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Zeta Functions and Casimir Energies on Infinite Symmetric Groups

by

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Abstract. We propose a definition of the Casimir energy of an infinite permutation using associated zeta function similar to the original Casimir energy. We show the negativity of the value in the case of a finite permuation. We investigate the case of windmill permutations from the view point of zeta functions of one variable or two variables. We look at the existence of the Euler product expression for zeta functions associated to permutations also.

1. Introduction

The Casimir energy (zero-point energy) is originated in physics (c.f. Milton [M]). Its generalization to a Riemann surface has been obtained in [KW1]. In this paper, we propose a definition of the Casimir energy of infinite permutations in the same spirit of the original Casimir energy, i.e., zeta regularization.

Let $\sigma : \mathbf{N} \to \mathbf{N}$ be the bijection between the set $\mathbf{N}$ of natural numbers. A naive definition of the Casimir energy of σ is

$$\mathrm{Cas}(\sigma) = \text{“}\sum_{n=1}^{\infty} n\sigma(n)\text{”}.$$

In order to justify the definition above, we introduce the zeta function of σ as

$$\zeta_\sigma(s) = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s}. \tag{1}$$

If $\zeta_\sigma(s)$ converges on some domain and has an analytic continuation to $s = -1$, then we define the Casimir energy $\mathrm{Cas}(\sigma)$ by

$$\mathrm{Cas}(\sigma) = \zeta_\sigma(-1).$$

Note that not all σ has the Casimir energy (see Example 20).

Let $S_\omega = \mathrm{Aut}(\mathbf{N})$. Let S_∞ be the subgroup of S_ω consisting of the elements σ such that $\sigma(n) = n$ for all but a finite number of $n \in \mathbf{N}$.

THEOREM 1. *Let $\sigma \in S_\infty$. Then, the zeta function $\zeta_\sigma(s)$ converges absolutely in* $\mathrm{Re}(s) > 2$, *has an analytic continuation to the entire s-plane as a meromorphic function,* *and is holomorphic except for a simple pole of residue* 1 *at* $s = 2$. *The Casimir energy is given by*

$$\mathrm{Cas}(\sigma) = -\frac{1}{2}\sum_{n\in\mathbf{N}}(\sigma(n) - n)^2 . \tag{2}$$

In particular, *the Casimir energy satisfies* $\mathrm{Cas}(\sigma) \le 0$. *The equality* $\mathrm{Cas}(\sigma) = 0$ *implies* $\sigma = \mathrm{id}$, *the identity.*

We introduce the following notion.

DEFINITION 2. Let k be a positive integer and $\tau \in S_k = \mathrm{Aut}(\{1, 2, \dots, k\})$. We define $\sigma = \sigma_\tau \in S_\omega$ by $\sigma(i + kj) = \tau(i) + kj$ for $i = 1, \dots, k$ and $j \ge 0$. Such a permutation σ_τ is called a *windmill* permutation.

This terminology is borrowed from the infinite orthogonal group.

THEOREM 3. *Let* σ *be a windmill permutation. Then* $\zeta_\sigma(s)$ *is a meromorphic function in* $s \in \mathbf{C}$ *and it is holomorphic except for a simple pole at* $s = 2$. *The Casimir energy has the following expression*:

$$\mathrm{Cas}(\sigma) = \frac{1}{2k}\sum_{i=1}^{k}(\sigma(i) - i)(k - i)i . \tag{3}$$

We give several examples of the Casimir energy of windmill permutations in Example 10. We also discuss the other special values of the zeta functions of σ at $s = 0$ and $s = 1$.

THEOREM 4. *Let* $\sigma \in S_\infty$. *Then*

(1) $\zeta_\sigma(0) = -\frac{1}{12}$.

(2) $\zeta_\sigma(1) = -\dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty}\frac{\sigma(n) - n}{n} \ge -\dfrac{1}{2}$. *The equality for the last inequality is valid only for the identity* $\sigma = \mathrm{id}$.

THEOREM 5. *Let* $\sigma \in S_\omega$ *be a windmill permutation coming from* $\tau \in S_k$.

(1) $\zeta_\sigma(0) = -\dfrac{1}{12} + \dfrac{1}{2k}\displaystyle\sum_{j=1}^{k}(\sigma(j) - j)^2 \ge -\dfrac{1}{12}$. *The equality for the last inequality is valid only for the identity* $\sigma = \mathrm{id}$.

(2) $\zeta_\sigma(1) = -\dfrac{1}{2} - \dfrac{1}{k}\displaystyle\sum_{j=1}^{k}(\sigma(j) - j)\frac{\Gamma'}{\Gamma}\left(\frac{j}{k}\right)$.

According to the naive definition Eq. (1), it seems that $\mathrm{Cas}(\sigma^{-1}) =$ "$\sum_{n\in\mathbf{N}} n\sigma^{-1}(n)$" would be equal to $\mathrm{Cas}(\sigma) =$ "$\sum_{n\in\mathbf{N}} \sigma(n)n$". This guess does not hold in general. More precisely, we have the following examples.

THEOREM 6. (1) $\mathrm{Cas}(\sigma) = \mathrm{Cas}(\sigma^{-1})$ *for all* $\sigma \in S_\infty$.

(2) *There is a windmill* $\sigma \in S_\omega$ *such that* $\mathrm{Cas}(\sigma) \neq \mathrm{Cas}(\sigma^{-1})$.
(3) $\zeta_\sigma(0) = \zeta_{\sigma^{-1}}(0)$ *for all* $\sigma \in S_\infty$.
(4) $\zeta_\sigma(0) = \zeta_{\sigma^{-1}}(0)$ *for all windmill* $\sigma \in S_\omega$.
(5) *There is an example of* σ *such that* $\zeta_\sigma(0)$ *and* $\zeta_{\sigma^{-1}}(0)$ *are defined but these values are different.*

One reasoning of the phenomenon (2) of Theorem 6 is given in Corollary 17, where we use the double Dirichlet series, see Definition 15.

Finally, we propose to examine a basic question, such as an existence of an Euler product, on the zeta function of a permutation introduced above. This question is partially answered in the final section.

2. On abscissa of zeta functions of a permutation

For $\sigma \in S_\omega$ let $\mathrm{abs}(\sigma)$ be the abscissa of absolute convergence of $\zeta_\sigma(s)$. By the general theory of Dirichlet series (see Hardy-Riesz [HR] Theorems 7 and 8) we know that

$$\mathrm{abs}(\sigma) = \limsup_{n\to\infty} \frac{\log(\sigma(1) + \cdots + \sigma(n))}{\log n} \geq 2$$

since $\sigma(1) + \cdots + \sigma(n) \geq 1 + \cdots + n$. We show that $\mathrm{abs}(\sigma)$ may take any number in $[2, +\infty)$.

THEOREM 7. *Let* $2 \leq c \leq +\infty$. *Then there exists a* $\sigma_c \in S_\omega$ *such that* $\mathrm{abs}(\sigma_c) = c$.

Proof. First, let

$$a_c(n) = \begin{cases} [2^{c^n}] & \text{if} \quad 2 \leq c < +\infty \\ 2^{2^{n^2}} & \text{if} \quad c = +\infty \end{cases}$$

for $n = 1, 2, 3, \ldots$. Then $1 < a_c(1) < a_c(2) < a_c(3) < \cdots$ are positive integers. Then it is easy to see that

$$\lim_{n\to\infty} \frac{\log(a_c(n+1))}{\log a_c(n)} = c\,.$$

Next we see that there exist (a lot of) $\sigma_c \in S_\omega$ satisfying

$$\begin{cases} \sigma_c(a_c(n)) = a_c(n+1), \quad n = 1, 2, \ldots, \\ \sigma_c(m) \leq m \text{ otherwise}\,. \end{cases}$$

One way to construct such an element σ_c is the following: Let $\sigma_c(a_c(n)) = a_c(n+1)$ for $n \geq 1$, and employ an order-preserving bijection between the complements.

Then we see that

$$\mathrm{abs}(\sigma_c) = \limsup_{N\to\infty} \frac{\log(\sigma_c(1) + \cdots + \sigma_c(N))}{\log N}$$

coincides with c. In fact

$$\lim_{n\to\infty} \frac{\log(\sigma_c(1) + \cdots + \sigma_c(a_c(n)))}{\log a_c(n)} = \lim_{n\to\infty} \frac{\log(\sigma_c(1) + \cdots + a_c(n+1))}{\log a_c(n)} = c\,.$$

Another way to obtain $\mathrm{abs}(\sigma_c) = c$ is to look directly at

$$\zeta_{\sigma_c}(s) = \sum_{n=1}^{\infty} \frac{a_c(n+1)}{a_c(n)^s} + \sum_{m} \frac{\sigma_c(m)}{m^s},$$

where the second term converges on $\mathrm{Re}(s) > 2$. □

3. The Casimir energies for finite/windmill permutations

In this section we calculate the Casimir energies of finite permutations and windmill permutations.

EXAMPLE 8. $\mathrm{Cas(id)} = 0$ since $\zeta_{\mathrm{id}}(s) = \zeta(s-1)$: $\mathrm{Cas(id)} = \zeta_{\mathrm{id}}(-1) = \zeta(-2) = 0$.

DEFINITION 9. For a natural number k, we denote by S_k the symmetric group of k letters, $\{1, 2, \dots, k\}$. Let S_∞ be the subgroup of S_ω consisting of the elements σ such that $\sigma(n) = n$ for all but a finite number of $n \in \mathbf{N}$. In other words, S_∞ is the union (the inductive limit) of S_k $(k = 1, 2, \dots)$ with respect to a natural inclusion $S_k \subset S_{k+1}$.

Proof of Theorem 1. For $\sigma \in S_\infty$,

$$\zeta_\sigma(s) = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \sum_{n=1}^{\infty} \frac{n}{n^s} + \sum_{n=1}^{\infty} \frac{\sigma(n)-n}{n^s} = \zeta(s-1) + \sum_{n=1}^{\infty} \frac{\sigma(n)-n}{n^s}.$$

This implies the analytic continuation of $\zeta_\sigma(s)$ since the final summation contains only a finite number of non-zero terms. To be precise, let $A = \{n \in \mathbf{N} \mid \sigma(n) \neq n\}$, then A is a finite set and

$$\zeta_\sigma(s) = \zeta(s-1) + \sum_{n \in A} (\sigma(n) - n) n^{-s}.$$

Hence, $\zeta_\sigma(s)$ is meromorphic in $s \in \mathbf{C}$, and it is holomorphic except for a simple pole at $s = 2$. Thus

$$\begin{aligned}\mathrm{Cas}(\sigma) &= \zeta(-2) + \sum_{n \in A} (\sigma(n) - n) n \\ &= 0 + \sum_{n \in A} (2\sigma(n)n - n^2 - \sigma(n)^2)/2 \\ &= -\sum_{n \in A} (\sigma(n) - n)^2/2, \qquad (4)\end{aligned}$$

which proves (2). Hence $\mathrm{Cas}(\sigma) = 0$ if and only if $A = \emptyset$, that is, $\sigma = \mathrm{id}$. □

Proof of Theorem 3. First of all,

$$\begin{aligned}\zeta_\sigma(s) &= \zeta(s-1) + \sum_{i=1}^{k}\sum_{j\geq 0}\frac{\tau(i)-i}{(i+kj)^s}\\ &= \zeta(s-1) + \frac{1}{k^s}\sum_{i=1}^{k}(\tau(i)-i)\zeta\left(s,\frac{i}{k}\right),\end{aligned}$$

where $\zeta(s,a)$ is the Hurwitz zeta function. Then the zeta function has an analytic continuation as stated above. The formula for the special value $\zeta(-1,a) = -B_2(a)/2 = -(a^2 - a + \frac{1}{6})/2$ implies that

$$\begin{aligned}\mathrm{Cas}(\sigma) &= -\frac{k}{2}\sum_{i=1}^{k}(\tau(i)-i)\left(\left(\frac{i}{k}\right)^2 - \left(\frac{i}{k}\right) + \frac{1}{6}\right)\\ &= \frac{1}{2k}\sum_{i=1}^{k}(\tau(i)-i)(k-i)i\\ &= \frac{1}{2k}\sum_{i=1}^{k}(\sigma(i)-i)(k-i)i\,.\end{aligned}$$

□

EXAMPLE 10. The Casimir energies of windmill permutations may take positive, negative and zero values. We give several examples using formula (3).

- For $k = 2$, $\tau = (12)$, then $\mathrm{Cas}(\sigma) = 1/4 > 0$.
- For $k = 4$, $\tau = (12)$, then $\mathrm{Cas}(\sigma) = -1/8 < 0$.
- For $k = 4$, $\tau = (34)$, then $\mathrm{Cas}(\sigma) = 3/8 > 0$.
- For $k = 3$, $\tau = (12)$, then $\mathrm{Cas}(\sigma) = 0$.
- For $k = 3$, $\tau = (123)$. Then $\mathrm{Cas}(\sigma) = 2/3$ and $\mathrm{Cas}(\sigma^{-1}) = 1/3$.

4. More on special values

The special values $\zeta_\sigma(s)$ at $s = -1$ have been discussed as the Casimir energy. In this section, we discuss the special values at $s = 0$ and $s = 1$.

Proof of Theorem 4. From

$$\zeta_\sigma(s) = \zeta(s-1) + \sum_{n=1}^{\infty}\frac{\sigma(n)-n}{n^s}$$

we have

$$\zeta_\sigma(0) = \zeta(-1) + \sum_{n=1}^{\infty}(\sigma(n)-n)$$

and

$$\zeta_\sigma(1) = \zeta(0) + \sum_{n=1}^{\infty} \frac{\sigma(n) - n}{n}\,.$$

Take N such that $\sigma \in S_N \subset S_\infty$. Then

$$\zeta_\sigma(0) = \zeta(-1) + \sum_{n=1}^{N}(\sigma(n) - n) = -\frac{1}{12}\,.$$

Moreover

$$\begin{aligned}
\zeta_\sigma(1) &= -\frac{1}{2} + \sum_{n=1}^{N} \frac{\sigma(n) - n}{n} \\
&= -\frac{1}{2} + N\left(\frac{1}{N}\sum_{n=1}^{N}\frac{\sigma(n)}{n} - 1\right) \\
&\geq -\frac{1}{2}\,,
\end{aligned}$$

where we used that

$$\frac{1}{N}\sum_{n=1}^{N}\frac{\sigma(n)}{n} \geq \left(\prod_{n=1}^{N}\frac{\sigma(n)}{n}\right)^{1/N} = 1\,.$$

□

Proof of Theorem 5. Recall that

$$\zeta_\sigma(s) = \zeta(s-1) + \sum_{j=1}^{k}(\tau(j) - j)k^{-s}\zeta\left(s, \frac{j}{k}\right).$$

Hence,

$$\begin{aligned}
\zeta_\sigma(0) &= -\frac{1}{12} + \sum_{j=1}^{k}(\tau(j) - j)\left(\frac{1}{2} - \frac{j}{k}\right) \\
&= -\frac{1}{12} - \frac{1}{k}\sum_{j=1}^{k}(\tau(j) - j)j
\end{aligned}$$

gives (1).

From the Laurent expansion for $\zeta(s, x)$ around $s = 1$ due to Lerch (cf. [KW2]) we see that

$$\zeta(s, x) = \frac{1}{s-1} - \frac{\Gamma'}{\Gamma}(x) + \varphi(s, x)$$

with $\varphi(1,x)=0$. Hence, we have

$$\sum_{j=1}^{k}(\tau(j)-j)\zeta\left(s,\frac{j}{k}\right)=-\sum_{j=1}^{k}(\tau(j)-j)\frac{\Gamma'}{\Gamma}\left(\frac{j}{k}\right)+\sum_{j=1}^{k}(\tau(j)-j)\varphi\left(s,\frac{j}{k}\right).$$

Thus, letting $s\to 1$ we get (2). □

An example

We give an example other than finite permutations or windmill permutations.

THEOREM 11. *Let $Q(m)=m^2$. We define $\sigma\in S_\omega$ by*

$$\begin{cases}\sigma(Q(m))=Q(m-1)+1 & \text{for } m=1,2,\ldots\\ \sigma(n)=n+1 & \text{otherwise}.\end{cases}$$

$$\begin{array}{c|ccc|ccccc|ccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & \cdots\\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 1 & 3 & 4 & 2 & 6 & 7 & 8 & 9 & 5 & 11 & 12 & \cdots \end{array}$$

Note that

$$\begin{cases}\sigma^{-1}(Q(m)+1)=Q(m+1) & \text{for } m=0,1,2,\ldots\\ \sigma^{-1}(n)=n+1 & \text{otherwise}.\end{cases}$$

Then

(1) $\zeta_\sigma(s)=\zeta(s-1)+\zeta(s)-2\zeta(2s-1)+\zeta(2s)$, *and is meromorphic on* **C**.

(2) $\zeta_\sigma(-1)=-\frac{1}{10}$, $\zeta_\sigma(0)=-\frac{11}{12}$.

(3)

$$\zeta_{\sigma^{-1}}(s)=\zeta(s-1)-\zeta(s)+1+3\cdot 2^{-s}+\sum_{k=0}^{\infty}(-1)^k\binom{s+k-1}{k}\left(2\zeta(2s+2k-1)+\zeta(2s+2k)-3\right),$$

and is meromorphic on **C**.

(4) $\zeta_{\sigma^{-1}}(-1)=-\frac{1}{15}$, $\zeta_{\sigma^{-1}}(0)=-\frac{1}{4}$.

Proof. (1)

$$\begin{aligned}\zeta_\sigma(s)&=\sum_{n=1}^{\infty}\frac{n+1}{n^s}+\sum_{n=1}^{\infty}\frac{\sigma(n)-n-1}{n^s}\\ &=\zeta(s-1)+\zeta(s)+\sum_{m=1}^{\infty}\frac{Q(m-1)-Q(m)}{Q(m)^s}\\ &=\zeta(s-1)+\zeta(s)-2\zeta(2s-1)+\zeta(2s).\end{aligned}$$

Note that $\mathrm{abs}(\sigma)=2$, and the last expression has a simple pole at $s=2$.

(2) We use (1).

$$\begin{aligned}\zeta_\sigma(-1) &= \zeta(-2) + \zeta(-1) - 2\zeta(-3) + \zeta(-2) \\ &= \zeta(-1) - 2\zeta(-3) = -\frac{1}{12} - \frac{2}{120} = -\frac{1}{10}. \\ \zeta_\sigma(0) &= \zeta(-1) + \zeta(0) - 2\zeta(-1) + \zeta(0) \\ &= 2\zeta(0) - \zeta(-1) = -1 + \frac{1}{12} = -\frac{11}{12}.\end{aligned}$$

(3)

$$\begin{aligned}\zeta_{\sigma^{-1}}(s) &= \sum_{n=1}^{\infty} \frac{n-1}{n^s} + \sum_{n=1}^{\infty} \frac{\sigma^{-1}(n) - n + 1}{n^s} \\ &= \zeta(s-1) - \zeta(s) + \sum_{m=0}^{\infty} \frac{Q(m+1) - Q(m)}{(Q(m)+1)^s} \\ &= \zeta(s-1) - \zeta(s) + \varphi(s),\end{aligned}$$

where we have set

$$\varphi(s) = \sum_{m=0}^{\infty} \frac{2m+1}{(m^2+1)^s}.$$

We have

$$\begin{aligned}\varphi(s) &= 1 + 3 \cdot 2^{-s} + \sum_{m=2}^{\infty} (2m+1)(m^2+1)^{-s} \\ &= 1 + 3 \cdot 2^{-s} + \sum_{m=2}^{\infty} m^{-2s}(2m+1) \sum_{k=0}^{\infty} \binom{-s}{k} m^{-2k} \\ &= 1 + 3 \cdot 2^{-s} + \sum_{k=0}^{\infty} \binom{-s}{k} \{2(\zeta(2s+2k-1) - 1) + (\zeta(2s+2k) - 1)\} \\ &= 1 + 3 \cdot 2^{-s} + \sum_{k=0}^{\infty} (-1)^k \binom{s+k-1}{k} \{2\zeta(2s+2k-1) + \zeta(2s+2k) - 3\} \\ &= 1 + 3 \cdot 2^{-s} + (2\zeta(2s-1) + \zeta(2s) - 3) \\ &\quad -s(2\zeta(2s+1) + \zeta(2s+2) - 3) + \frac{s(s+1)}{2}(2\zeta(2s+3) + \zeta(2s+4) - 3) - \cdots.\end{aligned}$$

This gives a meromorphic continuation of $\varphi(s)$ on $\mathbf{C}$. Note that $\zeta_{\sigma^{-1}}(s)$ has a simple pole at $s = 2$ and that $\mathrm{abs}(\sigma^{-1}) = 2$.

(4)

$$\begin{aligned}\varphi(0) &= 1+3+(2\zeta(-1)+\zeta(0)-3)-\operatorname*{Res}_{s=1}\zeta(s)\\ &= 1+2\times\left(-\frac{1}{12}\right)+\left(-\frac{1}{2}\right)-1=-\frac{2}{3}.\\ \zeta_{\sigma^{-1}}(0) &= \zeta(-1)-\zeta(0)+\varphi(0)\\ &= -\frac{1}{12}+\frac{1}{2}-\frac{2}{3}=-\frac{1}{4}.\end{aligned}$$

$$\begin{aligned}\varphi(-1) &= 1+3\cdot 2+(2\zeta(-3)+\zeta(-2)-3)+(2\zeta(-1)+\zeta(0)-3)-\frac{1}{2}\operatorname*{Res}_{s=1}\zeta(s)\\ &= 1+6+\left(2\times\frac{1}{120}-3\right)+\left(2\times\left(-\frac{1}{12}\right)+\left(-\frac{1}{2}\right)-3\right)-\frac{1}{2}\\ &= -\frac{3}{20}.\\ \zeta_{\sigma^{-1}}(-1) &= \zeta(-2)-\zeta(-1)+\varphi(-1)\\ &= \frac{1}{12}-\frac{3}{20}=-\frac{1}{15}.\end{aligned}$$

Note that the remaining summation over $k \geq 3$ converges absolutely and uniformly near $s = 0, -1$. □

5. Relation between $\mathrm{Cas}(\sigma)$ and $\mathrm{Cas}(\sigma^{-1})$

5.1. Zeta functions in two variables

Let us introduce

$$\zeta_\sigma(s,t) = \sum_{n=1}^{\infty}\frac{1}{\sigma(n)^s n^t}$$

for $\sigma \in S_\omega$. The convergence of $\zeta_\sigma(s,t)$ is discussed in Lemma 13 below. At this moment, we suppose the convergence and an analytic continuation as a meromorphic function in two variables (s,t). The Casimir energies are written as

$$\begin{aligned}\mathrm{Cas}(\sigma) &= \zeta_\sigma(s)|_{s=-1} = \zeta_\sigma(-1,s)|_{s=-1},\\ \mathrm{Cas}(\sigma^{-1}) &= \zeta_{\sigma^{-1}}(s)|_{s=-1} = \zeta_{\sigma^{-1}}(-1,s)|_{s=-1} = \zeta_\sigma(s,-1)|_{s=-1},\end{aligned} \tag{5}$$

where the last equality is an analytic continuation of the easy Lemma 14. Both of the right-hand sides seem to be equal to $\zeta(-1,-1)$. The validity is discussed in the later section.

EXAMPLE 12. For the identity $\mathrm{id} \in S_\infty$, $\zeta_{\mathrm{id}}(s,t) = \zeta(s+t)$. In the case $\sigma \in S_\infty$,

$$\zeta_\sigma(s,t) = \zeta(s+t) + \sum_{n\in\mathbf{N}}(\sigma(n)^{-s} - n^{-s})n^{-t}, \quad (\mathrm{Re}(s+t) > 1)$$

where the second term is a finite sum. It has an analytic continuation to the whole (s,t) space $\mathbf{C}^2$ and holomorphic at $(s,t) = (-1,-1)$. This implies

$$\mathrm{Cas}(\sigma^{-1}) = \zeta_\sigma(s,-1)|_{s=-1} = \zeta_\sigma(-1,-1) = \zeta_\sigma(-1,s)|_{s=-1} = \mathrm{Cas}(\sigma)\,.$$

We here prove the convergence and symmetry property quoted above.

LEMMA 13. *For $\sigma \in S_\omega$, the series $\zeta_\sigma(s,t)$ converges absolutely on $\{(s,t) \in \mathbf{C}^2 \mid \mathrm{Re}(s) \geq 0, \mathrm{Re}(t) \geq 0, \mathrm{Re}(s+t) > 1\}$.*

Proof. Let $A = \{n \in \mathbf{N} \mid \sigma(n) \geq n\}$ and $B = \{n \in \mathbf{N} \mid \sigma(n) < n\}$. Then $\mathbf{N} = A \cup B$ (a disjoint union). On $\mathrm{Re}(s) > 0$, we have

$$\left|\sum_{n\in A} \frac{1}{\sigma(n)^s n^t}\right| \leq \sum_{n\in A} \frac{1}{\sigma(n)^{\mathrm{Re}(s)} n^{\mathrm{Re}(t)}} \leq \sum_{n\in A} \frac{1}{n^{\mathrm{Re}(s+t)}} \leq \sum_{n\in \mathbf{N}} \frac{1}{n^{\mathrm{Re}(s+t)}}\,,$$

and similarly on $\mathrm{Re}(t) > 0$, we have

$$\left|\sum_{n\in B} \frac{1}{\sigma(n)^s n^t}\right| \leq \sum_{n\in B} \frac{1}{\sigma(n)^{\mathrm{Re}(s)} n^{\mathrm{Re}(t)}} \leq \sum_{n\in B} \frac{1}{\sigma(n)^{\mathrm{Re}(s+t)}} \leq \sum_{n\in \mathbf{N}} \frac{1}{n^{\mathrm{Re}(s+t)}}\,.$$

Both of them are bounded by $\zeta(\mathrm{Re}(s+t))$, which is convergent on $\mathrm{Re}(s+t) > 1$. □

LEMMA 14. *If $\zeta_\sigma(s,t)$ converges absolutely at (s,t), then $\zeta_{\sigma^{-1}}(t,s)$ converges at (t,s) and $\zeta_{\sigma^{-1}}(t,s) = \zeta_\sigma(s,t)$.*

5.2. Double Dirichlet series

We introduce

DEFINITION 15. We set

$$\zeta(s,t;x,y) = \sum_{m=0}^{\infty} \frac{1}{(m+x)^s (m+y)^t}$$

for $x, y > 0$.

It is obvious to have $\zeta(s,t;x,y) = \zeta(t,s;y,x)$ wherever both sides make sense.

LEMMA 16. (i)

$$\zeta(s,t;x,y) = \frac{1}{\Gamma(s)\Gamma(t)} \int_0^\infty \int_0^\infty \frac{e^{-ux-vy} u^{s-1} v^{t-1}}{1-e^{-u-v}} du\, dv\,.$$

This implies that $\zeta(s,t;x,y)$ is meromorphic in (s,t) on $\mathbf{C}^2$ for every fixed (x,y).

(ii) *Let $0 < x \leq 1$, $0 < y \leq 1$. Near $(s,t) = (-1,-1)$, we have*

$$\zeta(s,t;x,y) = \frac{(x-y)^3}{6} \frac{((s+1)-(s+1)^3)}{s+t+2} + (\textit{holomorphic})\,.$$

(iii)

$$\zeta(-1,s;x,y)|_{s=-1} - \zeta(s,-1;x,y)|_{s=-1} = \frac{1}{6}(y-x)^3\,. \qquad (6)$$

Proof. (i) follows from the standard argument. (iii) is an immediate consequence of (ii). For (ii),

$$\begin{aligned}
&\zeta(s,t;x,y)\\
&= x^{-s}y^{-t} + (1+x)^{-s}(1+y)^{-t} + \sum_{m=2}^{\infty} \frac{1}{m^{s+t}}\left(1+\frac{x}{m}\right)^{-s}\left(1+\frac{y}{m}\right)^{-t}\\
&= x^{-s}y^{-t} + (1+x)^{-s}(1+y)^{-t} + \sum_{m=2}^{\infty} \frac{1}{m^{s+t}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{-s}{i}\binom{-t}{j}\left(\frac{x}{m}\right)^{i}\left(\frac{y}{m}\right)^{j}\\
&= x^{-s}y^{-t} + (1+x)^{-s}(1+y)^{-t} + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(\zeta(s+t+i+j)-1)\binom{-s}{i}\binom{-t}{j}x^i y^j\\
&= x^{-s}y^{-t} + (1+x)^{-s}(1+y)^{-t} + \sum_{k=0}^{\infty}(\zeta(s+t+k)-1)F_k(s,t)\,,
\end{aligned}$$

where we set

$$F_k(s,t) = \sum_{i=0}^{k}\binom{-s}{i}\binom{-t}{k-i}x^i y^{k-i}\,.$$

Then

$$\begin{aligned}
\zeta(s,t;x,y) &= (\zeta(s+t+3)-1)F_3(s,t)\\
&\quad+\Big\{x^{-s}y^{-t} + (1+x)^{-s}(1+y)^{-t} + \sum_{k\neq 3}(\zeta(s+t+k)-1)F_k(s,t)\Big\}\,.
\end{aligned}$$

The second term converges and holomorphic at $(s,t) = (-1,-1)$. We will examine the first term.

$$\begin{aligned}
(\zeta(s+t+3)-1)F_3(s,t) &= \frac{F_3(s,-s-2)}{s+t+2}\\
&\quad+\left\{\left(\zeta(s+t+3)-\frac{1}{s+t+2}-1\right)F_3(s,t) + \frac{F_3(s,t)-F_3(s,-s-2)}{s+t+2}\right\}\,.
\end{aligned}$$

The second term is holomorphic at $(s,t) = (-1,-1)$. As for the first term, we have an equality for $k \in \mathbf{Z}_{\geq 0}$

$$F_k(s,-s-k+1) = \sum_{i=0}^{k}\binom{-s}{i}\binom{-s-i}{k-i}x^i(-y)^{k-i} = \binom{-s}{k}(x-y)^k\,.$$

In particular

$$\binom{-s}{3} = \frac{1}{6}(s+1)(1-(s+1)^2)$$

shows (ii). □

COROLLARY 17. *Consider the windmill permutation $\sigma \in S_\omega$ coming from $\tau \in S_k$.*

(i) *Assume* $\mathrm{Re}(s), \mathrm{Re}(t) > 1$. *Then*

$$\zeta_\sigma(s,t) = k^{-s-t} \sum_{i=1}^{k} \zeta\left(s,t; \frac{\tau(i)}{k}, \frac{i}{k}\right). \tag{7}$$

In particular, it has a meromorphic continuation to $\mathbf{C}^2$.

(ii)

$$\mathrm{Cas}(\sigma) - \mathrm{Cas}(\sigma^{-1}) = \frac{1}{6k} \sum_{i=1}^{k} (i - \sigma(i))^3.$$

Proof. (i)

$$\begin{aligned}
\zeta_\sigma(s,t) &= \sum_{n=1}^{\infty} \sigma(n)^{-s} n^{-t} \\
&= \sum_{i=1}^{k} \sum_{j=0}^{\infty} \sigma(i+kj)^{-s} (i+kj)^{-t} \\
&= \sum_{i=1}^{k} \sum_{j=0}^{\infty} (\tau(i)+kj)^{-s} (i+kj)^{-t} \\
&= k^{-s-t} \sum_{i=1}^{k} \sum_{j=0}^{\infty} \left(j + \frac{\tau(i)}{k}\right)^{-s} \left(j + \frac{i}{k}\right)^{-t} \\
&= k^{-s-t} \sum_{i=1}^{k} \zeta\left(s,t; \frac{\tau(i)}{k}, \frac{i}{k}\right).
\end{aligned}$$

(ii)

$$\begin{aligned}
&\mathrm{Cas}(\sigma) - \mathrm{Cas}(\sigma^{-1}) \\
&\quad = \zeta_\sigma(-1,s)|_{s=-1} - \zeta_\sigma(s,-1)|_{s=-1} \quad \text{by (5)} \\
&\quad = k^2 \sum_{i=1}^{k} \left(\zeta\left(-1,s; \frac{\tau(i)}{k}, \frac{i}{k}\right)\Big|_{s=-1} - \zeta\left(s,-1; \frac{\tau(i)}{k}, \frac{i}{k}\right)\Big|_{s=-1} \right) \quad \text{by (7)} \\
&\quad = k^2 \sum_{i=1}^{k} \frac{1}{6} \left(\frac{i-\tau(i)}{k}\right)^3 \quad \text{by (6)} \\
&\quad = \frac{1}{6k} \sum_{i=1}^{k} (i - \sigma(i))^3.
\end{aligned}$$

□

For example, if $k = 3$ and $\tau = (123)$, then $\mathrm{Cas}(\sigma) - \mathrm{Cas}(\sigma^{-1}) = 1/3$, which is compatible with Example 10. Summarize the above, the non-commutativity $\mathrm{Cas}(\sigma) \neq$

$\mathrm{Cas}(\sigma^{-1})$ of the Casimir energies of windmill permutations comes from the 'undefined point' of the meromorphic function with two variables.

Proof of Theorem 6. The statement (1) follows from Equation (2) in Theorem 1. An example for (2) is given in Example 10. Theorem 4(1) implies the statement (3), Theorem 5(1) implies the statement (4), and Theorem 11 implies the statement (5). □

6. Euler product problem

We may consider the following natural question; List up all σ whose zeta function $\zeta_\sigma(s)$ has an Euler product, equivalently, multiplicative $\sigma \in S_\omega$.

THEOREM 18. (1) *Let* $\sigma \in S_\infty$. *Then* $\zeta_\sigma(s)$ *has an Euler product if and only if* $\sigma = \mathrm{id}$.

(2) *There are 'a lot of'* $\sigma \in S_\omega$ *whose* $\zeta_\sigma(s)$ *have Euler products.*

Proof. (1) The 'if' part follows from the fact $\zeta_{\mathrm{id}}(s) = \zeta(s-1)$. Suppose $\sigma \in S_\infty$ is multiplicative. Then there exists a natural number N such that $\sigma \in S_N \subset S_\infty$. Then for $i \le N$ and a prime $p > N$, we have $pi = \sigma(pi) = \sigma(p)\sigma(i) = p\sigma(i)$. This shows $\sigma(i) = i$, that is, σ is the identity.

(2) We can give one method to construct such elements. Let $P = \{2, 3, 5, \dots\}$ be the set of all primes. For a given bijective map $\lambda \in \mathrm{Aut}(P)$, we define a multiplicative σ_λ by

$$\sigma_\lambda(p^e) = \lambda(p)^e \quad (e = 1, 2, \dots).$$

It is easy to see that $\sigma_{(\lambda^{-1})}$ is the inverse of σ_λ, which implies $\sigma_\lambda \in S_\omega$. Then

$$\zeta_{\sigma_\lambda}(s) = \prod_p \left(1 - \lambda(p)p^{-s}\right)^{-1}.$$

□

For a concrete example of such a $\lambda \in \mathrm{Aut}(P)$, we may pose several questions.

EXAMPLE 19. Let $\lambda \in \mathrm{Aut}(P)$ be an involution such as

$$2 \leftrightarrow 3, \quad 5 \leftrightarrow 7, \quad 11 \leftrightarrow 13, \quad 17 \leftrightarrow 19, \quad 23 \leftrightarrow 29, \dots.$$

Then

$$\zeta_{\sigma_\lambda}(s) = \left(1 - 3\cdot 2^{-s}\right)^{-1} \times \left(1 - 2\cdot 3^{-s}\right)^{-1} \times \left(1 - 7\cdot 5^{-s}\right)^{-1} \times \left(1 - 5\cdot 7^{-s}\right)^{-1} \times \cdots.$$

Is this meromorphically continued to whole $s \in \mathbf{C}$?

EXAMPLE 20. (Higher Euler product.)

We also give a slight modification of the above construction. We define a multiplicative σ by

$$\sigma(p^{2k}) = p^{2k-1}, \sigma(p^{2k-1}) = p^{2k} \qquad (k = 1, 2, \dots).$$

Then $\sigma \in S_\omega$ is an involution, that is, $\sigma^2 = \mathrm{id}$. The corresponding zeta function is

$$\begin{aligned}\zeta_\sigma(s) &= \prod_p \left(1 + \sum_{k=1}^{\infty}\left(\frac{p^{2k}}{p^{(2k-1)s}} + \frac{p^{2k-1}}{p^{2ks}}\right)\right) \\ &= \prod_p \left(1 + (p^s + p^{-1})\frac{p^{2-2s}}{1 - p^{2-2s}}\right) \\ &= \prod_p \frac{1 + p^{2-s} - (p^2 - p)p^{-2s}}{1 - p^{2-2s}} \\ &= \zeta(2s-2)\prod_p \left(1 + p^{2-s} - (p^2 - p)p^{-2s}\right).\end{aligned}$$

The last expression converges for $\mathrm{Re}(s) > 3$ and has an analytic continuation to $\mathrm{Re}(s) > 2$ as a meromorphic function. The $\mathrm{Re}(s) = 2$ is the natural boundary, c.f. [K1, 2]. In particular, we have no definition of the Casimir energy $\mathrm{Cas}(\sigma)$ for such a σ.

EXAMPLE 21. (Twin primes.)

The third example concerns with twin primes. We define $\lambda_{\mathrm{twin}} \in \mathrm{Aut}(P)$ by exchanging twin primes ≥ 5;

$$\begin{array}{cccccccccccc} 2 & 3 & 5 & 7 & 11 & 13 & 17 & 19 & 23 & 29 & 31 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 2 & 3 & 7 & 5 & 13 & 11 & 19 & 17 & 23 & 31 & 29 & \cdots \end{array}$$

We denote $\sigma_{\lambda_{\mathrm{twin}}} \in S_\omega$ by σ_{twin} for simplicity. Then

$$\zeta_{\sigma_{\mathrm{twin}}}(s) = \zeta(s-1) \times \prod_{p+2,\, p \geq 5:\ \text{primes}} \frac{(1 - p^{1-s})(1 - (p+2)^{1-s})}{(1 - (p+2)p^{-s})(1 - p(p+2)^{-s})}.$$

Is this meromorphic on $\mathbf{C}$ or of natural boundary type ? If it is the latter case, then we have a proof that there are infinitely many twin primes.

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