

Generalized Whittaker Functions of the Degenerate Principal Series Representations of $SL(3, \mathbf{R})$

by

Taku ISHII and Takayuki ODA

(Received August 28, 2005)

(Revised September 15, 2005)

Introduction

This paper is an appendix to the memorial talk for Tsuneo Arakawa [3]. As discussed in the section of Hecke in [3], the Epstein zeta functions define certain Eisenstein series, i.e., automorphic forms on $SL(n, \mathbf{Z}) \backslash SL(n, \mathbf{R})$. The representations of $SL(n, \mathbf{R})$ generated by the right translations of these Eisenstein series are very degenerate principal series representations. Though the Epstein zeta functions themselves are quite familiar objects, but the number of papers which discuss them as automorphic forms is not so many in the literature. And the Fourier expansions of these Epstein-Eisenstein series as automorphic forms on $SL(n, \mathbf{Z}) \backslash SL(n, \mathbf{R})$ have been obtained by direct computation of integrals. In this paper we want to discuss them in more conceptual way for $n = 3$.

When $n = 3$, there are three types of parabolic subgroups in $SL(3, \mathbf{Q})$, and accordingly three types of corresponding Fourier expansions. In either case, it is fundamental to know Whittaker functions and generalized Whittaker functions. In this paper we have multiplicity-free results on (degenerate) Whittaker functions and generalized Whittaker functions on $SL(3, \mathbf{R})$ belonging to the degenerate principal series and find an explicit formula for them.

Our main results are Theorems 4.6, 4.7 (multiplicity-freeness of degenerate Whittaker models), and Theorems 5.6, 5.7. At least for the non-spherical cases, these results seem to be new. In section 6 we discuss three types of Fourier expansions of the automorphic forms belonging to the spherical principal series representations.

Note that the meaning of the classical result by Siegel [9, Chapter 1 §5, p. 46–55] on Epstein zeta functions with spherical harmonic polynomials becomes clear, since they correspond to the other vectors with different K -types in the automorphic representations generated by the original Epstein zeta function. Therefore we can know Fourier expansions of these generalized Epstein zeta functions, too.

ACKNOWLEDGEMENT. We thank Weng Lin, because his enthusiasm in non-abelian zeta functions for GL_3 was a strong impetus for us to consider the results of this paper.

1. Preliminaries

1.1. Degenerate principal series

Let $Z(g, s)$ be the Epstein zeta function on $G = SL(3, \mathbf{R})$, defined by

$$Z(g, s) := \sum_{\mathbf{m} \in \mathbf{Z}^3 \setminus \{0\}} (\mathbf{m} g^t g^t \mathbf{m})^{-s} \quad (g \in G, \operatorname{Re}(s) > 3/2).$$

Then it is an Eisenstein series of class 1 associated with the maximal parabolic subgroup

$$P_1 := \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\}$$

of G . Then the right translations of this with respect to G generate a degenerate principal series representation of G . We want to describe the (generalized) Whittaker functions belonging to this representation.

To define this representation, we firstly specify a Langlands decomposition $P_1 = N_1 M_1 A_1$ of P_1 by

$$N_1 := \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \in G \right\} \cong \mathbf{R}^2,$$

$$M_1 := \left\{ \begin{pmatrix} h & \mathbf{0} \\ \mathbf{0} & \det(h)^{-1} \end{pmatrix} \in G \mid \det h = \pm 1 \right\} \cong SL(2, \mathbf{R}) \times \{\pm 1\},$$

$$A_1 := \{\operatorname{diag}(r, r, r^{-2}) \mid r > 0\}.$$

Let $\sigma_1 \in \widehat{M}_1$ be a character of M_1 , and $\nu_1 \in \operatorname{Hom}_{\mathbf{R}}(\mathfrak{a}_1, \mathbf{C}) = \mathfrak{a}_1^* \otimes_{\mathbf{R}} \mathbf{C}$ a linear form on $\mathfrak{a}_1 = \operatorname{Lie}(A_1)$ which is identified with a complex number by evaluation of it at the element $H_1 = \operatorname{diag}(1, 1, -2) \in \mathfrak{a}_1$. Let ρ_1 be the half-sum of the positive roots in $\mathfrak{n}_1 = \operatorname{Lie}(N_1)$. Then $\rho_1 = \frac{1}{2}(3 + 3) = 3$. Thus for $a_1 = \operatorname{diag}(r, r, r^{-2}) \in A_1$, we have $e^{(\nu_1 + \rho_1)(\log a_1)} = a_1^{\nu_1 + \rho_1} = r^{\nu_1 + 3}$.

DEFINITION 1.1. Put

$$\pi(\sigma_1, \nu_1) := \operatorname{Ind}_{P_1}^G(\sigma_1 \otimes e^{\nu_1 + \rho_1} \otimes 1_{N_1}).$$

Then the representation space of $\pi(\sigma_1, \nu_1)$ is given by

$$\{f : G \rightarrow \mathbf{C}, \text{ measurable} \mid f(m_1 n_1 a_1 x) = \sigma_1(m_1) a_1^{\nu_1 + \rho_1} f(x) \text{ a.e.} \\ \text{for } (x, m_1, n_1, a_1) \in G \times M_1 \times N_1 \times A_1, f|_K \in L^2(K)\}.$$

We call this representation the *spherical degenerate principal series*, if σ_1 is the trivial character 1_{M_1} , and the *non-spherical degenerate principal series* if σ_1 is the determinant representation \det_{M_1} of $SL(2, \mathbf{R}) \times \{\pm 1\}$.

1.2. Whittaker model and generalized Whittaker model

For an irreducible admissible representation π of G , (generalized) Whittaker model for π is a realization of π in an induced module $C^\infty\text{-Ind}_N^G(\psi)$, where

$$N = \left\{ n(x_{12}, x_{13}, x_{23}) = \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix} \mid x_{12}, x_{13}, x_{23} \in \mathbf{R} \right\}$$

is a maximal unipotent subgroup of G and $\psi \in \widehat{N}$.

Since N is the Heisenberg group of dimension 3, the unitary dual \widehat{N} consists of unitary characters and infinite-dimensional representations. An infinite-dimensional irreducible unitary representation of N is uniquely determined via its central character on the center Z of N , by the theorem of Stone and von Neumann. We recall the construction and the basic properties of such representations (cf. [1]).

Since Z is identified with its commutator subgroup $[N, N]$, passing to the associated Lie algebras, we have an equality $\mathfrak{z} = \text{Lie}(Z) = [\mathfrak{n}, \mathfrak{n}]$. The theory of coadjoint orbit method (the Kirillov theory) tells that the unitary characters of N are parametrized by the subspace $(\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^* (\cong \mathbf{R}^2)$ of \mathfrak{n}^* . Here $*$ means the linear dual over \mathbf{R} . And the infinite-dimensional representations are parametrized by nonzero elements in $\mathfrak{z}^* = [\mathfrak{n}, \mathfrak{n}]^* \cong \mathbf{R}$. Thus we have to start with a nonzero linear form on \mathfrak{z} :

$$l : xE_{13} \mapsto cx \in \mathbf{R} \quad (c \in \mathbf{R} \cong \mathfrak{z}^*, c \neq 0).$$

Here E_{ij} is the matrix unit with 1 at (i, j) -th entry and 0 at other entries. This induces a bilinear form on \mathfrak{n}^* :

$$B_l(X, Y) = l([X, Y]) \quad (X, Y \in \mathfrak{n}^*).$$

Then \mathfrak{z} is the radical of this bilinear form, and the restriction of B_l to $\mathfrak{n}_1 = \text{Lie}(N_1)$ is trivial, and the quotient $\mathfrak{n}_1/\mathfrak{z}$ is a maximally totally isotropic subspace in $\mathfrak{n}/\mathfrak{z}$ with respect to B_l . Namely \mathfrak{n}_1 is a polarization subalgebra or a maximal subordinate subalgebra for l (cf. [1, §§1.3, p. 27–28]). Let us define a unitary character $\chi_{l, N_1} : N_1 \rightarrow U(1)$ of N_1 by

$$\chi_{l, N_1}(\exp(Y)) := \exp(2\pi\sqrt{-1}\tilde{l}(Y)) \quad (Y \in \mathfrak{n}_1).$$

Here \tilde{l} is an extension of l to \mathfrak{n}_1 such that

$$\tilde{l} : \mathfrak{n}_1 \ni xE_{13} + yE_{23} \mapsto c(x + my) \in \mathbf{R}$$

with $m \in \mathbf{R}$. Then the induced representation $\pi_{l, N_1} = \text{Ind}_{N_1}^N(\chi_{l, N_1})$ is the Schrödinger representation with representation space $L^2(N_1 \backslash N) \cong L^2(\mathbf{R})$ ([1, §§2.2, Example 2.2.6]).

PROPOSITION 1.2. (i) *The unitary character of N is of the form*

$$\psi(n(x_{12}, x_{13}, x_{23})) = \exp\{2\pi\sqrt{-1}(c_1x_{12} + c_2x_{23})\}$$

with $c_1, c_2 \in \mathbf{R}$.

(ii) *The infinite-dimensional unitary representation of N is realized on $L^2(\mathbf{R})$ as*

$$\psi(n(x_{12}, x_{13}, x_{23}))\phi(s) = \exp\{2\pi\sqrt{-1}c(x_{13} + (s + m)x_{23})\}\phi(s + x_{12})$$

for $\phi \in L^2(\mathbf{R})$ with $c \in \mathbf{R} \setminus \{0\}$ and $m \in \mathbf{R}$.

DEFINITION 1.3. Fix $\psi \in \widehat{N}$ as in Proposition 1.2. For an irreducible admissible representation π of G , we denote by π_∞ the subspace of smooth vectors in π and consider the intertwining space

$$\mathrm{Hom}_{(\mathfrak{g}, K)}(\pi_\infty, C^\infty\text{-Ind}_N^G(\psi))$$

between (\mathfrak{g}, K) -modules ($\mathfrak{g} = \mathrm{Lie}(G)$, $K = SO(3)$). For a nonzero intertwining operator I and a vector $f \in \pi_\infty$, the image $I(f)$ is called the *Whittaker function* if ψ is the unitary character and the *generalized Whittaker function* if ψ is the infinite-dimensional unitary representation.

REMARK 1. Because of the Iwasawa decomposition $G = NAK$ with

$$A = \{\mathrm{diag}(a_1, a_2, a_3) \in G\},$$

if we specify the K -type of f then $I(f)$ is determined by its restriction $I(f)|_A$ to A , which is called the (A -)radial part of the (generalized) Whittaker function.

2. Representations of K

2.1. Irreducible K -modules

As is well-known the finite-dimensional irreducible representations of the maximal compact subgroup $K = SO(3)$ is constructed from those of $SU(2)$. Let (τ_{2l}, V_{2l}) ($l = 0, 1, 2, \dots$) be the $(2l + 1)$ -dimensional irreducible representation of K corresponding to the l -th symmetric tensor of the standard representation of $SU(2)$ and $\{v_k \mid 0 \leq k \leq 2l\}$ the standard basis of V_{2l} . Then we have

$$\begin{cases} \tau_{2l}(K_{23})v_k = \sqrt{-1}(-l + k)v_k, \\ \tau_{2l}(K_{13} - \sqrt{-1}K_{12})v_k = (2l - k)v_{k+1}, \\ \tau_{2l}(K_{13} + \sqrt{-1}K_{12})v_k = -k v_{k-1} \end{cases}$$

with $K_{ij} = E_{ij} - E_{ji} \in \mathfrak{k} = \mathrm{Lie}(K)$ ($1 \leq i < j \leq 3$). See [5, §2] for the details. We note that τ_2 is equivalent to the tautological representation $K \rightarrow GL(3, \mathbf{C})$.

2.2. Some irreducible components of $\tau_i \otimes \tau_j$ ($i, j = 2, 4$)

For our later use, we want to specify the standard basis of the unique irreducible constituent τ_4 in the tensor product $\tau_2 \otimes \tau_2$, τ_2 and τ_4 in $\tau_2 \otimes \tau_4$, and τ_4 in $\tau_4 \otimes \tau_4$ by the following 4 lemmas. The proofs are similar to that of [5, Lemma 2.1].

LEMMA 2.1. Let $\{v_i \mid 0 \leq i \leq 2\}$ be the standard basis of (τ_2, V_2) . Define a set of elements $\{w'_i \mid 0 \leq i \leq 4\}$ in $\tau_2 \otimes \tau_2$ by

$$\begin{aligned} w'_0 &= v_0 \otimes v_0, \\ w'_1 &= \frac{1}{2}(v_0 \otimes v_1 + v_1 \otimes v_0), \\ w'_2 &= \frac{1}{6}(v_0 \otimes v_2 + 4v_1 \otimes v_1 + v_2 \otimes v_0), \\ w'_3 &= \frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1), \\ w'_4 &= v_2 \otimes v_2. \end{aligned}$$

Then it defines a set of standard basis in $\tau_4 \hookrightarrow \tau_2 \otimes \tau_2$, which is unique up to a common scalar multiple.

LEMMA 2.2 ([5, Lemma 2.1]). *Let $\{v_i \mid 0 \leq i \leq 2\}$ and $\{w_j \mid 0 \leq j \leq 4\}$ be the standard basis of (τ_2, V_2) and (τ_4, V_4) , respectively. Then the elements*

$$\begin{aligned} v'_0 &= v_0 \otimes w_2 - 2v_1 \otimes w_1 + v_2 \otimes w_0, \\ v'_1 &= v_0 \otimes w_3 - 2v_1 \otimes w_2 + v_2 \otimes w_1, \\ v'_2 &= v_0 \otimes w_4 - 2v_1 \otimes w_3 + v_2 \otimes w_2 \end{aligned}$$

define a set of standard basis in $\tau_2 \hookrightarrow \tau_2 \otimes \tau_4$, which is unique up to a common scalar multiple.

LEMMA 2.3. *Let $\{w_j \mid 0 \leq j \leq 4\}$ be the standard basis of (τ_4, V_4) . Then the standard basis $\{W_j \mid 0 \leq j \leq 4\}$ of the irreducible τ_4 -isotypic component in $\tau_4 \otimes \tau_4$ is, up to constant multiple, given by*

$$\begin{aligned} W_0 &= w_0 \otimes w_2 - 2w_1 \otimes w_1 + w_2 \otimes w_0, \\ W_1 &= \frac{1}{2}(w_0 \otimes w_3 - w_1 \otimes w_2 - w_2 \otimes w_1 + w_3 \otimes w_0), \\ W_2 &= \frac{1}{6}(w_0 \otimes w_4 + 2w_1 \otimes w_3 - 6w_2 \otimes w_2 + 2w_3 \otimes w_1 + w_4 \otimes w_0), \\ W_3 &= \frac{1}{2}(w_1 \otimes w_4 - w_2 \otimes w_3 - w_3 \otimes w_2 + w_4 \otimes w_1), \\ W_4 &= w_2 \otimes w_4 - 2w_3 \otimes w_3 + w_4 \otimes w_2. \end{aligned}$$

LEMMA 2.4. *Let $\{v_i \mid 0 \leq i \leq 2\}$ and $\{w_j \mid 0 \leq j \leq 4\}$ be the standard basis of (τ_2, V_2) and (τ_4, V_4) , respectively. Then the standard basis $\{w'_j \mid 0 \leq j \leq 4\}$ of the irreducible τ_4 -isotypic component in $\tau_2 \otimes \tau_4$ is, up to constant multiple, given by*

$$\begin{aligned} w'_0 &= v_0 \otimes w_1 - v_1 \otimes w_0, \\ w'_1 &= \frac{1}{4}(3v_0 \otimes w_2 - 2v_1 \otimes w_1 - v_2 \otimes w_0), \\ w'_2 &= \frac{1}{2}(v_0 \otimes w_3 - v_2 \otimes w_1), \\ w'_3 &= \frac{1}{4}(v_0 \otimes w_4 + 2v_1 \otimes w_3 - 3v_2 \otimes w_2), \\ w'_4 &= v_1 \otimes w_4 - v_2 \otimes w_3. \end{aligned}$$

2.3. The K -module isomorphism between $\mathfrak{p}_{\mathbb{C}}$ and V_4

We denote by $\mathfrak{p}_{\mathbb{C}}$ the complexification of the orthogonal complement \mathfrak{p} of \mathfrak{k} with respect to the Killing form, on which the group K acts via the adjoint action $\text{Ad}_{\mathfrak{p}_{\mathbb{C}}}$. Note that E_{ii} and $E_{ij} + E_{ji}$ are considered as elements in \mathfrak{p} . We set $H_{ij} = E_{ii} - E_{jj}$ for $i \neq j$.

LEMMA 2.5 ([5, Lemma 2.2]). *Via the unique isomorphism between V_4 and $\mathfrak{p}_{\mathbb{C}}$ as K -modules we have the identification*

$$\begin{aligned} w_0 &= -2\{H_{23} - \sqrt{-1}(E_{23} + E_{32})\} & =: X_0, \\ w_1 &= \sqrt{-1}\{(E_{12} + E_{21}) - \sqrt{-1}(E_{13} + E_{31})\} & =: X_1, \\ w_2 &= \frac{2}{3}(H_{12} + H_{13}) & =: X_2, \end{aligned}$$

$$\begin{aligned} w_3 &= \sqrt{-1}\{(E_{12} + E_{21}) + \sqrt{-1}(E_{13} + E_{31})\} =: X_3, \\ w_4 &= -2\{H_{23} + \sqrt{-1}(E_{23} + E_{32})\} =: X_4. \end{aligned}$$

2.4. The elementary functions on K

Let us consider the tautological representation of $K: k \mapsto (s_{ij}(k))_{1 \leq i, j \leq 3}$. Then we have 9 functions $s_{ij}(k)$ on K . For each fixed i , the space generated by $\{s_{i1}, s_{i2}, s_{i3}\}$ defines a representation of K isomorphic to τ_2 in $L^2(K)$. Since $\tau_4 \hookrightarrow \tau_2 \otimes \tau_2$, we want to investigate the quadratic polynomials in s_{ij} to obtain a natural K -monomorphism $\tau_4 \hookrightarrow L^2((M_1 \cap K) \backslash K)$.

By inspection, the left $(M_1 \cap K)$ -invariant quadratic functions in s_{ij} are generated by

$$\begin{cases} t'_{ab}(k) := s_{1a}(k) \cdot s_{1b}(k) + s_{2a}(k) \cdot s_{2b}(k), \\ t_{ab}(k) := s_{3a}(k) \cdot s_{3b}(k). \end{cases}$$

Since $t'_{ab}(k) + t_{ab}(k) = \delta_{ab}$ (δ_{ab} being the Kronecker delta), it suffices to consider 6 functions $\{t_{ab}(k)\}_{1 \leq a, b \leq 3}$. Moreover $\sum_{a=1}^3 t_{aa}(k) = 1$ implies that

$$\dim_{\mathbb{C}} \left\{ \sum_{1 \leq a, b \leq 3} \mathbb{C} t_{ab} \right\} / \mathbb{C} f_0 \leq 5.$$

Here $f_0 \equiv 1$ on K is a natural generator of τ_0 . To have the standard basis, we recall that

$$\begin{aligned} v_0 &= \sqrt{-1}(s_{32} - \sqrt{-1}s_{33}) := f_{2,0}, \\ v_1 &= s_{31} := f_{2,1}, \\ v_2 &= \sqrt{-1}(s_{32} + \sqrt{-1}s_{33}) := f_{2,2} \end{aligned}$$

generate a K -module isomorphic to τ_2 in $L^2(K)$ with standard basis $\{v_0, v_1, v_2\}$. Notice that there is the decomposition $\tau_2 \otimes \tau_2 \cong \tau_4 \oplus \tau_2 \oplus \tau_0$.

LEMMA 2.6. *The standard basis $\{w_k \mid 0 \leq k \leq 4\}$ of $\tau_4 \hookrightarrow L^2((M_1 \cap K) \backslash K)$ is given as follows:*

$$\begin{aligned} w_0 &= \{\sqrt{-1}(s_{32} - \sqrt{-1}s_{33})\}^2 = s_{33}^2 - s_{32}^2 + 2\sqrt{-1}s_{32}s_{33} =: f_{4,0}, \\ w_1 &= \sqrt{-1}(s_{32} - \sqrt{-1}s_{33})s_{31} = s_{31}s_{33} + \sqrt{-1}s_{31}s_{32} =: f_{4,1}, \\ w_2 &= -\frac{1}{3}(s_{32}^2 + s_{33}^2) + \frac{2}{3}s_{31}^2 = \frac{1}{3}(2s_{31}^2 - s_{32}^2 - s_{33}^2) =: f_{4,2}, \\ w_3 &= s_{31}\sqrt{-1}(s_{32} + \sqrt{-1}s_{33}) = -s_{31}s_{33} + \sqrt{-1}s_{31}s_{32} =: f_{4,3}, \\ w_4 &= \{\sqrt{-1}(s_{32} + \sqrt{-1}s_{33})\}^2 = s_{33}^2 - s_{32}^2 - 2\sqrt{-1}s_{32}s_{33} =: f_{4,4}. \end{aligned}$$

REMARK 2. The values at the unity $e \in K$ of $f_{4,k}$ are given by

$$(f_{4,0}(e), f_{4,1}(e), f_{4,2}(e), f_{4,3}(e), f_{4,4}(e)) = (1, 0, -\frac{1}{3}, 0, 1).$$

3. The (\mathfrak{g}, K) -modules structures around the minimal K -types

3.1. The K -types

The representation space of $\pi(\sigma_1, \nu_1)$ is isomorphic to

$$L^2_{\sigma_1}(K) = \{f \in L^2(K) \mid f(mk) = \sigma_1(m)f(k) \text{ for all } m \in M \cap K, k \in K\}$$

as K -modules and we have a Hilbert space direct sum decomposition of K -modules:

$$L^2_{\sigma_1}(K) = \begin{cases} \widehat{\bigoplus}_{m=0}^{\infty} \tau_{4m} & \text{if } \sigma_1 = 1_{M_1}, \\ \widehat{\bigoplus}_{m=0}^{\infty} \tau_{4m+2} & \text{if } \sigma_1 = \det_{M_1}. \end{cases}$$

3.2. The eigenvalues on the (\mathfrak{g}, K) -modules structures around the minimal K -types

Hereafter the action of $X \in \mathfrak{g}$ via the representation π on the vectors $f_{4,k}$ is denoted by $Xf_{4,k}$ omitting the symbol π . Firstly we recall the Iwasawa decomposition of elements in \mathfrak{p} .

LEMMA 3.1. *The five elements $H_{12}, H_{23}, E_{ij} + E_{ji}$ ($1 \leq i < j \leq 3$) of \mathfrak{p} have the following Iwasawa decomposition:*

$$H_{ij} = 0 + H_{ij} + 0, \quad E_{ij} + E_{ji} = 2E_{ij} + 0 + (-K_{ij})$$

with respect to $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$ ($\mathfrak{a} = \text{Lie}(A)$).

LEMMA 3.2. *The actions of the elements in \mathfrak{k} are given by*

$$\begin{aligned} K_{12}f_{2,0} &= \sqrt{-1}f_{2,1}, & K_{13}f_{2,0} &= f_{2,1}, & K_{23}f_{2,0} &= -\sqrt{-1}f_{2,0}, \\ K_{12}f_{2,1} &= \frac{\sqrt{-1}}{2}(f_{2,0} + f_{2,2}), & K_{13}f_{2,1} &= \frac{1}{2}(f_{2,2} - f_{2,0}), & K_{23}f_{2,1} &= 0, \\ K_{12}f_{2,2} &= \sqrt{-1}f_{2,1}, & K_{13}f_{2,2} &= -f_{2,1}, & K_{23}f_{2,2} &= \sqrt{-1}f_{2,2}, \end{aligned}$$

in τ_2 . Similarly in τ_4 we have

$$\begin{aligned} K_{12}f_{4,0} &= 2\sqrt{-1}f_{4,1}, & K_{13}f_{4,0} &= 2f_{4,1}, & K_{23}f_{4,0} &= 2\sqrt{-1}f_{4,0}, \\ K_{12}f_{4,1} &= \frac{\sqrt{-1}}{2}(3f_{4,2} + f_{4,0}), & K_{13}f_{4,1} &= \frac{1}{2}(3f_{4,2} - f_{4,0}), & K_{23}f_{4,1} &= \sqrt{-1}f_{4,1}, \\ K_{12}f_{4,2} &= \sqrt{-1}(f_{4,3} + f_{4,1}), & K_{13}f_{4,2} &= f_{4,3} - f_{4,1}, & K_{23}f_{4,2} &= 0, \\ K_{12}f_{4,3} &= \frac{\sqrt{-1}}{2}(f_{4,4} + 3f_{4,2}), & K_{13}f_{4,3} &= \frac{1}{2}(f_{4,4} - 3f_{4,2}), & K_{23}f_{4,3} &= -\sqrt{-1}f_{4,3}, \\ K_{12}f_{4,4} &= 2\sqrt{-1}f_{4,3}, & K_{13}f_{4,4} &= -2f_{4,3}, & K_{23}f_{4,4} &= 2\sqrt{-1}f_{4,4}. \end{aligned}$$

3.2.1. The spherical case.

PROPOSITION 3.3. *Define a set $\{f'_{4,k} \mid 0 \leq k \leq 4\}$ by*

$$f'_{4,k} = X_k f_0.$$

Then it is a standard basis in $\tau_4 \hookrightarrow H_\pi$, and we have

$$f'_{4,k} = -(\nu_1 + \rho_1)f_{4,k} \quad (0 \leq k \leq 4).$$

Proof. From Lemma 2.5, $\{f'_{4,k} \mid 0 \leq k \leq 4\}$ is a standard basis in τ_4 . In particular, there is a common scalar λ such that $f'_{4,k} = \lambda f_{4,k}$ ($0 \leq k \leq 4$). To find the scalar λ , it suffices to compute the values $f'_{4,0}$ at the identity $e \in K$. Since $K_{23}f_0 = 0$ and $E_{23}f_0 = 0$, we have $f'_{4,0}(e) = -2H_{23}f_0(e)$. Note that $H_{23} \cong \frac{1}{2}\text{diag}(1, 1, -2) \pmod{\mathfrak{m}_1}$ to get $H_{23}f_0(e) = \frac{1}{2}(v_1 + \rho_1)$. Hence $f'_{4,0}(e) = \lambda = -(v_1 + \rho_1)$ as desired. \square

Similarly we can prove the following:

PROPOSITION 3.4. *Define a set $\{f''_{4,k} \mid 0 \leq k \leq 4\}$ by*

$$\begin{aligned} f''_{4,0} &= X_2 f_{4,0} - 2X_3 f_{4,1} + X_4 f_{4,2}, \\ f''_{4,1} &= \frac{1}{2}(X_3 f_{4,0} - X_2 f_{4,1} - X_1 f_{4,2} + X_0 f_{4,3}), \\ f''_{4,2} &= \frac{1}{6}(X_4 f_{4,0} + 2X_3 f_{4,1} - 6X_2 f_{4,2} + 2X_1 f_{4,3} + X_0 f_{4,4}), \\ f''_{4,3} &= \frac{1}{2}(X_4 f_{4,1} - X_3 f_{4,2} - X_2 f_{4,3} + X_1 f_{4,4}), \\ f''_{4,4} &= X_4 f_{4,2} - 2X_3 f_{4,3} + X_2 f_{4,4}. \end{aligned}$$

Then it is a standard basis of the unique τ_4 -component of the K -module $\mathfrak{p} \cdot \tau_4$, and we have

$$f''_{4,k} = \frac{2}{3}v_1 f_{4,k} \quad (0 \leq k \leq 4).$$

3.2.2. *The non-spherical case.* As in Proposition 3.3, we can prove the following:

PROPOSITION 3.5. *Define a set $\{\tilde{f}_{2,k} \mid 0 \leq k \leq 2\}$ by*

$$\tilde{f}_{2,k} = X_{k+2}f_{2,0} - 2X_{k+1}f_{2,1} + X_k f_{2,2}.$$

Then we have

$$\tilde{f}_{2,k} = \frac{4}{3}v_1 f_{2,k} \quad (0 \leq k \leq 2).$$

Since there exists no K -type τ_4 in π for the non-spherical case, we obtain the following:

PROPOSITION 3.6. *We have*

$$\begin{aligned} X_1 f_{2,0} - X_0 f_{2,1} &= 0, \\ 3X_2 f_{2,0} - 2X_1 f_{2,1} - X_0 f_{2,2} &= 0, \\ X_3 f_{2,0} - X_1 f_{2,2} &= 0, \\ X_4 f_{2,0} + 2X_3 f_{2,1} - 3X_2 f_{2,2} &= 0, \\ X_4 f_{2,1} - X_3 f_{2,2} &= 0. \end{aligned}$$

4. Whittaker functions

4.1. The Whittaker realizations of the (\mathfrak{g}, K) -modules structures around minimal K -types

The computation of Whittaker realizations of representations of G is done by using the Iwasawa decomposition $X = X_N + X_A + X_K$ of an element $X \in \mathfrak{g}$ with respect to $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$. Here we recollect the formulae of A -radial parts of the actions of some standard elements in \mathfrak{a} and \mathfrak{n} . The \mathfrak{k} -action is given in Lemma 3.2.

We use the coordinates $y_1 = a_1/a_2$, and $y_2 = a_2/a_3 = a_1 a_2^2$ for an element $a = \text{diag}(a_1, a_2, a_3)$ of A , corresponding to the simple roots, and the associated Euler operators are denoted by $\partial_i = y_i \frac{\partial}{\partial y_i}$ ($i = 1, 2$).

LEMMA 4.1 ([5, Lemma 4.1, 4.2]). *Let Φ be an element of $C^\infty\text{-Ind}_N^G(\psi)$, and $\Phi_A = \Phi_A(y_1, y_2)$ its restriction to A . Let $\rho_A(X)$ be the A -radial part of an operator $X \in \mathfrak{a} \oplus \mathfrak{n}$.*

- (i) $\rho_A(H_{12} + H_{13})\Phi_A = 3\partial_1\Phi_A$ and $\rho_A(H_{23})\Phi_A = (-\partial_1 + 2\partial_2)\Phi_A$.
- (ii) $\rho_A(E_{12})\Phi_A = 2\pi\sqrt{-1}c_1y_1\Phi_A$, $\rho_A(E_{23})\Phi_A = 2\pi\sqrt{-1}c_2y_2\Phi_A$, and $\rho_A(E_{13})\Phi_A = 0$.

4.1.1. *The spherical case.* Let I be a nonzero Whittaker functional from the spherical degenerate principal series $\pi(1_{M_1}, \nu_1)$ to $C^\infty\text{-Ind}_N^G(\psi)$, i.e., let

$$0 \neq I \in \text{Hom}_{(\mathfrak{g}, K)}(\pi(1_{M_1}, \nu_1), C^\infty\text{-Ind}_N^G(\psi)).$$

For the K -fixed vector f_0 and the functions $f_{4,k}$ ($0 \leq k \leq 4$) defined in Lemma 2.6, we set

$$\varphi_0(a) := I(f_0)|_A(a), \quad \varphi_{4,k}(a) := I(f_{4,k})|_A(a) \quad (a \in A).$$

Then we have

PROPOSITION 4.2. *If we denote by $\varphi_4 := (\varphi_{4,0}, \varphi_{4,1}, \varphi_{4,2}, \varphi_{4,3}, \varphi_{4,4})$, the functions φ_0 and φ_4 satisfy the system of the following partial differential equations:*

$$(W1) \quad -\frac{1}{2}(\nu_1 + \rho_1)\varphi_4 = {}^t(\Delta_-, -2\pi c_1y_1, \partial_1, -2\pi c_1y_1, \Delta_+)\varphi_0,$$

$$(W2) \quad \mathcal{M}\varphi_4 = \frac{2}{3}\nu_1\varphi_4,$$

where the matrix differential operator \mathcal{M} is

$$\begin{pmatrix} 2(\partial_1 - 1) & 8\pi c_1y_1 & 2\Delta_- & 0 & 0 \\ -2\pi c_1y_1 & -(\partial_1 - 1) & 2\pi c_1y_1 & \Delta_- + 1 & 0 \\ \frac{1}{3}(\Delta_+ + 2) & -\frac{4}{3}\pi c_1y_1 & -2(\partial_1 - 1) & -\frac{4}{3}\pi c_1y_1 & \frac{1}{3}(\Delta_- + 2) \\ 0 & \Delta_+ + 1 & 2\pi c_1y_1 & -(\partial_1 - 1) & -2\pi c_1y_1 \\ 0 & 0 & 2\Delta_+ & 8\pi c_1y_1 & 2(\partial_1 - 1) \end{pmatrix}$$

and

$$\Delta_\pm = \partial_1 - 2\partial_2 \pm 4\pi c_2y_2.$$

Proof. We apply Lemmas 3.1, 3.2 and 4.1 to Propositions 3.3 and 3.4 to get (W1) and (W2), respectively. \square

4.1.2. *The non-spherical case.* For nonzero element I in $\text{Hom}_{(\mathfrak{g}, K)}(\pi(\det_{M_1}, \nu_1), C^\infty\text{-Ind}_N^G(\psi))$ and the functions $f_{2,k}$ ($0 \leq k \leq 2$), put

$$\varphi_{2,k}(a) := I(f_{2,k})|_A(a) \quad (a \in A).$$

Then the Whittaker realizations of Propositions 3.5 and 3.6 are given as follows:

PROPOSITION 4.3. *Under the same symbol Δ_\pm as in Proposition 4.2, we have*

$$(W3) \quad \begin{pmatrix} \partial_1 - 1 - \frac{2}{3}\nu_1 & 4\pi c_1 y_1 & \Delta_- + 1 \\ -2\pi c_1 y_1 & -2(\partial_1 - 1) - \frac{2}{3}\nu_1 & -2\pi c_1 y_1 \\ \Delta_+ + 1 & 4\pi c_1 y_1 & \partial_1 - 1 - \frac{2}{3}\nu_1 \end{pmatrix} \begin{pmatrix} \varphi_{2,0} \\ \varphi_{2,1} \\ \varphi_{2,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$(W4) \quad \begin{pmatrix} 2\pi c_1 y_1 & \Delta_- & 0 \\ 3\partial_1 - 1 & 4\pi c_1 y_1 & -(\Delta_- + 1) \\ 2\pi c_1 y_1 & 0 & -2\pi c_1 y_1 \\ -(\Delta_+ + 1) & 4\pi c_1 y_1 & 3\partial_1 - 1 \\ 0 & \Delta_+ & 2\pi c_1 y_1 \end{pmatrix} \begin{pmatrix} \varphi_{2,0} \\ \varphi_{2,1} \\ \varphi_{2,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

4.2. The determination of the solutions φ_0 and $\{\varphi_{2,k}\}_{(k=0,1,2)}$

From now on we assume that $\nu_1 + \rho_1 \neq 0$.

4.2.1. The spherical case.

PROPOSITION 4.4. *Modulo the equation (W1), the system (W2) is equivalent to the following equations:*

$$\begin{aligned} (A) \quad & (4\pi c_2 y_2)(\partial_1 - \frac{1}{6}\nu_1 - \frac{1}{2})\varphi_0 = 0, \\ (B) \quad & \{(\partial_1 - \frac{1}{6}\nu_1 - \frac{1}{2})(-\partial_1 + 2\partial_2) + (2\pi c_1 y_1)^2\}\varphi_0 = 0, \\ (C_\pm) \quad & (2\pi c_1 y_1)(2\partial_2 - 1 + \frac{1}{3}\nu_1 \pm 4\pi c_2 y_2)\varphi_0 = 0, \\ (D) \quad & \{-\partial_1(3\partial_1 - 3 + \nu_1) + (-\partial_1 + 2\partial_2)^2 - 2(-\partial_1 + 2\partial_2) \\ & \quad + 2(2\pi c_1 y_1)^2 - (4\pi c_2 y_2)^2\}\varphi_0 = 0. \end{aligned}$$

Proof. Replace the vector φ_4 in (W2) by using (W1). □

LEMMA 4.5. *If $c_1 c_2 \neq 0$ (the non-degenerate case), the solution φ_0 of the equations in Proposition 4.4 is trivial, i.e., $\varphi_0 = 0$.*

Proof. It is immediate from (A) and (B). □

Thus we have to consider only “degenerate cases”: (I) $c_1 \neq 0$, $c_2 = 0$, (II) $c_1 = 0$, $c_2 \neq 0$, and (III) $c_1 = c_2 = 0$.

The case (I): The equation (C) implies $\varphi_0(y_1, y_2) = C_0(y_1) \cdot y_2^{-\nu_1/6+1/2}$ with some function $C_0(y_1)$ in y_1 . Then the equation (D) leads

$$\left\{ \left(y_1 \frac{d}{dy_1} - \frac{\nu_1}{6} - \frac{1}{2} \right) \left(y_1 \frac{d}{dy_1} + \frac{\nu_1}{3} - 1 \right) - (2\pi c_1 y_1)^2 \right\} C_0(y_1) = 0.$$

If we put $C_0(y_1) = y_1^{-\nu_1/12+3/4} \tilde{C}_0(y_1)$, the above equation is reduced to Bessel's differential equation and therefore we get

$$\tilde{C}_0(y_1) = CK_{\frac{1}{4}(\nu_1-1)}(2\pi|c_1|y_1) + C'I_{\frac{1}{4}(\nu_1-1)}(2\pi|c_1|y_1).$$

The case (II) can be similarly done.

The case (III): The equations (A) and (C) tell nothing. By (B) and (D), we have a system of partial differential equations with constant coefficients:

$$\begin{cases} (\partial_1 - \frac{1}{6}\nu_1 - \frac{1}{2})(-\partial_1 + 2\partial_2)\varphi_0 = 0, \\ \{-\partial_1(3\partial_1 - 3 + \nu_1) + (-\partial_1 + 2\partial_2)^2 - 2(-\partial_1 + 2\partial_2)\}\varphi_0 = 0. \end{cases}$$

We can readily find that this is a holonomic system of rank 4 with regular singularities along the divisors $y_1 = 0$ and $y_2 = 0$ of normal crossing at the origin $(0, 0)$ and the fundamental solutions are

$$1, \quad y_1^{-\frac{1}{3}\nu_1+1} y_2^{-\frac{1}{6}\nu_1+\frac{1}{2}}, \quad y_1^{\frac{1}{6}\nu_1+\frac{1}{2}} y_2^{\frac{1}{3}\nu_1+1}, \quad y_1^{\frac{1}{6}\nu_1+\frac{1}{2}} y_2^{-\frac{1}{6}\nu_1+\frac{1}{2}}.$$

Summing up the computations above, we have the following main result for the spherical case.

THEOREM 4.6 (Multiplicity-free theorem). *We have the following A-radial part $\varphi_0(y_1, y_2)$ for Whittaker function belonging to the spherical degenerate principal series:*

- (i) *If the character $\psi \in \hat{N}$ is non-degenerate, i.e., $c_1c_2 \neq 0$, then $\varphi_0(y_1, y_2)$ is identically zero.*
- (ii) *If $c_1 \neq 0$, $c_2 = 0$, we have*

$$\varphi_0(y_1, y_2) = y_1^{-\frac{1}{12}\nu_1+\frac{3}{4}} y_2^{-\frac{1}{6}\nu_1+\frac{1}{2}} (CK_{\frac{1}{4}(\nu_1-1)}(2\pi|c_1|y_1) + C'I_{\frac{1}{4}(\nu_1-1)}(2\pi|c_1|y_1)).$$

In particular, the unique solution of moderate growth at infinity is given by $C' = 0$ in the above.

- (iii) *If $c_1 = 0$, $c_2 \neq 0$, we have*

$$\varphi_0(y_1, y_2) = y_1^{\frac{1}{6}\nu_1+\frac{1}{2}} y_2^{\frac{1}{12}\nu_1+\frac{3}{4}} (CK_{\frac{1}{4}(\nu_1+1)}(2\pi|c_2|y_2) + C'I_{\frac{1}{4}(\nu_1+1)}(2\pi|c_2|y_2)).$$

In particular, the unique solution of moderate growth at infinity is given by $C' = 0$ in the above.

- (iv) *If $c_1 = c_2 = 0$, we have*

$$\varphi_0(y_1, y_2) = C \cdot 1 + C'(y_1^2 y_2)^{-\frac{1}{6}\nu_1+\frac{1}{2}} + C''(y_1 y_2^2)^{\frac{1}{6}\nu_1+\frac{1}{2}} + C^{(3)} y_1^{\frac{1}{6}\nu_1+\frac{1}{2}} y_2^{-\frac{1}{6}\nu_1+\frac{1}{2}}.$$

4.2.2. The non-spherical case. We can determine the vector of Whittaker functions $\{\varphi_{2,0}, \varphi_{2,1}, \varphi_{2,2}\}$ by solving (W3) and (W4). The precise is left to the reader.

THEOREM 4.7 (Multiplicity-free theorem). *We have the following A-radial part $\varphi_{2,k} = \varphi_{2,k}(y_1, y_2)$ ($k = 0, 1, 2$) for the Whittaker function with the minimal K-type τ_2 belonging to the non-spherical degenerate principal series:*

- (i) *If the character $\psi \in \hat{N}$ is non-degenerate, i.e., $c_1c_2 \neq 0$, the vector of functions $(\varphi_{2,0}, \varphi_{2,1}, \varphi_{2,2})$ is identically zero.*

(ii) If $c \neq 0$, $c_2 = 0$, we have $\varphi_{2,0} = \varphi_{2,2}$ and

$$\begin{pmatrix} \varphi_{2,0}(y_1, y_2) \\ \varphi_{2,1}(y_1, y_2) \end{pmatrix} = y_1^{-\frac{1}{12}v_1 + \frac{4}{3}} y_2^{-\frac{1}{6}v_1 + \frac{1}{2}} \\ \times \left\{ C \begin{pmatrix} K_{\frac{1}{4}v_1 - \frac{2}{3}}(2\pi|c_1|y_1) \\ -\operatorname{sgn}(c_1)K_{\frac{1}{4}v_1 + \frac{1}{3}}(2\pi|c_1|y_1) \end{pmatrix} + C' \begin{pmatrix} I_{\frac{1}{4}v_1 - \frac{2}{3}}(2\pi|c_1|y_1) \\ -\operatorname{sgn}(c_1)I_{\frac{1}{4}v_1 + \frac{1}{3}}(2\pi|c_1|y_1) \end{pmatrix} \right\}.$$

In particular, the unique solution of moderate growth at infinity is given by $C' = 0$ in the above.

(iii) If $c_1 = 0$, $c_2 \neq 0$, we have $\varphi_{2,1} = 0$ identically and $\varphi_{2,0} = \frac{1}{2}(\varphi_+ + \varphi_-)$, $\varphi_{2,2} = \frac{1}{2}(\varphi_+ - \varphi_-)$ with

$$\begin{pmatrix} \varphi_+(y_1, y_2) \\ \varphi_-(y_1, y_2) \end{pmatrix} = y_1^{\frac{1}{6}v_1 + \frac{1}{2}} y_2^{\frac{1}{12}v_1 + \frac{5}{4}} \\ \times \left\{ C \begin{pmatrix} K_{\frac{1}{4}(v_1+3)}(2\pi|c_2|y_2) \\ \operatorname{sgn}(c_2)K_{\frac{1}{4}(v_1-1)}(2\pi|c_2|y_2) \end{pmatrix} + C' \begin{pmatrix} I_{\frac{1}{4}(v_1+3)}(2\pi|c_2|y_2) \\ \operatorname{sgn}(c_2)I_{\frac{1}{4}(v_1-1)}(2\pi|c_2|y_2) \end{pmatrix} \right\}.$$

In particular, the unique solution of moderate growth at infinity is given by $C' = 0$ in the above.

(iv) If $c_1 = c_2 = 0$, we have

$$\begin{pmatrix} \varphi_{2,0}(y_1, y_2) \\ \varphi_{2,1}(y_1, y_2) \\ \varphi_{2,2}(y_1, y_2) \end{pmatrix} = C y_1^{\frac{1}{6}v_1 + \frac{1}{2}} y_2^{-\frac{1}{6}v_1 + \frac{1}{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C' y_1^{\frac{1}{6}v_1 + \frac{1}{2}} y_2^{\frac{1}{3}v_1 + 1} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ + C'' y_1^{-\frac{1}{3}v_1 + 1} y_2^{-\frac{1}{6}v_1 + \frac{1}{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

5. Generalized Whittaker functions

5.1. The generalized Whittaker realizations of the (\mathfrak{g}, K) -modules structures

The action of \mathfrak{n} on the space of C^∞ -vectors, i.e., on the Schwartz space \mathcal{S} is described as follows:

LEMMA 5.1. *Via the Schrödinger representation, the operators E_{12} , E_{23} and E_{13} in \mathfrak{n} act on $\mathcal{S}(\mathbf{R}) \subset L^2(\mathbf{R})$ as follows:*

$$E_{13}f(s) = 2\pi\sqrt{-1}cf(s), \quad E_{12}f(s) = \frac{d}{ds}f(s), \\ E_{23}f(s) = 2\pi\sqrt{-1}c(s+m)f(s)$$

for $f \in \mathcal{S}(\mathbf{R})$.

By the parameter shift $s \mapsto s - m$ the case of general m is reduced to the case $m = 0$. From now on we assume that $m = 0$.

5.1.1. *The spherical case.*

PROPOSITION 5.2. *We use the same symbol φ_0 and φ_4 for the radial part of the generalized Whittaker functions as §4.1.1. Set*

$$L_{\pm} = \partial_1 - 2\partial_2 \pm 4\pi c y_2 s, \quad \Lambda_{\pm} = y_1 \frac{\partial}{\partial s} \pm 2\pi c y_1 y_2.$$

Then we have

$$(GW1) \quad -\frac{1}{2}(v_1 + \rho_1) \varphi_4 = {}^t(L_-, \sqrt{-1}\Lambda_+, \partial_1, \sqrt{-1}\Lambda_-, L_+) \varphi_0,$$

$$(GW2) \quad \mathcal{M} \varphi_4 = \frac{2}{3} v_1 \varphi_4,$$

with

$$\mathcal{M} = \begin{pmatrix} 2(\partial_1 - 1) & -4\sqrt{-1}\Lambda_+ & 2L_- & 0 & 0 \\ \sqrt{-1}\Lambda_- & -(\partial_1 - 1) & -\sqrt{-1}\Lambda_+ & (L_- + 1) & 0 \\ \frac{1}{3}(L_+ + 2) & \frac{2}{3}\sqrt{-1}\Lambda_- & -2(\partial_1 - 1) & \frac{2}{3}\sqrt{-1}\Lambda_+ & \frac{1}{3}(L_- + 2) \\ 0 & L_+ + 1 & -\sqrt{-1}\Lambda_- & -(\partial_1 - 1) & \sqrt{-1}\Lambda_+ \\ 0 & 0 & 2L_+ & -4\sqrt{-1}\Lambda_- & 2(\partial_1 - 1) \end{pmatrix}.$$

5.1.2. *The non-spherical case.* Here are generalized Whittaker realization of the formulae in Propositions 3.3 and 3.4.

PROPOSITION 5.3. *Let I be a nonzero generalized Whittaker functional. Then, under the same symbols L_{\pm} and Λ_{\pm} as in Proposition 5.2, we have the following system of differential equations for three functions $\varphi_k = I(f_{2,k})|_A$ ($k = 0, 1, 2$):*

$$(GW3) \quad \begin{pmatrix} \partial_1 - \frac{2}{3}v_1 - 1 & -2\sqrt{-1}\Lambda_+ & L_- + 1 \\ \sqrt{-1}\Lambda_- & -2(\partial_1 + \frac{1}{3}v_1 - 1) & \sqrt{-1}\Lambda_+ \\ L_+ + 1 & -2\sqrt{-1}\Lambda_- & \partial_1 - \frac{2}{3}v_1 - 1 \end{pmatrix} \begin{pmatrix} \varphi_{2,0} \\ \varphi_{2,1} \\ \varphi_{2,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$(GW4) \quad \begin{pmatrix} \sqrt{-1}\Lambda_+ & -L_- & 0 \\ 3\partial_1 - 1 & -2\sqrt{-1}\Lambda_+ & -(L_- + 1) \\ \Lambda_- & 0 & -\Lambda_+ \\ -(L_+ + 1) & -2\sqrt{-1}\Lambda_- & 3\partial_1 - 1 \\ 0 & -L_+ & \sqrt{-1}\Lambda_- \end{pmatrix} \begin{pmatrix} \varphi_{2,0} \\ \varphi_{2,1} \\ \varphi_{2,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

5.2. **Explicit formula for the generalized Whittaker functions**

5.2.1. *The spherical case.* By Proposition 5.2, we have the following:

PROPOSITION 5.4. *Modulo the equation (GW1), the system (GW2) is equivalent to the following equations:*

$$(A) \quad [(2\partial_1 - 1 - \frac{1}{3}v_1)(\partial_1 - 2\partial_2) + 2y_1^2\{(\frac{\partial}{\partial s})^2 + (2\pi c y_2)^2\}]\varphi_0 = 0,$$

$$(B) \quad (4\pi c y_2)\{2(\partial_1 - 1 - \frac{1}{3}v_1)s - 2y_1^2\frac{\partial}{\partial s}\}\varphi_0 = 0,$$

$$(C) \quad y_1(2\pi c y_2)(\partial_1 - \partial_2 + \frac{1}{2} + \frac{1}{6}v_1 + s\frac{\partial}{\partial s})\varphi_0 = 0,$$

$$(D) \quad y_1\{(\partial_2 - \frac{1}{2} + \frac{1}{6}v_1)\frac{\partial}{\partial s} - (2\pi c y_2)^2 s\}\varphi_0 = 0,$$

$$(E) \quad [2y_1^2\{(\frac{\partial}{\partial s})^2 - (2\pi cy_2)^2\} + 3(\partial_1 - 1 + \frac{1}{3}\nu_1)\partial_1 - (\partial_1 - 2\partial_2)^2 - 2(\partial_1 - 2\partial_2) + (4\pi cy_2s)^2]\varphi_0 = 0.$$

PROPOSITION 5.5. *If we put $\varphi_0(y_1, y_2; s) = y_1^{\nu_1/6+1/2} y_2^{-\nu_1/6+1/2} \psi_0(y_1, y_2; s)$, then the system in Proposition 5.4 is reduced to the following:*

$$(B') \quad \left\{ \frac{\partial}{\partial(y_1^2)} - \frac{\partial}{\partial(s^2)} \right\} \psi_0 = 0,$$

$$(C') \quad \left\{ y_1^2 \frac{\partial}{\partial(y_1^2)} + s^2 \frac{\partial}{\partial(s^2)} - y_2^2 \frac{\partial}{\partial(y_2^2)} + \frac{\nu_1 + 1}{4} \right\} \psi_0 = 0,$$

$$(D') \quad \left\{ \frac{\partial}{\partial(s^2)} \frac{\partial}{\partial(y_2^2)} - (\pi c)^2 \right\} \psi_0 = 0,$$

$$(F') \quad \left\{ \frac{\partial}{\partial(y_1^2)} \frac{\partial}{\partial(y_2^2)} - (\pi c)^2 \right\} \psi_0 = 0.$$

Let us solve the above system. Apply the Euler operator $\partial_{(y_2^2)} = y_2^2 \frac{\partial}{\partial(y_2^2)}$ to (C') and utilize (D') and (F') to get

$$\left\{ \partial_{(y_2^2)}^2 - \left(\frac{\nu_1 + 1}{4} \right) \partial_{(y_2^2)} - (\pi cy_2)^2 (y_1^2 + s^2) \right\} \psi_0 = 0.$$

Set $\psi_0 = (y_2^2)^{\frac{1}{8}(\nu_1+1)} \tilde{\psi}_0$. Then we have

$$\left[\partial_2^2 - \left\{ \left(\frac{\nu_1 + 1}{4} \right)^2 + \left(2\pi cy_2 \sqrt{y_1^2 + s^2} \right)^2 \right\} \right] \tilde{\psi}_0 = 0.$$

Here we used $\partial_{(y_2^2)} = \frac{1}{2}\partial_2$. Thus the solution $\tilde{\psi}_0(y_2; y_1, s)$ can be written as

$$C_1(y_1, s) K_{\frac{1}{4}(\nu_1+1)} \left(2\pi |c| y_2 \sqrt{y_1^2 + s^2} \right) + C_2(y_1, s) I_{\frac{1}{4}(\nu_1+1)} \left(2\pi |c| y_2 \sqrt{y_1^2 + s^2} \right).$$

Here $C_i(y_1, s)$ ($i = 1, 2$) are functions in y_1, s . In view of (B'), $C_i(y_1, s)$ should be of the form $C_i(y_1^2 + s^2)$ with one variable function $C_i(t)$. From (C') we have

$$\frac{d}{dt} C_i(t) = -\frac{1}{8}(\nu_1 + 1)t C_i(t)$$

and therefore we obtain the following:

THEOREM 5.6 (Multiplicity-free theorem). *We have the following A-radial part $\varphi_0(y_1, y_2; s)$ for the generalized Whittaker function belonging to the spherical degenerate principal series:*

$$\varphi_0(y_1, y_2; s) = y_1^{\frac{1}{6}\nu_1 + \frac{1}{2}} y_2^{-\frac{1}{6}\nu_1 + \frac{1}{2}} \left(\frac{y_2}{\sqrt{y_1^2 + s^2}} \right)^{\frac{1}{4}(\nu_1+1)} \\ \times \left\{ C K_{\frac{1}{4}(\nu_1+1)} \left(2\pi |c| y_2 \sqrt{y_1^2 + s^2} \right) + C' I_{\frac{1}{4}(\nu_1+1)} \left(2\pi |c| y_2 \sqrt{y_1^2 + s^2} \right) \right\}.$$

In particular, the unique solution of moderate growth at infinity is given by $C' = 0$ in the above, up to constant multiple.

5.2.2. *The non-spherical case.* In the same way as in the spherical case, we can show the following from (GW3) and (GW4):

THEOREM 5.7 (Multiplicity-free theorem). *We have*

$$\begin{cases} \varphi_{2,0}(y_1, y_2; s) = \frac{1}{2} \left(\frac{1}{2\pi c y_2} \frac{\partial}{\partial s} + 1 \right) \varphi(y_1, y_2; s), \\ \varphi_{2,1}(y_1, y_2; s) = \frac{1}{4\pi c \sqrt{-1}} \frac{y_1}{y_2 s} \frac{\partial}{\partial s} \varphi(y_1, y_2; s), \\ \varphi_{2,2}(y_1, y_2; s) = \frac{1}{2} \left(\frac{1}{2\pi c y_2} \frac{\partial}{\partial s} - 1 \right) \varphi(y_1, y_2; s), \end{cases}$$

with

$$\begin{aligned} \varphi(y_1, y_2; s) &= y_1^{\frac{1}{6}v_1 + \frac{1}{2}} y_2^{-\frac{1}{6}v_1 + \frac{3}{2}} \left(\frac{y_2}{\sqrt{y_1^2 + s^2}} \right)^{\frac{1}{4}(v_1 - 1)} \\ &\quad \times \left\{ C K_{\frac{1}{4}(v_1 - 1)} \left(2\pi |c| y_2 \sqrt{y_1^2 + s^2} \right) + C' I_{\frac{1}{4}(v_1 - 1)} \left(2\pi |c| y_2 \sqrt{y_1^2 + s^2} \right) \right\}. \end{aligned}$$

In particular, the solution with moderate growth condition when $y_1, y_2 \rightarrow \infty$ is given unique up to scalar multiple by $C' = 0$ in the above.

5.3. Another realization of Schrödinger representation and generalized Whittaker functions

The change of polarization algebra from \mathfrak{n}_1 to

$$\mathfrak{n}_2 := \left\{ \begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

induces an intertwining isomorphism which is realized by the following Fourier transformation.

DEFINITION 5.8. For $f \in L^2(\mathbf{R})$, set

$$f^*(t) = \mathcal{F}_c(f)(t) := \int_{\mathbf{R}} f(s) \exp(2\pi \sqrt{-1} c s t) ds \quad (ds \text{ the Lebesgue measure}).$$

The integration by part, and the change of the order of differentiation and integration imply that

$$\mathcal{F}_c \left(\frac{d}{ds} f \right) = -(2\pi \sqrt{-1} c t) \mathcal{F}_c(f), \quad \frac{d}{dt} \mathcal{F}_c(f) = \mathcal{F}_c(2\pi \sqrt{-1} c s \cdot f).$$

Passing to the differential vectors in the dual $L^2(\mathbf{R})$, we have the following:

LEMMA 5.9. Under the above realization of the Schrödinger representation with respect to \mathfrak{n}_2 , the action of an element in \mathfrak{n} is given by

$$E_{13}f^*(t) = 2\pi\sqrt{-1}cf^*(t), \quad E_{12}f^*(t) = -2\pi\sqrt{-1}ctf^*(t),$$

$$E_{23}f^*(t) = \frac{d}{dt}f^*(t)$$

for $f^* \in \mathcal{S}(\mathbf{R})$.

PROPOSITION 5.10. For the A -radial part φ_0 of the generalized Whittaker function given in §5.1.1 and §5.2.1, set $\varphi^*(t) = \mathcal{F}_c(\varphi_0)$. Then the system of partial differential equations for φ^* is obtained from that of φ_0 by the replacement of the symbols:

$$y_1 \leftrightarrow y_2, \quad \nu_1 \leftrightarrow -\nu_1, \quad s \mapsto t, \quad L_+ \leftrightarrow L_-.$$

Therefore $\varphi^*(t) = \varphi^*(y_1, y_2; t)$ with moderate growth property is

$$C y_1^{\frac{1}{6}\nu_1 + \frac{1}{2}} y_2^{-\frac{1}{6}\nu_1 + \frac{1}{2}} \left(\frac{y_1}{\sqrt{y_2^2 + t^2}} \right)^{\frac{1}{4}(-\nu_1 + 1)} K_{\frac{1}{4}(-\nu_1 + 1)} \left(2\pi|c|y_1\sqrt{y_2^2 + t^2} \right)$$

with some constant C .

To determine the constant C we utilize the following formula ([2, 6.726.4, p. 730]):

$$\int_{\mathbf{R}} \alpha^{\pm\nu} (x^2 + \beta^2)^{\mp\frac{1}{2}\nu} K_{\nu} \left(\alpha\sqrt{x^2 + \beta^2} \right) \exp(\sqrt{-1}xy) dx$$

$$= \sqrt{2\pi} \beta^{\frac{1}{2}\mp\nu} (y^2 + \alpha^2)^{\pm\frac{1}{2}\nu - \frac{1}{4}} K_{\frac{1}{2}\mp\nu} \left(\beta\sqrt{y^2 + \alpha^2} \right)$$

for $\text{Re}(\alpha), \text{Re}(\beta) > 0$. Take the upper sign in the above formula. Then, by the change of variables

$$x = \sqrt{2\pi c}s, \quad y = \sqrt{2\pi c}t, \quad \alpha = \sqrt{2\pi c}y_2, \quad \beta = \sqrt{2\pi c}y_1, \quad \nu = \frac{1}{4}(\nu_1 + 1),$$

we have $C = |c|^{-\frac{1}{2}}$, i.e.,

$$\int_{\mathbf{R}} \left(\frac{y_2}{\sqrt{y_1^2 + s^2}} \right)^{\frac{1}{4}(\nu_1 + 1)} K_{\frac{1}{4}(\nu_1 + 1)} \left(2\pi|c|y_2\sqrt{y_1^2 + s^2} \right) \exp(2\pi\sqrt{-1}cst) ds$$

$$= |c|^{-\frac{1}{2}} \left(\frac{y_1}{\sqrt{y_2^2 + t^2}} \right)^{\frac{1}{4}(-\nu_1 + 1)} K_{\frac{1}{4}(-\nu_1 + 1)} \left(2\pi|c|y_1\sqrt{y_2^2 + t^2} \right).$$

5.4. Generalized Whittaker functions with respect to maximal parabolic subgroups

The unitary characters of the abelian unipotent radical $N_1 = \{n(0, x_{13}, x_{23}) \mid x_{13}, x_{23} \in \mathbf{R}\}$ of the maximal parabolic subgroup P_1 of G are exhausted by

$$\chi_{c,d} : n(0, x_{13}, x_{23}) \mapsto \exp\{2\pi\sqrt{-1}(cx_{13} + dx_{23})\} \quad (c, d \in \mathbf{R}).$$

The unitary induction $\text{Ind}_{N_1}^N(\chi_{c,d})$ from N_1 to N gives a Schrödinger representation ψ of N . Then by the transitivity of the induction, we have

$$\text{Hom}_{(\mathfrak{g},K)}(\pi, \text{Ind}_N^G(\psi)) \cong \text{Hom}_{(\mathfrak{g},K)}(\pi, \text{Ind}_{N_1}^G(\chi_{c,d})).$$

Therefore we have a result analogous to Theorems 5.6 and 5.7. More precisely we can show the following:

THEOREM 5.11. *Fix the double coset decomposition*

$$G = N_1 B_1 K$$

with

$$B_1 = \{n(s, 0, 0) \mid s \in \mathbf{R}\} \cdot A.$$

Then for a non-zero intertwining operator $\tilde{I} \in \text{Hom}_{(\mathfrak{g},K)}(\pi, \text{Ind}_{N_1}^G(\chi_{c,d}))$ the B_1 -‘radial’ parts $\Phi(y_1, y_2; s)$ of $\tilde{I}(f_0)$ or $\tilde{I}(f_{*,k})$ ($f_0 \in \pi(1_{M_1}, \nu_1)$ or $f_{*,k} \in \pi(\det_{M_1}, \nu_1)$) are given by the functions in Theorems 5.6 and 5.7 with the replacement $s \mapsto s + d/c$.

6. Fourier expansions

Let us review some of the immediate implications of the local multiplicity-free theorems 4.6, 4.7, 5.6, 5.7 and 5.11 in the previous sections for the Fourier expansions of automorphic forms on $\Gamma \backslash G$ ($\Gamma = SL(3, \mathbf{Z})$) belonging to the spherical degenerate principal series representation of $G = SL(3, \mathbf{R})$.

6.1. General forms of Fourier expansions

Analogous situation was, probably firstly in the literature, legitimately investigated by Ishikawa [4], and later extended to some important and fundamental cases by Narita [6].

6.1.1. *Fourier expansion along the maximal parabolic subgroup P_1 .* Let F be a right K -invariant automorphic form on $\Gamma \backslash G$ whose right G -translations generate a (\mathfrak{g}, K) -module isomorphic to the spherical degenerate principal series $\pi(1_{M_1}, \nu_1)$. For

$$\begin{cases} n_1 = n(0, x_{13}, x_{23}) \in N_1, \\ a = (y_1 y_2^2)^{-1/3} \text{diag}(y_1 y_2, y_2, 1) \in A, \\ b_1 = n(x_{12}, 0, 0) \cdot a \in B_1, \end{cases}$$

the function $F(n_1 b_1)$ in the variable n_1 is periodic under the lattice $\Gamma \cap N_1$ in N_1 . Hence by Theorem 5.11, we have the Fourier expansion:

$$F(n_1 b_1) = \sum_{(m_2, m_3) \in \mathbf{Z}^2} F_{(m_2, m_3)}(b_1) \chi_{(m_2, m_3)}(n_1),$$

where $F_{(m_2, m_3)}$ are functions independent of the variables x_{13}, x_{23} . The nature of each term $F_{(m_2, m_3)}$ is different depending on the value of the parameter c of the central character.

If $m_3 = 0$, then $F_{(m_2, 0)}(b_1)$ is periodic in x_{12} modulo \mathbf{Z} , hence

$$F_{(m_2,0)}(b_1) = \sum_{m_1 \in \mathbf{Z}} F_{(m_1, m_2, 0)}(a) \exp(2\pi \sqrt{-1} m_1 x_{12}),$$

with functions $F_{(m_1, m_2, 0)}$ depending only on $a \in A$. By the equivariance property of $F_{(m_1, m_2, 0)}$ with respect to N , we have

$$F_{(m_1, m_2, 0)}(a) = c_{(m_1, m_2, 0)}(F) W_{(m_1, m_2)}(a) \quad (c_{(m_1, m_2, 0)}(F) \in \mathbf{C}),$$

where $W_{(m_1, m_2)}(a)$ is the radial part of the degenerate Whittaker function associated with the character:

$$n(x_{12}, x_{13}, x_{23}) \mapsto \exp\{2\pi \sqrt{-1}(m_1 x_{12} + m_2 x_{23})\},$$

which is specified by Theorem 4.6.

If $m_3 \neq 0$, by Theorem 5.11 we have

$$F_{(m_2, m_3)}(b_1) = c_{(m_2, m_3)}(F) G W_{m_3}(y_1, y_2; x_{12} + m_2/m_3) \quad (c_{(m_2, m_3)}(F) \in \mathbf{C}).$$

Here the function $G W_{m_3}$ is the radial part of the generalized Whittaker function in Theorem 5.6. Summing up, we obtain the following:

PROPOSITION 6.1. *The Fourier expansion along P_1 is of the form*

$$\begin{aligned} F(n_1 b_1) &= \sum_{i=1}^4 c_{(0,0,0),i}(F) W_i(y_1, y_2) \\ &+ \sum_{\substack{(m_1, m_2) \in \mathbf{Z}^2 \setminus \{(0,0)\} \\ m_1 m_2 = 0}} c_{(m_1, m_2, 0)}(F) W_{(m_1, m_2)}(y_1, y_2) \\ &\quad \times \exp\{2\pi \sqrt{-1}(m_1 x_{12} + m_2 x_{23})\} \\ &+ \sum_{m_3 \in \mathbf{Z} \setminus \{0\}} \sum_{m_2 \in \mathbf{Z}} c_{(m_2, m_3)}(F) G W_{m_3}(y_1, y_2; x_{12} + m_2/m_3) \\ &\quad \times \exp\{2\pi \sqrt{-1}(m_2 x_{23} + m_3 x_{13})\}, \end{aligned}$$

with Fourier coefficients $c_{(0,0,0),i}(F)$, $c_{(m_1, m_2, 0)}(F)$ and $c_{(m_2, m_3)}(F)$. Here $W_i(y_1, y_2)$ and $W_{(m_1, m_2)}(y_1, y_2)$ are the radial parts of the moderate growth (degenerate) Whittaker functions;

$$\begin{aligned} W_1(y_1, y_2) &= 1, & W_2(y_1, y_2) &= (y_1^2 y_2)^{-\frac{1}{6}v_1 + \frac{1}{2}}, \\ W_3(y_1, y_2) &= (y_1 y_2^2)^{\frac{1}{6}v_1 + \frac{1}{2}}, & W_4(y_1, y_2) &= y_1^{\frac{1}{6}v_1 + \frac{1}{2}} y_2^{-\frac{1}{6}v_1 + \frac{1}{2}}, \end{aligned}$$

$$W_{(m_1, m_2)}(y_1, y_2) = \begin{cases} y_1^{-\frac{1}{12}v_1 + \frac{3}{4}} y_2^{-\frac{1}{6}v_1 + \frac{1}{2}} K_{\frac{1}{4}(v_1-1)}(2\pi |m_1| y_1) & \text{if } m_1 \neq 0 \text{ and } m_2 = 0, \\ y_1^{\frac{1}{6}v_1 + \frac{1}{2}} y_2^{\frac{1}{12}v_1 + \frac{3}{4}} K_{\frac{1}{4}(v_1+1)}(2\pi |m_2| y_2) & \text{if } m_1 = 0 \text{ and } m_2 \neq 0, \end{cases}$$

and $G W_{m_3}(y_1, y_2; s) = G W_{m_3}(y_1, y_2; s; v_1)$ is the generalized Whittaker function

$$\begin{aligned}
 &GW_{m_3}(y_1, y_2; s; \nu_1) \\
 &= y_1^{\frac{1}{6}\nu_1 + \frac{1}{2}} y_2^{-\frac{1}{6}\nu_1 + \frac{1}{2}} \left(\frac{y_2}{\sqrt{y_1^2 + s^2}} \right)^{\frac{1}{4}(\nu_1 + 1)} K_{\frac{1}{4}(\nu_1 + 1)} \left(2\pi |m_3| y_2 \sqrt{y_1^2 + s^2} \right).
 \end{aligned}$$

Note that the function $F(n_1 b_1)$ in the variable n_1 is invariant under $N \cap \Gamma$. Then we have

LEMMA 6.2. *If $m_3 \neq 0$, then we have $c_{(m_2, m_3)}(F) = c_{(m'_2, m_3)}(F)$ if $m_2 \equiv m'_2 \pmod{m_3}$.*

6.1.2. *Fourier expansion along the minimal parabolic subgroup P_0 .* Now we can regard our Fourier expansion along P_1 as that along the standard minimal parabolic subgroup P_0 of G . Similarly as [6, Theorem 9.2, p. 575], we introduce *Whittaker-theta series*:

DEFINITION 6.3. For $c \neq 0$ and $m \in \mathbf{Z}$, set

$$\begin{aligned}
 \Theta_{\frac{m}{c}}(n_1 b_1) &= \Theta_{\frac{m}{c}}(y_1, y_2; x_{12}, x_{23}; \nu_1) \\
 &:= \sum_{k \in \mathbf{Z}} GW_c(y_1, y_2; x_{12} + m/c + k; \nu_1) \exp\{2\pi \sqrt{-1}(m + ck)x_{23}\}.
 \end{aligned}$$

Then the Fourier expansion of $F(n_1 b_1)$ along P_0 is written as follows (cf. [6, Theorem 9.6]):

PROPOSITION 6.4.

$$\begin{aligned}
 F(n_1 b_1) &= \sum_{i=1}^4 c_{(0,0,0),i}(F) W_i(y_1, y_2) \\
 &+ \sum_{\substack{(m_1, m_2) \in \mathbf{Z}^2 \setminus \{(0,0)\} \\ m_1 m_2 = 0}} c_{(m_1, m_2, 0)}(F) W_{(m_1, m_2)}(y_1, y_2) \\
 &\times \exp\{2\pi \sqrt{-1}(m_1 x_{12} + m_2 x_{23})\} \\
 &+ \sum_{m_3 \in \mathbf{Z} \setminus \{0\}} \sum_{m_2 \in \mathbf{Z}/m_3 \mathbf{Z}} c_{(m_2, m_3)}(F) \Theta_{\frac{m_2}{m_3}}(y_1, y_2; x_{12}, x_{23}; \nu_1) \\
 &\times \exp(2\pi \sqrt{-1} m_3 x_{13}).
 \end{aligned}$$

6.1.3. *Poisson summation formula with displacement.* This is a preparation for the next subsection. Let \mathbf{R}^* be the Pontriagin dual of \mathbf{R} . For $\varphi \in \mathcal{S}(\mathbf{R})$ and $\varphi^* \in \mathcal{S}(\mathbf{R}^*)$, we set

$$\begin{aligned}
 (\sigma_a \varphi)(s) &:= \varphi(s + a) \quad (a \in \mathbf{R}), \\
 (\tau_b \varphi^*)(t) &:= \varphi^*(t + b) \quad (b \in \mathbf{R}).
 \end{aligned}$$

Then we immediately have

$$\exp(-2\pi \sqrt{-1} cat) \tau_b(\mathcal{F}_c(\varphi)) = \exp(2\pi \sqrt{-1} cab) \mathcal{F}_c(\sigma_a \varphi \cdot \exp(2\pi \sqrt{-1} cbs)).$$

Apply the Poisson summation formula for the pair of the mutually dual variables (s, t) . Then we have

$$\begin{aligned} & \exp(2\pi\sqrt{-1}cab) \sum_{k \in \mathbf{Z}} \varphi(a+k) \exp(2\pi\sqrt{-1}cbk) \\ &= \sum_{l \in \mathbf{Z}} \varphi^*\left(b + \frac{l}{c}\right) \exp\left(-2\pi\sqrt{-1}ca \cdot \frac{l}{c}\right). \end{aligned}$$

We may call this *the Poisson summation formula with displacement*. Now we change the parameters by

$$a \mapsto x_{12} + \frac{m}{c}, \quad b \mapsto x_{23}.$$

Then the above formula yields

$$\begin{aligned} & \exp(2\pi\sqrt{-1}cx_{12}x_{23}) \sum_{k \in \mathbf{Z}} \varphi\left(x_{12} + \frac{m}{c} + k\right) \exp\{2\pi\sqrt{-1}(ck+m)x_{23}\} \\ &= \sum_{l \in \mathbf{Z}} \varphi^*\left(x_{23} + \frac{l}{c}\right) \exp\left\{-2\pi\sqrt{-1}\left(x_{12} + \frac{m}{c}\right)l\right\}. \end{aligned}$$

Since we have $GW_c^*(y_1, y_2; x_{12}; \nu_1) = |c|^{-\frac{1}{2}}GW_c(y_2, y_1; x_{12}; -\nu_1)$ in §5.4, in terms of Whittaker-theta function, we can write this as follows:

PROPOSITION 6.5. For $c \neq 0$ and $m \in \mathbf{Z}$,

$$\begin{aligned} & \exp(2\pi\sqrt{-1}cx_{12}x_{23})\Theta_{\frac{m}{c}}(y_1, y_2; x_{12}, x_{23}; \nu_1) \\ &= |c|^{-\frac{1}{2}} \sum_{l \in \mathbf{Z}/c\mathbf{Z}} \exp\left(-2\pi\sqrt{-1}\frac{ml}{c}\right)\Theta_{\frac{l}{c}}(y_2, y_1; x_{23}, -x_{12}; -\nu_1). \end{aligned}$$

This formula is applied to have a relation between the Fourier expansions with respect to P_1 and P_2 in the next subsection.

6.1.4. *Comparison with Fourier expansion along P_2 .* A main result of Narita [6] is to compare the Fourier expansions along various maximal parabolic subgroups utilizing the most ‘rough’ Fourier expansion along the minimal parabolic subgroup. We can do similarly in our case, to compare the Fourier expansions along P_1 and P_2 , with

$$P_2 := \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in G \right\}.$$

But there appears a new feature here, i.e., the Poisson summation formula. As in §6.1.1, the Fourier expansion of F along P_2 is given as follows:

PROPOSITION 6.6. For

$$\begin{cases} n_2 = n(x_{12}, x_{13}, 0), \\ b_2 = n(0, 0, x_{23}) \cdot (y_1 y_2^2)^{-1/3} \text{diag}(y_1 y_2, y_2, 1), \end{cases}$$

we have a Fourier expansion along P_2 :

$$\begin{aligned}
 F(n_2 b_2) &= \sum_{i=1}^4 \bar{c}_{(0,0,0),i}(F) W_i(y_1, y_2) \\
 &+ \sum_{\substack{(m_1, m_2) \in \mathbf{Z}^2 \setminus \{(0,0)\} \\ m_1 m_2 = 0}} \bar{c}_{(m_1, m_2, 0)}(F) W_{(m_1, m_2)}(y_1, y_2) \\
 &\times \exp\{2\pi \sqrt{-1}(-m_1 x_{12} + m_2 x_{23})\} \\
 &+ \sum_{m_3 \in \mathbf{Z} \setminus \{0\}} \sum_{m_1 \in \mathbf{Z}} \bar{c}_{(m_1, m_3)}(F) G W_{m_3}(y_2, y_1; x_{23} + m_1/m_3; -v_1) \\
 &\times \exp\{2\pi \sqrt{-1}(-m_1 x_{12} + m_3 x_{13})\},
 \end{aligned}$$

with Fourier coefficients $\bar{c}_{(0,0,0),i}(F)$, $\bar{c}_{(m_1, m_2, 0)}(F)$ and $\bar{c}_{(m_1, m_3)}(F)$.

By applying Proposition 6.5 to the last term in Proposition 6.1 and comparing with Proposition 6.6, we obtain the following:

PROPOSITION 6.7. For $m_3 \neq 0$,

$$\bar{c}_{(m_1, m_3)}(F) = |m_3|^{-1/2} \sum_{m_2 \in \mathbf{Z}/m_3 \mathbf{Z}} c_{(m_2, m_3)}(F) \exp\left(-2\pi \sqrt{-1} \frac{m_1 m_2}{m_3}\right).$$

6.2. The case of the Epstein zeta function

In this section we discuss the Fourier expansion of Epstein zeta function in our formulation of Fourier expansions. The statements themselves are nothing new, but historically speaking this was the original problem.

Let $Z(s, Y)$ be the Epstein zeta function of degree 3

$$Z(s, Y) = \frac{1}{2} \sum_{\mathbf{m} \in \mathbf{Z}^3 \setminus \{0\}} (\mathbf{m} Y {}^t \mathbf{m})^{-s}.$$

Here $Y = g {}^t g$ with

$$g = y_1^{-1/3} y_2^{-2/3} \begin{pmatrix} y_1 y_2 & y_2 x_{12} & x_{13} \\ & y_2 & x_{23} \\ & & 1 \end{pmatrix} \in G.$$

It is known that $Z(s, Y)$ converges absolutely for $\text{Re}(s) > 3/2$ and is continued to a meromorphic function of s and satisfies Epstein's functional equation

$$(\natural) \quad \pi^{-s} \Gamma(s) Z(s, Y) = (\det Y)^{-1} \pi^{-(\frac{3}{2}-s)} \Gamma(\frac{3}{2}-s) Z(\frac{3}{2}-s, Y^{-1}).$$

The Fourier expansion of $Z(s, Y)$ is given by Terras [10]. We refer [10, Theorem 1] with $(n_1, n_2) = (2, 1)$ (resp. $(1, 2)$) to get the Fourier expansion along P_1 (resp. P_2):

PROPOSITION 6.8. Under the notation in Proposition 6.1 with $v_1 = 4s - 3$, Fourier coefficients $c_*(s) = c_*(Z(s, Y))$ are given as follows:

$$\begin{aligned} c_{(0,0,0),1}(s) &= 0, & c_{(0,0,0),2}(s) &= \frac{\pi \Gamma(s-1)}{\Gamma(s)} \zeta(2s-2), \\ c_{(0,0,0),3}(s) &= \zeta(2s), & c_{(0,0,0),4}(s) &= \frac{\sqrt{\pi} \Gamma(s-1/2)}{\Gamma(s)} \zeta(2s-1), \end{aligned}$$

$$c_{(m_1,0,0)}(s) = \frac{2\pi^s}{\Gamma(s)} |m_1|^{1-s} \sigma_{2s-2}(|m_1|) \quad (m_1 \neq 0), \quad c_{(0,m_2,0)}(s) = 0 \quad (m_2 \neq 0),$$

$$c_{(0,m_3)}(s) = 0, \quad c_{(m_2,m_3)}(s) = \frac{\pi^s}{\Gamma(s)} |m_3|^{-s+1/2} \sigma_{2s-1}(|m_2|, |m_3|) \quad (m_2, m_3 \neq 0),$$

where $\sigma_\nu(n) = \sum_{d|n} d^\nu$ is the divisor function and (m, n) means the g.c.d of m and n .

PROPOSITION 6.9. Under the notation in Proposition 6.6 with $v_1 = 4s - 3$, Fourier coefficients $\bar{c}_*(s) = \bar{c}_*(Z(s, Y))$ are given as follows:

$$\bar{c}_{(0,0,0),i}(s) = c_{(0,0,0),i}(s) \quad (1 \leq i \leq 4),$$

$$\bar{c}_{(m_1,0,0)}(s) = 0 \quad (m_1 \neq 0), \quad \bar{c}_{(0,m_2,0)}(s) = \frac{2\pi^s}{\Gamma(s)} |m_2|^{s-\frac{1}{2}} \sigma_{1-2s}(|m_2|) \quad (m_2 \neq 0),$$

$$\bar{c}_{(0,m_3)}(s) = 0, \quad \bar{c}_{(m_1,m_3)}(s) = \frac{\pi^s}{\Gamma(s)} |m_3|^{s-1} \sigma_{2-2s}(|m_1|, |m_3|) \quad (m_1, m_3 \neq 0).$$

REMARK 3. (i) The comparison of these two Fourier expansions along P_1 and P_2 is equivalent to the functional equation (†).

(ii) Fumihiro Sato [8] investigates Fourier coefficients of Eisenstein series with one parameter with respect to $SL(2n, \mathbf{Z})$ associated with the parabolic subgroup of type (n, n) .

(iii) By using the result of Hiroshi Oda and Toshio Oshima [7], it seems to be possible to extend our results to degenerate principal series of $SL(m, \mathbf{R})$ associated with maximal parabolic subgroup $P_{n,m-n}$ (personal communication by T. Oshima).

References

- [1] Corwin, L., Greenleaf, F. P.: Representations of nilpotent Lie groups and their applications, Part 1: Basic Theory and examples, Cambridge Univ. Press. 1989.
- [2] Gradshteyn, I. and Ryzhik, I.: Tables of integrals, series, and products (Corrected and enlarged edition), Academic Press, New York, (1980).
- [3] Ishii, T. and Oda, T.: A short history on investigation of the special values of zeta and L -functions of totally real number fields, to appear in *Automorphic forms and zeta functions, Proceedings of the conference in memory of Tsuneo Arakawa*.
- [4] Ishikawa, Y.: The generalized Whittaker functions for $SU(2, 1)$ and the Fourier expansion of automorphic forms, *J. Math. Sci. Univ. Tokyo* **6** (1999), no. 3, 477–526.
- [5] Manabe, H., Ishii, T., and Oda, T.: Principal series Whittaker functions on $SL(3, \mathbf{R})$, *Japan. J. Math. (N.S.)* **30** (2004), 183–226.
- [6] Narita, H.: Fourier expansion of holomorphic Siegel modular forms with respect to the minimal parabolic subgroup, *Math. Z.* **231** (1999), 557–588.
- [7] Oda, H. and Oshima, T.: Minimal polynomials and annihilators of generalized Verma modules of the scalar type, to appear in *J. of Lie Theory*.

- [8] Sato, F.: Fourier coefficients of Eisenstein series of GL_n , local densities of square matrices and subgroups of finite abelian groups, *Comment. Math. Univ. St. Pauli* **54** (2005), 33–48.
- [9] Siegel, C. L.: Advanced Analytic Number Theory. Tata Institute of Fundamental Research, Bombay, (1961).
- [10] Terras, A.: Bessel series expansions of the Epstein zeta function and the functional equation. *Trans. Amer. Math. Soc.* **183** (1973), 477–486.
- [11] Whittaker, E. T., Watson, G. N.: A course of modern analysis. Cambridge University Press, (1902).

Chiba Institute of Technology
2–1–1 Shibazono, Narashino,
Chiba 275–0023, Japan
e-mail: ishii.taku@it-chiba.ac.jp

Graduate School of Mathematical Sciences,
The University of Tokyo, Komaba
3–8–1, Meguro Ward, Tokyo 153–8914, Japan
e-mail: takayuki@ms.u-tokyo.ac.jp