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Primary Abelian Σ-Groups of Countable Length $\geq \omega.2$ are not Necessarily Summable

by

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Abstract. It is shown via a construction that for each countable ordinal $\alpha \geq \omega$ there exists a p-primary abelian Σ-group of length $\alpha + \omega$ which is not summable. In other words an example is given such that G being a p-torsion Σ-group with length $\alpha + \omega \geq \omega.2$ does not imply that the subgroup G^{p^α} is a coproduct of cyclics, i.e. it is not a Σ-group. Thus the class of all p-torsion abelian Σ-groups (with limit lengths L so that $\omega.2 \leq$L$\leq \Omega$) properly contains the class of all summable p-groups.

0. Introduction

In 1961, J. Irwin and E. Walker have introduced in [12] an important class of mixed abelian groups called *Σ-groups*. Later on, in 1964, K. Honda defined in [11] a new class of primary abelian groups named by him *principle groups* (usually termed in the algebraic literature as *summable p-groups*; see for instance [10], [14] and [7]). It is not difficult to be seen by the definitions and ([10] or [7], p. 125, Proposition 84.4) that every summable p-group is a Σ-group.

A question of some interest is of whether these two classes of abelian p-groups are independent, motivated of the fact we have argued in [2] that every p-primary abelian Σ-group of length not exceeding $\omega + n$, where $n \in N_0 = N \bigcup \{0\}$, is summable. The case for uncountable length Ω is examined by us in [5], and more especially we have proved that these two sorts of groups are absolutely different in this situation.

The main purpose of the present brief article is to give the answer to this problem for p-torsion multiplicatively written abelian groups with lengths $< \Omega$. All further notations and terminology will be the same as in the cited in the bibliography papers and classical books of L. Fuchs [7].

For a convenience of the readers, we emphasize certain basic facts and results necessary for the successful evidence of the central attainment.

PROPOSITION ([9], Theorem 3). *A countable ascending union of isotype summable subgroups is a summable group.*

PROPOSITION ([3]). *A countable ascending union of pure Σ-subgroups is a Σ-group.*

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We recall that the classes of summable p-groups and p-primary Σ-groups coincide in the separable case. More precisely all of them are coproducts of cyclics. Nevertheless, in the case of groups having (limit) lengths strictly more than ω, the situation is quite complicated.

Well, we concentrate on the constructing of a Σ-p-group of countable length that is not summable. This shall be made in the next paragraph.

1. A countable union of summable p-groups which is not summable

It is documented by us in [4, 5, 6] that if A is a summable p-group, then the same holds true even for $A^{p^{\alpha}}$ over each ordinal number α. However, this is not the case for Σ-groups when $\alpha \geq \omega$ as it will be demonstrated below. We establish in the sequel this crucial observation by using a special construction. The exhibition of a concrete example that such a group exists, we leave to the reader which must consult with ([8], Section 2).

And so, our major affirmation states as follows:

THEOREM (STRUCTURE). *Let G be an abelian p-group satisfying the following conditions*:

(1) $G = \bigcup_{n<\omega} G_n$, $G_n \subseteq G_{n+1}$ *and* G_n *is a summable pure subgroup of* G *with length* $\alpha+n$ *for some arbitrary but fixed ordinal* α: $\omega \leq \alpha < \Omega$ *and for each* $n < \omega$;

(2) $G_n \cap G^{p^{\alpha}} = G_n^{p^{\alpha}}$ *and* $|G_n^{p^{\alpha}}| = \aleph_1$ *for each* $n < \omega$;

(3) $G_n^{p^{\alpha}} \cap G^{p^{\alpha+m}} \neq 1$ *for each* $m < \omega$.

Then $G^{p^{\alpha}}$ *is not a coproduct of cyclic groups.*

Proof. It is constructive. Suppose α is an arbitrary but fixed countable ordinal and suppose $G = \bigcup_{n<\omega} G_n$, where $G_n \leq G_{n+1}$, all G_n are summable p-groups of length $\alpha+n$ so that G_n are pure in G and $G_n \cap G^{p^{\alpha}} = G_n^{p^{\alpha}}$ (in particular G_n may be assumed to be weakly p^{α}-pure in G) while $G_n^{p^{\alpha}} \cap G^{p^{\alpha+n}} \neq 1$ for each $n \in N$. Under these suppositions, we routine observe that G_n are not isotype subgroups of G that is equivalent to $G_n^{p^{\alpha}}$ are not pure in $G^{p^{\alpha}}$. Indeed, $G_n \cap G^{p^{\alpha+n}} \neq 1 = G_n^{p^{\alpha+n}}$, otherwise $G_n^{p^{\alpha}} \cap G^{p^{\alpha+n}} = 1$, contrary to the hypothesis.

Moreover, let $|G_n^{p^{\alpha}}| = \aleph_1$ for every $n < \omega$ whence $G_n^{p^{\alpha}} = \bigcup_{\gamma<\Omega} A_{\gamma}^{(n)}$, where $A_{\gamma}^{(n)} \leq G_n^{p^{\alpha}}$ are subgroups possessing the following properties: $A_{\gamma}^{(n)} \subseteq A_{\gamma+1}^{(n)}$, $A_{\beta}^{(n)} = \bigcup_{\gamma<\beta} A_{\gamma}^{(n)}$ for each limit ordinal $\beta < \Omega$, $|A_{\gamma}^{(n)}| \leq \aleph_0$ and $\mathrm{A}_{\gamma}^{(n)}$ are bounded at p^n. Besides, because $G_n \subseteq G_{n+1}$ and $A_{\gamma}^{(n+1)}$ are bounded by p^{n+1}, we can additionally take the inclusion $A_{\gamma}^{(n)} \subseteq A_{\gamma}^{(n+1)}$ for all $n < \omega$ and $\gamma < \Omega$.

Therefore, since $G^{p^{\alpha}} = \bigcup_{n<\omega}[G_n \cap G^{p^{\alpha}}] = \bigcup_{n<\omega} G_n^{p^{\alpha}}$, we detect that $G^{p^{\alpha}} = \bigcup_{n<\omega} [\bigcup_{\gamma<\Omega} A_{\gamma}^{(n)}] = \bigcup_{\gamma<\Omega}[\bigcup_{n<\omega} A_{\gamma}^{(n)}] = \bigcup_{\gamma<\Omega} B_{\gamma}$ by putting $B_{\gamma} = \bigcup_{n<\omega} A_{\gamma}^{(n)}$. Thereby $B_{\gamma} \subseteq B_{\gamma+1} \leq G^{p^{\alpha}}$ and $B_{\beta} = \bigcup_{\gamma<\beta} B_{\gamma}$ for each limit $\beta < \Omega$. If $G^{p^{\alpha}}$ has length $> \omega$, we are done. We now study the remaining case of length $\leq \omega$. We note then that B_γ is an unbounded coproduct of cyclics by referring to the first Pruefer theorem, documented in ([7]), since $|B_\gamma| \leq \aleph_0$ and length $(B_\gamma) \leq$ length $(G^{p^\alpha}) \leq \omega$; without harm of generality, we may presume that length $(G^{p^\alpha}) = \omega$ whence length $(G) = \alpha + \omega$. Thus $(G_n^{p^\alpha})^{p^n} = G_n^{p^{\alpha+n}} = 1$ and $(G^{p^\alpha})^{p^\omega} = G^{p^{\alpha+\omega}} = 1$. Certainly, the possibility $G_n^{p^\alpha} \cap G^{p^{\alpha+n}} = 1$ has been excluded from the assumptions of ours since otherwise the classical Kulikov's criterion ([6]) applies to get that G^{p^α} is a coproduct of cyclic groups.

Exploiting now the commentary from the introduction, all G_n are Σ-groups. Consequently, according to the above listed Proposition from [3], G should be a Σ-group, as well.

On the other hand, we shall deduce that G^{p^α} (notice that $|G^{p^\alpha}| \leq \aleph_1$) is not summable, i.e. it is not a coproduct of cyclic groups by making use of [7]. Hence by virtue of [4], [5] or [6], G is not itself summable, as desired.

In fact, since $B_\gamma \cap G^{p^{\alpha+m}} = [\bigcup_{n<\omega} A_\gamma^{(n)}] \cap [\bigcup_{k<\omega} G_k^{p^{\alpha+m}}] = \bigcup_{n<\omega} \bigcup_{k<\omega} [\ A_\gamma^{(n)} \cap G_k^{p^{\alpha+m}}]$ for every $m < \omega$ and since for some $\gamma < \Omega$ the subgroups $A_\gamma^{(n)}$ are not pure in G^{p^α} (otherwise it is a plain exercise to extract that $G_n^{p^\alpha}$ must be pure in G^{p^α} which is wrong), we can choose that either $\bigcup_{n<\omega} \bigcup_{k<\omega} [A_\gamma^{(n)} \cap G_k^{p^{\alpha+m}}] = \bigcup_{n<\omega} \bigcup_{k<\omega} [A_\gamma^{(n)} \cap \bigcup_{\delta<\Omega} A_\delta^{(k)p^m}] = \bigcup_{n<\omega} [A_\gamma^{(n)} \cap \bigcup_{\delta<\Omega} A_\delta^{(n)p^m}] = \bigcup_{n<\omega} \bigcup_{\delta<\Omega} [A_\gamma^{(n)} \cap A_\delta^{(n)p^m}] \neq \bigcup_{n<\omega} A_\gamma^{(n)p^m} = [\bigcup_{n<\omega} A_\gamma^{(n)}]^{p^m}$, i.e. equivalently $B_\gamma \cap G^{p^{\alpha+m}} \neq B_\gamma^{p^m}$, for some $m < \omega$, or $B_{\gamma+1}/B_\gamma$ is not separable when γ is a limit ordinal. Henceforth B_γ is either not pure in G^{p^α} or it is not closed in $B_{\gamma+1}$ whenever γ is a limit ordinal (see, for instance, [8], Theorem 1).

Utilizing now the well-known standard back-and-forth method due to Hill-Megibben (see, e.g., [8], Theorem 2), we shall show that G^{p^α} is not a coproduct of cyclics. In order to do this, suppose the reverse, i.e. that $G^{p^\alpha} = \sum_{i \in I} C_i$, where C_i are cyclics for each $i \in I$; the index set I is arbitrary. Let $\delta(0) = 1$ and let $I(1)$ be the unique minimal subset of I such that $B_{\delta(0)} \subseteq \sum_{i \in I(1)} C_i$. Since $B_{\delta(0)}$ is countable, so is $I(1)$. Thus, $H_1 = \sum_{i \in I(1)} C_i$ is countable and is, therefore, contained in B_δ for some countable δ. Let $\delta(1)$ be the first ordinal greater than $\delta(0)$ such that $H_1 \subseteq B_{\delta(1)}$. Because $B_{\delta(1)}$ is countable, there exists a unique minimal countable subset $I(2)$ of I so that $B_{\delta(1)} \subseteq \sum_{i \in I(2)} C_i$. Set $H_2 = \sum_{i \in I(2)} C_i$, and let $\delta(2)$ be the smallest ordinal greater than $\delta(1)$ such that $H_2 \subseteq B_{\delta(2)}$. Inductively, define $\delta(n)$ and $I(n)$, for every positive integer n, in the way that we have indicated. Since $\{\delta(n)\}$ is a strictly increasing sequence of countable ordinals, $\delta(\omega) = sup\{\delta(n)\}_{n<\omega}$ is a countable limit ordinal. Suppose $I(\omega) = \cup\cdot_{n<\omega} I(n)$. Since $B_{\delta(\omega)} = \bigcup_{n<\omega} B_{\delta(n)}$ and $\sum_{i \in I(n)} C_i \subseteq B_{\delta(n)} \subseteq \sum_{i \in I(n+1)} C_i$, we conclude that $\bigcup_{n<\omega} \sum_{i \in I(n)} C_i \subseteq \bigcup_{n<\omega} B_{\delta(n)} \subseteq \bigcup_{n<\omega} \sum_{i \in I(n+1)} C_i$, i.e. that $B_{\delta(\omega)} = \sum_{i \in I(\omega)} C_i$. Therefore, $B_{\delta(\omega)}$ is a direct factor of G^{p^α}, whence its pure subgroup. Furthermore, $G^{p^\alpha}/B_{\delta(\omega)} \cong \sum_{i \in I \setminus I(\omega)} C_i$, a direct factor of the separable G^{p^α}. It simple follows now that $B_{\delta(\omega)+1}/B_{\delta(\omega)}$, being a subgroup of $G^{p^\alpha}/B_{\delta(\omega)}$, is separable. But these two derivations contradict our assumptions from the text. Hence the claim.

So, the theorem is proved after all. °°°

The following consequences are immediate.

COROLLARY 1. *For every ordinal number* α *with* $\omega \leq \alpha < \Omega$ *there is a* Σ*-*p*-group* G *of length* $\alpha + \omega$ *such that* G^{p^α} *is not a* Σ*-group and* G *is not a summable group.*

COROLLARY 2. *For every countable limit ordinal* $\lambda > \omega$ *there exists a* Σ*-group* G *of length* λ *which is not summable.*

We point out that the following group criterion for summability, established in [4], namely that the abelian p-group G is summable $\Leftrightarrow$ G is a Σ-p-group and G^{p^ω} is summable, contrasts with the above given example.

The following are actual.

REMARK 1. Our main assertion listed above shows that the conditions for isotypity on the subgroups of the union required in ([9], p. 150, Theorem 3) are necessary and cannot be dropped—see also the Introduction. Moreover, it solves in negative our final question posed in [3].

REMARK 2. Following [1], it is easy to select an abelian p-group A with a countable basic subgroup such that $A^{p^\omega} \cong A/A^{p^\omega}$ are unbounded torsion-complete groups. But whether or not such an A is a Σ-group is difficult to prove and is unknown yet. However, in this way, an example due to Megibben [13] assures that there is a p-primary Σ-group G of length $\omega.2$ such that G^{p^ω} is unbounded torsion-complete. That is why such a group G is not summable. In our example, above obtained, G^{p^ω} need not be torsion-complete and can be an arbitrary separable group.

We conclude the work with two left-open problems (see [3], too) remain unanswered, namely:

PROBLEM 1. *Let* $T = \bigcup_{n<\omega} T_n$, $T_n \leq T_{n+1}$ *are pure in* T *and all* T_n *are torsion-complete* p*-groups. Then, what is the criterion illustrating* T *to be also torsion-complete?*

PROBLEM 2. *Let* $T = \bigcup_{n<\omega} T_n$, $T_n \leq T_{n+1}$ *and all* T_n *are bounded at* p^n*. Then, under what circumstances* T *is not a coproduct of cyclics and, in particular, is unbounded torsion-complete?*

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