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Some Asymptotic Formulas Involving Primes in Arithmetic Progressions

by

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Abstract. Let $\pi(x,q,a)$ be the number of primes $\le x$ that are congruent to a modulo q. We show that for any real numbers $\alpha_1,\cdots,\alpha_n$, not all zero, any distinct positive numbers $\beta_1,\cdots,\beta_n$, and any integers $q_1,\cdots,q_n$, $a_1,\cdots,a_n$, with $q_1,\cdots,q_n\ge 1$ and a_j relatively prime to q_j, for $1\le j\le n$, there exists an integer m, with $-1\le m\le n$, such that the limit $\lim_{x\to\infty} x\log^m x\sum_{j=1}^n \frac{\alpha_j}{\pi(\beta_j x,q_j,a_j)}$ exists, is finite and non-zero. Thus the sum has constant sign whenever $x\ge x_0$, for some positive real number x_0. The size of x_0 is given explicitly in terms of the parameters. Another consequence is a fact that witness in favor of the irregularity of $\pi(x,q,a)$. This states that the functions $\frac{x}{\pi(\beta x,q,a)}$ and $\frac{1}{\pi(\beta x,q,a)}$ are neither concave, nor convex.

1. Introduction

The Prime Number Theorem that gives the asymptotic behavior of $\pi(x)$, the function that counts the number of primes $\le x$, states that $\pi(x)\sim \mathrm{li}(x)$, as $x\to\infty$. Relying on extensive numerical computations, the formula was conjectured by Gauss in his early teens, and proved in a slightly stronger form, for the first time by J. Hadamard and Ch. de la Vallée-Poussin in 1896. Since the logarithmic integral $\mathrm{li}(x)=\mathrm{li}(e)+\int_e^x \frac{1}{\log t}\,dt$, in which $\mathrm{li}(e)=1.895\cdots$, is a hardly tractable transcendental function, during the ages PNT was stated in many different forms. Correcting a conjecture of Legendre, one of them is:

$$\pi(x)=\frac{x}{\log x-1-\frac{c_1}{\log x}-\cdots-\frac{c_n(1+a_n(x))}{\log^n x}}\,, \tag{1}$$

where $c_1,c_2,\cdots,c_n$ are integers given by the recurrence: $c_1=1$ and

$$c_n+1!c_{n-1}+2!c_{n-2}+\cdots+(n-1)!c_1=n\cdot n!\,,\quad \text{for } n\ge 2\,, \tag{2}$$

and $\lim_{x\to\infty}a_n(x)=0$ (see Panaitopol [5]). The coefficients c_n grow fast with n: $c_2=3$, $c_3=13$, $c_4=71$, $c_5=461$, $c_6=3447$, $c_7=29093,\cdots$, $c_{100}\approx 9.238\cdot 10^{159}$ and $\log c_n\gg n$ as $n\to\infty$.

Using the asymptotic formula (1), an estimate for the sum

$$S(x)=\sum_{n=2}^{[x]}\frac{1}{\pi(n)}$$

has been obtained in [5], improving on an earlier result of De Konink and Ivić [2]. Panaitopol [6] investigated the asymptotic behavior of the sum

$$S(\boldsymbol{\alpha}, \boldsymbol{\beta}; x) = \sum_{j=1}^{n} \frac{\alpha_j}{\pi(\beta_j x)}.$$

The components of $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$ are assumed to be any real numbers, not all zero, and those of $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_n)$ any distinct positive real numbers. It turns out that the size of the main term of $S(\boldsymbol{\alpha}, \boldsymbol{\beta}; x)$ is $x^{-1} \log^{-m} x$, for some integer m. More precisely, there exists an integer m, with $-1 \leq m \leq n$, such that the limit

$$\lim_{x \to \infty} \big(x \log^m x\big) S(\boldsymbol{\alpha}, \boldsymbol{\beta}; x)$$

exists, is finite and non-zero. As a consequence of this result, it follows that the sign of $S(\boldsymbol{\alpha}, \boldsymbol{\beta}; x)$ is constant for x large enough.

In the present paper we consider a similar problem for the counting function of the number of primes in an arithmetic progression. Given $a \geq 0$ and $q \geq 1$ relatively prime, let $\pi(x, q, a)$ be the number of primes $\leq x$ that are congruent to a modulo q. Our object is to study the asymptotic behavior of the sum

$$S(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}, \boldsymbol{a}; x) = \sum_{j=1}^{n} \frac{\alpha_j}{\pi(\beta_j x, q_j, a_j)}, \tag{3}$$

where the components of $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$ are real numbers, not all zero, and $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_n)$ with $\beta_1 > 0, \cdots, \beta_n > 0$, while the components of $\boldsymbol{q} = (q_1, \cdots, q_n)$ and $\boldsymbol{a} = (a_1, \cdots, a_n)$ are integers such that, for any j with $1 \leq j \leq n$, $q_j > 0$ and a_j is relatively prime with q_j. In particular, we are interested to see whether the sign of $S(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}, \boldsymbol{a}; x)$ is constant when x is large enough.

We remark that not always such a regularity of the sign exists. As was discovered a century and a half ago by Chebyshev, it appears that, at least for small values of x, the distribution of primes is biased towards certain arithmetic progressions. Chebyshev observed that there are more primes not exceeding x that are congruent to $3 \pmod 4$ than primes congruent to $1 \pmod 4$. Thus, if we let $n = 2, \alpha_1 = 1, \alpha_2 = -1, \beta_1 = \beta_2 = 1, q_1 = q_2 = 4, a_1 = 1$ and $a_2 = 3$, the sum

$$S(1, -1,\ 1, 1,\ 4, 4,\ 1, 3;\ x) = \frac{1}{\pi(x, 4, 1)} - \frac{1}{\pi(x, 4, 3)} \tag{4}$$

is positive for small values of x. The calculations of Leech [3] show that $x = 26861$ is the first value of x for which $\pi(x, 4, 1) > \pi(x, 4, 3)$. Earlier, Littlewood [4] proved that both sets $\{x\colon\ \pi(x, 4, 1) < \pi(x, 4, 3)\}$ and $\{x\colon\ \pi(x, 4, 3) < \pi(x, 4, 1)\}$ are unbounded, whence the sum $S(1, -1,\ 1, 1,\ 4, 4,\ 1, 3; x)$ fails to have constant sign for x large enough. For more on Chebyshev's bias, see Rubinstein and Sarnak [7] and the references therein.

In the following, in order to avoid situations as we saw that may occur for sums as in (4), we assume that the components of $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_n)$ are all distinct. The next theorem proves that when this happens, the total "noise" superimposed in $S(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}, \boldsymbol{a}; x)$ diminishes just enough to assure an asymptotic behavior.

THEOREM 1. *Let $\alpha_1, \cdots, \alpha_n$ be real numbers, not all zero, let $\beta_1, \cdots, \beta_n$ be distinct positive numbers, and let $q_1, \cdots, q_n$, $a_1, \cdots, a_n$ be integers, with $q_1, \cdots, q_n \geq 1$ and $(a_j, q_j) = 1$, for $1 \leq j \leq n$. Then there exists an integer m, with $-1 \leq m \leq n$, such that the limit*

$$\lim_{x\to\infty} \big(x \log^m x\big) S(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}, \boldsymbol{a}; x)$$

exists, is finite and non-zero.

This gives immediately:

COROLLARY 1. *Under the hypotheses of Theorem 1, there exists $x_0 = x_0(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}, \boldsymbol{a}) > 0$ such that the sign of $S(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}, \boldsymbol{a}; x)$ is constant for $x \geq x_0$.*

Then, a natural problem is to find x_0 as small as possible. In principle, x_0 might be quite large. A reason that motivates this possibility is that for one of the Dirichlet characters χ, whose conductor divides one of the moduli $q_1, \cdots, q_n$, the associated Dirichlet L-function $L(s, \chi)$ might have a Siegel zero. For instance, suppose χ is a primitive Dirichlet character $\pmod q$ and $L(s, \chi)$ has a Siegel zero, that is a zero exceptionally close to $s = 1$. Then, for x not very large in terms of q, the value of $\pi(x, q, a)$ will be about two times larger than normal for half of the residue classes $a \pmod q$ with $(a, q) = 1$, while for the other half, $\pi(x, q, a)$ will be much smaller than normal. Thus, if a_1 and a_2 lie in the first and respectively in the second of these halfs, the sum

$$S(1, -1,\ 1, 2,\ q, q,\ a_1, a_2;\ x) = \frac{1}{\pi(x, q, a_1)} - \frac{1}{\pi(2x, q, a_2)} \tag{5}$$

will be negative when x is not too large. But, as $x \to \infty$, the sum $S(1, -1,\ 1, 2,\ q, q,\ a_1, a_2;\ x)$ becomes positive. Hence in this case x_0 will have to be quite large. We shall address the problem of finding the size of $x_0(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}, \boldsymbol{a})$ in Section 4, and in Theorem 3 we shall give explicitly an acceptable value of x_0.

In the last section we give another application of Theorem 1. This is somehow in opposition with Corollary 1, which may be interpreted as an argument in favor of the regularity of $\pi(\beta x, q, a)$. We show that from the same Theorem 1 it follows that none of the functions $\frac{x}{\pi(\beta x, q, a)}$ and $\frac{1}{\pi(\beta x, q, a)}$ is concave or convex.

2. An asymptotic formula for $\pi(x, q, a)$

By the Siegel-Walfisz Theorem (cf. Davenport [1, Chapter 22]), we know that given a positive number N, there is a constant $c(N) > 0$ such that

$$\psi(x, q, a) = \frac{x}{\varphi(q)} + O(xe^{-c(N)\sqrt{\log x}}), \tag{6}$$

uniformly for any positive integer q in the range

$$q \leq \log^N x\,, \tag{7}$$

and any integer a relatively prime with q. Here $\psi(x,q,a)$ is the Chebyshev step function defined by

$$\psi(x,q,a) = \sum_{\substack{n\leq x\\ n\equiv a \pmod q}} \Lambda(n)\,,$$

where $\Lambda(n)$ is the von Mangoldt function,

$$\Lambda(n) = \begin{cases} \log p\,, & \text{if } n = p^k,\ p \text{ prime}\\ 0\,, & \text{otherwise}\,. \end{cases}$$

Translated for $\pi(x,q,a)$, the theorem of Siegel-Walfisz states that for any N there is a constant $c(N) > 0$ such that

$$\pi(x,q,a) = \frac{1}{\varphi(q)}\mathrm{li}(x) + O(xe^{-c(N)\sqrt{\log x}})\,, \tag{8}$$

uniformly for any q in the range (7) and any a relatively prime to q. For any positive integer k, we denote

$$I_k(x) = \int_e^x \frac{dt}{\log^k t}\,. \tag{9}$$

Thus $\mathrm{li}(x) = I_1(x) + O(1)$. Integrating by parts the integral (9), we find that

$$I_k(x) = \frac{x}{\log^k x} + kI_{k+1}(x) - e\,. \tag{10}$$

Multiplying with $(k-1)!$, we put (10) in the form

$$(k-1)!\,I_k(x) = \frac{(k-1)!\,x}{\log^k x} + k!\,I_{k+1}(x) + O\big((k-1)!\big)\,. \tag{11}$$

By a repeated application of (11), we see that, for any positive integer r,

$$\begin{aligned} I_1(x) &= \frac{x}{\log x} + I_2(x) + O(1)\\ &= \frac{x}{\log x} + \frac{x}{\log^2 x} + 2I_3(x) + O(1) = \cdots\\ &= \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2!\,x}{\log^3 x} + \cdots\\ &\quad + \frac{r!\,x}{\log^{r+1} x} + \frac{(r+1)!\,x}{\log^{r+2} x} + (r+2)!\,I_{r+3}(x) + O\big((r+1)!\big)\,. \end{aligned} \tag{12}$$

In what follows, we assume that r is bounded in the range

$$1 \leq r \leq \frac{\log x}{2}\,. \tag{13}$$

Then, from (12) we derive

$$\begin{aligned}\mathrm{li}(x) &= \sum_{k=1}^{r+1} \frac{(k-1)!\,x}{\log^k x} + O\left(\frac{(r+1)!\,x}{\log^{r+2} x}\right) \\ &= \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2!}{\log^2 x} + \cdots + \frac{r!}{\log^r x} + O\left(\frac{(r+1)!}{\log^{r+1} x}\right)\right).\end{aligned} \tag{14}$$

Inverting the series

$$1 + \frac{1}{\log x} + \frac{2!}{\log^2 x} + \cdots + \frac{r!}{\log^r x} + \cdots$$

considered as a power series in the variable $1/\log x$, with integer coefficients, we see that its inverse has the form

$$1 - \frac{1}{\log x} - \frac{c_1}{\log^2 x} - \cdots - \frac{c_r}{\log^{r+1} x} - \cdots,$$

and the coefficients $c_1, c_2, \cdots, c_r, \cdots$ satisfy the recurrence relation (2). Next we truncate the series and consider the product

$$\begin{aligned}P_r(x) &= \left(1 + \frac{1}{\log x} + \frac{2!}{\log^2 x} + \cdots + \frac{(r+1)!}{\log^{r+1} x}\right)\left(1 - \frac{1}{\log x} - \frac{c_1}{\log^2 x} - \cdots - \frac{c_r}{\log^{r+1} x}\right) \\ &= 1 + \frac{D_{r+2}}{\log^{r+2} x} + \frac{D_{r+3}}{\log^{r+3} x} + \cdots + \frac{D_{2r+2}}{\log^{2r+2} x}.\end{aligned} \tag{15}$$

Here each D_j is a sum of at most $r+2$ terms, and any such term has the form $c_l s!$, with $l + s + 1 = j$. It follows easily that $D_j = O(j!)$ for any j. Under the assumption (13), we find that

$$P_r(x) = 1 + O\left(\frac{(r+2)!}{\log^{r+2} x}\right). \tag{16}$$

Putting together the equalities (14), (15) and (16), we get

$$\begin{aligned}&\frac{\log x}{x}\left(1 - \frac{1}{\log x} - \frac{c_1}{\log^2 x} - \cdots - \frac{c_r}{\log^{r+1} x}\right)\mathrm{li}(x) \\ &= \left(1 + \frac{1}{\log x} + \cdots + \frac{(r+1)!}{\log^{r+2} x} + O\left(\frac{(r+2)!}{\log^{r+2} x}\right)\right) \\ &\times \left(1 - \frac{1}{\log x} - \frac{c_1}{\log^2 x} - \cdots - \frac{c_r}{\log^{r+1} x}\right) \\ &= 1 + O\left(\frac{(r+2)!}{\log^{r+2} x}\right),\end{aligned}$$

which yields

$$\mathrm{li}(x) = \frac{x}{\log x - 1 - \frac{c_1}{\log x} - \cdots - \frac{c_r}{\log^r x} + O\left(\frac{(r+2)!}{\log^{r+1} x}\right)}. \tag{17}$$

Now we assume that N is a fixed positive number and $0 < c(N) \le 1$ is a constant satisfying (8). We allow q, a, r and $x > e^e$ to vary subject to the constraint (7), and additionally we suppose that

$$r \le \frac{c(N)\sqrt{\log x}}{2\log\log x}. \tag{18}$$

Then r will also satisfy condition (13). Using (17) in (8), we get the required asymptotic formula for $\pi(x, q, a)$. We state it in the next theorem.

THEOREM 2. *Fix an $N > 0$ and a constant $c(N) \in (0, 1)$ satisfying* (8)*. Then, for any positive integers q and r, any integer a relatively prime to q, and any real number $x > e^e$ satisfying* (7) *and* (18)*, we have*

$$\pi(x, q, a) = \frac{x}{\varphi(q)\Big(\log x - 1 - \frac{c_1}{\log x} - \cdots - \frac{c_r}{\log^r x} + O\Big(\frac{(r+2)!}{\log^{r+1} x}\Big)\Big)}. \tag{19}$$

3. Proof of Theorem 1

A prerequisite needed for the proof of Theorem 1 is an appropriate formula for $1/\pi(\beta x, q, a)$. This is the scope of the following lemma.

LEMMA 1. *Fix an $N > 0$ and a constant $c(N) \in (0, 1)$ satisfying* (8)*. Let q, r be positive integers, let a be an integer, relatively prime to q, and let $\beta > 0$. Then, for any real number x satisfying the four constraints*

$$|\log \beta| \le \sqrt{\log x}\log\log x\,, \quad q \le \log^N(\beta x)\,, \quad \min\{x, \beta x\} \ge e^e$$

and

$$r \le \frac{c(N)}{2}\min\left\{\frac{\sqrt{\log(\beta x)}}{\log\log(\beta x)}, \frac{\sqrt{\log x}}{\log\log x}\right\},$$

we have

$$\frac{1}{\pi(\beta x, q, a)} = \varphi(q)\Big(\frac{\log x}{\beta x} + \frac{\log\beta - 1}{\beta x} + \sum_{i=1}^{r}\frac{t_i(\beta)}{x\log^i x} + O\Big(\frac{1 + |\log\beta|^r}{\beta}\cdot\frac{(r+2)!}{x\log^{r+1} x}\Big)\Big),$$

where

$$t_i(\beta) = \sum_{j=0}^{i-1}(-1)^{i-j}c_{j+1}\binom{i-1}{j}\frac{\log^{i-j-1}\beta}{\beta}$$

and $c_1, c_2, \cdots$, are defined by the recurrence relation (2)*.*

Proof. Notice first that if $\beta = 1$ the conclusion follows from (19). Next, replacing x by βx in (19), it follows that

$$\begin{aligned}\frac{1}{\pi(\beta x, q, a)} = \varphi(q)\Bigg(&\frac{\log \beta x}{\beta x} - \frac{1}{\beta x} - \frac{c_1}{\beta x \log \beta x} - \cdots - \frac{c_r}{\beta x \log^r \beta x} \\ &+ O\bigg(\frac{(r+2)!}{\beta x \log^{r+1}(\beta x)}\bigg)\Bigg).\end{aligned} \tag{20}$$

Note that since

$$\frac{|\log \beta|}{\log x} \le \frac{\log\log x}{\sqrt{\log x}} \le \frac{1}{2r}\,,$$

we have

$$\frac{\log^{r+1}(\beta x)}{\log^{r+1} x} = \left(\frac{\log x + \log \beta}{\log x}\right)^{r+1} = \left(1 + \frac{\log \beta}{\log x}\right)^{r+1} = O(1),$$

and

$$\frac{\log^{r+1} x}{\log^{r+1}(\beta x)} = \left(1 + \frac{\log \beta}{\log x}\right)^{-r-1} = O(1).$$

Therefore the error term on the right side of (20) may be replaced by $O\big(\frac{(r+2)!}{\beta x \log^{r+1} x}\big)$. For any $i \ge 1$, a generic term in the expansion (20) can be written as

$$\frac{-c_i}{\beta x \log^i \beta x} = \frac{-c_i}{\beta x (\log x + \log \beta)^i} = \frac{-c_i}{\beta x \log^i x \big(1 + \frac{\log \beta}{\log x}\big)^i}\,.$$

On the other hand, when $|y| < 1$, the binomial formula gives

$$\frac{1}{(1+y)^i} = 1 - \binom{i}{1} y + \binom{i+1}{2} y^2 + \cdots + (-1)^m \binom{i+m-1}{m} y^m + \cdots\,,$$

so, for $y = \frac{\log \beta}{\log x}$ we get

$$\begin{aligned}\frac{-c_i}{\beta x \log^i \beta x} = &-\frac{c_i}{\beta} \cdot \frac{1}{x \log^i x} + \frac{c_i \binom{i}{1} \log \beta}{\beta} \cdot \frac{1}{x \log^{i+1} x} - \frac{c_i \binom{i+1}{2} \log^2 \beta}{\beta} \cdot \frac{1}{x \log^{i+2} x} \\ &+ \cdots + (-1)^{r-i+1} \frac{c_i \binom{r-1}{r-i} \log^{r-i} \beta}{\beta} \cdot \frac{1}{x \log^r x} \\ &+ O\bigg(\frac{i^2 r!}{(r-i+1)!} \cdot \frac{1 + |\log \beta|^r}{\beta x \log^{r+1} x}\bigg).\end{aligned}$$

By combining this with (20), we obtain

$$\begin{aligned}\frac{1}{\pi(\beta x, q, a)} &= \varphi(q)\biggl(\frac{\log x}{\beta x} + \frac{\log\beta - 1}{\beta x} - \frac{c_1}{\beta x\log x} + \frac{c_1\binom{1}{0}\log\beta - c_2\binom{1}{1}}{\beta x\log^2 x} + \cdots \\ &\quad + \frac{(-1)^r c_1\binom{r-1}{0}\log^{r-1}\beta + (-1)^{r-1}c_2\binom{r-1}{1}\log^{r-2}\beta + \cdots - c_r\binom{r-1}{r-1}}{\beta x\log^r x} \\ &\quad + O\Bigl(\frac{1+|\log\beta|^r}{\beta}\cdot\frac{(r+2)!}{x\log^{r+1}x}\Bigr)\biggr),\end{aligned}$$

which is the required formula. This completes the proof of the lemma. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1. The proof goes along the same lines as the proof of Theorem 2.1 from [6]. Let $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n, q_1, \cdots, q_n, a_1, \cdots, a_n$ be as in the statement of the theorem. Let x be large enough so that Lemma 1 is applicable with q, a, r, β replaced by q_j, a_j, n, β_j for any $j \in \{1, 2, \cdots, n\}$. We then have

$$\begin{aligned}\sum_{j=1}^{n}\frac{\alpha_j}{\pi(\beta_j x, q_j, a_j)} &= \sum_{j=1}^{n}\alpha_j\varphi(q_j)\biggl(\frac{\log x}{\beta_j x} + \frac{\log\beta_j - 1}{\beta_j x} + \sum_{k=1}^{n}\frac{t_k(\beta_j)}{x\log^k x} \\ &\quad + O\Bigl(\frac{1+|\log\beta_j|^n}{\beta_j}\cdot\frac{(n+2)!}{x\log^{n+1}x}\Bigr)\biggr) \\ &= \Bigl(\sum_{j=1}^{n}\frac{\alpha_j\varphi(q_j)}{\beta_j}\Bigr)\frac{\log x}{x} + \Bigl(\sum_{j=1}^{n}\frac{\alpha_j\varphi(q_j)(\log\beta_j - 1)}{\beta_j}\Bigr)\frac{1}{x} \\ &\quad + \sum_{k=1}^{n}\Bigl(\sum_{j=1}^{n}\alpha_j\varphi(q_j)\,t_k(\beta_j)\Bigr)\frac{1}{x\log^k x} \\ &\quad + O\Bigl(\sum_{j=1}^{n}|\alpha_j|\varphi(q_j)\cdot\frac{1+|\log\beta_j|^n}{\beta_j}\cdot\frac{(n+2)!}{x\log^{n+1}x}\Bigr).\end{aligned} \tag{21}$$

We claim that not all the coefficients of $\log x/x$, $1/x$ and $1/(x\log^k x)$, for $1 \le k \le n$, in the right-hand side of relation (21) vanish. For if

$$U_{-1}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}) := \sum_{j=1}^{n}\frac{\alpha_j\varphi(q_j)}{\beta_j} = 0\,,$$

$$U_0(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}) := \sum_{j=1}^{n}\frac{\alpha_j\varphi(q_j)(\log\beta_j - 1)}{\beta_j} = 0\,,$$

$$U_k(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}) := \sum_{j=1}^{n}\alpha_j\varphi(q_j)t_k(\beta_j) = 0\,, \quad \text{for} \quad k = 1, 2, \cdots, n\,,$$

proceeding recursively, we also have:

$$- c_1 \sum_{j=1}^{n} \frac{\alpha_j \varphi(q_j)}{\beta_j} = 0\,,$$

$$c_1 \binom{1}{0} \sum_{j=1}^{n} \frac{\alpha_j \varphi(q_j)}{\beta_j} \log \beta_j - c_2 \binom{1}{1} \sum_{j=1}^{n} \frac{\alpha_j \varphi(q_j)}{\beta_j} = 0,$$

$$\vdots$$

$$(-1)^n c_1 \binom{n-1}{0} \sum_{j=1}^{n} \alpha_j \varphi(q_j) \frac{\log^{n-1} \beta_j}{\beta_j} + (-1)^{n-1} c_2 \binom{n-1}{1} \sum_{j=1}^{n} \alpha_j \varphi(q_j) \frac{\log^{n-2} \beta_j}{\beta_j}$$

$$+ \cdots - c_n \binom{n-1}{n-1} \sum_{j=1}^{n} \frac{\alpha_j \varphi(q_j)}{\beta_j} = 0.$$

Then, it follows that

$$(\mathcal{S}) \begin{cases} \displaystyle\sum_{j=1}^{n} \frac{\alpha_j \varphi(q_j)}{\beta_j} = 0\,, \\ \displaystyle\sum_{j=1}^{n} \frac{\alpha_j \varphi(q_j)}{\beta_j} \log \beta_j = 0\,, \\ \quad\vdots \\ \displaystyle\sum_{j=1}^{n} \frac{\alpha_j \varphi(q_j)}{\beta_j} \log^{n-1} \beta_j = 0\,. \end{cases}$$

We consider $(\mathcal{S})$ as an homogeneous linear system with indeterminates

$$\frac{\alpha_1 \varphi(q_1)}{\beta_1}, \quad \frac{\alpha_2 \varphi(q_2)}{\beta_2}, \cdots, \frac{\alpha_n \varphi(q_n)}{\beta_n}\,.$$

The determinant of this system is a Vandermonde determinant. Since $\log \beta_i \neq \log \beta_j$ for $i \neq j$, the system has only the trivial solution

$$\frac{\alpha_1 \varphi(q_1)}{\beta_1} = \frac{\alpha_2 \varphi(q_2)}{\beta_2} = \cdots = \frac{\alpha_n \varphi(q_n)}{\beta_n} = 0\,.$$

But this contradicts our hypothesis that not all α_j with $1 \leq j \leq n$ vanish. Therefore, at least one of the equations in $(\mathcal{S})$ fails and this proves our claim.

Let $i_0 \in [1, n]$ be the smallest integer for which

$$\sum_{j=1}^{n} \frac{\alpha_j \varphi(q_j)}{\beta_j} \log^{i_0 - 1} \beta_j \neq 0\,.$$

Then, for x large enough, the term corresponding to i_0 will dominate the remaining terms on the right hand side of (21). Thus, putting $m = i_0 - 2$ if $i_0 = 1, 2$ and $m = i_0$ if $i_0 \geq 3$, we conclude that the limit

$$\lim_{x\to\infty} x \log^m x \sum_{j=1}^{n} \frac{\alpha_j}{\pi(\beta_j x, q_j, a_j)}$$

exists and it is non-zero. This completes the proof of Theorem 1.

4. The size of $x_0(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}, \boldsymbol{a})$

We now turn to the problem of finding a number x_0, depending on a set of numbers $\{\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n, q_1, \cdots, q_n, a_1, \cdots, a_n\}$ as in Theorem 1, with the property that the sign of $S(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}, \boldsymbol{a}; x)$ is constant for $x \geq x_0$. We saw in the proof of Theorem 1 that the sums

$$T_i(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}) := \sum_{j=1}^{n} \frac{\alpha_j \varphi(q_j) \log^{i-1} \beta_j}{\beta_j}, \quad i = 1, 2, \cdots, n,$$

play an important role in this problem. More precisely, if i_0 is the smallest positive integer for which $T_{i_0}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}) \neq 0$, then for x large enough, $S(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}, \boldsymbol{a}; x)$ has the same sign as $(-1)^{i_0} T_{i_0}$ for $i_0 \geq 2$, and has the sign of T_1 if $i_0 = 1$. Let us remark that there are three types of conditions subject to which $x_0(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}, \boldsymbol{a})$ should respond:

A. The hypotheses of Lemma 1.

B. The first non-zero term should dominate the other components of the main term on the right side of (21).

C. The first non-zero term should dominate the $O(\cdot)$ term in (21).

Next we treat each item separately and eventually we put the results together.

A. The application of Lemma 1 a number n times in the beginning of the proof of Theorem 1 imposes a series of conditions that are all satisfied if $x \geq x_1$, where

$$x_1 := \max_{1\leq j\leq n} \left(\left(1 + \frac{1}{\beta_j}\right) \max\left\{ e^{e^{\frac{\log q_j}{N}}};\ (\beta_j + 1)e^e;\ e^{\log^2 \beta_j};\ e^{\frac{(8n)^4}{c(N)^4}} \right\} \right). \tag{22}$$

B. We assume that (22) holds, and look for further constraints x needs to satisfy. Dropping for simplicity the parameters in the notation $U_j(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q})$, the main term on the right-hand side of (21) is

$$M := U_{-1}\frac{\log x}{x} + U_0 \frac{1}{x} + U_1 \frac{1}{x \log x} + \cdots + U_n \frac{1}{x \log^n x}. \tag{23}$$

Here if $i_0 = 1$, then $U_{-1} = T_1 \neq 0$. If $i_0 = 2$, then $U_{-1} = 0$ and $U_0 = T_2 \neq 0$. If $i_0 \geq 3$, then $U_{-1} = U_0 = U_1 = \cdots = U_{i_0-1} = 0$, and $U_{i_0} = (-1)^{i_0} T_{i_0} \neq 0$. Let

$$U = U(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}) := \max_{-1\leq j\leq n} \big(|U_j(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q})|, 1\big).$$

In order for the first non-zero term in (23) to dominate the sum of the others, it suffices to have

$$|T_{i_0}|\frac{1}{x\log^{i_0} x} \geq \frac{2U}{x\log^{i_0+1} x}\cdot\frac{1-\frac{1}{\log^{n-i_0} x}}{1-\frac{1}{\log x}},$$

and this happens if we suppose that $x \geq x_2$, where

$$x_2 := e^{\frac{2U}{|T_{i_0}|}}. \tag{24}$$

C. In order for the first non-zero term to dominate the error term on the right side of (21), it suffices to let x be large enough so that

$$|T_{i_0}|\log^{n+1-i_0} x \geq C(n+2)!\sum_{j=1}^{n}|\alpha_j|\varphi(q_j)\frac{1+|\log\beta_j|^n}{\beta_j},$$

for a suitable absolute constant $C > 0$, implied by the $O(\cdot)$ symbol on the right side of (21). The above inequality holds for $x \geq x_3$, where

$$x_3 := \exp\left(\left(\frac{C(n+2)!}{|T_{i_0}|}\sum_{j=1}^{n}|\alpha_j|\varphi(q_j)\frac{1+|\log\beta_j|^n}{\beta_j}\right)^{1/(n+1-i_0)}\right).$$

We now let

$$x_0 := \max\{x_1, x_2, x_3\}, \tag{25}$$

with x_1, x_2, x_3 defined as above. Then, for $x \geq x_0$ the first non-zero term on the right side of (21) will dominate the other terms, and hence $S(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}, \boldsymbol{a}; x)$ will have the same sign as the first non-zero term. We state the result in the next theorem.

THEOREM 3. *Fix an $N > 0$ and a constant $0 < c(N) \leq 1$ satisfying* (8)*. Let $\alpha_1, \cdots, \alpha_n$ be real numbers, not all zero, let $\beta_1, \cdots, \beta_n$ be distinct positive numbers, and let $q_1, \cdots, q_n$, $a_1, \cdots, a_n$ be integers with $q_1, \cdots, q_n \geq 1$ and a_j relatively prime with q_j, for $1 \leq j \leq n$. Denote by i_0 the smallest positive integer for which $T_{i_0}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}) \neq 0$, and let $x_0(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}, \boldsymbol{a})$ be defined by* (25)*. Then the sum $\sum_{j=1}^{n}\frac{\alpha_j}{\pi(\beta_j x, q_j, a_j)}$ has constant sign for $x \geq x_0$.*

5. Special cases

We conclude with an application of Theorem 1 that gets bounds for two finite sums involving the inverse of $\pi(\beta x, q, a)$, and then deduce a consequence that shows that neither convexity, nor concavity characterizes the functions $x/\pi(\beta x, q, a)$ and $1/\pi(\beta x, q, a)$.

THEOREM 4. *Let $n \geq 3$ be an integer, let $\beta_1, \beta_2, \cdots, \beta_{n-1}$ be distinct positive numbers with the property that none of them is equal to the arithmetic mean of the others, and let $q_1, \cdots, q_n$, $a_1, \cdots, a_n$ be integers with $q_1, \cdots, q_n \geq 1$ and a_j relatively prime with q_j, for $1 \leq j \leq n$. Then, for x large enough (or $x \geq x_0$, with x_0 as in Theorem 3), we have*

$$\sum_{i=1}^{n-1} \frac{\beta_i}{\varphi(q_i)\pi(\beta_i x, q_i, a_i)} < \frac{\beta_1 + \beta_2 + \cdots + \beta_{n-1}}{\varphi(q_n)\pi\left(\frac{\beta_1+\beta_2+\cdots+\beta_{n-1}}{n-1}x, q_n, a_n\right)}, \tag{26}$$

and

$$\sum_{i=1}^{n-1} \frac{1}{\varphi(q_i)\pi(\beta_i x, q_i, a_i)} > \frac{n-1}{\varphi(q_n)\pi\left(\frac{\beta_1+\beta_2+\cdots+\beta_{n-1}}{n-1}x, q_n, a_n\right)}. \tag{27}$$

Proof. We apply Theorem 1 for $\alpha_j = \frac{\beta_j}{\varphi(q_j)}$ when $1 \le j \le n-1$, $\alpha_n = -\frac{(n-1)\beta_n}{\varphi(q_n)}$ and $\beta_n = \frac{\beta_1+\cdots+\beta_{n-1}}{n-1}$. With the notations from the previous sections, one has firstly that $T_1(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}) = 0$, by our choice and secondly, $T_2(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}) < 0$ from classical inequalities. In view of the proof of Theorem 1, we know that $S(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q}, \boldsymbol{a}; x)$ has the same sign as $T_2(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{q})$, hence (26) holds. Similarly, if $\alpha_i = \frac{1}{(n-1)\varphi(q_i)}$, for any $1 \le i \le n-1$, $\alpha_n = \frac{-1}{\varphi(q_n)}$ and $\beta_n = \frac{\beta_1+\cdots+\beta_{n-1}}{n-1}$, we obtain (27). □

THEOREM 5. *Let q and a be integers with $q \ge 1$ and $(a, q) = 1$, and let $\beta > 0$. Then the functions $f(x) = x/\pi(\beta x, q, a)$ and $g(x) = 1/\pi(\beta x, q, a)$ are neither convex, nor concave.*

Proof. Let $r_1, r_2, \cdots, r_m, \cdots$, be the sequence of prime numbers that are $\equiv a \bmod q$. Choosing $x_1 = \frac{r_m - 1}{\beta}$ and $x_2 = \frac{r_m+1}{\beta}$, we have

$$f(x_1) + f(x_2) - 2f\left(\frac{x_1 + x_2}{2}\right) = \frac{r_m - 1}{\beta(m-1)} + \frac{r_m + 1}{\beta m} - \frac{2r_m}{\beta m} = \frac{r_m - 1}{\beta m(m-1)} > 0,$$

which, together with the first part of Theorem 4, imply that $f(x)$ is neither convex, nor concave.

For the second function, with $x_1,\ x_2$ as above and $\lambda = \frac{m-1}{m}$, we have

$$\lambda g(x_1) + (1-\lambda)g(x_2) - g\big(\lambda x_1 + (1-\lambda)x_2\big) = \frac{1}{m} + \frac{1}{m^2} - \frac{1}{m-1} < 0.$$

This shows that $g(x)$ is neither convex, nor concave, and concludes the proof of the theorem. □

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