. To get a symmetric $L(t)$, the method developed in T.Takebe [10] may be applied. The purpose of the present paper is to construct *explicitly* a discrete Lax pair with $L(t)$ symmetric in the frame work of functional analysis, and show some functional analytic properties of $L(t)$.

More precisely, we state our results. Let $\{q_n(t), r_n(t)\}$ $(n \in Z,\ t \in Z_\delta)$ be a solution of (1.1). Then setting

$$a_{jk}(t) = \begin{cases} \sqrt{1-\delta r_j(t)} & (k=j) \\ \delta\sqrt{q_j(t)} & (k=j+1) \\ 0 & (otherwise)\,, \end{cases} \tag{1.3}$$

we define (complex) infinite matrices $A(t)$, $L(t)$ by

$$A(t) = (a_{jk}(t)) \tag{1.4}$$

$$L(t) = A^*(t)A(t) \tag{1.5}$$

where $A^*(t)$ is the (complex) adjoint matrix of (complex) matrix $A(t)$.
It is easy to see that $L(t) = (l_{jk}(t))$ with

$$l_{jk}(t) = \begin{cases} \delta\sqrt{1-\delta r_{j-1}(t)}\,\left(\sqrt{q_{j-1}(t)}\right)^* & (k = j-1) \\ |1-\delta r_j(t)| + \delta^2 |q_{j-1}(t)| & (k = j) \\ \delta\left(\sqrt{1-\delta r_j(t)}\right)^*\sqrt{q_j(t)} & (k = j+1) \\ 0 & (otherwise) \end{cases}$$

where $(\alpha)^*$ means the complex conjugate of a complex number α. By direct calculation, we can see that the pair of infinite matrices $\{L(t), A(t)\}$ thus defined is a discrete Lax pair, namely it satisfies the discrete Lax equation (1.2) equivalent to (1.1).

We can now state our main result.

THEOREM 1.1. *Let* δ *be a fixed number with* $0 < \delta < 1$. *Let* $\{q_n(t), r_n(t)\}$ $(n \in Z,\ t \in Z_\delta)$ *be a solution of* (1.1) *such that for each* $t \in Z_\delta$

$$q_n(t) \to 1,\ \ r_n(t) \to 0 \quad (n \to \pm\infty)\,. \tag{1.6}$$

Suppose that

$$q_n(0) \neq 0,\ \ \delta r_n(0) \neq 1 \quad (n \in Z)\,. \tag{1.7}$$

Then the family of pairs of infinite matrices $\{L(t), A(t)\}$ *defined by* (1.3), (1.4), (1.5) *is a discrete Lax pair, and* $L(t)$ *is a bounded positive self-adjoint operator in the complex* $l^2(-\infty,\infty)$. *Moreover all eigenvalues of* $L(t)$ *are independent of* $t \in Z_\delta$, *and the relation*

$$A^*(t)A(t) = A(t-\delta)A^*(t-\delta) \tag{1.8}$$

must hold for each $t \in Z_\delta$.

REMARK 1.2. The invariance (with respect to t) of the eigenvalue $\lambda(t)$ of $L(t)$ follows formally from the following argument. If $\lambda(0)$ is the eigenvalue of $L(0)$, and $\varphi(0)$ the eigenvector corresponding to $\lambda(0)$, then it is easy to see that

$$\varphi(t) = A(t-\delta)\cdots A(\delta)A(0)\varphi(0)$$

satisfies

$$L(t)\varphi(t) = \lambda(0)\varphi(t)\,.$$

This does not imply that $\lambda(0)$ is the eigenvalue of $L(t)$. It is not clear that the $\varphi(t)$ defined above is nontrivial.

In the subsequent paper we will develop the inverse scattering method for the discrete Toda equation on the basis of the present results.

In the section 2, we state some properties (Theorem 2.5) connected with Jacobi operator. In the section 3, we prove Theorem 1.1 as an application of Theorem 2.5.

2. Jacobi operators

An infinite matrix (h_{ij}) is called a Jacobi matrix if $h_{ij} = 0$ for $|i - j| \geq 2$. In this section we state an abstract theorem on a Jacobi operator H associated with a Jacobi matrix (h_{ij}) in the complex $l^2(-\infty, \infty)$. For Jacobi operators and completely integrable nonlinear lattices, see G. Teschl [8].

LEMMA 2.1. *Let (h_{ij}) be a Jacobi matrix. Suppose that (h_{ij}) is bounded*:

$$|h_{ij}| \leq c \quad (i, j \in Z) \tag{2.1}$$

c being a constant independent of i, j. Then the Jacobi operator H associated with a Jacobi matrix (h_{ij}) is a bounded operator in $l^2(-\infty, \infty)$ with a bound:

$$||H|| \leq 3\,c\,. \tag{2.2}$$

Proof. For $x \in l^2(-\infty, \infty)$ we have

$$\begin{aligned} ||Hx||^2 &= \sum_{i=-\infty}^{\infty} |h_{i,i-1}x_{i-1} + h_{i,i}x_i + h_{i,i+1}x_{i+1}|^2 \\ &\leq 9c^2||x||^2\,, \end{aligned}$$

$(x = (x_i))$, from which (2.2) easily follows. This proves Lemma 2.1.

We next consider an upper triangular Jacobi matrix $A = (a_{ij})$:

$$a_{ij} = 0 \quad (j \leq i - 1, j \geq i + 2)\,.$$

In this section we assume that A satisfies the following :

i)

$$a_{jj} \neq 0 \quad (j \in Z) \tag{2.3}$$

ii)

$$\sup_n |a_{n,n}| < \infty \tag{2.4}$$

iii)

$$\limsup_{n \to +\infty} \frac{|a_{n,n+1}|}{|a_{n,n}|} < 1\,, \quad \limsup_{n \to -\infty} \frac{|a_{n-1,n}|}{|a_{n,n}|} < 1\,. \tag{2.5}$$

We have :

LEMMA 2.2. *Let A satisfy* (2.3), (2.4) *and* (2.5). *Then A is a bounded operator satisfying* :

$$N(A) = N(A^*) = \{0\} \tag{2.6}$$

where $N(A), N(A^*)$ denote the null spaces of A and its adjoint A^* (*in the complex* $l^2(-\infty, \infty)$), *respectively.*

Proof. Clearly A is a bounded operator by our assumptions (2.4), (2.5) and Lemma 2.1. We set $a_{n,n} = a_n$, and $a_{n,n+1} = b_n$ for simplicity. Suppose $x \in N(A)$, i.e., $Ax = 0$. Let k be any fixed integer, and denote the k-th element of Ax by $(Ax)_k$. Then $(Ax)_k = a_k x_k + b_k x_{k+1} = 0$, and hence

$$x_k = c_k c_{k+1} \cdots c_n x_{n+1} \tag{2.7}$$

$(n = k, k+1, \cdots)$ where $c_j = -b_j/a_j$. By (2.5) there is a θ $(0 < \theta < 1)$ and $n_0 (\geq k)$ such that $|c_j| < \theta$ $(|j| \geq n_0)$. Hence by (2.7)

$$|x_k| \leq |c_k| \cdots |c_{n_0-1}|\, \theta^{n-n_0+1}\, |x_{n+1}| \tag{2.8}$$

for large n. Since

$$|x_n| \to 0 \quad (n \to \pm\infty) \tag{2.9}$$

because of $x \in l^2(-\infty, \infty)$, it follows that the right-hand side of (2.8) tends to zero as $n \to \infty$. Thus $x_k = 0$, and so $x = 0$. This proves $N(A) = \{0\}$. We next show $N(A^*) = \{0\}$. If $x \in N(A^*)$, then $(A^*x)_k = (a_k)^* x_k + (b_{k-1})^* x_{k-1} = 0$. Setting $d_j = -b_{j-1}/a_j$, we can get

$$|x_k| \leq |d_k| \cdots |d_{-n_0+1}|\, \theta^{n-n_0+1} |x_{-n-1}| \tag{2.10}$$

for large n. In the same way as before, we get $x_k = 0$ by letting $n \to \infty$ in (2.10). This implies $N(A^*) = \{0\}$. Thus Lemma 2.2 is proved.

LEMMA 2.3.

$$l^2(-\infty, \infty) = \overline{R(A)}$$

where $R(A)$ denotes the range of A, and $\overline{R(A)}$ its closure.

Proof. We get the orthogonal decomposition :

$$l^2(-\infty, \infty) = N(A^*) \oplus \overline{R(A)}\,,$$

since $\overline{R(A)}^{\perp} = N(A^*)$ where $\overline{R(A)}^{\perp}$ is the orthogonal complements of $\overline{R(A)}$ in $l^2(-\infty, \infty)$. By Lemma 2.2, $N(A^*) = \{0\}$, and hence we get Lemma 2.3.

LEMMA 2.4. *Let L be a operator defined by*

$$L = A^*A\,. \tag{2.11}$$

Then L is a bounded positive self-adjoint operator in $l^2(-\infty, \infty)$.

Proof. L is clearly a non-negative bounded self-adjoint operator. Since $(Lx, x) = ||Ax||^2$, and since $N(A) = \{0\}$, it follows that L is positive. This proves Lemma 2.4.

THEOREM 2.5. *Let A satisfy the conditions* (2.3), (2.4), (2.5) *and L be defined in* (2.11). *If M is a bounded operator in $l^2(-\infty, \infty)$ satisfying the condition*

$$MA = AL \tag{2.12}$$

then

$$M = AA^*. \tag{2.13}$$

Further

$$\sigma_p(M) = \sigma_p(L) \tag{2.14}$$

where $\sigma_p(T)$ denotes the point spectrum of an operator T.

Proof. By (2.11) and (2.12),

$$(M - AA^*)Ax = 0, \quad x \in l^2(-\infty, \infty).$$

Since $R(A)$ is dense in $l^2(-\infty, \infty)$, it follows that $M - AA^* = 0$, proving (2.13). Let $\lambda \in \sigma_p(M)$. Then there is a non zero ϕ in $l^2(-\infty, \infty)$ with $M\phi = \lambda\phi$. Thus $A^*M\phi = \lambda A^*\phi$, and so, by (2.13)

$$LA^*\phi = A^*(AA^*)\phi = A^*M\phi = \lambda A^*\phi.$$

Since $N(A^*) = 0$, $A^*\phi$ is not zero. Thus λ is an eigenvalue of L, and $A^*\phi$ is a corresponding eigenvector. Thus $\sigma_p(M) \subset \sigma_p(L)$. Let $\lambda \in \sigma_p(L)$, and ϕ an eigenvector corresponding to λ. Then by (2.12) and (2.13),

$$MA\phi = A(A^*A)\phi = AL\phi = \lambda A\phi.$$

Hence λ is an eigenvalue for M, and $A\phi$ an eigenvector for M, since $A\phi$ is not zero because of (2.6). Hence $\sigma_p(L) \subset \sigma_p(M)$. This completes the proof of Theorem 2.5.

3. Proof of Theorem 1.1

We claim that $A(t)$ defined by (1.3) and (1.4) satisfies (2.3),(2.4),(2.5) for each $t \in Z_\delta$. It is easy to verify (2.4) and (2.5), since $\limsup_{n\to+\infty}|a_{n,n+1}|/|a_{n.n}| = \delta$ and $\limsup_{n\to-\infty}|a_{n-1,n}|/|a_{n.n}| = \delta$, by (1.6). We show that for each $t \in Z_\delta$

$$q_n(t) \neq 0, \quad \delta r_n(t) \neq 1 \quad (n \in Z). \tag{3.1}$$

By the assumption (1.7), (3.1) is true for $t = 0$. Suppose (3.1) holds for $t = s (\in Z_\delta)$. Clearly we have

$$q_{n-1}(t+\delta)\,(1 - \delta r_{n-1}(t+\delta)) = q_{n-1}(t)(1 - \delta r_n(t)) \tag{3.2}$$

for $t \in Z_\delta$ and $n \in Z$. Hence the right-hand side of (3.2) is, by the assumption, not zero for $t = s$, and so is the left-hand side. This implies (3.1) holds for $t = s + \delta$. Similarly the left-hand side is not zero for $t = s - \delta$, and so is the right-hand side. This implies (3.1) holds for $t = s - \delta$. This proves that (3.1) holds for all $t \in Z_\delta$.

It follows from Lemma 2.2 (2.6) and Lemma 2.4 that $L(t)$ defined by (1.5) is a bounded positive self-adjoint operator in $l^2(-\infty, \infty)$.

Applying Theorem 2.5 with $A = A(t)$, $L = L(t)$ and $M = L(t+\delta)$, we get

$$L(t+\delta) = A(t)A^*(t) \tag{3.3}$$

from which it follows that $\sigma_p(L(t))$ is independent of $t \in Z_\delta$, and the relation (1.8) easily follows from (1.5) and (3.3). This completes the proof of Theorem 1.1.

Acknowledgement. The authors thank Professor Saburo Kakei for suggesting the problem of developing the inverse scattering method for discrete Toda lattice equation.

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