COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 53, No. 1 2004

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171–8501 JAPAN

Certain Inequalities Satisfied by the Hermite Constants of Global Fields

To the memory of Professor Tsuneo Arakawa

by

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(Received March 11, 2004)

Introduction

For a positive integer n, the constant defined by

$$\gamma_n = \max_{g \in GL_n(\mathbf{R})} \min_{0 \neq x \in \mathbf{Z}^n} \frac{||gx||^2}{|\det g|^{2/n}}$$

is called Hermite's constant. As a generalization of γ_n, Icaza ([I]) and Thunder ([T]) introduced, independently, the constant $\gamma_n(k)$ for any algebraic number field k. (See Section 1 for its precise definition.) The constant $\gamma_n(\mathbf{Q})$ for the rational number field $\mathbf{Q}$ is none other than Hermite's constant γ_n. Recently, Vaaler ([V]) showed that $\gamma_n(k)$ is characterized as the best upper bound of an inequality concerning Siegel's lemma. After the work of [I] and [T], Ohno and the author ([O-W]) proved the inequality of the form

$$\gamma_n(k) \leq |D_k|^{1/d} \frac{\gamma_{nd}(\mathbf{Q})}{d}, \tag{1}$$

where $d = [k : \mathbf{Q}]$ and D_k denotes the absolute discriminant of k. Such inequality was first proved by Cohn ([C]) provided that $n = 2$ and k is a totally real number field of class number one in order to study the fundamental domains of Hilbert modular groups. The inequality (1) yields a better estimate of $\gamma_n(k)$ than the Minkowski type upper bound given by Icaza and Thunder. Baeza, etal ([BCIR]) used this kind of estimates to compute $\gamma_2(k)$ of some real quadratic fields k.

In a similar fashion to the algebraic number fields, one can define the constant $\gamma_n(k)$ for any function field k of one variable over a finite field. In this case, both the Minkowski–Hlawka type lower bound and the Minkowski type upper bound of $\gamma_n(k)$ were given in [Wa].

The purpose of this paper is to generalize the inequality (1) to any relative extension L of any global field K. For example, if K is an algebraic number field, then the following inequality will be proved:

$$\gamma_n(L) \leq \frac{|D_L|^{1/[L:\mathbf{Q}]}}{|D_K|^{1/[K:\mathbf{Q}]}} \frac{\gamma_{n[L:K]}(K)}{[L:K]^{\sharp(\mathfrak{V}_{K,\infty})/[K:\mathbf{Q}]}},$$

where $\sharp(\mathfrak{V}_{K,\infty})$ denotes the number of infinite places of K. Our method is an adelic analogue of the argument used in [O-W]. It has the advantage of which one needs not assume that the class number of K equals one.

1. Hermite's constant of a global field

Let k be a global field, i.e., an algebraic number field of finite degree over $\mathbf{Q}$ or an algebraic function field of one variable over a finite field. We denote by $\mathfrak{V}_k$, $\mathfrak{V}_{k,\infty}$ and $\mathfrak{V}_{k,f}$ the sets of all places of k, all infinite places of k and all finite places of k, respectively. For $v \in \mathfrak{V}_k$, let k_v be the completion of k at v and $|\cdot|_{k_v}$ be the absolute value of k_v normalized so that $|a|_{k_v} = \mu_v(aC)/\mu_v(C)$, where μ_v is a Haar measure of k_v and C is an arbitrary compact subset of k_v with nonzero measure. If v is finite, $\mathfrak{O}_{k_v}$ denotes the ring of integers in k_v. The adele ring of k is denoted by $\mathbf{A}_k$ and its idele norm is denoted by $|\cdot|_{\mathbf{A}_k}$, i.e., $|\cdot|_{\mathbf{A}_k} = \prod_{v\in\mathfrak{V}_k} |\cdot|_{k_v}$. We define the constant Δ_k as follows:

$$\Delta_k = \begin{cases} |D_k| & (k \text{ is an algebraic number field of absolute discriminant } D_k)\,. \\ q^{2g_k-2} & (k \text{ is a function field of genus } g_k \text{ and constant field } \mathbf{F}_q)\,. \end{cases}$$

We recall the definition of Hermite's constant. Let k^n be the n dimensional column vector space over k with standard basis $e_1, \cdots, e_n$. For $v \in \mathfrak{V}_k$ and $g_v \in GL_n(k_v)$, the local standard height $H_{k_v^n}$ and the local twisted height $H_{k_v^n, g_v}$ on k_v^n are defined as follows: for $a_v = \alpha_1 e_1 + \cdots + \alpha_n e_n \in k_v^n$,

$$H_{k_v^n}(a_v) = \begin{cases} \left(\sum_{i=1}^n |\alpha_i|_{k_v}^2\right)^{1/2} & (\text{if } v \text{ is real}) \\ \sum_{i=1}^n |\alpha_i|_{k_v} & (\text{if } v \text{ is complex}) \\ \sup_{1\le i\le n} |\alpha_i|_{k_v} & (\text{if } v \text{ is finite}) \end{cases}$$

and

$$H_{k_v^n, g_v}(a_v) = H_{k_v^n}(g_v a_v)\,.$$

The global twisted height $H_{k^n,g}$ on k^n for $g = (g_v) \in GL_n(\mathbf{A}_k)$ is defined to be the product of all $H_{k_v^n, g_v}$'s, namely,

$$H_{k^n,g}(a) = \prod_{v\in\mathfrak{V}_k} H_{k_v^n, g_v}(a) \quad (a \in k^n)\,.$$

Then, it is known that the following maximum exists:

$$\widehat{\gamma}_n(k) = \max_{g\in GL_n(\mathbf{A}_k)} \min_{0\neq a\in k^n} \frac{H_{k^n,g}(a)^2}{|\det g|_{\mathbf{A}_k}^{2/n}}\,.$$

If k is a number field, the constant $\gamma_n(k) = \widehat{\gamma}_n(k)^{1/[k:\mathbf{Q}]}$ is called the Hermite constant of k. In any case, $\widehat{\gamma}_n(k)$ is the same as the constant $\widetilde{\gamma}(GL_n, Q_1, k)^{2/n}$ introduced in [Wa].

§2. Main theorem

Let K be a global field and L/K a finite separable extension of degree $r = [L : K]$. We fix a basis $u_1, \cdots, u_r$ of L over K, and set

$$\Delta(u_1, \cdots, u_r) = \det(\mathrm{Tr}_{L/K}(u_i u_j))_{1 \le i,j \le r} .$$

For each $v \in \mathfrak{V}_K$, $\mathfrak{W}_v$ stands for the set of places $w \in \mathfrak{V}_L$ which lie above v. If $v \in \mathfrak{V}_{K,f}$ and $w \in \mathfrak{W}_v$, there is a basis $\varepsilon_1, \cdots, \varepsilon_{r_w}$ of $\mathfrak{O}_{L_w}$ over $\mathfrak{O}_{K_v}$, where r_w denotes $[L_w : K_v]$. The ideal

$$\mathfrak{D}_{L_w/K_v} = (\det(\mathrm{Tr}_{L_w/K_v}(\varepsilon_i \varepsilon_j))_{1 \le i,j \le r_w})\mathfrak{O}_{K_v}$$

is the relative discriminant of L_w/K_v. Thus the absolute value

$$\Delta_{L_w/K_v} = |\det(\mathrm{Tr}_{L_w/K_v}(\varepsilon_i \varepsilon_j))_{1 \le i,j \le r_w}|_{K_v}$$

is independent of the choice of a basis of $\mathfrak{O}_{L_w}$ over $\mathfrak{O}_{K_v}$. By [J-P, Theorem A], one has

$$\frac{\Delta_L}{\Delta_K^r} = \prod_{v \in \mathfrak{V}_{K,f}} \prod_{w \in \mathfrak{W}_v} \Delta_{L_w/K_v}^{-1} . \tag{2}$$

THEOREM. *The notations being as above, we have*

$$\widehat{\gamma}_n(L) \le \frac{\Delta_L}{\Delta_K^r} \left(\frac{\widehat{\gamma}_{nr}(K)}{r^{\sharp(\mathfrak{V}_{K,\infty})}} \right)^r .$$

Proof. Let $e_1, \cdots, e_n$ denote the standard basis of the column vector space L^n. We identify L^n with the K-vector space K^{nr} generated by the basis $u_i e_j$, $1 \le i \le r$, $1 \le j \le n$. For $v \in \mathfrak{V}_K$ and $g = (g_w) \in GL_n(\mathbf{A}_L)$, define the function $F_{v,g}$ on $L^n = K^{nr}$ by

$$F_{v,g}(x) = \sum_{w \in \mathfrak{W}_v} r_w H_{L_w^n, g_w}(x)^{2/r_w} , \quad \text{where } r_w = [L_w : K_v] .$$

From the arithmetic and geometric mean inequality, it follows

$$\prod_{w \in \mathfrak{W}_v} H_{L_w^n, g_w}(x)^{2/r} \le \frac{F_{v,g}(x)}{r} . \tag{3}$$

We will estimate $F_{v,g}(x)$ in the following.

(I) The case that $v \in \mathfrak{V}_{K,\infty}$ is a real place. The set $\mathfrak{W}_v$ is divided into two subsets $\mathfrak{W}_{v,1} = \{w \in \mathfrak{W}_v : L_w \cong \mathbf{R}\}$ and $\mathfrak{W}_{v,2} = \{w \in \mathfrak{W}_v : L_w \cong \mathbf{C}\}$. Then, by definition,

$$F_{v,g}(x) = \sum_{w \in \mathfrak{W}_{v,1}} H_{L_w^n, g_w}(x)^2 + \sum_{w \in \mathfrak{W}_{v,2}} 2 H_{L_w^n, g_w}(x) ,$$

which gives a quadratic form on K^{nr}. We determine the symmetric matrix corresponding to $F_{v,g}$. Let $\mathfrak{W}_{v,1} = \{w_1, \cdots, w_{r_1}\}$ and $\mathfrak{W}_{v,2} = \{w_{r_1+1}, \cdots, w_{r_1+r_2}\}$. Each $w \in \mathfrak{W}_v$ is identified with an embedding of L into $\mathbf{C}$. We define the r by r real matrix U_v and the nr by nr real matrix Ω_v as follows:

$$U_v = \begin{pmatrix} w_1(u_1) & w_1(u_2) & \cdots & w_1(u_r) \\ \vdots & \vdots & \ddots & \vdots \\ w_{r_1}(u_1) & w_{r_1}(u_2) & \cdots & w_{r_1}(u_r) \\ \mathrm{Re}(w_{r_1+1}(u_1)) & \mathrm{Re}(w_{r_1+1}(u_2)) & \cdots & \mathrm{Re}(w_{r_1+1}(u_r)) \\ \mathrm{Im}(w_{r_1+1}(u_1)) & \mathrm{Im}(w_{r_1+1}(u_2)) & \cdots & \mathrm{Im}(w_{r_1+1}(u_r)) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Re}(w_{r_1+r_2}(u_1)) & \mathrm{Re}(w_{r_1+r_2}(u_2)) & \cdots & \mathrm{Re}(w_{r_1+r_2}(u_r)) \\ \mathrm{Im}(w_{r_1+r_2}(u_1)) & \mathrm{Im}(w_{r_1+r_2}(u_2)) & \cdots & \mathrm{Im}(w_{r_1+r_2}(u_r)) \end{pmatrix}$$

and $\Omega_v = \mathrm{diag}(U_v, \cdots, U_v)$, where $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ stand for the real and the imaginary part of a given complex number z, respectively. One has

$$(\det U_v)^2 = 4^{-r_2}\Delta(u_1, \cdots, u_r)\,.$$

Let $a_j(s,t)$ be the (s,t)-component of the Hermitian matrix ${}^t\overline{g}_{w_j} g_{w_j}$ for $1 \le j \le r_1 + r_2$. If $r_1 + 1 \le j \le r_2$, we put

$$a'_j(s,t) = \begin{pmatrix} 2\mathrm{Re}(a_j(s,t)) & 2\mathrm{Im}(a_j(s,t)) \\ -2\mathrm{Im}(a_j(s,t)) & 2\mathrm{Re}(a_j(s,t)) \end{pmatrix}.$$

The r by r real matrix $D_{v,g}(s,t)$ and the nr by nr real symmetric matrix $D_{v,g}$ are defined as

$$D_{v,g}(s,t) = \mathrm{diag}(a_1(s,t), \cdots, a_{r_1}(s,t), a'_{r_1+1}(s,t), \cdots, a'_{r_1+r_2}(s,t))$$

and $D_{v,g} = (D_{v,g}(s,t))_{1\le s,t\le n}$. Then the symmetric matrix corresponding to $F_{v,g}$ is given by ${}^t\Omega_v D_{v,g}\Omega_v$. If a matrix $A_{v,g} \in GL_{nr}(K_v)$ is chosen so that ${}^tA_{v,g}A_{v,g} = {}^t\Omega_v D_{v,g}\Omega_v$, then

$$F_{v,g}(x) = H_{K_v^{nr}, A_{v,g}}(x)^2$$

holds for $x \in L^n = K^{nr}$. Here the height $H_{K_v^{nr}}$ on K_v^{nr} is the standard height with respect to the basis $\{u_i e_j \ : \ 1 \le i \le r,\ 1 \le j \le n\}$. Moreover, we have

$$|\det A_{v,g}|_{K_v}^2 = |\Delta(u_1, \cdots, u_r)|_{K_v}^n \prod_{w\in\mathfrak{W}_v} |\det g_w|_{L_w}^2\,.$$

(II) The case that $v \in \mathfrak{V}_{K,\infty}$ is a complex place. Let $\mathfrak{W}_v = \{w_1, \cdots, w_r\}$. If we define the Hermitian form $G_{v,g}$ on K^{nr} by

$$G_{v,g}(x) = \sum_{w\in\mathfrak{W}_v} H_{L_w^n, g_w}(x)\,,$$

then the inequality

$$F_{v,g}(x) \le G_{v,g}(x)^2$$

holds for all $x \in K^{nr}$. Let $a_j(s,t)$ be the (s,t)-component of the Hermitian matrix ${}^t\overline{g_{w_j}}\, g_{w_j}$ for $1 \le j \le r$. In a similar fashion to (I), the r by r complex matrices U_v, $D_{v,g}(s,t)$ and the nr by nr complex matrices Ω_v, $D_{v,g}$ are defined as follows:

$$U_v = (w_i(u_j))_{1\le i,j\le r}\,, \quad \Omega_v = \mathrm{diag}(U_v, \cdots, U_v)\,,$$
$$D_{v,g}(s,t) = \mathrm{diag}(a_1(s,t), \cdots, a_r(s,t))\,, \quad D_{v,g} = (D_{v,g}(s,t))_{1\le s,t\le n}\,.$$

Then the Hermitian matrix corresponding to $G_{v,g}$ is given by ${}^t\overline{\Omega_v} D_{v,g}\Omega_v$. There is a matrix $A_{v,g} \in GL_{nr}(K_v)$ such that ${}^t\overline{A_{v,g}}A_{v,g} = {}^t\overline{\Omega_v}D_{v,g}\Omega_v$. If $H_{K_v^{nr}}$ denotes the standard height on the vector space K_v^{nr} spanned by $\{u_i e_j \ : \ 1 \le i \le r,\ 1 \le j \le n\}$, then one has

$$G_{v,g}(x) = H_{K_v^{nr}, A_{v,g}}(x)$$

for $x \in K^{nr}$, and

$$|\det A_{v,g}|^2_{K_v} = |\Delta(u_1, \cdots, u_r)|^n_{K_v} \prod_{w\in\mathfrak{W}_v} |\det g_w|^2_{L_w}\,.$$

(III) The case that $v \in \mathfrak{V}_{K,f}$. On the one hand, the space $L^n \otimes_K K_v$ is naturally identified with the vector space K_v^{nr} spanned by $\{u_i e_j \,:\, 1 \le i \le r,\ 1 \le j \le n\}$. Then $H_{K_v^{nr}}$ stands for the standard height on K_v^{nr}. On the other hand, the K_v-linearly extension of the diagonal embedding $L^n \hookrightarrow \bigoplus_{w\in\mathfrak{W}_v} L_w^n$ gives rise to the ring isomorphism ϕ : $L^n \otimes_K K_v \to \bigoplus_{w\in\mathfrak{W}_v} L_w^n$. Let $P_w : L^n \otimes_K K_v \to L_w^n$ be the composition of ϕ and the projection from $\bigoplus_{w\in\mathfrak{W}_v} L_w^n$ onto L_w^n. We define another height $G_{v,g}$ on $K_v^{nr} = L^n \otimes_K K_v$ by

$$G_{v,g}(x) = \sup_{w\in\mathfrak{W}_v} (H_{L_w^n, g_w}(P_w(x))^{1/r_w})\,.$$

Then, the inequality

$$F_{v,g}(x) \le r G_{v,g}(x)^2$$

holds for all $x \in K^{nr}$. Since $G_{v,g}$ is a norm on K_v^{nr} whose values are contained in the cyclic group generated by the order of the residual field of K_v, by [We, Chap. II, §1, Proposition 3], there exists $A_{v,g} \in GL_{nr}(K_v)$ such that $G_{v,g} = H_{K_v^{nr}, A_{v,g}}$. We would like to determine $|\det A_{v,g}|_v^2$. Let $\varepsilon_1^w, \cdots, \varepsilon_{r_w}^w$ be a basis of $\mathfrak{O}_{L_w}$ over $\mathfrak{O}_{K_v}$. Regarding each L_w^n as a K_v-vector space spanned by $\{\varepsilon_i^w e_j \ : \ 1 \le i \le r_w,\ 1 \le j \le n\}$, we have the regular representation $\rho : \prod_{w\in\mathfrak{W}_v} GL_n(L_w) \longrightarrow GL_{nr}(K_v)$. We set $\widehat{g}_v = \phi^{-1}\circ\rho((g_w)_{w\in\mathfrak{W}_v})\circ\phi$. Then,

$$\det \widehat{g}_v = \prod_{w\in\mathfrak{W}_v} \mathrm{Nr}_{L_w/K_v}(\det g_w)\,.$$

Since the set of $x \in K_v^{nr}$ such that $G_{v,g}(x) \le 1$ is equal to the $\mathfrak{O}_{K_v}$-lattice

$$\phi^{-1}\Bigl(\bigoplus_{w\in\mathfrak{W}_v} g_w^{-1}\mathfrak{O}_{L_w}^n\Bigr) = \widehat{g}_v^{-1}\phi^{-1}\Bigl(\bigoplus_{w\in\mathfrak{W}_v} \mathfrak{O}_{L_w}^n\Bigr),$$

$A_{v,g}$ is characterized by

$$(A_{v,g})^{-1}\mathfrak{O}_{K_v}^{nr} = \widehat{g_v}^{-1}\phi^{-1}\left(\bigoplus_{w\in\mathfrak{W}_v}\mathfrak{O}_{L_w}^n\right).$$

Let $B_w : L_w^n \times L_w^n \to K_v$ denote the bilinear form defined as

$$B_w\left(\sum_{j=1}^n \alpha_j e_j, \sum_{j=1}^n \beta_j e_j\right) = \sum_{j=1}^n \mathrm{Tr}_{L_w/K_v}(\alpha_j\beta_j).$$

Then $B = \bigoplus_{w\in\mathfrak{W}_v} B_w$ is a bilinear form on $\bigoplus_{w\in\mathfrak{W}_v} L_w^n$ and satisfies

$$B\left(\phi(\sum_{j=1}^n a_j e_j), \phi\left(\sum_{j=1}^n b_j e_j\right)\right) = \sum_{j=1}^n \mathrm{Tr}_{L/K}(a_j b_j) \quad \left(\sum_{j=1}^n a_j e_j,\ \sum_{i=j}^n b_j e_j \in L^n\right).$$

By computing the Gram matrices of both basis $\{u_i e_j \ : \ 1 \le i \le r,\ 1 \le j \le n\}$ and $\{\phi^{-1}(\varepsilon_i^w e_j) \ : \ 1 \le i \le r_w,\ w \in \mathfrak{W}_v,\ 1 \le j \le n\}$ of K_v^{nr} with respect to B, we obtain

$$\begin{aligned}&\Delta(u_1, \cdots, u_r)^n \det {}^t A_{v,g}^{-1} A_{v,g}^{-1}\\ &= \textstyle\prod_{w\in\mathfrak{W}_v} \det(\mathrm{Tr}_{L_w/K_v}(\varepsilon_s^w \varepsilon_t^w)_{1\le s,t\le r_w})^n \cdot \det {}^t\widehat{g_v}^{-1}\widehat{g_v}^{-1},\end{aligned}$$

and hence

$$|\det A_{v,g}|_{K_v}^2 = \frac{|\Delta(u_1, \cdots, u_r)|_{K_v}^n \prod_{w\in\mathfrak{W}_v} |\det g_w|_{L_w}^2}{\prod_{w\in\mathfrak{W}_v} \Delta_{L_w/K_v}^n}.$$

Putting (3), (I), (II) and (III) together, we obtain

$$\begin{aligned}\min_{0\ne x\in L^n} \frac{\prod_{w\in\mathfrak{V}_L} H_{L_w^n, g_w}(x)^{2/r}}{|\det g|_{\mathbf{A}_L}^{2/(nr)}} &\le \frac{1}{r^{\sharp(\mathfrak{V}_{K,\infty})}} \min_{0\ne x\in K^{nr}} \frac{\prod_{v\in\mathfrak{V}_K} H_{K_v^{nr}, A_{v,g}}(x)^2}{\prod_{w\in\mathfrak{V}_L} |\det g_w|_{L_w}^{2/(nr)}}\\ &= \frac{1}{r^{\sharp(\mathfrak{V}_{K,\infty})}} \frac{|\Delta(u_1, \cdots, u_r)|_{\mathbf{A}_K}^{1/r}}{\prod_{v\in\mathfrak{V}_{K,f}} \prod_{w\in\mathfrak{W}_v} \Delta_{L_w/K_v}^{1/r}}\\ &\quad\times \min_{0\ne x\in K^{nr}} \frac{\prod_{v\in\mathfrak{V}_K} H_{K_v^{nr}, A_{v,g}}(x)^2}{\prod_{v\in\mathfrak{V}_K} |\det A_{v,g}|_{K_v}^{2/(nr)}}.\end{aligned}$$

Therefore, by (2) and the product formula,

$$\left(\min_{0\ne x\in L^n} \frac{\prod_{w\in\mathfrak{V}_L} H_{L_w^n, g_w}(x)^2}{|\det g|_{\mathbf{A}_L}^{2/n}}\right)^{1/r} \le \frac{\Delta_L^{1/r}}{\Delta_K} \frac{\widehat{\gamma}_{nr}(K)}{r^{\sharp(\mathfrak{V}_{K,\infty})}}.$$

Since the right hand side is independent of $g \in GL_n(\mathbf{A}_L)$, this concludes that

$$\widehat{\gamma}_n(L) \le \frac{\Delta_L}{\Delta_K^r} \left(\frac{\widehat{\gamma}_{nr}(K)}{r^{\sharp(\mathfrak{V}_{K,\infty})}}\right)^r.$$

This completes the proof.

If K is an algebraic number filed, then Theorem immediately implies

$$\frac{\gamma_n(L)}{\Delta_L^{1/[L:\mathbf{Q}]}} \leq \frac{1}{r^{\sharp(\mathfrak{V}_{K,\infty})/[K:\mathbf{Q}]}} \cdot \frac{\gamma_{nr}(K)}{\Delta_K^{1/[K:\mathbf{Q}]}} .$$

If K is a function field, Theorem is described as

$$\left(\frac{\widehat{\gamma}_n(L)}{\Delta_L}\right)^n \leq \left(\frac{\widehat{\gamma}_{nr}(K)}{\Delta_K}\right)^{nr} .$$

It is known by [Wa, Theorem 7] that $\widehat{\gamma}_n(K)$ is a power of q_K^2 with $1 \leq \widehat{\gamma}_n(K) \leq q_K^{2g_K}$, where q_K denotes the order of the constant field of K.

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