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Complex Pisot Numeration Systems

by

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Abstract. Using a real number $\beta > 1$, a positive real number x can be represented by $x = \sum_{n=k_0}^{\infty} b_n \beta^{-n}$, $b_n \in \{0, 1, 2, \dots, [\beta]\}$, which is so-called the β-expansion. This primitive numerical representation of the real number x proposes various topics in the field of number theory, ergodic theory, dynamical system theory, and tiling theory, etc. In particular, for an algebraic integer $\beta\,(> 1)$, many properties of the β-transformation are studied in [A], [Bl], [Re], [P], [IT], [Sol], [Sch], [T], etc. However, it seems to be unclear whether for a complex number z there exists the algorithm which induces the complex λ-expansion as $z = \sum_{n=k_0}^{\infty} a_n \lambda^{-n}$, $a_n \in \Gamma$ where Γ is the finite digit set of $\mathbb{Z}[\lambda]$ by using a complex algebraic integer λ. In this paper, by using a complex Pisot number $\lambda \in \mathbb{C}\backslash\mathbb{R}$, $|\lambda| > 1$, we give the algorithm which induces the complex Pisot λ-expansion.

0. Introduction

The purpose of this paper is to show the algorithm to produce the complex λ-expansion of $z \in \mathbb{C}$, $z = \sum_{n=k_0}^{\infty} a_n \lambda^{-n}$, $a_n \in \Gamma$ where Γ is the finite digit set of $\mathbb{Z}[\lambda]$. For this purpose, we introduce a complex Pisot number λ.

DEFINITION 0.1. A complex number $\lambda \in \mathbb{C}\backslash\mathbb{R}$ is a complex Pisot number if λ is the algebraic integer of the minimal polynomial $p(x) = x^d - k_1 x^{d-1} - \cdots - k_{d-1}x - k_d$, $k_i \in \mathbb{Z}$ $(1 \le i \le d)$ whose roots $\lambda\,(= \lambda_1)$, $\overline{\lambda}\,(= \lambda_2)$, $\lambda_3, \dots, \lambda_d$ satisfy

$$|\lambda| = \left|\overline{\lambda}\right| > 1 > |\lambda_i| \quad (3 \le i \le d)\,. \tag{0.1}$$

If $k_d = \pm 1$, λ is said to be *unimodular*. In this paper, we assume that λ is an unimodular complex Pisot number. Let A be the $d \times d$ integer matrix whose characteristic polynomial coincides with $p(x)$ and λ is a complex Pisot number of $p(x)$. We call A the *complex Pisot matrix* of λ.

We consider that a complex Pisot matrix A of λ is the linear transformation on the d-dimensional Euclidean space $\mathbb{R}^d$, therefore A has the 2-dimensional A-invariant expanding plane P_e and the $(d-2)$-dimensional A-invariant contracting plane P_c. Using P_e and P_c, $\mathbb{R}^d$ is decomposed into P_e and P_c, i.e., $\mathbb{R}^d = P_e \oplus P_c$. Then, let us define the projection $\pi_e : \mathbb{R}^d \to P_e$ (resp. $\pi_c : \mathbb{R}^d \to P_c$) along P_c (resp. P_e) by $\pi_e \boldsymbol{x} = x_1$ (resp. $\pi_c \boldsymbol{x} = x_2$) for $\boldsymbol{x} = x_1 + x_2 \in \mathbb{R}^d$, where $x_1 \in P_e$ and $x_2 \in P_c$.

DEFINITION 0.2. For a complex Pisot number λ, we assume that we can find the finite family of compact sets $\mathcal{P} = \{\gamma_j\}_{j\in I}$ of P_e with the finite integer vector sequence $\left\{\boldsymbol{f}_k^{(j)}\right\}_{1\le k\le l_j}$, $\boldsymbol{f}_k^{(j)} \in \mathbb{Z}^d$ and the finite index sequence $\left\{V_k^{(j)}\right\}_{1\le k\le l_j}$, $V_k^{(j)} \in I$, where I is an index set, satisfying

(N1) $\mu_e(\gamma_j) > 0$, $\mathrm{cl}\left(\mathrm{int}(\gamma_j)\right) = \gamma_j$, and $\mu_e(\partial\gamma_j) = 0$
where μ_e is the Lebesgue measure on P_e, $\mathrm{int}(Y)$ and $\mathrm{cl}(Y)$ are the interior and the closure of a set Y respectively, and $\partial Y := Y\backslash\mathrm{int}(Y)$;

(N2) for each $j \in I$, the following set equation holds:

$$A\gamma_j = \bigcup_{k=1}^{l_j}\left(\gamma_{V_k^{(j)}} + \pi_e \boldsymbol{f}_k^{(j)}\right) \quad \text{(disjoint)} \tag{0.2}$$

where "$\bigcup_k Y_k$ (disjoint)" means that $\mathrm{int}(Y_k) \cap \mathrm{int}(Y_{k'}) = \emptyset$ if $k \neq k'$;

(N3) $\gamma := \bigcup_{j\in I}\gamma_j$ (disjoint).

Then, we say that the pair $(A, \mathcal{P})$ is the complex Pisot numeration system of λ.

Note. In this paper, the index set I is chosen as $I = \{1, 2, 3\}$, $\{1\wedge 2, 1\wedge 3, 2\wedge 3\}$, etc.

From the complex Pisot numeration system $(A, \mathcal{P})$ of λ, we obtain the numerical expression of $\boldsymbol{x} \in \gamma$ by

$$\boldsymbol{x} = \sum_{n=1}^{\infty} A^{-n}\left(\pi_e \boldsymbol{f}_{k_{n-1}}^{(j_{n-1})}\right) \tag{0.3}$$

where the double positive integer sequence $\left(\binom{j_0}{k_0}\binom{j_1}{k_1}\cdots\binom{j_n}{k_n}\cdots\right)$ is given by the following process: for $\boldsymbol{x}_0 = \boldsymbol{x} \in \gamma_{j_0} \subset \gamma$, there exists $\binom{j_0}{k_0}$ such that $A\boldsymbol{x}_0 \in \gamma_{V_{k_0}^{(j_0)}} + \pi_e \boldsymbol{f}_{k_0}^{(j_0)}$ by (0.2) and then, put $\boldsymbol{x}_1 := A\boldsymbol{x}_0 - \pi_e \boldsymbol{f}_{k_0}^{(j_0)} \in \gamma_{V_{k_0}^{(j_0)}}$ and $j_1 := V_{k_0}^{(j_0)}$. Using $\boldsymbol{x}_1 \in \gamma_{j_1}$ and the existence $\binom{j_1}{k_1}$ such that $A\boldsymbol{x}_1 \in \gamma_{V_{k_1}^{(j_1)}} + \pi_e \boldsymbol{f}_{k_1}^{(j_1)}$, we obtain $\boldsymbol{x}_2 := A\boldsymbol{x}_1 - \pi_e \boldsymbol{f}_{k_1}^{(j_1)} \in \gamma_{j_2}$ and $j_2 := V_{k_1}^{(j_1)}$, and so on.

Moreover, there exists the linear map $\phi_e : P_e \to \mathbb{C}$ satisfying

$$\phi_e(A\boldsymbol{x}) = \lambda\phi_e(\boldsymbol{x}) \quad \text{and} \quad \phi_e(\pi_e \boldsymbol{e}_1) = 1,$$

so the numerical expression (0.3) can be represented as the complex Pisot λ-expansion by

$$z = \phi_e(\boldsymbol{x}) = \sum_{n=1}^{\infty} a_{\binom{j_{n-1}}{k_{n-1}}}\lambda^{-n} \in \phi_e(\gamma) \subset \mathbb{C}, \tag{0.4}$$

where $a_{\binom{j_n}{k_n}} = \phi_e\left(\pi_e \boldsymbol{f}_{k_n}^{(j_n)}\right)$. The precise definitions of these expressions (0.3), (0.4), and ϕ_e are found in the section 1.

In this paper, we discuss when we can find the complex Pisot numeration system $(A, \mathcal{P})$ of λ, in other words, for the complex Pisot matrix A, we give the way how to find the finite family of compact sets $\mathcal{P} = \{\gamma_j\}_{j \in I}$ satisfying (N1), (N2), (N3) in Definition 0.2.

We introduce three classes of the complex Pisot number which has the complex Pisot numeration system. The first class introduced in the section 2 is that the inverse of the complex Pisot matrix A of λ is the non-negative 3×3 integer matrix. The second class introduced in the section 3 is that the complex Pisot matrix of λ is the 3×3 companion matrix whose characteristic polynomial is $p(x) = x^3 - ax^2 - bx \pm 1$. In this class, we give the sufficient condition of $a, b \in \mathbb{Z}$ for existence of the complex Pisot numeration system of λ. And the third class introduced in the section 4 is that the complex Pisot matrix of λ is the 4×4 companion matrix whose characteristic polynomial is $p(x) = x^4 - ax^3 - bx^2 - cx \pm 1$. In this class, we give the sufficeint condition of $a, b, c \in \mathbb{Z}$ for existence of the complex Pisot numeration system of λ.

1. Complex Pisot expansions

1.1. Expanding transformations

Let us start to give the precise definition of the λ-expansion in this section again. For this purpose, let us start to give the following definition.

DEFINITION 1.1. Let $(A, \mathcal{P})$ be an complex Pisot numeration system of λ and let $B \subset P_e$ be the union of the boundary set of each compact set γ_j, i.e., $B := \bigcup_{j \in I} \partial \gamma_j$. We define the expanding transformation $T_A : \gamma \backslash B \to \gamma \backslash B$ by

$$T_A(\boldsymbol{x}) := A\boldsymbol{x} - \pi_e \boldsymbol{f}_{k_0}^{(j_0)} \quad \text{if} \quad \boldsymbol{x} \in \operatorname{int}(\gamma_{j_0}) \text{ and } A\boldsymbol{x} \in \operatorname{int}\left(\gamma_{V_{k_0}^{(j_0)}}\right) + \pi_e \boldsymbol{f}_{k_0}^{(j_0)},$$

and for $T_A(\boldsymbol{x}) \in \operatorname{int}\left(\gamma_{V_{k_0}^{(j_0)}}\right)$, the iteration of T_A is defined by

$$T_A^n(\boldsymbol{x}) := AT_A^{n-1}(\boldsymbol{x}) - \pi_e \boldsymbol{f}_{k_{n-1}}^{(j_{n-1})} \quad \text{if} \quad \begin{cases} T_A^{n-1}(\boldsymbol{x}) \in \operatorname{int}(\gamma_{j_{n-1}}) \\ \text{and} \\ AT_A^{n-1}(\boldsymbol{x}) \in \operatorname{int}\left(\gamma_{V_{k_{n-1}}^{(j_{n-1})}}\right) + \pi_e \boldsymbol{f}_{k_{n-1}}^{(j_{n-1})} \end{cases}.$$

If $AT_A^{n-1}(\boldsymbol{x}) - \pi_e \boldsymbol{f}_{k_{n-1}}^{(j_{n-1})} \in B$, then the iteration will be stopped. By the definition of the null set $Nu := \left\{\boldsymbol{x} \in \gamma \,\middle|\, \exists n : T_A^{n-1}(\boldsymbol{x}) \in B\right\}$, the iteration T_A^n is well-defined for all n for μ_e-almost all $\boldsymbol{x} \in \gamma$.

From Definition 1.1, for μ_e-almost all $\boldsymbol{x} \in \gamma$, there uniquely exists the sequence $w(\boldsymbol{x}) := \left(\binom{j_0}{k_0}\binom{j_1}{k_1}\cdots\binom{j_n}{k_n}\cdots\right)$ satisfying $T_A^n(\boldsymbol{x}) \in \operatorname{int}(\gamma_{j_n})$ and $AT_A^n(\boldsymbol{x}) \in \operatorname{int}(\gamma_{j_{n+1}}) + \pi_e \boldsymbol{f}_{k_n}^{(j_n)}$, $\boldsymbol{x}$ can be represented by

$$\boldsymbol{x} = \sum_{n=1}^{\infty} A^{-n}\left(\pi_e \boldsymbol{f}_{k_{n-1}}^{(j_{n-1})}\right). \tag{1.5}$$

We call (1.5) the *numerical representation* of $\boldsymbol{x}$.

1.2. λ-expansion

LEMMA 1.2. *Let λ be a unimodular complex Pisot number, let A be a complex Pisot matrix of λ, and let $\boldsymbol{u}_1, \boldsymbol{u}_2$ be the eigenvectors corresponding to $\lambda, \overline{\lambda}$ respectively. Put $\boldsymbol{v}_1 := \frac{\boldsymbol{u}_2+\boldsymbol{u}_1}{2}$, $\boldsymbol{v}_2 := \frac{\boldsymbol{u}_2-\boldsymbol{u}_1}{2i}$, then $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a base of P_e satisfying*

$$A\boldsymbol{v}_1 = c\boldsymbol{v}_1 + d\boldsymbol{v}_2, \quad A\boldsymbol{v}_2 = -d\boldsymbol{v}_1 + c\boldsymbol{v}_2$$

where $\lambda = c + di$. Moreover, there exists a linear map $\phi_e : \mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2) (= P_e) \to \mathbb{C}$ satisfying the following properties:

(1) $\phi_e(A\boldsymbol{x}) = \lambda\phi_e(\boldsymbol{x})$ *for* $x \in \mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)$;

(2) $\phi_e(\pi_e\boldsymbol{e}_1) = 1$.

Proof. It is easy to see that for $\mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2) \ni \boldsymbol{y} = y_1\boldsymbol{v}_1 + y_2\boldsymbol{v}_2$, $y_1, y_2 \in \mathbb{R}$, we have

$$A[\boldsymbol{v}_1\ \boldsymbol{v}_2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = [\boldsymbol{v}_1\ \boldsymbol{v}_2]\begin{bmatrix} c & -d \\ d & c \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Let us define

$$\begin{array}{cccc} \phi_e: & \mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2) & \to & \mathbb{C} \\ & \Downarrow & & \Downarrow \\ & \boldsymbol{y} = y_1\boldsymbol{v}_1 + y_2\boldsymbol{v}_2 & \mapsto & \alpha_e(y_1 + y_2 i), \end{array}$$

where $\alpha_e \in \mathbb{C}$ is a constant. Then, we see that ϕ_e is the linear map and that ϕ_e satisfies $\phi_e(A\boldsymbol{y}) = \lambda\phi_e(\boldsymbol{y})$. Moreover, if we choose the constant $\alpha_e = \frac{1}{x_1^{(1)} + x_2^{(1)} i}$ for $\pi_e\boldsymbol{e}_1 = x_1^{(1)}\boldsymbol{v}_1 + x_2^{(1)}\boldsymbol{v}_2$, then ϕ_e satisfies the properties (1) and (2). □

Hence, we obtain the following representaion: for $\phi \circ \mu_e$-almost all $z \in \phi_e(\gamma) \subset \mathbb{C}$, z can be represented by

$$z = \sum_{n=1}^{\infty} a_{\binom{j_{n-1}}{k_{n-1}}} \lambda^{-n} \tag{1.6}$$

where $a_{\binom{j_n}{k_n}} = \phi_e\left(\pi_e \boldsymbol{f}_{k_n}^{(j_n)}\right)$. It is the λ-*expansion* with the finite digits $\left\{\phi_e\left(\pi_e \boldsymbol{f}_k^{(j)}\right) \,\middle|\, j \in I,\ 1 \le k \le l_j\right\}$.

By the way, if A is given by

$$A = \begin{bmatrix} 0 & & & & O & & k_d \\ 1 & 0 & & & & & k_{d-1} \\ 0 & 1 & 0 & & & & k_{d-2} \\ & 0 & \ddots & \ddots & & & \vdots \\ & & \ddots & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & 0 & k_2 \\ & O & & & 0 & 1 & k_1 \end{bmatrix}, \tag{1.7}$$

which is called a *companion matrix* of $p(x)$, then we see that $a_{\binom{j_n}{k_n}} \in \mathbb{Z}[\lambda]$. In fact, from

$$\phi_e(\pi_e \boldsymbol{e}_k) = \phi_e\left(\pi_e A^{k-1}\boldsymbol{e}_1\right) \overset{\text{by Lemma 1.2 (1)}}{=} \lambda^{k-1}\phi_e(\pi_e \boldsymbol{e}_1) \overset{\text{by Lemma 1.2 (2)}}{=} \lambda^{k-1},$$

we know that for ${}^t[m_1\ m_2\ \cdots\ m_d] \in \mathbb{Z}^d$,

$$\phi_e\left(\pi_e{}^t[m_1\ m_2\ \cdots m_d]\right) = m_1 + m_2\lambda + \ldots + m_d\lambda^{d-1} \in \mathbb{Z}[\lambda]\ .$$

Therefore, if the complex Pisot matrix A is isomorphic to the companion matrix C of λ, i.e.,

$$\exists B \in GL(d, \mathbb{Z}) : B^{-1}AB = C\,,$$

then we can find ϕ_e such that $\phi_e(\pi_e \boldsymbol{u}) \in \mathbb{Z}[\lambda]$ for $\boldsymbol{u} \in \mathbb{Z}^d$, that is, we obtain the representation (1.6) in the sense of $a_{\binom{j_n}{k_n}} \in \mathbb{Z}[\lambda]$.

1.3. The graph of the admissible edge sequence

Let us define the directed multigraph $G_\lambda = (V, E, i, t)$ consisting of a finite set V of vertices, a countable set of directed edges E and two functions $i, t : E \to V$. For each edge $e \in E$, $i(e)$ is the initial vertex of e and $t(e)$ is the the terminal vertex of e. From the finite sequence $\left\{V_k^{(j)}\right\}_{1\le k\le l_j}$, $V_k^{(j)} \in I$ in Definition 0.2 (N2), we define V, E, i, t as follows:

$$V := I,\ E := \left\{ \binom{j}{k} \,\middle|\, j \in V,\ 1 \le k \le l_j \right\},$$

$$\begin{array}{ccc} i : E & \to & V \\ \Psi & & \Psi \\ \binom{j}{k} & \mapsto & j, \end{array} \qquad \begin{array}{ccc} t : E & \to & V \\ \Psi & & \Psi \\ \binom{j}{k} & \mapsto & V_k^{(j)}. \end{array} \qquad j_p \xrightarrow{\binom{j_p}{k_p}} V_{k_p}^{(j_p)} = j_{p+1}$$

From the directed multigraph G_λ, we obtain the one-sided edge-admissible symbolic space $\Omega_\lambda^{(j)}$ $(1 \le j \le N)$:

$$\begin{aligned} \Omega_\lambda^{(j)} &= \left\{ \left(\binom{j_0}{k_0}\binom{j_1}{k_1}\cdots\right) \,\middle|\, j_0 = j \in V,\ t\binom{j_p}{k_p} = i\binom{j_{p+1}}{k_{p+1}} \right\} \\ &= \left\{ \left(\binom{j_0}{k_0}\binom{j_1}{k_1}\cdots\right) \,\middle|\, j_0 = j \in V,\ t\binom{j_p}{k_p} = j_{p+1} \right\}. \end{aligned} \tag{1.8}$$

Moreover, we know that for μ_e-almost all $x \in \gamma$, the sequence $w(x) = \left(\binom{j_0}{k_0}\binom{j_1}{k_1}\cdots\binom{j_n}{k_n}\cdots\right)$ given by (1.5) is the admissible sequence of G_λ.

Let us define the labeling $\mathcal{L} : E \to \pi_e\mathbb{Z}^d$ and the map $\varphi : \Omega_\lambda^{(j)} \to P_e$ by

$$\mathcal{L}\left(\binom{j}{k}\right) := \pi_e f_k^{(j)}, \quad \varphi\left(\binom{j_0}{k_0}\binom{j_1}{k_1}\cdots\right) := \sum_{n=1}^{\infty} A^{-n}\left(\pi_e \boldsymbol{f}_{k_{n-1}}^{(j_{n-1})}\right),$$

then we have the following proposition.

PROPOSITION 1.3. *If G_λ is irreducible, then $\varphi\left(\Omega_\lambda^{(j)}\right) = \gamma_j$ for $j \in I$.*

Proof. It is easy to see that the set $\left\{\varphi\left(\Omega_{\lambda}^{(j)}\right)\right\}_{j\in I}$ is the family of the compact sets and satisfies the set equation (0.2) (see [Ed]). On the other hand, we see that $\gamma_j \backslash Nu \subset \varphi\left(\Omega_{\lambda}^{(j)}\right)$, $\gamma_j \subset \mathrm{cl}\left(\gamma_j \backslash Nu\right)$ and so $\gamma_j \subset \varphi\left(\Omega_{\lambda}^{(j)}\right)$. Therefore, from the uniqueness of attractors by the graph-directed iterated function system theorem [MW], we have $\varphi\left(\Omega_{\lambda}^{(j)}\right) = \gamma_j$. □

2. Complex Pisot numeration systems from Pisot unimodular substitutions

In this section, we give a survey how we obtain the complex Pisot numeration system from an unimodular Pisot substitution with three letters.

Let $\mathcal{A} = \{1, 2, 3\}$ be an alphabet and $\mathcal{A}^* = \bigcup_{n\geq 0}^{\infty} \mathcal{A}_n$ the set of finite words. A substitution σ is a map $\sigma : \mathcal{A} \to \mathcal{A}^*$. Let $M_\sigma = \left(m_{ij}\right)_{1\leq i,j\leq 3}$ be the incidence matrix of σ, i.e., m_{ij} is the number of occurences of i in $\sigma(j)$. In this paper, we assume that

(i) M_σ is *primitive*, i.e., there exists a positive integer n_0 such that $M_\sigma^{n_0} > O$;

(ii) M_σ is *unimodular*, i.e., $\det M_\sigma = \pm 1$;

(iii) σ is a *complex Pisot substitution*, i.e., the eigenvalues μ, μ', μ'' of M_σ satisfy

$$\mu > 1 > \left|\mu'\right|, \left|\mu''\right|, \quad \mu', \mu'' \in \mathbb{C}\backslash\mathbb{R}.$$

Under the assumption (i), (ii), (iii), let us define the matrix $A := M_\sigma^{-1}$. Then the root λ of the characteristic polynomial $p(x)$ of A is $\frac{1}{\mu'}$ and it is the complex Pisot number. Therefore there exist two invariant subspaces of A, that is, one is the 2-dimensional A-invariant expanding plane P_e and another is the 1-dimensional A-invariant contractive line P_c generated by the real eigenvector of A, and the Euclidean space $\mathbb{R}^3$ is decomposed into P_e and P_c, i.e., $\mathbb{R}^3 = P_e \oplus P_c$.

By the way, for the substitution σ whose incidence matrix M_σ satisfies the assumption (i), (ii), (iii), it is known that there exists the infinite sequence w of $\{1, 2, 3\}$ which is periodic with respect to σ, i.e., $\exists m : \sigma^m(w) = w$. Put $w = s_1 s_2 \cdots s_k \cdots$, and let us define the set δ_i by the *projection method*:

$$\delta_i := \mathrm{cl}\left(\pi_e \{ f(s_1 s_2 \cdots s_{k-1}) \mid \exists k \in \mathbb{N} : s_k = i\}\right) \subset P_e \quad \text{for} \quad i = 1, 2, 3$$

where $s_0 = \varepsilon$ (the empty word), $f : \mathcal{A}^* \to \mathbb{Z}^3$ is the abelianization map given by $f(\varepsilon) = \mathbf{0}$, $f(i) = \boldsymbol{e}_i$, $i = 1, 2, 3$, and $f(w_1 w_2 \cdots w_k) := \sum_{n=1}^{k} f(w_n)$ for $w_1 w_2 \cdots w_k \in \mathcal{A}^*$. We call the family $\{\delta_i\}_{i=1,2,3}$ the *atomic surfaces of* σ.

Then we have the following theorem.

THEOREM 2.1 ([AI], [IR], [FFIW]). *Let σ be an unimodular Pisot substitution of three letters and M_σ the incidence matrix of σ. Then atomic surfaces $\{\delta_i\}_{i=1,2,3}$ satisfy the following properties*:

(1) $\mu_e(\delta_i) > 0$, $\delta_i = \mathrm{cl}(\mathrm{int}(\delta_i))$, *and* $\mu_e(\partial\delta_i) = 0$;

(2) $$M_\sigma^{-1}\delta_i = \bigcup_{j=1}^{3} \bigcup_{k : W_k^{(j)} = i} \left(\delta_j + M_\sigma^{-1}\left(\pi_e f\left(P_k^{(j)}\right)\right)\right) \quad (\textit{disjoint})$$

where $\sigma(j) = W_1^{(j)} W_2^{(j)} \cdots W_{l_j}^{(j)}$ *and* $P_k^{(j)}$ *is the prefix of* $W_k^{(j)}$, *i.e.,* $P_k^{(j)} = W_1^{(j)} W_2^{(j)} \cdots W_{k-1}^{(j)}$;

(3) *If* σ *satisfies the strong coincidence condition, i.e., there exist* n *and* k *such that* $\sigma^n(i)$, $i = 1, 2, 3$ *have the same* k*-th letter and their prefixes of the length* $k-1$ *of* $\sigma^n(i)$ *have the same image under the abelianization map* f, *then* $\delta = \bigcup_{i=1}^{3} \delta_i$ *is disjoint.*

The formula (2) in Theorem 2.1 says that the set $A\delta_i$ is generated by the union of the set ($\delta_j +$ translation). Therefore, we can rewrite the formula (2) as (2′) as follows: there exists the finite integer vector sequence $\left\{\boldsymbol{f}_h^{(i)}\right\}_{1\le h\le l_i}$, $\boldsymbol{f}_h^{(i)} \in \mathbb{Z}^3$ such that

$$\left\{\boldsymbol{f}_1^{(i)}, \ldots, \boldsymbol{f}_{l_i}^{(i)}\right\} = \left\{Af\left(P_k^{(j)}\right) \,\middle|\, j = 1, 2, 3 \text{ and } k : W_k^{(j)} = i\right\}$$

and the finite index sequence $\left\{V_h^{(i)}\right\}_{1\le h\le l_i}$, $V_h^{(i)} \in \{1, 2, 3\}$ such that

$$(2') \quad A\delta_i = \bigcup_{h=1}^{l_i} \left(\delta_{V_h^{(i)}} + \pi_e \boldsymbol{f}_h^{(i)}\right) \quad \left(= \bigcup_{j=1}^{3} \bigcup_{k:W_k^{(j)}=i} \left(\delta_j + \pi_e A\left(f\left(P_k^{(j)}\right)\right)\right).\right.$$

Hence, by this rewriting, we see that the pair $(A, \mathcal{P})$, which is constructed by the matrix $A = M_\sigma^{-1}$ and the family of compact sets $\mathcal{P} = \{\delta_i\}_{1\le i\le 3}$, is the complex Pisot numeration system.

REMARK 2.2. In the next section, by using $E_2(\theta)$, the compact set γ_i will be introduced by

$$\gamma_i := \lim_{n\to\infty} M_\sigma^{-n} \pi_e E_2(\theta)^n (\boldsymbol{e}_i, j \wedge k)$$

where θ is the mirror image of the inverse of σ, i.e., $\theta := \overleftarrow{\left(\sigma^{-1}\right)}$ and σ is a substitution (see [AI], [SAI], [E]). We see that $\gamma_i = -\delta_i$ holds.

EXAMPLE 2.3 (Rauzy substitution: [Ra], [AI], [IK]). Let σ be $\sigma : 1 \mapsto 12$, $2 \mapsto 13$, $3 \mapsto 1$ and the incidence matrix of σ $M_\sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Then, $A = M_\sigma^{-1}$ satisfies the complex Pisot condition, i.e., $\lambda = -0.771845 + 1.11514i$, $\overline{\lambda} = -0.771845 - 1.11514i$, $\lambda_3 = 0.543689$. Moreover, the family of compact sets $\mathcal{P} = \{\delta_1, \delta_2, \delta_3\}$, which is given by the projection method, i.e.,

$$\delta_i = \mathrm{cl}\left(\{\pi_e f(s_1 s_2 \cdots s_{k-1}) \mid \exists k \in \mathbb{N}, s_k = i\}\right),$$

satisfies not only the following set equations:

$$A\delta_1 = \delta_1 \cup \delta_2 \cup \delta_3, \quad A\delta_2 = \delta_1 + \pi_e \boldsymbol{e}_3, \quad A\delta_3 = \delta_2 + \pi_e \boldsymbol{e}_3$$

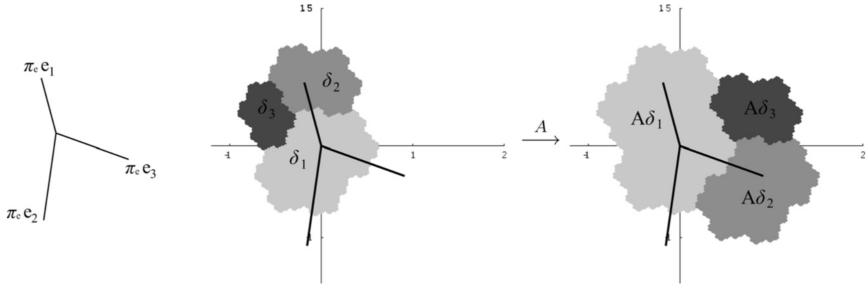


FIGURE 1. $\mathcal{P} = \{\delta_i\}_{i=1,2,3}$ and $A\mathcal{P}$.

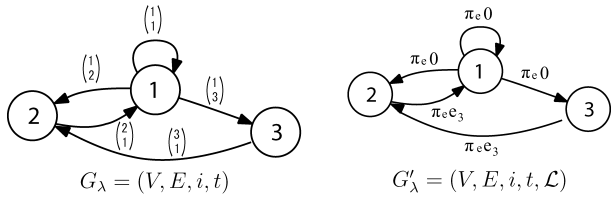


FIGURE 2. The directed multigraph G_λ and the labeled G'_λ of Example 2.3.

(see Figure 1), i.e., the property (N2), but also the properties (N1) and (N3) of Definition 0.2 where $w = s_1 s_2 \cdots = \lim_{n\to\infty} \sigma^n (1)$ is the fixed point of σ. Therefore, we see that $(A, \mathcal{P})$ is the complex Pisot numeration system of λ.

On this example, the directed multigraph G_λ and the labeld graph G'_λ are given by Figure 2.

Therefore, $\phi \circ \mu$-almost all $z \in \phi_e(\gamma)$ can be represented by (1.6): $z = \sum_{n=1}^{\infty} a_{\binom{j_{n-1}}{k_{n-1}}} \lambda^{-n}$ where $a_{\binom{j_{n-1}}{k_{n-1}}}$ given by the property $\phi_e(A\boldsymbol{x}) = \lambda \phi_e(\boldsymbol{x})$ and $\phi_e(\pi_e \boldsymbol{e}_3) = \phi_e(\pi_e A\boldsymbol{e}_1) = \lambda \phi_e(\pi_e \boldsymbol{e}_1) = \lambda$ as follows:

$$a_{\binom{j_{n-1}}{k_{n-1}}} = \phi_e\left(\pi_e \boldsymbol{f}_{k_{n-1}}^{(j_{n-1})}\right) = \begin{cases} 0 & \text{if } \binom{j_{n-1}}{k_{n-1}} = \binom{1}{*} \\ \lambda & \text{if } \binom{j_{n-1}}{k_{n-1}} = \binom{2}{*} \text{ or } \binom{3}{*} \end{cases}.$$

3. Complex Pisot numeration systems from 3×3 unimodular complex Pisot companion matrices

3.1. Classifying of 3×3 unimodular complex Pisot companion matrices

In this section, we give the complex Pisot numeration system generated by a 3×3 unimodular complex Pisot companion matrix .

Let A be the 3×3 companion matrix whose characteristic polynomial is $p_{\mp}(x) = x^3 - ax^2 - bx \pm 1$, $a, b \in \mathbb{Z}$. Let us consider two types of matrices, called (type -1) and (type $+1$) respectively, as follows:

$$A_- = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix} : \text{ (type } -1) \ , \quad A_+ = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix} : \text{ (type } +1)$$

$$p_-(x) = x^3 - ax^2 - bx + 1 \,, \qquad p_+(x) = x^3 - ax^2 - bx - 1 \,.$$

For each matrix, we will examine the property of algebraic integers $\lambda_1, \lambda_2, \lambda_3$ of $p_\mp(x)$.

PROPOSITION 3.1 (for type -1). *The roots $\lambda_1, \lambda_2, \lambda_3$ of $p_-(x) = x^3 - ax^2 - bx + 1$ satisfy the conditions*:

$$-1 < \lambda_3 < 0 \,, \quad \lambda_1, \lambda_2 \in \mathbb{C}\backslash\mathbb{R} \,, \quad |\lambda_1| = |\lambda_2| > 1 > |\lambda_3| \tag{3.9}$$

if and only if the coordinate of a and b satisfies the following:

(1) $-a + b < 0$;

(2) (i) $a^2 + 3b \leq 0$ *or* (ii) *if* $a^2 + 3b > 0$ *then* $27 - 4a^3 - 18ab - a^2b^2 - 4b^3 > 0$ (*see Figure* 3). *Moreover*, *Re* $(\lambda_1) > 0$ $(resp. Re\ (\lambda_1 < 0))$ *if and only if*

(3) $a \geq 0$ *(resp.* $a < 0$*)*,

where "Re (z)*" means the real part of* $z \in \mathbb{C}$.

PROPOSITION 3.2 (for type $+1$). *The roots $\lambda_1, \lambda_2, \lambda_3$ of $p_+(x) = x^3 - ax^2 - bx - 1$ satisfy the conditions*:

$$0 < \lambda_3 < 1 \,, \quad \lambda_1, \lambda_2 \in \mathbb{C}\backslash\mathbb{R} \,, \quad |\lambda_1| = |\lambda_2| > 1 > |\lambda_3| \tag{3.10}$$

if and only if the coordinate of a and b satisfies the following:

(1) $a + b > 0$;

(2) (i) $a^2 + 3b \leq 0$ *or* (ii) *if* $a^2 + 3b > 0$, *then* $27 + 4a^3 + 18ab - a^2b^2 - 4b^3 > 0$ (*see Figure* 3). *Moreover*, *Re* $(\lambda_1) < 0$ (*resp. Re* $(\lambda_1) > 0$) *if and only if*

(3) $a \leq 0$ (*resp.* $a > 0$).

Before we prove Propositions 3.1 and 3.2, we prepare the following lemma.

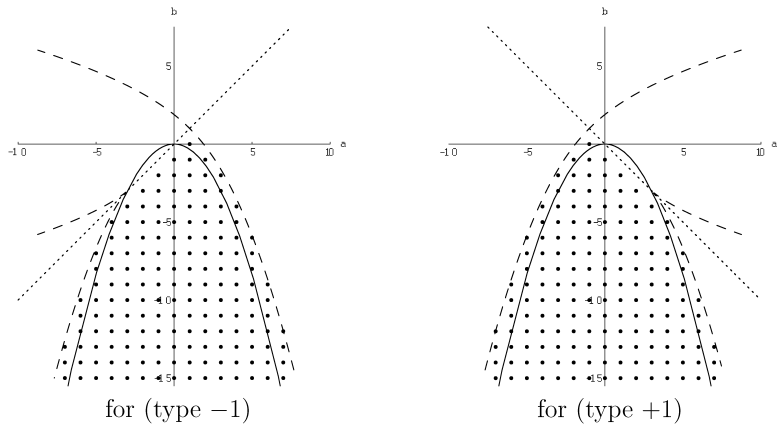


FIGURE 3. The condition of (a, b) satisfying (3.9) and (3.10).

LEMMA 3.3. *For the roots* $\lambda_1, \lambda_2, \lambda_3$ *of* $p_{\mp}(x)$ *satisfying* (3.9) *or* (3.10), $\lambda_1 + \lambda_2 \neq 0$, *i.e.,* λ_1 *and* λ_2 *are not purely imaginary numbers.*

Proof. From the relation between λ_1 and λ_2, the real parts of λ_1 and λ_2 is $\frac{\lambda_1+\lambda_2}{2}$. So, λ_1 and λ_2 are purely imaginary numbers if and only if $\lambda_1 + \lambda_2 = 0$. From the relation between $p_{\mp}(x)$ and roots, we know that $\lambda_1 + \lambda_2 + \lambda_3 = a$. Suppose that λ_1, λ_2 are purely imaginary numbers, then, $a = \lambda_3$. However, from the assumption $-1 < \lambda_3 < 0$ of (3.9) or $0 < \lambda_3 < 1$ of (3.10), it contradicts the fact that a is an integer. Therefore $\lambda_1 + \lambda_2 \neq 0$, i.e., λ_1 and λ_2 are not purely imaginary numbers. □

Proof of Proposition 3.1. About (1), (2): It is easy to see that the roots of $p_-(x)$ satisfy the condition (3.9) if and only if

(i) $p_-(-1) < 0$;

(ii) (ii-1) $D \leq 0$ or (ii-2) if $D > 0$ then $p_-(s)\,p_-(t) > 0$,

where D is the discriminant of $p'_-(x) = 3x^2 - 2ax - b$ and s, t are the roots of $p'_-(x)$. The conditions (i) and (ii) are explicitly given by (I) and (II) respectively:

(I) $-a + b < 0$;

(II) (II-1) $D = a^2 + 3b \leq 0$
or (II-2) if $D > 0$ then $p_-(s)\,p_-(t) = \frac{1}{27}\left(27 - 4a^3 - 18ab - a^2b^2 - 4b^3\right) > 0$

(see Figure 3).

About (3): Put $x^3 - ax^2 - bx + 1 = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3) = 0$. Then, we know that $\lambda_1 + \lambda_2 = a - \lambda_3$. From Lemma 3.3, we see that $a - \lambda_3 > 0$ implies $a \geq 0$. Conversely, we know that $a \geq 0$ implies $a - \lambda_3 > 0$. □

We get the proof of Proposition 3.2 analogously.

COROLLARY 3.4 (for type -1). *For the roots* $\lambda_1, \lambda_2, \lambda_3$ *of* $p_-(x) = x^3 - ax^2 - bx + 1$,

(1) *the condition* $p_-(1) = 2 - a - b > 0$ *is the necessary condition of* (3.9);

(2) *the condition* $p_-(-1) = -a + b < 0$ *is the necessary condition of* (3.9).

Therefore, from $a, b \in \mathbb{Z}$, *we see that* $b \leq 0$ *is the necessary condition of* (1), (2).

COROLLARY 3.5 (for type $+1$). *For the roots* $\lambda_1, \lambda_2, \lambda_3$ *of* $p_+(x) = x^3 - ax^2 - bx - 1$,

(1) *the condition* $p_+(1) = -a - b > 0$ *is the necessary condition of* (3.10);

(2) *the condition* $p_+(-1) = -2 - a + b < 0$ *is the necessary condition of* (3.10).

Therefore, from $a, b \in \mathbb{Z}$, *we see that* $b \leq 0$ *is the necessary condition of* (1), (2).

We call $A_{\mp}$ the unimodular complex Pisot *companion matrices* if the characteristic polynomial of $A_{\mp}$ conincides with $p_{\mp}(x)$ and the roots $\lambda_1, \lambda_2, \lambda_3$ of $p(x)$ satisfying the condition (3.9) in Proposition 3.1 or (3.10) in Proposition 3.2, i.e.,

$$\begin{aligned}
&(\text{type} - 1) &:\quad& -1 < \lambda_3 < 0\,, &\quad& \lambda_1, \lambda_2 \in \mathbb{C}\backslash\mathbb{R}\,, &\quad& |\lambda_1| = |\lambda_2| > 1\,,\\
&(\text{type} + 1) &:\quad& 1 > \lambda_3 > 0\,, && \lambda_1, \lambda_2 \in \mathbb{C}\backslash\mathbb{R}\,, && |\lambda_1| = |\lambda_2| > 1\,.
\end{aligned}$$

Let $\boldsymbol{u}_i$, $(1 \le i \le 3)$ be the eigenvectors of λ_i respectively. Put $\boldsymbol{v}_1 := \frac{\boldsymbol{u}_2+\boldsymbol{u}_1}{2}$, $\boldsymbol{v}_2 := \frac{\boldsymbol{u}_2-\boldsymbol{u}_1}{2i}$, and $\boldsymbol{v}_3 := \boldsymbol{u}_3$, then, by Lemma 1.2, we obtain the following properties:

$$A\,[\boldsymbol{v}_1\ \boldsymbol{v}_2] = [\boldsymbol{v}_1\ \boldsymbol{v}_2]\begin{bmatrix} c & -d \\ d & c \end{bmatrix}, \quad \mathbb{R}^3 = P_e \oplus P_c\,,$$

where $\lambda_1 = c + di$, $P_e = \mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)$, and $P_c = \mathcal{L}(\boldsymbol{v}_3)$. Let $\pi_e : \mathbb{R}^3 \to P_e$ be the projection along P_c and let us denote the counter clockwise angle between $\pi_e\boldsymbol{e}_i$ and $\pi_e\boldsymbol{e}_j$ by $\arg(i \wedge j)$. Then we have the following lemmas.

LEMMA 3.6 (for type -1). *If $a \ge 0$, then $0 < \arg(1\wedge 2)\,, \arg(2\wedge 3) < \frac{\pi}{2}$, and if $a < 0$, then $\frac{\pi}{2} < \arg(1\wedge 2)\,, \arg(2\wedge 3) < \pi$ (see Figure 4).*

Proof. Assume that $a \ge 0$, then we see that Re $(\lambda\,(=\lambda_1)) > 0$ by Proposition 3.1 (3). On the other hand, we know that

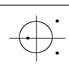
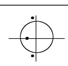
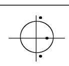
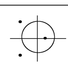
$$\phi_e(\pi_e\boldsymbol{e}_1) = 1\,, \quad \phi_e(\pi_e\boldsymbol{e}_2) = \phi_e(A\pi_e\boldsymbol{e}_1) = \lambda\,, \quad \phi_e(\pi_e\boldsymbol{e}_3) = \lambda^2\,.$$

Therefore, it is clear that $0 < \arg(\lambda) < \frac{\pi}{2}$ and $0 < \arg\left(\lambda^2\right) < \pi$. Moreover, we also know that $\phi_e : P_e \to \mathbb{C}$ is linear and bijective. Therefore, we see that $0 < \arg(1\wedge 2)\,, \arg(2\wedge 3) < \frac{\pi}{2}$. The case of $a < 0$ is proved analogously. □

LEMMA 3.7 (for type +1). *If $a \le 0$, then $\frac{\pi}{2} < \arg(1\wedge 2)\,, \arg(2\wedge 3) < \pi$ and if $a > 0$, then $0 < \arg(1\wedge 2)\,, \arg(2\wedge 3) < \frac{\pi}{2}$ (see Figure 4).*

We get the proof by the analogous discussion of Lemma 3.6.

From Lemmas 3.6 and 3.7, we classify the characteristic polynomial $p_{\mp}(x)$ into four classes such that

	(type –1)		(type +1)	
A	$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix}$		$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix}$	
$p(x)$	$x^3 - ax^2 - bx + 1$		$x^3 - ax^2 - bx - 1$	
Signature of λ_3	$-$		$+$	
The distribution of eigenvalues of A				
Signature of $Re(\lambda_1)$	$+$	$-$	$+$	$-$
a	$a \ge 0$	$a < 0$	$a > 0$	$a \le 0$
The distribution of $\{\pi_e\boldsymbol{e}_i\}_{i=1,2,3}$	V_0	V_1	V_0	V_1
Name	type (–1,0)	type (–1,1)	type (+1,0)	type (+1,1)

Let $A_{\mp}$ be the 3×3 unimodular complex Pisot companion matrix, let λ_3 be the real eigenvalue of $A_{\mp}$, and let $\boldsymbol{v}^* = \left[v_1^*\ v_2^*\ v_3^*\right]$ and $\boldsymbol{v} = {}^t\,[v_1\ v_2\ v_3]$ be the row and column eigenvectors of λ_3, i.e.,

$$\boldsymbol{v}^* A = \lambda_3\boldsymbol{v}^*, \quad A_{\mp}\boldsymbol{v} = \lambda_3\boldsymbol{v}\,.$$

Then, $\boldsymbol{v}^*$ and $\boldsymbol{v}$ are explicitly given by

$$\boldsymbol{v}^* = \left[1\ \lambda_3\ \lambda_3^2\right], \quad \boldsymbol{v} = {}^t\left[\mp\frac{1}{\lambda_3}\ \ \lambda_3 - a\ \ 1\right]. \tag{3.11}$$

$V_0 = \{1 \wedge 2, 1 \wedge 3, 2 \wedge 3\}$ $V_1 = \{1 \wedge 2, 3 \wedge 1, 2 \wedge 3\}$

FIGURE 4. The distribution of $\{\pi_e \boldsymbol{e}_i\}_{i=1,2,3}$ for V_0 and V_1.

Therefore, by using (3.11), we have the following lemma.

LEMMA 3.8.

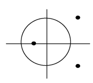
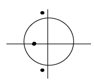
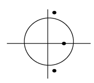
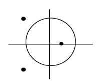
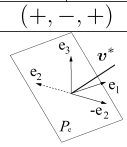
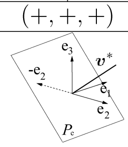
Name	type(−1,0)	type(−1,1)	type(+1,0)	type(+1,1)
$\operatorname{sgn}(\boldsymbol{v}^*) = (\operatorname{sgn}(v_1^*), \operatorname{sgn}(v_2^*), \operatorname{sgn}(v_3^*))$	$(+,-,+)$		$(+,+,+)$	
$\operatorname{sgn}(\boldsymbol{v}) = (\operatorname{sgn}(v_1), \operatorname{sgn}(v_2), \operatorname{sgn}(v_3))$	$(+,-,+)$	$(+,+,+)$	$(+,-,+)$	$(+,+,+)$

where $\operatorname{sgn}(v) = +$ *if* $v > 0$ *and* $\operatorname{sgn}(v) = -$ *if* $v < 0$.

And P_e is characterized by $\boldsymbol{v}^*$ as follows.

LEMMA 3.9. $P_e = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \mid \langle \boldsymbol{x}, \boldsymbol{v}^* \rangle = 0 \right\}$.

Then, we have the following table:

Name	type (–1,0)	type (–1,1)	type (+1,0)	type (+1,1)
A	$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix}$		$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix}$	
$p(x)$	$x^3 - ax^2 - bx + 1$		$x^3 - ax^2 - bx - 1$	
Signature of λ_3	$-$		$+$	
The distribution of eigenvalues of A				
Signature of $Re(\lambda_1)$	$+$	$-$	$+$	$-$
a	$a \geq 0$	$a < 0$	$a > 0$	$a \leq 0$
Distributions of $\{\pi_e \boldsymbol{e}_i\}_{i=1,2,3}$	V_0	V_1	V_0	V_1
Signature of $\boldsymbol{v}^*$	$(+,-,+)$		$(+,+,+)$	
Image of $\{\boldsymbol{e}_i\}_{i=1,2,3}$				
Signature of $\boldsymbol{v}$	$(+,-,+)$	$(+,+,+)$	$(+,-,+)$	$(+,+,+)$

3.2. Stepped planes and quasi-periodic tilings of P_e

For the 2-dimensional A-invariant expanding plane P_e, we introduce the stepped plane in this section.

For $\boldsymbol{x} \in \mathbb{R}^3$ and $i, j \in \{\pm 1, \pm 2, \pm 3\}$, let us define the 2-dimentional *unit face* $(\boldsymbol{x}, i \wedge j)$ given by

$$(\boldsymbol{x}, i \wedge j) := \left\{ \boldsymbol{x} + \lambda\,(\operatorname{sgn}(i))\,\boldsymbol{e}_{|i|} + \mu\,(\operatorname{sgn}(j))\,\boldsymbol{e}_{|j|} \;\middle|\; 0 \le \lambda, \mu \le 1 \right\}$$

(see Figure 5).

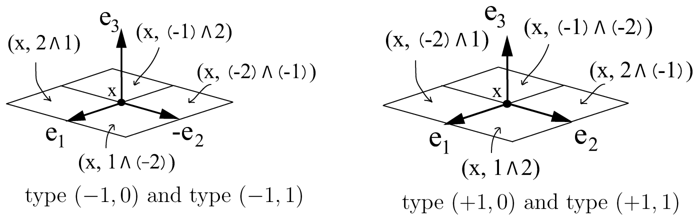


FIGURE 5. 2-dimensional unit face $(\boldsymbol{x}, i \wedge j)$ for type $(-1, 0)$, type $(-1, 1)$, type $(+1, 0)$, and type $(+1, 1)$.

Using Lemma 3.8 and Lemma 3.9, let us define the stepped plane of P_e as follows.

DEFINITION 3.10 (for type $(-1, 0)$ and type $(-1, 1)$, i.e., in the case of $\boldsymbol{v}^* = (+, -, +)$). Let us define the sets of unit faces $S_-^{\geq}$, $S_-^{>}$ of P_e as follows:

$$\begin{aligned}
S_-^{\geq} := &\left\{ (\boldsymbol{x}, (-2) \wedge 1) \;\middle|\; \boldsymbol{x} \in \mathbb{Z}^3, \langle \boldsymbol{x}, \boldsymbol{v}^* \rangle \geq 0, \langle \boldsymbol{x} - \boldsymbol{e}_3, \boldsymbol{v}^* \rangle < 0 \right\} \\
&\cup \left\{ (\boldsymbol{x}, 1 \wedge 3) \;\middle|\; \boldsymbol{x} \in \mathbb{Z}^3, \langle \boldsymbol{x}, \boldsymbol{v}^* \rangle \geq 0, \langle \boldsymbol{x} + \boldsymbol{e}_2, \boldsymbol{v}^* \rangle < 0 \right\} \\
&\cup \left\{ (\boldsymbol{x}, 3 \wedge (-2)) \;\middle|\; \boldsymbol{x} \in \mathbb{Z}^3, \langle \boldsymbol{x}, \boldsymbol{v}^* \rangle \geq 0, \langle \boldsymbol{x} - \boldsymbol{e}_1, \boldsymbol{v}^* \rangle < 0 \right\}, \\
S_-^{>} := &\left\{ (\boldsymbol{x}, (-2) \wedge 1) \;\middle|\; \boldsymbol{x} \in \mathbb{Z}^3, \langle \boldsymbol{x}, \boldsymbol{v}^* \rangle > 0, \langle \boldsymbol{x} - \boldsymbol{e}_3, \boldsymbol{v}^* \rangle \leq 0 \right\} \\
&\cup \left\{ (\boldsymbol{x}, 1 \wedge 3) \;\middle|\; \boldsymbol{x} \in \mathbb{Z}^3, \langle \boldsymbol{x}, \boldsymbol{v}^* \rangle > 0, \langle \boldsymbol{x} + \boldsymbol{e}_2, \boldsymbol{v}^* \rangle \leq 0 \right\} \\
&\cup \left\{ (\boldsymbol{x}, 3 \wedge (-2)) \;\middle|\; \boldsymbol{x} \in \mathbb{Z}^3, \langle \boldsymbol{x}, \boldsymbol{v}^* \rangle > 0, \langle \boldsymbol{x} - \boldsymbol{e}_1, \boldsymbol{v}^* \rangle \leq 0 \right\}.
\end{aligned}$$

DEFINITION 3.11 (for type $(+1, 0)$ and type $(+1, 1)$, i.e., in the case of $\boldsymbol{v}^* = (+, +, +)$). Let us define the sets of unit faces $S_+^{\geq}$, $S_+^{>}$ of P_e as follows:

$$S_+^{\geq} := \left\{ (\boldsymbol{x}, i \wedge j) \;\middle|\; \begin{array}{l} \boldsymbol{x} \in \mathbb{Z}^3,\ \{i, j, k\} = \{1, 2, 3\},\ i \wedge j \in \{1 \wedge 2, 3 \wedge 1, 2 \wedge 3\}, \\ \langle \boldsymbol{x}, \boldsymbol{v}^* \rangle \geq 0,\ \langle \boldsymbol{x} - \boldsymbol{e}_k, \boldsymbol{v}^* \rangle < 0 \end{array} \right\},$$

$$S_+^{>} := \left\{ (\boldsymbol{x}, i \wedge j) \;\middle|\; \begin{array}{l} \boldsymbol{x} \in \mathbb{Z}^3,\ \{i, j, k\} = \{1, 2, 3\},\ i \wedge j \in \{1 \wedge 2, 3 \wedge 1, 2 \wedge 3\}, \\ \langle \boldsymbol{x}, \boldsymbol{v}^* \rangle > 0,\ \langle \boldsymbol{x} - \boldsymbol{e}_k, \boldsymbol{v}^* \rangle \leq 0 \end{array} \right\}.$$

DEFINITION 3.12. We define the family of finite sets of unit faces $\mathcal{G}_-^{\geq}$ of $S_-^{\geq}$, called the patch, which is generated as a finite formal sum of unit faces as follows:

$$\mathcal{G}_-^{\geq} := \left\{ \sum_{\lambda \in \Lambda} (\boldsymbol{x}, i \wedge j)_\lambda \;\middle|\; \begin{array}{l} \#\Lambda < +\infty,\ (\boldsymbol{x}, i \wedge j)_\lambda \in S_-^{\geq}, \\ (\boldsymbol{x}_\lambda, i \wedge j)_\lambda \neq (\boldsymbol{x}_\lambda, i \wedge j)_{\lambda'} \text{ if } \lambda \neq \lambda' \end{array} \right\}$$

(see Figure 6). The other cases $\mathcal{G}_{-}^{>}$, $\mathcal{G}_{+}^{\geq}$, $\mathcal{G}_{+}^{>}$ are defined analogously.

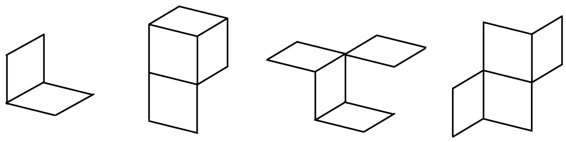


FIGURE 6. Examples of patches.

DEFINITION 3.13. Using $S_{-}^{\geq}$, we define the surfaces $\mathscr{S}_{-}^{\geq}$ of $S_{-}^{\geq}$ called the stepped plane of P_e as follows:

$$\mathscr{S}_{-}^{\geq} \quad := \quad \bigcup_{(\boldsymbol{x}, i \wedge j) \in S_{-}^{\geq}} (\boldsymbol{x}, i \wedge j)$$

(see Figure 7). $(\boldsymbol{x}, i \wedge j) \in S_{-}^{\geq}$ is called the *unit face* of the stepped plane located at $\boldsymbol{x}$. The other cases $\mathscr{S}_{-}^{>}$, $\mathscr{S}_{+}^{\geq}$, $\mathscr{S}_{+}^{>}$ are defined analogously.

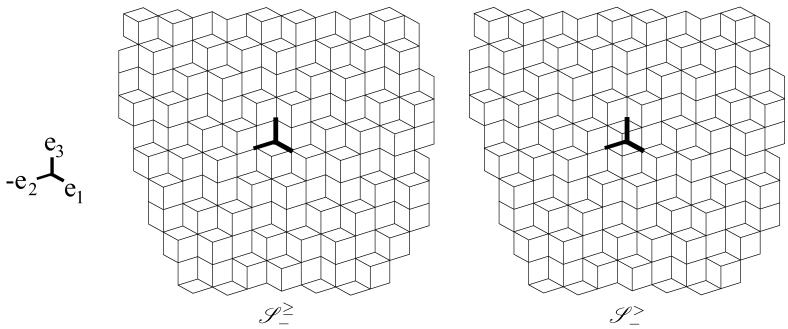


FIGURE 7. Stepped planes $\mathscr{S}_{-}^{\geq}$ and $\mathscr{S}_{-}^{>}$.

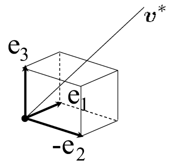
REMARK 3.14. (1) $\pi_e \mathscr{S}_{-}^{\geq}, \pi_e \mathscr{S}_{-}^{>}, \pi_e \mathscr{S}_{+}^{\geq}, \pi_e \mathscr{S}_{+}^{>} = P_e$.

(2) The fact that $\boldsymbol{x} \in P_e$ and $\boldsymbol{x} \in \mathbb{Z}^3$ implies $\boldsymbol{x} = \boldsymbol{0}$. Since $\boldsymbol{v}^* =^t \begin{bmatrix} 1 \ \lambda \ \lambda^2 \end{bmatrix}$ is rationally independent, i.e., if $l + m\lambda + n\lambda^2 = 0$ for some m, n, then $(l, m, n) = (0, 0, 0)$.

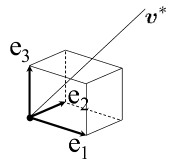
(3) We see that $S_{-}^{\geq} \supset \{(\boldsymbol{0}, (-2) \wedge 1), (\boldsymbol{0}, 1 \wedge 3), (\boldsymbol{0}, 3 \wedge (-2))\}$ and $S_{-}^{>} \supset \{(\boldsymbol{e}_3, (-2) \wedge 1), (-\boldsymbol{e}_2, 1 \wedge 3), (\boldsymbol{e}_1, 3 \wedge (-2))\}$. Moreover, we have

$$S_{-}^{\geq} \backslash S_{-}^{>} = \{(\boldsymbol{0}, (-2) \wedge 1), (\boldsymbol{0}, 1 \wedge 3), (\boldsymbol{0}, 3 \wedge (-2)), (\boldsymbol{e}_3, (-2) \wedge 1), (-\boldsymbol{e}_2, 1 \wedge 3), (\boldsymbol{e}_1, 3 \wedge (-2))\}$$

(see Figure 8).



$S^{\geq}_{-}\backslash S^{>}_{-}$



$S^{\geq}_{+}\backslash S^{>}_{+}$

FIGURE 8.

(4) We see that $S^{\geq}_{+} \supset \{(\mathbf{0}, 1 \wedge 2), (\mathbf{0}, 3 \wedge 1), (\mathbf{0}, 2 \wedge 3)\}$ and $S^{>}_{+} \supset \{(\boldsymbol{e}_3, 1 \wedge 2), (\boldsymbol{e}_2, 3 \wedge 1), (\boldsymbol{e}_1, 2 \wedge 3)\}$. Moreover, we have

$$S^{\geq}_{+}\backslash S^{>}_{+} = \{(\mathbf{0}, 1 \wedge 2), (\mathbf{0}, 3 \wedge 1), (\mathbf{0}, 2 \wedge 3), (\boldsymbol{e}_3, 1 \wedge 2), (\boldsymbol{e}_2, 3 \wedge 1), (\boldsymbol{e}_1, 2 \wedge 3)\}.$$

(see Figure 8).

(5) For unit faces of $S^{\geq}_{-}$ and $S^{>}_{-}$, we consider the rearrangement such as

$$(\boldsymbol{x}, (-2) \wedge 1) = (\boldsymbol{x} - \boldsymbol{e}_2, 1 \wedge 2), \ (\boldsymbol{x}, 3 \wedge (-2)) = (\boldsymbol{x} - \boldsymbol{e}_2, 2 \wedge 3). \tag{3.12}$$

Then, using the rearrangement (3.12), $S^{\geq}_{-}$ and $S^{>}_{-}$ are rewritten by

$$\begin{aligned}
S^{\geq}_{-} := & \left\{(z, 1 \wedge 2) \ \middle|\ \langle -\boldsymbol{e}_2, \boldsymbol{v}^* \rangle \leq \langle z, \boldsymbol{v}^* \rangle < \langle -\boldsymbol{e}_2 + \boldsymbol{e}_3, \boldsymbol{v}^* \rangle \right\} \\
& \cup \left\{(z, 1 \wedge 3) \ \middle|\ 0 \leq \langle z, \boldsymbol{v}^* \rangle < \langle -\boldsymbol{e}_2, \boldsymbol{v}^* \rangle \right\} \\
& \cup \left\{(z, 2 \wedge 3) \ \middle|\ \langle -\boldsymbol{e}_2, \boldsymbol{v}^* \rangle \leq \langle z, \boldsymbol{v}^* \rangle < \langle -\boldsymbol{e}_2 + \boldsymbol{e}_1, \boldsymbol{v}^* \rangle \right\}, \\
S^{>}_{-} := & \left\{(z, 1 \wedge 2) \ \middle|\ \langle -\boldsymbol{e}_2, \boldsymbol{v}^* \rangle < \langle z, \boldsymbol{v}^* \rangle \leq \langle -\boldsymbol{e}_2 + \boldsymbol{e}_3, \boldsymbol{v}^* \rangle \right\} \\
& \cup \left\{(z, 1 \wedge 3) \ \middle|\ 0 < \langle z, \boldsymbol{v}^* \rangle \leq \langle -\boldsymbol{e}_2, \boldsymbol{v}^* \rangle \right\} \\
& \cup \left\{(z, 2 \wedge 3) \ \middle|\ \langle -\boldsymbol{e}_2, \boldsymbol{v}^* \rangle < \langle z, \boldsymbol{v}^* \rangle \leq \langle -\boldsymbol{e}_2 + \boldsymbol{e}_1, \boldsymbol{v}^* \rangle \right\}.
\end{aligned}$$

For the characterization of the faces which generate the stepped plane, we prepare the following notations.

NOTATION 1 (for type $(-1, 0)$, type $(-1, 1)$). For the set of unit faces of $S^{\geq}_{-}$, $S^{>}_{-}$ which generate the stepped plane $\mathscr{S}^{\geq}_{-}$, $\mathscr{S}^{>}_{-}$ respectively, let us denote the segments $I^{\geq}_{-}$ $(i \wedge j)$, $I^{>}_{-}$ $(i \wedge j)$ of $\mathcal{L}(\boldsymbol{v}) = P_c$, $i \wedge j \in V_0$ as follows:

$$\begin{aligned}
I^{\geq}_{-}(1 \wedge 2) &:= [\pi_c(-\boldsymbol{e}_2), \pi_c(-\boldsymbol{e}_2 + \boldsymbol{e}_3))_c \\
&:= \{\alpha \pi_c(-\boldsymbol{e}_2 + \boldsymbol{e}_3) + (1 - \alpha)\pi_c(-\boldsymbol{e}_2) \mid 0 \leq \alpha < 1\}, \\
I^{\geq}_{-}(1 \wedge 3) &:= [\pi_c \mathbf{0}, \pi_c(-\boldsymbol{e}_2))_c \\
&:= \{\alpha \pi_c(-\boldsymbol{e}_2) \mid 0 \leq \alpha < 1\}, \\
I^{\geq}_{-}(2 \wedge 3) &:= [\pi_c(-\boldsymbol{e}_2), \pi_c(-\boldsymbol{e}_2 + \boldsymbol{e}_1))_c \\
&:= \{\alpha \pi_c(-\boldsymbol{e}_2 + \boldsymbol{e}_1) + (1 - \alpha)\pi_c(-\boldsymbol{e}_2) \mid 0 \leq \alpha < 1\}, \\
I^{>}_{-}(1 \wedge 2) &:= (\pi_c(-\boldsymbol{e}_2), \pi_c(-\boldsymbol{e}_2 + \boldsymbol{e}_3)]_c \\
&:= \{\alpha \pi_c(-\boldsymbol{e}_2 + \boldsymbol{e}_3) + (1 - \alpha)\pi_c(-\boldsymbol{e}_2) \mid 0 < \alpha \leq 1\}, \\
I^{>}_{-}(1 \wedge 3) &:= (\pi_c \mathbf{0}, \pi_c(-\boldsymbol{e}_2)]_c
\end{aligned}$$

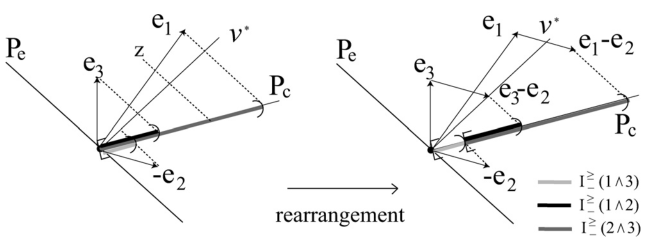


FIGURE 9. $I^{\geq}_{-}(i \wedge j), i \wedge j \in V_0$.

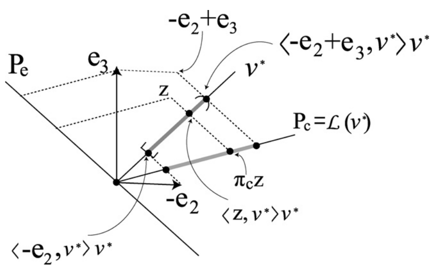


FIGURE 10.

$$
\begin{aligned}
&:= \{\alpha\pi_c(-\boldsymbol{e}_2) \mid 0 < \alpha \leq 1\}, \\
I^{>}_{-}(2 \wedge 3) &:= (\pi_c(-\boldsymbol{e}_2)\ , \pi_c(-\boldsymbol{e}_2 + \boldsymbol{e}_1)]_c \\
&:= \{\alpha\pi_c(-\boldsymbol{e}_2 + \boldsymbol{e}_1) + (1-\alpha)\pi_c(-\boldsymbol{e}_2) \mid 0 < \alpha \leq 1\}
\end{aligned}
$$

(see Figure 9).

NOTATION 2 (for type $(+1, 0)$, type $(+1, 1)$). For the set of unit faces $S^{\geq}_{+}$, $S^{>}_{+}$, which generate the stepped plane $\mathscr{S}^{\geq}_{+}$, $\mathscr{S}^{>}_{+}$ respectively, let us denote the segments of $I^{\geq}_{+}(i \wedge j)$, $I^{>}_{+}(i \wedge j)$ of $\mathcal{L}(\boldsymbol{v}) = P_c$, $i \wedge j \in V_1$ as follows:

$$
\begin{aligned}
I^{\geq}_{+}(i \wedge j) &:= [\pi_c\mathbf{0}, \pi_c\boldsymbol{e}_k)_c := \{\alpha\pi_c\boldsymbol{e}_k \mid 0 \leq \alpha < 1\}, \\
I^{>}_{+}(i \wedge j) &:= (\pi_c\mathbf{0}, \pi_c\boldsymbol{e}_k]_c := \{\alpha\pi_c\boldsymbol{e}_k \mid 0 < \alpha \leq 1\}
\end{aligned}
$$

where $\{i, j, k\} = \{1, 2, 3\}$.

Using $I^{\geq}_{-}(i \wedge j)$, $I^{>}_{-}(i \wedge j)$, $i \wedge j \in V_0$ and $I^{\geq}_{+}(i \wedge j)$, $I^{>}_{+}(i \wedge j)$, $i \wedge j \in V_1$, we can characterize the faces of stepped plane.

LEMMA 3.15. *Under the assumption* $\operatorname{sgn}(\boldsymbol{v}) = \operatorname{sgn}(\boldsymbol{v}^*)$, *i.e.*, *under type* $(-1, 0)$ *or type* $(+1, 1)$, $(\boldsymbol{z}, i \wedge j) \in S^{\geq}_{-}$ *(resp.* $(\boldsymbol{z}, i \wedge j) \in S^{>}_{-}$, $(\boldsymbol{x}, i \wedge j) \in S^{\geq}_{+}$, $(\boldsymbol{x}, i \wedge j) \in S^{>}_{+}$*) if and only if* $\pi_c\boldsymbol{z} \in I^{\geq}_{-}(i \wedge j)$ *(resp.* $\pi_c\boldsymbol{z} \in I^{>}_{-}(i \wedge j)$, $\pi_c\boldsymbol{x} \in I^{\geq}_{+}(i \wedge j)$, $\pi_c\boldsymbol{x} \in I^{>}_{+}(i \wedge j)$*) (see Figure* 10*).*

Proof. From the definition of $S^{\geq}_{-}$, it is clear that $S^{\geq}_{-} \ni (\boldsymbol{z}, 1 \wedge 2)$ if and only if $\langle -\boldsymbol{e}_2, \boldsymbol{v}^* \rangle \leq \langle \boldsymbol{z}, \boldsymbol{v}^* \rangle < \langle -\boldsymbol{e}_2 + \boldsymbol{e}_3, \boldsymbol{v}^* \rangle$. Let $\pi_* : \mathbb{R}^3 \to \mathcal{L}(\boldsymbol{v}^*)$ be the projection along P_e. Then, from the fact that $\pi_*(\boldsymbol{z}) = \langle \boldsymbol{z}, \boldsymbol{v}^* \rangle \boldsymbol{v}^*$, we can write that $\langle \boldsymbol{z}, \boldsymbol{v}^* \rangle \in [\langle -\boldsymbol{e}_2, \boldsymbol{v}^* \rangle, \langle -\boldsymbol{e}_2 +$

$\boldsymbol{e}_3, \boldsymbol{v}^* \rangle)$ on $\mathcal{L}(\boldsymbol{v}^*)$ where $|\boldsymbol{v}^*| = 1$. From $\boldsymbol{z} = \pi_c \boldsymbol{z} + \pi_e \boldsymbol{z}$, $\langle \pi_c \boldsymbol{z}, \boldsymbol{v}^* \rangle \in [\langle \pi_c(-\boldsymbol{e}_2), \boldsymbol{v}^* \rangle$, $\langle \pi_c(-\boldsymbol{e}_2 + \boldsymbol{e}_3), \boldsymbol{v}^* \rangle)$ on $\mathcal{L}(\boldsymbol{v}^*)$. Moreover, $\mathrm{sgn}(\boldsymbol{v})= \mathrm{sgn}(\boldsymbol{v}^*)$, we see that $\pi_c \boldsymbol{z} \in [\pi_c(-\boldsymbol{e}_2)$, $\pi_c(-\boldsymbol{e}_2 + \boldsymbol{e}_3))_c$. We can prove the other cases analogously. □

Hereafter, let us discuss only type $(-1, 0)$ or type $(+1, 1)$ cases whose classes are characterized by $\mathrm{sgn}(\boldsymbol{v}) = \mathrm{sgn}(\boldsymbol{v}^*)$.

DEFINITION 3.16. Let us define the set of projected unit faces of $S_-^{\geq}$ as follows:

$$T_-^{\geq} := \left\{ \pi_e(\boldsymbol{z}, i \wedge j) \;\middle|\; (\boldsymbol{z}, i \wedge j) \in S_-^{\geq} \right\} .$$

The other cases of $T_-^{>}$, $T_+^{\geq}$, $T_+^{>}$ are defined analogously.

DEFINITION 3.17. Using $T_-^{\geq}$, the tiling of P_e is defined by

$$\mathscr{T}_-^{\geq} := \bigcup_{\pi_e(\boldsymbol{z}, i \wedge j) \in T_-^{\geq}} \pi_e(\boldsymbol{z}, i \wedge j) .$$

The other cases $\mathscr{T}_-^{>}$, $\mathscr{T}_+^{\geq}$, $\mathscr{T}_+^{>}$ are defined analogously.

Then, by the property $\mathrm{sgn}(\boldsymbol{v}) = \mathrm{sgn}(\boldsymbol{v}^*)$, we have the following proposition.

PROPOSITION 3.18. *If* $\mathrm{sgn}(\boldsymbol{v}) = \mathrm{sgn}(\boldsymbol{v}^*)$, *then* $\mathscr{T}_-^{\geq}$, $\mathscr{T}_-^{>}$ *are the quasi-periodic tilings of* P_e *by proto-tiles* $\{\pi_e(\boldsymbol{0}, i \wedge j) \mid i \wedge j \in \{1 \wedge 2, 1 \wedge 3, 2 \wedge 3\} = V_0\}$ *and* $\mathscr{T}_+^{\geq}$, $\mathscr{T}_+^{>}$ *are the quasi-periodic tilings of* P_e *by proto-tiles* $\{\pi_e(\boldsymbol{0}, i \wedge j) \mid i \wedge j \in \{1 \wedge 2, 3 \wedge 1, 2 \wedge 3\} = V_1\}$.

The proof is obtained by Theorem 3.8 in [IO2] analogously.

REMARK 3.19. We are interested in the projection $\pi_e S_-^{\geq}$ (resp. $\pi_e S_-^{>}, \pi_e S_+^{\geq}, \pi_e S_+^{>}$). But in the cases of type $(-1, 1)$ and type $(+1, 0)$, $\pi_e S_-^{\geq}$ (resp. $\pi_e S_-^{>}$, $\pi_e S_+^{\geq}$, $\pi_e S_+^{>}$) are not tilings but coverings because of the property $\mathrm{sgn}(\boldsymbol{v}^*) \neq \mathrm{sgn}(\boldsymbol{v})$. Therefore, we will discuss only the case of $\mathrm{sgn}(\boldsymbol{v}^*) = \mathrm{sgn}(\boldsymbol{v})$, i.e., type $(-1, 0)$ and type $(+1, 1)$. It is unclear about the case of $\mathrm{sgn}(\boldsymbol{v}^*) \neq \mathrm{sgn}(\boldsymbol{v})$ now. We will try to introduce the existence of the numeration system and the tiling property in the different paper.

Finally, we give the definition of the positive oriented face as follows.

DEFINITION 3.20. The unit face $(\boldsymbol{x}, i \wedge j)$ located at $\boldsymbol{x}$ is positive oriented if $i \wedge j \in V_0$ for type $(-1, 0)$ and if $i \wedge j \in V_1$ for type $(+1, 1)$.

REMARK 3.21. In the case of type $(-1, 0)$, let us assume that $i \wedge j = (-2) \wedge 1$ for the unit face $(\boldsymbol{x}, i \wedge j)$. Then, we can rearrange it as $(\boldsymbol{x} - \boldsymbol{e}_2, 1 \wedge 2)$, so it is the poritive oriented by Definition 3.20. Thus if the rearrangement face is positive oriented, we also say that the non-arrangement face is positive oriented.

3.3. 2-dimensional extension $E_2(\sigma)$

Let us introduce the automorphisms σ_- and σ_+ on the free group $F\langle 1, 2, 3 \rangle$ whose incidence matrices are A_- and A_+ respectively as follows:

$$\sigma_- : \begin{array}{ccc} 1 & \to & 2 \\ 2 & \to & 3 \\ 3 & \to & 3^a 1^{-1} 2^b \end{array} , \quad \sigma_+ : \begin{array}{ccc} 1 & \to & 2 \\ 2 & \to & 3 \\ 3 & \to & 1 2^b 3^a \end{array} .$$

Using the automorphism σ $(= \sigma_-$ or $\sigma_+)$, let us introduce the 2-*dimensional extension* $E_2(\sigma)$ *of* σ on the family of patches generated by the symbolic faces of the set V_e $(= V_0$ or $V_1)$: for each $i \wedge j \in V_e$,

$$\begin{aligned}
E_2(\sigma)(\mathbf{0}, i \wedge j) &:= (\mathbf{0}, \sigma(i) \wedge \sigma(j)) \\
&:= \sum_{\substack{1 \le k \le l_i \\ 1 \le l \le l_j}} \left(f\left(P_k^{(i)}\right) + f\left(P_l^{(j)}\right), W_k^{(i)} \wedge W_l^{(j)} \right) \qquad (3.13) \\
E_2(\sigma)(\boldsymbol{x}, i \wedge j) &:= A\boldsymbol{x} + E_2(\sigma)(\mathbf{0}, i \wedge j) \\
E_2(\sigma)\left(\sum_{\lambda} (\boldsymbol{x}, i \wedge j)_\lambda \right) &:= \sum_{\lambda} E_2(\sigma)(\boldsymbol{x}, i \wedge j)_\lambda
\end{aligned}$$

where $f : F\langle 1, 2, 3\rangle \to \mathbb{Z}^3$ is the homomorphism satisfying $f(\varepsilon) = \mathbf{0}$, $f(i) = \boldsymbol{e}_i$, $\sigma(i) = W_1^{(i)} W_2^{(i)} \cdots W_{l_i}^{(i)}$, $P_k^{(i)}$ is the prefix of $W_k^{(i)}$, i.e., $P_k^{(i)} = W_1^{(i)} \cdots W_{k-1}^{(i)}$ and $\boldsymbol{y} + (\mathbf{0}, i \wedge j) = (\boldsymbol{y}, i \wedge j)$ (see [AFHI]).

If 2-dimensional extension $E_2(\sigma)$ given by (3.13) satisfies the property that all faces of $E_2(\sigma)(\mathbf{0}, i \wedge j)$ are positive orientated, we say that $E_2(\sigma)$ has the *positive orientation property* (the *POP-property* for simplicity).

Then, we have the following proposition.

PROPOSITION 3.22. *$E_2(\sigma_-)$ and $E_2(\sigma_+)$ have the POP-property.*

Proof. $E_2(\sigma_-)(\mathbf{0}, i \wedge j)$, $i \wedge j \in V_0$ can be explicitly given by

$$\begin{aligned}
E_2(\sigma_-)(\mathbf{0}, 1 \wedge 2) &= (\mathbf{0}, 2 \wedge 3) \\
E_2(\sigma_-)(\mathbf{0}, 1 \wedge 3) &= \left(\mathbf{0}, 2 \wedge 3^a\right) + \left(f\left(3^a\right), 2 \wedge 1^{-1} \right) \\
&\overset{(*)}{=} \left(\sum_{k=1}^{a} ((k-1)\boldsymbol{e}_3, 2 \wedge 3) \right) + (a\boldsymbol{e}_3 - \boldsymbol{e}_1, 1 \wedge 2) \\
E_2(\sigma_-)(\mathbf{0}, 2 \wedge 3) &= \left(f\left(3^a\right), 3 \wedge 1^{-1} \right) + \left(f\left(3^a\right) + f\left(1^{-1}\right), 3 \wedge 2^b \right) \\
&\overset{(*)}{=} (a\boldsymbol{e}_3 - \boldsymbol{e}_1, 1 \wedge 3) + \sum_{k=1}^{-b} (a\boldsymbol{e}_3 - \boldsymbol{e}_1 - k\boldsymbol{e}_2, 2 \wedge 3) \ .
\end{aligned}$$

Here, the technical manner $(*)$ means that the "*rearrangement*" is used. It is clear that all faces of $E_2(\sigma_-)(\mathbf{0}, i \wedge j)$, $i \wedge j \in V_0$ are positive oriented, i.e., $E_2(\sigma_-)$ has the POP-property. We get the proof of $E_2(\sigma_+)$ analogously. □

3.4. Invariant stepped plane generated by $E_2(\sigma_\pm)$

For $\boldsymbol{t} \in \mathbb{R}^3$, let us consider the plane $P_e(\boldsymbol{t}) = P_e + \boldsymbol{t}$, the stepped plane $\mathscr{S}_-^{\geq}(\boldsymbol{t}) = \mathscr{S}_-^{\geq} + \boldsymbol{t}$ of $P_e(\boldsymbol{t})$. Moreover, the element $(\boldsymbol{z}, i \wedge j) \in S_-^{\geq}(\boldsymbol{t}) = S_-^{\geq} + \boldsymbol{t}$ can be characterised by $\pi_c \boldsymbol{z} \in I_-^{\geq}(i \wedge j)(\boldsymbol{t})$ from Lemma 3.15 where $I_-^{\geq}(i \wedge j)(\boldsymbol{t}) := I_-^{\geq}(i \wedge j) + \pi_c \boldsymbol{t}$. The cases of $\mathscr{S}_-^{>}(\boldsymbol{t})$, $S_-^{>}(\boldsymbol{t})$, and $I_-^{>}(i \wedge j)(\boldsymbol{t})$ is defined analogously.

Now, let us consider the existence problem of the $E_2(\sigma)$-invariant stepped plane.

DEFINITION 3.23 (for type $(-1, 0)$). Let $s \in \mathbb{R}^3$ be the solution satisfying

$$s + e_3 - e_2 = As + ae_3 - e_1 , \tag{3.14}$$

which is given by $s = {}^t\left[\frac{b-1}{2-a-b}, \frac{(a-1)(b-1)}{2-a-b}, \frac{a-1}{2-a-b}\right]$. Using s, let us consider the plane $P_e(s)$ and $S^{\geq}_{-}(s)$ of $P_e(s)$, moreover let us define the $\mathcal{U}^{\geq}_{-}(s)$ and $\mathcal{U}^{>}_{-}(s)$, which are called the seed, as follows:

$$\mathcal{U}^{\geq}_{-}(s) := (s - e_2, 1 \wedge 2) + (s, 1 \wedge 3) + (s - e_2, 2 \wedge 3) ,$$
$$\mathcal{U}^{>}_{-}(s) := (s - e_2 + e_3, 1 \wedge 2) + (s - e_2, 1 \wedge 3) + (s - e_2 + e_1, 2 \wedge 3) .$$

Then, from the definition of $I^{\geq}_{-}(i \wedge j)$, $i \wedge j \in V_0$, $\pi_c(s - e_2) \in I^{\geq}_{-}(1 \wedge 2)(s)$, $\pi_c s \in I^{\geq}_{-}(1 \wedge 3)(s)$, $\pi_c(s - e_2) \in I^{\geq}_{-}(2 \wedge 3)(s)$. Therefore, we see that $\mathcal{U}^{\geq}_{-}(s) \in \mathcal{G}^{\geq}_{-}(s)$ where $\mathcal{G}^{\geq}_{-}(s) = \mathcal{G}^{\geq}_{-} + s$. By the analogous discussion, we get $\mathcal{U}^{>}_{-}(s) \in \mathcal{G}^{>}_{-}(s)$, where $\mathcal{G}^{>}_{-}(s) = \mathcal{G}^{>}_{-} + s$.

DEFINITION 3.24 (for type (+1,1)). Let us define the $\mathcal{U}^{\geq}_{+}$ and $\mathcal{U}^{>}_{+}$, which are called the seed, as follows:

$$\mathcal{U}^{\geq}_{+} := (\mathbf{0}, 1 \wedge 2) + (\mathbf{0}, 1 \wedge 3) + (\mathbf{0}, 2 \wedge 3) ,\ \mathcal{U}^{>}_{+} := (e_3, 1 \wedge 2) + (e_2, 3 \wedge 1) + (e_1, 2 \wedge 3) .$$

Then, it is easy to see that

$$\mathcal{U}^{\geq}_{+} \in \mathcal{G}^{\geq}_{+} , \quad \mathcal{U}^{>}_{+} \in \mathcal{G}^{>}_{+}$$

by the anlogous discussion above.

REMARK 3.25. We usually treat $\mathcal{U}^{\geq}_{-}(s), \mathcal{U}^{>}_{-}(s), \mathcal{U}^{\geq}_{+}, \mathcal{U}^{>}_{+}$ as patches, i.e., $\mathcal{U}^{\geq}_{-}(s) \in \mathcal{G}^{\geq}_{-}(s)$, but we sometimes treat them three distinct unit faces, i.e., $\mathcal{U}^{\geq}_{-}(s) \subset S^{\geq}_{-}(s)$.

LEMMA 3.26 (for type $(-1, 0)$). *Using* s *satisfying* (3.14), *we get the following relations*:

$$E_2(\sigma_-)\mathcal{U}^{\geq}_{-}(s) \succ \mathcal{U}^{>}_{-}(s) , \quad E_2(\sigma_-)\mathcal{U}^{>}_{-}(s) \succ \mathcal{U}^{\geq}_{-}(s)$$

where $\delta \succ \gamma$ *means that the patch* γ *is the subpatch of the patch* δ. *In other words*, $\delta \succ \gamma$ *means that if* $(z, i \wedge j) \in \gamma$, *then* $(z, i \wedge j) \in \delta$.

LEMMA 3.27 (for type $(+1, 1)$). *The following relations hold*:

$$E_2(\sigma_+)\mathcal{U}^{\geq}_{+} \succ \mathcal{U}^{\geq}_{+} , \quad E_2(\sigma_+)\mathcal{U}^{>}_{+} \succ \mathcal{U}^{>}_{+} .$$

The proofs of Lemmas 3.26 and 3.27 are given by checking of $E_2(\sigma_-)\mathcal{U}^{\geq}_{-}(s)$, $E_2(\sigma_-)\mathcal{U}^{>}_{-}(s)$, $E_2(\sigma_+)\mathcal{U}^{\geq}_{+}$, $E_2(\sigma_+)\mathcal{U}^{>}_{+}$ explicitly.

PROPOSITION 3.28 (for type $(-1, 0)$). *Using* s *satisfying* (3.14), *let us consider two seeds* $\mathcal{U}^{\geq}_{-}(s)$ *and* $\mathcal{U}^{>}_{-}(s)$. *Then, the following properties hold*:

(1) $\mathcal{U}^{\geq}_{-}(s) \in \mathcal{G}^{\geq}_{-}(s)$, $\mathcal{U}^{>}_{-}(s) \in \mathcal{G}^{>}_{-}(s)$;
(2) $E_2(\sigma_-)\mathcal{U}^{\geq}_{-}(s) \succ \mathcal{U}^{>}_{-}(s)$, $E_2(\sigma_-)\mathcal{U}^{>}_{-}(s) \succ \mathcal{U}^{\geq}_{-}(s)$;
(3) $E_2(\sigma_-)\mathcal{U}^{\geq}_{-}(x) - \mathcal{U}^{>}_{-}(s) = E_2(\sigma_-)\mathcal{U}^{>}_{-}(s) - \mathcal{U}^{\geq}_{-}(s)$;
(4) $S^{\geq}_{-}(s) \setminus \mathcal{U}^{\geq}_{-}(s) \ni (z, i \wedge j)$ *implies* $E_2(\sigma_-)(z, i \wedge j) \in \mathcal{G}^{\geq}_{-}(s)$;
$S^{>}_{-}(s) \setminus \mathcal{U}^{>}_{-}(s) \ni (z, i \wedge j)$ *implies* $E_2(\sigma_-)(z, i \wedge j) \in \mathcal{G}^{>}_{-}(s)$;

(5) $(z, i \wedge j), (z', (i \wedge j)') \in S_-^{\geq}(s)$ (or $S_-^{>}(s)$), $(z, i \wedge j) \neq (z', (i \wedge j)')$ *imply* $\nexists (w, k \wedge l) : (w, k \wedge l) \in E_2(\sigma_-)(z, i \wedge j)$ *and* $(w, k \wedge l) \in E_2(\sigma_-)(z' (i \wedge j)')$.

(6) *For any* $(z, i \wedge j) \in S_-^{\geq} \backslash \mathcal{U}_-^{\geq}(s)$, *there exists* $(y, k \wedge l) \in S_-^{\geq}(s) \backslash \mathcal{U}_-^{\geq}(s)$ *such that* $E_2(\sigma_-)(y, k \wedge l) \ni (z, i \wedge j)$.

Proof. (1) is clear from the definition of $\mathcal{U}_-^{\geq}(s)$ and $\mathcal{U}_-^{>}(s)$. (2) is proved in Lemma 3.26. (3) is clear from the definition of $E_2(\sigma_-)$. For (4). Let us assume that $S_-^{\geq}(s) \backslash \mathcal{U}_-^{\geq}(s) \ni (z, 1 \wedge 2)$, i.e., by Lemma 3.15, let us assume that $\pi_c z \in (\pi_c(s - e_2), \pi_c(s - e_2 + e_3))_c = \mathrm{int}\left(I_-^{\geq}(1 \wedge 2)(s)\right)$. From $E_2(\sigma_-)(z, 1 \wedge 2) = (Az, 2 \wedge 3)$, we want to show that $\pi_c(Az) \in (\pi_c(s - e_2), \pi_c(s - e_2 + e_1))_c = \mathrm{int}\left(I_-^{\geq}(2 \wedge 3)(s)\right)$. From the assumption $\pi_c z \in (\pi_c(s - e_2), \pi_c(s - e_2 + e_3))_c$ and from the properties that A is the linear contracting map on P_c and $\det A = -1$, we have

$$\begin{aligned}\pi_c(Az) &\in (\pi_c A(s - e_2 + e_3), \pi_c A(s - e_2))_c \\ &= (\pi_c(be_2), \pi_c(-ae_3 + e_1))_c + \pi_c(s - e_2).\end{aligned}$$

On the other hand, from the condition (3.9) and $a \geq 0$, i.e., roughly speaking $a \geq 0$, $b \geq 0$, we know that

$$(\pi_c(be_2), \pi_c(-ae_3 + e_1))_c \subset (\pi_c \mathbf{0}, \pi_c e_1)_c .$$

Therefore, we get $\pi_c Az \in \mathrm{int}\left(I_-^{\geq}(2 \wedge 3)(s)\right)$. We prove the other cases, i.e., $(z, 1 \wedge 3)$, $(z, 2 \wedge 3) \in S_-^{\geq}(s) \backslash \mathcal{U}_-^{\geq}(s)$, analogously. In particular, we use the properties such that $|\pi_c e_1| : |\pi_c e_2| : |\pi_c e_3| = 1 : |\lambda_3| : |\lambda_3^2|$, $1 > a\lambda_3^2$, and $1 > b\lambda_3$. For (5). We assume that $\exists (w, k \wedge l) \in S_-^{\geq}(s) : (w, k \wedge l) \in E_2(\sigma_-)(z, i \wedge j) \cap E_2(\sigma_-)(z', (i \wedge j)')$. It is enough to consider $k \wedge l = 2 \wedge 3$. Let consider the case that $i \wedge j = 1 \wedge 2$ and $(i \wedge j)' = 1 \wedge 3$. From the definition of $E_2(\sigma_-)$, we assume that there exists j $(1 \leq j \leq a)$: $Az = Az' + (j - 1)e_3$. By the way, $(z, 1 \wedge 2) \in S_-^{\geq}(s)$ $(z', 1 \wedge 3) \in S_-^{>}(s)$, so $\pi_c z \in I_-^{\geq}(1 \wedge 2)(s)$ and $\pi_c z \in I_-^{\geq}(1 \wedge 3)(s)$. From $1 \leq j \leq a$, $a \geq 0$, we know that $0 \leq j - 1 \leq a - 1$. This fact contradicts the assumption that there exists j $(1 \leq j \leq a) : Az = Az' + (j - 1)e_3$. We prove other cases, i.e., $\{(z, i \wedge j), (z', (i \wedge j)')\} \in \{\{(z, 1 \wedge 2), (z', 1 \wedge 2)\}$, $\{(z, 1 \wedge 2), (z', 2 \wedge 3)\}$, $\{(z, 1 \wedge 3), (z', 1 \wedge 3)\}$, $\{(z, 1 \wedge 3), (z', 2 \wedge 3)\}$, $\{(z, 2 \wedge 3), (z', 2 \wedge 3)\}\}$, analogously. For (6). The proof is obtained by the analogous discussion with Lemma 2.3 in [IO2]. □

On type $(+1, 1)$, we obtain anlogous result.

PROPOSITION 3.29 (for type $(+1, 1)$). *Let us consider two seeds* $\mathcal{U}_+^{\geq}$ *and* $\mathcal{U}_+^{>}$. *Then, the following properties hold*:

(1) $\mathcal{U}_+^{\geq} \in \mathcal{G}_+^{\geq}$, $\mathcal{U}_+^{>} \in \mathcal{G}_+^{>}$;

(2) $E_2(\sigma_+)\mathcal{U}_+^{\geq} \succ \mathcal{U}_+^{\geq}$, $E_2(\sigma_+)\mathcal{U}_+^{>} \succ \mathcal{U}_+^{>}$;

(3) $E_2(\sigma_+)\mathcal{U}_+^{\geq} - \mathcal{U}_+^{>} = E_2(\sigma_+)\mathcal{U}_+^{>} - \mathcal{U}_+^{\geq}$;

(4) $S_+^{\geq} \backslash \mathcal{U}_+^{\geq} \ni (x, i \wedge j)$ *implies* $E_2(\sigma_+)(x, i \wedge j) \in \mathcal{G}_+^{\geq}$;
$S_+^{>} \backslash \mathcal{U}_+^{>} \ni (x, i \wedge j)$ *implies* $E_2(\sigma_+)(x, i \wedge j) \in \mathcal{G}_+^{>}$;

(5) $(\boldsymbol{x}, i \wedge j), (\boldsymbol{x}', (i \wedge j)') \in S_+^{\geq}$ *(or* $S_+^{>}$*)*, $(\boldsymbol{x}, i \wedge j) \neq (\boldsymbol{x}', (i \wedge j)')$ *imply* $\nexists (\boldsymbol{w}, k \wedge l) : (\boldsymbol{w}, k \wedge l) \in E_2(\sigma_+)(\boldsymbol{x}, i \wedge j)$ *and* $(\boldsymbol{w}, k \wedge l) \in E_2(\sigma_+)$ $(\boldsymbol{x}' (i \wedge j)')$.

(6) *For any* $(\boldsymbol{x}, i \wedge j) \in S_+^{\geq} \backslash \mathcal{U}_+^{\geq}$, *there exists* $(\boldsymbol{y}, k \wedge l) \in S_+^{\geq} \backslash \mathcal{U}_+^{\geq}$ *such that* $E_2(\sigma_+)(\boldsymbol{y}, k \wedge l) \ni (\boldsymbol{x}, i \wedge j)$.

COROLLARY 3.30. $E_2(\sigma_-)^2 \mathcal{U}_-^{\geq}(\boldsymbol{s}) \succ \mathcal{U}_-^{\geq}(\boldsymbol{s})$ *and* $E_2(\sigma_-)^2 \mathcal{U}_-^{>}(\boldsymbol{s}) \succ \mathcal{U}_-^{>}(\boldsymbol{s})$ *where* $\boldsymbol{s}$ *is satisfying* (3.14).

Hence, we see that there exist the invariant quasi-periodic tilings of P_e by $E_2(\sigma_\pm)$ (see [AI], [FIR]).

3.5. Complex Pisot numeration systems from 3×3 unimodular complex Pisot companion matrices

Let us discuss only the existence of complex Pisot numeration system on type $(-1, 0)$ and type $(+1, 1)$.

Let us define $T_-^{\geq}(\boldsymbol{s})$, $\mathcal{T}_-^{\geq}(\boldsymbol{s})$ as

$$T_-^{\geq}(\boldsymbol{s}) := T_-^{\geq} + \pi_c \boldsymbol{s}, \quad \mathcal{T}_-^{\geq}(\boldsymbol{s}) := \mathcal{T}_-^{\geq} + \pi_c \boldsymbol{s}$$

where $\boldsymbol{s}$ is the solution of (3.14). Then from the definition of $S_-^{\geq}(\boldsymbol{s})$ and the property of $\operatorname{sgn}(\boldsymbol{v}) = \operatorname{sgn}(\boldsymbol{v}^*)$, we see that $\mathcal{T}_-^{\geq}(\boldsymbol{s})$ is a quasi-periodic tiling of P_e with prototiles $\{\pi_e \ (\boldsymbol{0}, i \wedge j) \mid i \wedge j \in V_0\}$. Since the non-periodicity of the tiling comes from the irreducibility of $p(x)$ and the quasi-periodicity of the tiling comes from the fact that the tiling is constructed by the projection π_e of the $\boldsymbol{s}$-translated stepped plane $\mathcal{S}_-^{\geq}(\boldsymbol{s})$. The other cases $T_-^{>}(\boldsymbol{s})$, $\mathcal{T}_-^{>}(\boldsymbol{s})$, $T_+^{\geq}$, $\mathcal{T}_+^{\geq}$, $T_+^{>}$, and $\mathcal{T}_+^{>}$ are discussed analogously.

Let $\pi_e \mathcal{G}_-^{\geq}(\boldsymbol{s})$ (resp. $\pi_e \mathcal{G}_-^{>}(\boldsymbol{s})$) be the family of patches of $T_-^{\geq}(\boldsymbol{s}) = T_-^{\geq} + \pi_c \boldsymbol{s}$ (resp. $T_-^{>}(\boldsymbol{s})$), we proved that the operator $\pi_e E_2(\sigma_-)$ is the *tiling* substitution from $\pi_e \mathcal{G}_-^{\geq}(\boldsymbol{s})$ (resp. $\pi_e \mathcal{G}_-^{>}(\boldsymbol{s})$) to $\pi_e \mathcal{G}_-^{>}(\boldsymbol{s})$ (resp. $\pi_e \mathcal{G}_-^{\geq}(\boldsymbol{s})$) and that that the operator $\pi_e E_2(\sigma_+)$ is the *tiling* substitution from $\pi_e \mathcal{G}_+^{\geq}$ (resp. $\pi_e \mathcal{G}_+^{>}$) to $\mathcal{G}_+^{\geq}$ (resp. $\mathcal{G}_+^{>}$) in the subsection 3.4.

Finally, we arrive at the following theorem.

THEOREM 3.31. (1) *Let us define* $\gamma_{i \wedge j, -}$ *by*

$$\gamma_{i \wedge j, -} := \lim_{n \to \infty} A^{-n} \pi_e E_2(\sigma_-)^n (\boldsymbol{s}_{i \wedge j}, i \wedge j)$$

where $(\boldsymbol{s}_{i \wedge j}, i \wedge j) \in \mathcal{U}_-^{\geq}(\boldsymbol{s})$ *or* $\mathcal{U}_-^{>}(\boldsymbol{s})$. *Moreover, we assume that there exist the new seed* $\mathcal{U}_-^{\geq'}(\boldsymbol{s}')$ *such that*

(i) $\mathcal{U}_-^{\geq'}(\boldsymbol{s}') = \mathcal{U}_-^{\geq}(\boldsymbol{s}) + \boldsymbol{u}$ *for some* $\boldsymbol{u} \in \mathbb{Z}^3$;

(ii) $E_2(\sigma_-)^2 \mathcal{U}_-^{\geq'}(\boldsymbol{s}') \succ \mathcal{U}_-^{\geq'}(\boldsymbol{s}')$;

(iii) $\bigcup_{n=1}^{\infty} E_2(\sigma_-)^{2n} \mathcal{U}_-^{\geq'}(\boldsymbol{s}') = P_e$.

Then $(A_-, \mathcal{P}_-)$, $\mathcal{P}_- = \{\gamma_{i \wedge j, -} \mid i \wedge j \in V_0\}$ *satisfies the properties for the complex Pisot numeration system, i.e.,* (N1), (N2), (N3) *in Definition* 0.2 *hold.*

(2) *Let us define* $\gamma_{i \wedge j, +}$ *by*

$$\gamma_{i \wedge j, +} := \lim_{n \to \infty} A^{-n} \pi_e E_2(\sigma_+)^n (\boldsymbol{x}_{i \wedge j}, i \wedge j)$$

where $\left(\boldsymbol{x}_{i\wedge j}, i\wedge j\right)\in\mathcal{U}_+^{\geq}$ *or* $\mathcal{U}_+^{>}$. *Moreover, we assume that there exist the new seed* $\mathcal{U}_+^{\geq'}\left(\boldsymbol{x}'\right)$ *such that*

(i) $\mathcal{U}_+^{\geq'}\left(\boldsymbol{x}'\right)=\mathcal{U}_+^{\geq}+\boldsymbol{x}'$ *for some* $\boldsymbol{x}'\in\mathbb{Z}^3$;

(ii) $E_2(\sigma_+)\,\mathcal{U}_-^{\geq'}\left(\boldsymbol{x}'\right)\succ\mathcal{U}_-^{\geq'}\left(\boldsymbol{x}'\right)$;

(iii) $\bigcup_{n=1}^{\infty}E_2(\sigma_+)^n\,\mathcal{U}_-^{\geq'}\left(\boldsymbol{x}'\right)=P_e$.

Then $(A_+,\mathcal{P}_+)$, $\mathcal{P}_+=\left\{\gamma_{i\wedge j,+}\mid i\wedge j\in V_1\right\}$ *satisfies the properties for the complex Pisot numeration system, i.e.*, (N1), (N2), (N3) *in Definition* 0.2 *hold.*

Proof. For (1). The property (N1) for $\gamma'_{i\wedge j}=\lim_{n\to\infty}A^{-n}E_2(\sigma_-)^n\left(\boldsymbol{s}'_{i\wedge j}, i\wedge j\right)$, $\left(\boldsymbol{s}'_{i\wedge j}, i\wedge j\right)\in\mathcal{U}_-^{\geq'}\left(\boldsymbol{s}'\right)$ holds from the assumption (ii) and (iii). Therefore, we see that the property (N1) holds for $\gamma_{i\wedge j}$ (c.f. Theorem 1.5 in [EIR] using Theorem 5.3 in [LW]). We prove the property (N2) analogously with the proof of Corollary 2 in [AI]. From the relation $E_2(\sigma_-)(\boldsymbol{0}, i\wedge j)$ given by the proof of Proposition 3.22, we obtain the set equations of $\left\{\widehat{\gamma}_{i\wedge j}\,\middle|\,\widehat{\gamma}_{i\wedge j}=\lim_{n\to\infty}E_2(\sigma_-)(\boldsymbol{0}, i\wedge j)\right\}$ by $A\widehat{\gamma}_{i\wedge j}=\bigcup_{\substack{1\leq k\leq l_i\\ 1\leq l\leq l_j}}\left(\gamma_{W_k^{(i)}\wedge W_l^{(j)}}+f\left(P_k^{(i)}\right)+f\left(P_l^{(j)}\right)\right)$. On the other hand, $\gamma_{i\wedge j}$ is written by the translation of $\widehat{\gamma}_{i\wedge j}$. Therefore, we obtain that $A\gamma_{i\wedge j}$ is written by the sum of the translation of the elements by $\left\{\gamma_{i\wedge j}\mid i\wedge j\in V_0\right\}$. To prove the property (N3), we must show that $E_2(\sigma)$ satisfies the strongly coincidence condition (see [AI]). By the way, the strongly coincidence condition is geometrically given by the following: there exist n, $i\wedge j$, $\boldsymbol{y}$, and $\boldsymbol{t}\in\mathbb{R}^3$ (resp. n', $(i\wedge j)'$, $\boldsymbol{y}'$, and $\boldsymbol{t}'$) such that

$$E_2(\sigma_-)^n(\boldsymbol{y}, i\wedge j)\succ\boldsymbol{t}+\mathcal{U}_-^{\geq}(\boldsymbol{s})\left(\text{resp. } E_2(\sigma_+)^{n'}\left(\boldsymbol{y}', (i\wedge j)'\right)\succ\boldsymbol{t}'+\mathcal{U}_+^{\geq}\right).$$

This conditions holds in the case of $n=2$, $i\wedge j=2\wedge 3$, $\boldsymbol{y}=\boldsymbol{s}-\boldsymbol{e}_2$, and $\boldsymbol{t}=\boldsymbol{0}$ in Lemma 3.26. (2) is proved analogously and see Remark 3.33. □

COROLLARY 3.32. *Let* $\mathscr{T}_-^{\geq,*}(\boldsymbol{s}):=\left\{\gamma_{i\wedge j,-}+\pi_e\boldsymbol{z}\mid i\wedge j\in V_0, (\boldsymbol{z}, i\wedge j)\in T_-^{\geq}(\boldsymbol{s})\right\}$. *Then,* $\mathscr{T}_-^{\geq,*}(\boldsymbol{s})$ *is a quasi-periodic self-similar tiling of* P_e *by the linear transformation A. The other cases* $\mathcal{T}_-^{>,*}(\boldsymbol{s})$, $\mathcal{T}_+^{\geq,*}$, $\mathcal{T}_+^{>,*}$ *are discussed analogously.*

REMARK 3.33. For type $(+1, 1)$ in Theorem 3.31 , we have obtained the numeration system $\left(A_+,\mathcal{P}=\left\{\gamma_{i\wedge j,+}\right\}_{i\wedge j\in V_1}\right)$ where

$$\gamma_{i\wedge j,+}=\lim_{n\to\infty}A_e^{-n}\pi_eE_2(\sigma_+)^n\left(\boldsymbol{x}_{i\wedge j}, i\wedge j\right)$$

for $\left(\boldsymbol{x}_{i\wedge j}, i\wedge j\right)\in\mathcal{U}_+^{\geq}$ or $\mathcal{U}_+^{>}$ where σ_+ is given by $\sigma_+:\begin{array}{l}1\to 2\\ 2\to 3\\ 3\to 12^b3^a\end{array}$ and $A_{\sigma_+}=\begin{bmatrix}0&0&1\\1&0&b\\0&1&a\end{bmatrix}$. Let us consider the automorphism $\tau=\left(\overleftarrow{\sigma_+}\right)^{-1}$, which is the inverse of the mirror image of σ_+, then,

$$\tau : \begin{array}{l} 1 \to 1^{-b}2^{-a}3 \\ 2 \to 1 \\ 3 \to 2\,. \end{array}$$

From the condition of (a, b) satisfying the complex Pisot condition (3.10), i.e., (1), (2), (3) in Proposition 3.2, it is clear that the automorphism τ is a substitution. Therefore we know that the limit set $\gamma_{i\wedge j,+}$ is also obtained as $\gamma_{i\wedge j,+} = -\delta_k$, $\{1, 2, 3\} = \{i, j, k\}$ by Theorem 2.1, in other words, the numeration system produced from the class of type $(+1, 1)$ is the numeration system produced from the unimodular Pisot substitution (c.f. Remark 2.2 discussed in the section 2).

EXAMPLE 3.34. Let us consider the case $(a, b) = (1, 0)$ of type $(-1, 0)$, i.e., $A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and its characteristic polynomial is $x^3 - x^2 + 1$. The eigenvalues of A are $\lambda_1 = 0.877439+0.744862i, \lambda_2 = 0.877439-0.744862i, \lambda_3 = -0.754878$ (see Figure 11) and V_0 is given by $V_0 = \{1 \wedge 2,\ 1 \wedge 3,\ 2 \wedge 3\}$.

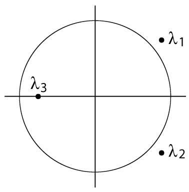


FIGURE 11

Let the automorphism $\sigma : \begin{cases} 1 \to 2 \\ 2 \to 3 \\ 3 \to 31^{-1} \end{cases}$, then $E_2(\sigma)$ is given by

$$\begin{aligned} E_2(\sigma)(\mathbf{0}, 1 \wedge 2) &= (\mathbf{0}, \sigma(1) \wedge \sigma(2)) = (\mathbf{0}, 2 \wedge 3) \\ E_2(\sigma)(\mathbf{0}, 1 \wedge 3) &= \left(\mathbf{0}, 2 \wedge 31^{-1}\right) = (\mathbf{0}, 2 \wedge 3) + \left(\boldsymbol{e}_3, 2 \wedge 1^{-1}\right) \\ &\overset{(*)}{=} (\mathbf{0}, 2 \wedge 3) + ((\boldsymbol{e}_3 - \boldsymbol{e}_1), 1 \wedge 2) \\ E_2(\sigma)(\mathbf{0}, 2 \wedge 3) &= \left(\mathbf{0}, 3 \wedge 31^{-1}\right) = \left(\boldsymbol{e}_3, 3 \wedge 1^{-1}\right) \overset{(*)}{=} ((\boldsymbol{e}_3 - \boldsymbol{e}_1), 1 \wedge 3) \end{aligned}$$

where $(*)$ means the rearrangement and it satisfies the POP-property (see Figure 12).

Let $\mathcal{U}_{-}^{\geq}(-\boldsymbol{e}_1)$ be

$$\mathcal{U}_{-}^{\geq}(-\boldsymbol{e}_1) := ((-\boldsymbol{e}_1 - \boldsymbol{e}_2 + \boldsymbol{e}_3), 1 \wedge 2) + ((-\boldsymbol{e}_1 - \boldsymbol{e}_2), 1 \wedge 3) + (-\boldsymbol{e}_2, 2 \wedge 3),$$

then $\mathcal{U}_{-}^{\geq}(-\boldsymbol{e}_1)$ satisfies

$$E_2(\sigma)^2\, \mathcal{U}_{-}^{\geq}(-\boldsymbol{e}_1) \succ \mathcal{U}_{-}^{\geq}(-\boldsymbol{e}_1)$$

(see Figure 13).

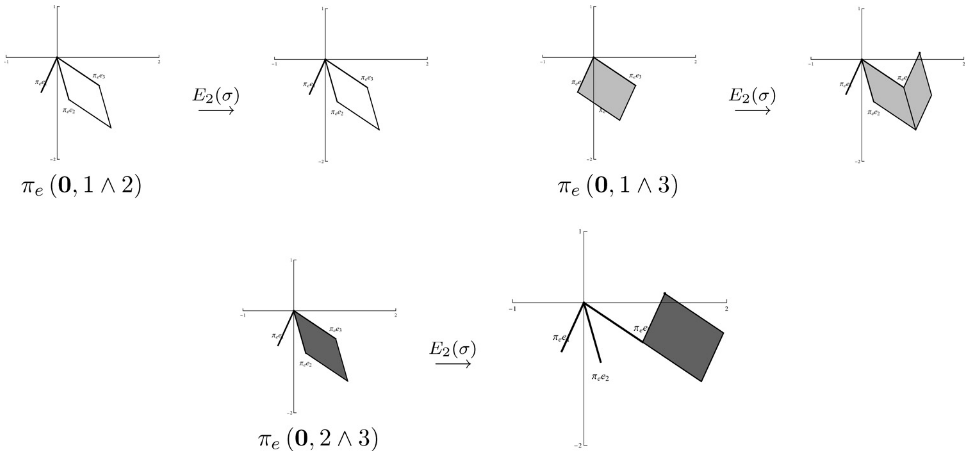


FIGURE 12. $\pi_e E_2\,(\sigma)\,(\mathbf{0}, i \wedge j)$ in Example 3.34: $\sigma : 1 \mapsto 2,\, 2 \mapsto 3,\, 3 \mapsto 31^{-1}$.

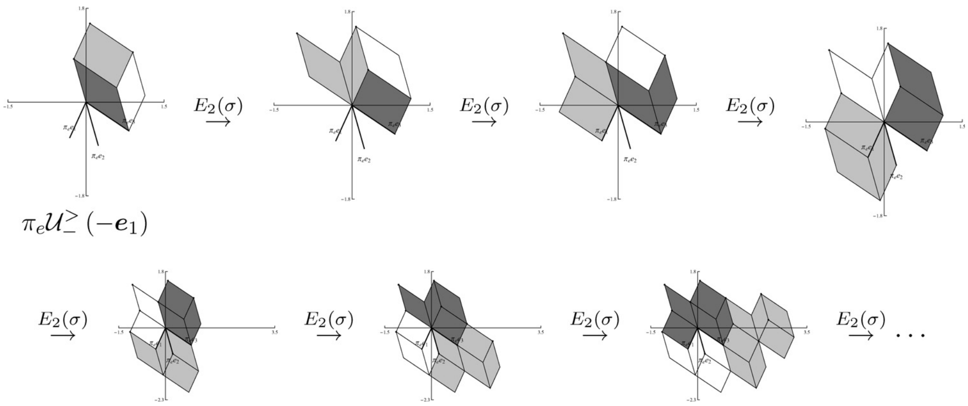


FIGURE 13. $\pi_e E_2\,(\sigma)^n\,\mathcal{U}_{-}^{\geq}\,(-\boldsymbol{e}_1)$, $n = 0, 1, 2, \ldots, 6$.

Let us define $\mathscr{T}_{-}^{>}\,(-\boldsymbol{e}_1)$ as follows: the tiling $\mathscr{T}_{-}^{>}\,(-\boldsymbol{e}_1)$ generated by the projection π_e of the stepped plane $\mathscr{S}_{-}^{\geq}\,(-\boldsymbol{e}_1)$ satisfies the following properties:

$$T_{-}^{\geq}\,(-\boldsymbol{e}_1) := \left\{\pi_e\,(z, i \wedge j) \,\middle|\, (z, i \wedge j) \in E_2\,(\sigma)^n\,\mathcal{U}_{-}^{\geq}\,(-\boldsymbol{e}_1) \text{ for some } n\right\},$$

$$\mathscr{T}_{-}^{>}\,(-\boldsymbol{e}_1) := \bigcup_{\pi_e(z, i\wedge j) \in T_{-}^{\geq}(-\boldsymbol{e}_1)} \pi_e\,(z, i \wedge j)\,(= P_e)$$

on this example (see Figure 14). The analogous proof can be obtained by the method of $*$-connected in [AFHI], or C-covered propery in [IO1].

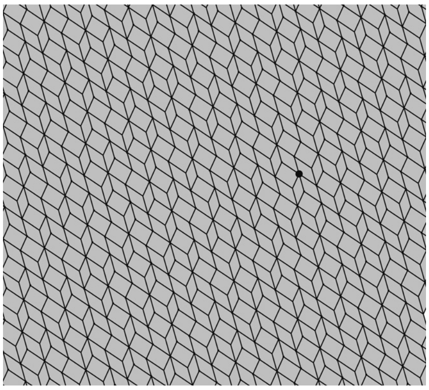


FIGURE 14. $\mathcal{T}_{-}^{\geq}(-\boldsymbol{e}_1)$ where the black dot is the origin point.

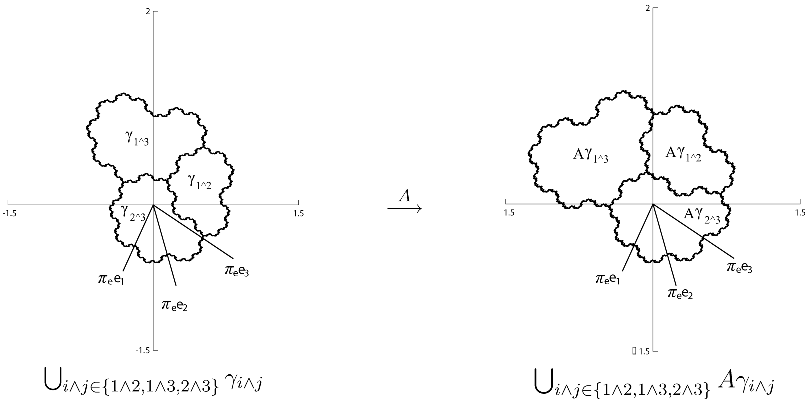


FIGURE 15. The set equations.

Let us the limit set $\gamma_{i\wedge j} = \lim_{n\to\infty} A^{-n}\pi_e E_2(\sigma)^n\left(\boldsymbol{s}_{i\wedge j}, i\wedge j\right)$ for $\left(\boldsymbol{s}_{i\wedge j}, i\wedge j\right)\in \mathcal{U}_{-}^{\geq}(-\boldsymbol{e}_1)$. Then, $\mathcal{P}=\left\{\gamma_{i\wedge j}\right\}_{i\wedge j\in V_0}$ satisfies not only the following set equations

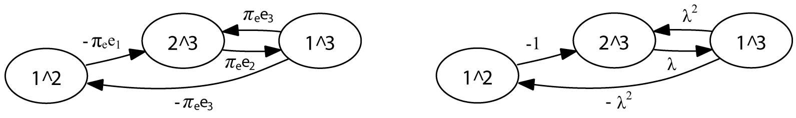
$$\begin{aligned}A\gamma_{1\wedge 2} &= \gamma_{2\wedge 3}-\pi_e\boldsymbol{e}_1\\ A\gamma_{1\wedge 3} &= (\gamma_{1\wedge 2}-\pi_e\boldsymbol{e}_3)\cup(\gamma_{2\wedge 3}-\pi_e\boldsymbol{e}_3)\\ A\gamma_{2\wedge 3} &= \gamma_{1\wedge 3}+\pi_e\boldsymbol{e}_2\end{aligned}$$

(see Figure 15), i.e., the property (N2), but also the properties (N1) and (N3) of Definition 0.2. Therefore, we see that $(A,\mathcal{P})$ is the the complex Pisot numeration system of λ.

Then, the labeled graph $(V,E,i,t,\mathcal{L})$ on the example is given by Figure 16. From the fact that

$$\phi(\pi_e\boldsymbol{e}_1)=1,\ \phi(\pi_e\boldsymbol{e}_2)=\lambda,\ \phi(\pi_e\boldsymbol{e}_3)=\lambda^2,$$

we see that the labeled graph $(V,E,i,t,(\phi\mathcal{L}))$ is given by Figure 16.



The graph $(V, E, i, t, \mathcal{L})$ The graph $(V, E, i, t, (\phi\mathcal{L}))$

FIGURE 16. The graph of $(a, b) = (1, 0)$ for type $(-1, 0)$.

Therefore, let $\Omega_{i\wedge j}$ be the label-admissible symbolic space which is starting from the vertex $i \wedge j$ by the labeled graph $(V, E, i, t, (\phi\mathcal{L}))$ and let its element be $(a_0, a_1, a_2, \ldots)$, $a_i \in \{-1, \lambda, \lambda^2, -\lambda^2\}$, then $z \in \phi_e \left(\bigcup_{i\wedge j \in V_e} \gamma_{i\wedge j}\right)$ is represented by $z = \sum_{n=1}^{\infty} a_{n-1}\lambda^{-n}$ where $a_n = \phi_e \left(\pi_e \boldsymbol{f}_{k_{n-1}}^{(j_{n-1})}\right)$.

4. Complex Pisot numeration systems from 4×4 unimodular complex Pisot companion matrices

4.1. Setting

In this section, we discuss how we obtain the complex Pisot numeration system from a 4×4 unimodular complex Pisot companion matrix (see [AFHI], [FIR], [F]).

Let $A_\pm$ be the 4×4 companion matrix whose characteristic polynomial is $p_\pm(x) = x^4 - ax^3 - bx^2 - cx \mp 1$, $a, b, c \in \mathbb{Z}$, i.e.,

$$A_\pm = \begin{bmatrix} 0 & 0 & 0 & \pm 1 \\ 1 & 0 & 0 & c \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a \end{bmatrix}.$$

We assume that the algebraic integers λ_i $(1 \leq i \leq 4)$ of $p_\pm(x)$ satisfy the *non-Pisot hyperbolic condition, i.e.,*

$$|\lambda_1| \geq |\lambda_2| > 1 > |\lambda_3| \geq |\lambda_4| .$$

Under the non-Pisot hyperbolic condition, let $\boldsymbol{u}_i (1 \leq i \leq 4)$ be the eigenvectors of λ_i respectively and put the corresponding vectors $\boldsymbol{v}_i$ of eigenvectors

$$\begin{cases} \boldsymbol{v}_1 = \dfrac{\boldsymbol{u}_2 + \boldsymbol{u}_1}{2}, \quad \boldsymbol{v}_2 = \dfrac{\boldsymbol{u}_2 - \boldsymbol{u}_1}{2i} & \text{if} \quad \lambda_1, \lambda_2 \in \mathbb{C}\backslash\mathbb{R} \\ \boldsymbol{v}_1 = \boldsymbol{u}_1, \quad \boldsymbol{v}_2 = \boldsymbol{u}_2 & \text{if} \quad \lambda_1, \lambda_2 \in \mathbb{R} \end{cases},$$

$$\begin{cases} \boldsymbol{v}_3 = \dfrac{\boldsymbol{u}_4 + \boldsymbol{u}_3}{2}, \quad \boldsymbol{v}_4 = \dfrac{\boldsymbol{u}_4 - \boldsymbol{u}_3}{2i} & \text{if} \quad \lambda_3, \lambda_4 \in \mathbb{C}\backslash\mathbb{R} \\ \boldsymbol{v}_3 = \boldsymbol{u}_3, \quad \boldsymbol{v}_4 = \boldsymbol{u}_4 & \text{if} \quad \lambda_3, \lambda_4 \in \mathbb{R} \end{cases},$$

Then, the linear transformation A has the 2-dimesional A-invariant expanding plane P_e spanned by $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ and the 2-dimensional A-invariant contracting plane P_c spanned by $\{\boldsymbol{v}_3, \boldsymbol{v}_4\}$. Using P_e and P_c, $\mathbb{R}^d$ is decomposed by P_e and P_c, i.e., $\mathbb{R}^d = P_e \oplus P_c$. Then, let us define the projection $\pi_e : \mathbb{R}^4 \to P_e$, $\pi_e(x\boldsymbol{v}_1 + y\boldsymbol{v}_2 + z\boldsymbol{v}_3 + w\boldsymbol{v}_4) := x\boldsymbol{v}_1 + y\boldsymbol{v}_2$

(resp. $\pi_c : \mathbb{R}^4 \to P_c$, $\pi_c\,(x\boldsymbol{v}_1 + y\boldsymbol{v}_2 + z\boldsymbol{v}_3 + w\boldsymbol{v}_4) := z\boldsymbol{v}_3 + w\boldsymbol{v}_4$ be the projection to P_c). Moreover, $P_e \circ A = A \circ P_e$ (resp. $P_c \circ A = A \circ P_c$) holds.

Using the representation by

$$[\boldsymbol{e}_1\ \boldsymbol{e}_2\ \boldsymbol{e}_3\ \boldsymbol{e}_4] = [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_3\ \boldsymbol{v}_4]\left[x_{ji}\right]_{1\le j,i\le 4}\ , \tag{4.15}$$

$\pi_e\boldsymbol{e} \in P_e$ (resp. $\pi_c\boldsymbol{e} \in P_c$) of the projected canonical basis $\{\boldsymbol{e}_i\}_{1\le i\le 4}$ are given by

$$\pi_e\boldsymbol{e}_i = x_{1i}\boldsymbol{v}_1 + x_{2i}\boldsymbol{v}_2 \simeq [x_{1i}, x_{2i}]^t$$
$$(\text{resp. } \pi_c\boldsymbol{e}_i = x_{3i}\boldsymbol{v}_3 + x_{4i}\boldsymbol{v}_4 \simeq [x_{3i}, x_{4i}]^t)\ .$$

DEFINITION 4.1. The set of the projected canonical basis $\{\pi_e\boldsymbol{e}_i\}_{1\le i\le 4}$ has the good star property if $\pi_e\boldsymbol{e}_i = w\pi_e\boldsymbol{e}_j$ for some real number $w \ne 0$ implies $i = j$. We define the good star property for the set $\{\pi_c\boldsymbol{e}_i\}_{1\le i\le 4}$ analogously.

Using $\{\pi_e\boldsymbol{e}_i\}_{1\le i\le 4}$ (resp. $\{\pi_c\boldsymbol{e}_i\}_{1\le i\le 4}$) with the good star property, we uniquely obtain the proto-tiles set V_e (resp. V_c) consisting of six symbolic faces whose orientation are positive denoted by

$$V_e := \left\{ i \wedge j \;\middle|\; \begin{array}{l} i \ne j,\ i, j \in \{1, \ldots, 4\}\,,\ i \wedge j \text{ be the positive oriented parallelogram} \\ \text{generated by } \pi_e\boldsymbol{e}_i \text{ and } \pi_e\boldsymbol{e}_j \text{ where } i \wedge j \text{ is chosen if the counterclockwise} \\ \text{angle } \alpha \text{ between } \pi_e\boldsymbol{e}_i \text{ and } \pi_e\boldsymbol{e}_j \text{ satisfies } 0 < \alpha < \pi \end{array} \right\}.$$

The case of V_c is defined by $\{\pi_c\boldsymbol{e}_i\}_{1\le i\le 4}$ analogously.

Let a pair $(\boldsymbol{x},\ i \wedge j) \in \mathbb{Z}^4 \times V_e$ (resp. V_c) be the positive oriented parallelogram $i \wedge j$ located at $\boldsymbol{x}$, i.e.,

$$(\boldsymbol{x},\ i \wedge j) := \left\{\boldsymbol{x} + \mu\boldsymbol{e}_i + \nu\boldsymbol{e}_j\ \mid 0 \le \mu, \nu \le 1\right\}$$

(see Figure 17).

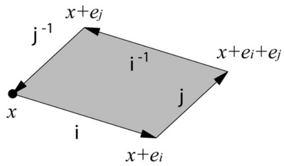


FIGURE 17. $(\boldsymbol{x}, i \wedge j)$.

Let σ (resp. θ) be the automorphism on the free group $F\,\langle 1, 2, 3, 4\rangle$ given by

$$\sigma : \begin{cases} 1 \to 2 \\ 2 \to 3 \\ 3 \to 4 \\ 4 \to 2^c3^b4^a1^{\mp 1} \end{cases},\ \sigma^{-1} : \begin{cases} 1 \to 4^{-1}1^c2^b3^a \\ 2 \to 1 \\ 3 \to 2 \\ 4 \to 3 \end{cases},\ \theta := \overleftarrow{\sigma^{-1}} : \begin{cases} 1 \to 3^a2^b1^c4^{-1} \\ 2 \to 1 \\ 3 \to 2 \\ 4 \to 3 \end{cases}.$$

Using the automorphism σ (resp. θ), the 2-dimentional extension $E_2\,(\sigma)$ (resp. $E_2\,(\theta)$) on the patches of the symbolic faces of V_e on P_e (resp. V_c on P_c) is defined analogously with (3.13) in the section 3.

From now on, we try to get the sufficient conditions of $a, b, c \in \mathbb{Z}$ satisfying the following properties:

(S1) The eigenvalues λ_i of A satisfy the *hyperbolic* non-Pisot condition:

$$|\lambda_1 (= \lambda)| \geq |\lambda_2| > 1 > |\lambda_3| \geq |\lambda_4| ;$$

(S2) $\{\pi_e \boldsymbol{e}_i\}_{1 \leq i \leq 4}$ (resp. $\{\pi_c \boldsymbol{e}_i\}_{1 \leq i \leq 4}$) satisfy the *good star property* (described later);

(S3) The 2-*dimensional extension* $E_2(\sigma)$ (resp. $E_2(\theta)$) has the POP-property.

4.2. Computer experiments

For the companion matrix A_-:

$$A_- = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & c \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a \end{bmatrix}, \quad -15 \leq a, b, c \leq 15 ,$$

by the computer experiments, we observe the following facts:

(1) For $-15 \leq a, b, c \leq 15$, the automorphism σ (resp. θ) satisfies all of the properties (S1), (S2), (S3) if and only if $b = 0$, $c = -a - 1, -a, -a + 1$.

More precisely,

(2) See the table in the section 4.5;

(3) There is no automorphism σ (resp. θ) satisfying all of the properties (S1), (S2), (S3) associated with the companion matrix A_+

$$A_+ = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & c \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a \end{bmatrix}, \quad -7 \leq a, b, c \leq 7 .$$

The condition that $b = 0$ and $c \in \{-a - 1, -a, -a + 1\}$ for the companion matrix A_- of $p(x) = x^4 - ax^3 - bx^2 - cx + 1$ seems to be the necessary and sufficient condition satisfying that $E_2(\sigma)$ has the POP-property.

4.3. Theorem

We will prove the following theorem in this section.

THEOREM 4.2. *Let A be the companion matrix of $p(x) = x^4 - ax^3 - cx + 1$ by*

$$A_- = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a \end{bmatrix}$$

satisfying $a, c \in \mathbb{Z}$ and $c \in \{-a - 1, -a, -a + 1\}$. Then

(1) *the automorphism σ (resp. θ) associated with A (resp. A^{-1}) satisfies all of the properties* (S1), (S2), (S3). *In particular*, (a, c) *satisfies*

$$(a, c) \in \left\{ \begin{array}{l} (-3, 4), (-2, 3), (-2, 2), (-2, 1), (-1, 2), (-1, 1), (-1, 0), (0, 1), \\ (0, -1), (1, 0), (1, -1), (1, -2), (2, -1), (2, -2), (2, -3), (3, -4) \end{array} \right\}, \tag{4.16}$$

then, the eigenvalues λ_i $(1 \le i \le 4)$ *of* A *satisfy the complex Pisot condition*:

$$|\lambda_1| = |\lambda_2| > 1 > |\lambda_3|, \ |\lambda_4|, \ \text{and} \ \lambda_2 = \overline{\lambda_1}.$$

(*see the table in the section* 4.5);

(2) *Under the assumption that there exists* $\mathcal{U}_e$ (*resp.* $\mathcal{U}_c$) *which is the family of* ${}_4C_2 = 6$ *pieces symbolic distinct faces satisfying*

(i) *there exists* k (*resp.* k') *such that*

$$E_2(\sigma)^k \mathcal{U}_e \quad \succ \mathcal{U}_e, \quad \mathcal{U}_e = \sum_{i \wedge j \in V_e} \left(\pi_e \boldsymbol{x}_{i \wedge j}, i \wedge j\right)$$

$$\left(\text{resp. } E_2(\theta)^{k'} \mathcal{U}_c \quad \succ \mathcal{U}_c, \quad \mathcal{U}_c = \sum_{i \wedge j \in V_c} \left(\pi_c \boldsymbol{x}_{i \wedge j}, i \wedge j\right)\right)$$

(ii) $\bigcup_{n=1}^{\infty} E_2(\sigma)^{nk} \mathcal{U}_e = P_e$ $\left(\text{resp. } \bigcup_{n=1}^{\infty} E_2(\theta)^{nk'} \mathcal{U}_c = P_c\right)$, *then, the compact set*

$$\gamma_{i \wedge j, e} := \lim_{n \to \infty} A^{-n} \pi_e E_2(\sigma)^n \left(\boldsymbol{x}_{i \wedge j}, i \wedge j\right)$$

$$\left(\text{resp. } \gamma_{i \wedge j, c} := \lim_{n \to \infty} A^{-n} \pi_c E_2(\theta)^n \left(\boldsymbol{x}_{i \wedge j}, i \wedge j\right)\right)$$

for each $\left(\boldsymbol{x}_{i \wedge j}, i \wedge j\right) \in \mathcal{U}_e$ (*resp.* $\mathcal{U}_c$).

Then $\mathcal{P} = \left\{\gamma_{i \wedge j, e}\right\}_{i \wedge j \in V_e}$ (*resp.* $\left\{\gamma_{i \wedge j, c}\right\}_{i \wedge j \in V_c}$) *satisfies a set equation and the properties* (N1), (N2), (N3), *then* $(A, \mathcal{P})$ *has the complex Pisot numeration system.*

To prove Theorem 4.2, we prepare a few lemmas.

LEMMA 4.3. *For* $p(x) = x^4 - ax^3 - cx + 1$, $a, c \in \mathbb{Z}$, $c \in \{-a-1, -a, -a+1\}$,

(1) *if* $a \ge 5$, *then* $p(x)$ *has two real roots in the interval* $(-1, 0)$;

(2) *if* $a \le -5$, *then* $p(x)$ *has two real roots in the interval* $(0, 1)$.

Proof. For (1). It is easy to see that $p(0) = 1$, $p(-1) = 2 + a + c \ge 1$, $p\left(-\frac{1}{2}\right) = \frac{1}{16}(17 + 2a + 8c) < 0$. Therefore, the statement holds. (2) can be obtained analogously by the observation of $p(0)$, $p\left(\frac{1}{2}\right) < 0$, and $p(1) \ge 1$. □

LEMMA 4.4. *For* $p(x) = x^4 - ax^3 - cx + 1$, $a, c \in \mathbb{Z}$, $c \in \{-a-1, -a, -a+1\}$, *the roots* λ_i $(1 \le i \le 4)$ *of* $p(x)$ *are distributed as follows*:

(1) *if* $a \ge 5$, *then*

$$-1 < \lambda_3 < \lambda_4 < 0 < 1 < \lambda_2 < \lambda_1;$$

(2) *if* $a \le -5$, *then*

$$\lambda_1 < \lambda_2 < -1 < 0 < \lambda_4 < \lambda_3 < 1.$$

Proof. For (1). By Lemma 4.3, we know that $p(x)$ has two real roots in the interval $(-1, 0)$. Let $q(x) = x^4 - cx^3 - ax + 1$, then $q(x)$ satisfies the condition (2) in Lemma 4.3. Therefore there exist two roots μ_1 and μ_2 of $q(x)$ in $(0, 1)$. From the fact that $\lambda_1 := \frac{1}{\mu_1}$, $\lambda_2 := \frac{1}{\mu_2}$ are the roots of $p(x)$, we see that λ_1, λ_2 satisfy the relation $1 < \lambda_2 < \lambda_1$. (2) is obtained by the analogous discussion. □

LEMMA 4.5. *Let A be the companion matrix whose characteristic polynomial $p(x) = x^4 - ax^3 - cx + 1$, $a, c \in \mathbb{Z}$, $c \in \{-a-1, -a, -a+1\}$ and $\boldsymbol{v}_i$ $(1 \le i \le 4)$ be the corresponding vectors of the eigenvectors discussed in* (4.15). *Then the signature of x_{j1}, $(1 \le j \le 4)$ in* (4.15) *are given by*

(1) *if $a \ge 5$, then*

$$(\mathrm{sgn}(x_{11}), \mathrm{sgn}(x_{21}), \mathrm{sgn}(x_{31}), \mathrm{sgn}(x_{41})) = (+, -, -, +);$$

(2) *if $a \le -5$, then*

$$(\mathrm{sgn}(x_{11}), \mathrm{sgn}(x_{21}), \mathrm{sgn}(x_{31}), \mathrm{sgn}(x_{41})) = (-, +, +, -).$$

Proof. From Lemma 4.4, $\boldsymbol{v}_i$ $(1 \le i \le 4)$ of (4.15) are the eigenvectors themselves. The eigenvector $\boldsymbol{v}_i$ $(1 \le i \le 4)$ is given by

$$\boldsymbol{v}_i = {}^t\left[-\frac{1}{\lambda_i},\ -\frac{1}{\lambda_i^2} + \frac{c}{\lambda_i},\ -\frac{1}{\lambda_i^3} + \frac{c}{\lambda_i^2},\ 1\right].$$

Therefore we have

$$\begin{aligned}
x_{11} &= -\frac{\lambda_1^3\left(\lambda_2\lambda_3\lambda_4 c^2 - (\lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4)\,c + \lambda_2 + \lambda_3 + \lambda_4\right)}{(-\lambda_1 + \lambda_2)(-\lambda_1 + \lambda_3)(-\lambda_1 + \lambda_4)}\\
x_{21} &= -\frac{\lambda_2^3\left(\lambda_1\lambda_3\lambda_4 c^2 - (\lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_3\lambda_4)\,c + \lambda_1 + \lambda_3 + \lambda_4\right)}{(-\lambda_2 + \lambda_1)(-\lambda_2 + \lambda_3)(-\lambda_2 + \lambda_4)}\\
x_{31} &= -\frac{\lambda_3^3\left(\lambda_1\lambda_2\lambda_4 c^2 - (\lambda_1\lambda_2 + \lambda_1\lambda_4 + \lambda_2\lambda_4)\,c + \lambda_1 + \lambda_2 + \lambda_4\right)}{(-\lambda_3 + \lambda_1)(-\lambda_3 + \lambda_2)(-\lambda_3 + \lambda_4)}\\
x_{41} &= -\frac{\lambda_4^3\left(\lambda_1\lambda_2\lambda_3 c^2 - (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\,c + \lambda_1 + \lambda_2 + \lambda_3\right)}{(-\lambda_4 + \lambda_1)(-\lambda_4 + \lambda_2)(-\lambda_4 + \lambda_3)}.
\end{aligned}$$

On the other hand, we know the relations between roots and coeffcients for $p(x)$ such that

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = a \tag{4.17}$$

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = 0 \tag{4.18}$$

$$\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4 = c \tag{4.19}$$

$$\lambda_1\lambda_2\lambda_3\lambda_4 = 1. \tag{4.20}$$

By the way,

$$\begin{aligned}
\lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 &= \lambda_1(\lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4)\frac{1}{\lambda_1}\\
&\overset{(4.19)}{=} (c - \lambda_2\lambda_3\lambda_4)\frac{1}{\lambda_1}\\
&\overset{(4.20)}{=} \left(c - \frac{1}{\lambda_1}\right)\frac{1}{\lambda_1}.
\end{aligned} \tag{4.21}$$

Thus we have

$$\lambda_1^3\left(\lambda_2\lambda_3\lambda_4 c^2 - (\lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4)\,c + \lambda_2 + \lambda_3 + \lambda_4\right)$$

$$\overset{(4.17),(4.20),(4.21)}{=} \lambda_1^3 \left(\frac{1}{\lambda_1} c^2 - \left(c - \frac{1}{\lambda_1} \right) \frac{1}{\lambda_1} c + (a - \lambda_1) \right)$$
$$= \quad \lambda_1^3 \cdot \frac{1}{\lambda_1^3} = 1 \,.$$

Therefore, $x_{11} = -\frac{1}{(-\lambda_1+\lambda_2)(-\lambda_1+\lambda_3)(-\lambda_1+\lambda_4)}$. By Lemma 4.4, if $a \geq 5$, it is clear that $x_{11} > 0$. For x_{j1}, $j = 2, 3, 4$, we can discuss analogously. We get the proof for (2) analogously. □

LEMMA 4.6. *For $p(x) = x^4 - ax^3 - cx + 1, a, c \in \mathbb{Z}, c \in \{-a-1, -a, -a+1\}$, the following properties hold*:

(1) *If $a \geq 5$, then $\{\pi_e \boldsymbol{e}_i\}_{1 \leq i \leq 4}$ satisfies the good star property and the proto-tiles set V_e is given by*

$$V_e = \{1 \wedge 2,\ 1 \wedge 3,\ 1 \wedge 4,\ 2 \wedge 3,\ 2 \wedge 4,\ 3 \wedge 4\}$$

called $V(0)$ in the table in the section 4.5. *Moreover*, $E_2(\sigma)$ *has the POP property. On the other hand, $\{\pi_c \boldsymbol{e}_i\}_{1 \leq i \leq 4}$ satisfies the good star property and the proto-tile set V_c is given by*

$$V_c = \{2 \wedge 1,\ 1 \wedge 3,\ 4 \wedge 1,\ 3 \wedge 2,\ 2 \wedge 4,\ 4 \wedge 3\}$$

called $V(2)$ in the table in the section 4.5. *Moreover*, $E_2(\theta)$ *has the POP property.*

(2) *If $a \leq -5$, then $\{\pi_e \boldsymbol{e}_i\}_{1 \leq i \leq 4}$ satisfies the good star property and the proto-tiles set V_e is given by*

$$V_e = \{2 \wedge 1,\ 1 \wedge 3,\ 4 \wedge 1,\ 3 \wedge 2,\ 2 \wedge 4,\ 4 \wedge 3\}$$

called $V(2)$ in the table in the section 4.5. *Moreover*, $E_2(\sigma)$ *has the POP property. On the other hand, we see that $\{\pi_c \boldsymbol{e}_i\}_{1 \leq i \leq 4}$ satisfies the good star property and the proto-tiles set V_c is given by*

$$V_c = \{1 \wedge 2,\ 1 \wedge 3,\ 1 \wedge 4,\ 2 \wedge 3,\ 2 \wedge 4,\ 3 \wedge 4\}$$

called $V(0)$ in the section 4.5. *Moreover*, $E_2(\theta)$ *has the POP property.*

Proof. For (1). From Lemma 4.4, $\boldsymbol{v}_i$ $(1 \leq i \leq 4)$ of (4.15) are the eigenvectors themselves and we know that $P_e = \mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)$ and $P_c = \mathcal{L}(\boldsymbol{v}_3, \boldsymbol{v}_4)$. From the notation of the inverse matrix of $[\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4]$, we have $\pi_e \boldsymbol{e}_1 = x_{11}\boldsymbol{v}_1 + x_{21}\boldsymbol{v}_2$ and we know that

$$\begin{aligned}
\pi_e \boldsymbol{e}_2 &= \pi_e A \boldsymbol{e}_1 = \lambda_1 x_{11} \boldsymbol{v}_1 + \lambda_2 x_{21} \boldsymbol{v}_2 \\
\pi_e \boldsymbol{e}_3 &= \pi_e A \boldsymbol{e}_2 = \lambda_1^2 x_{11} \boldsymbol{v}_1 + \lambda_2^2 x_{21} \boldsymbol{v}_2 \\
\pi_e \boldsymbol{e}_4 &= \pi_e A \boldsymbol{e}_3 = \lambda_1^3 x_{11} \boldsymbol{v}_1 + \lambda_2^3 x_{21} \boldsymbol{v}_2 \,.
\end{aligned}$$

Therefore, $\{\pi_e \boldsymbol{e}_i\}_{1 \leq i \leq 4}$ satisfies the good star property since $1 < \lambda_2 < \lambda_1$ and $x_{11} \geq 0$, $x_{21} \leq 0$ by Lemma 4.5. So, we obtain Figure 18 and we see that the proto-tiles set is given by

$$V_e = \{1 \wedge 2,\ 1 \wedge 3,\ 1 \wedge 4,\ 2 \wedge 3,\ 2 \wedge 4,\ 3 \wedge 4\} = V(0) \,.$$

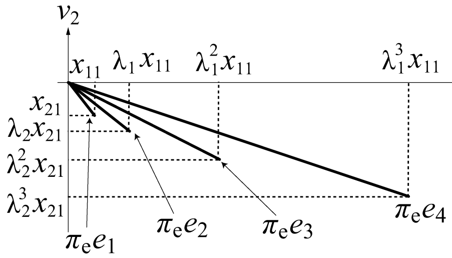


FIGURE 18. The image of $\{\pi_e \boldsymbol{e}_i\}_{1\le i\le 4}$in $a \ge 5$.

Now let us operate the 2-dimensional extension $E_2(\sigma)$ to the proto-tiles of V_e, then we obtain the following:

$$
\begin{aligned}
E_2(\sigma)(\mathbf{0}, 1\wedge 2) &= (\mathbf{0}, \sigma(1)\wedge\sigma(2)) = (\mathbf{0}, 2\wedge 3)\\
E_2(\sigma)(\mathbf{0}, 1\wedge 3) &= (\mathbf{0}, \sigma(1)\wedge\sigma(3)) = (\mathbf{0}, 2\wedge 4)\\
E_2(\sigma)(\mathbf{0}, 1\wedge 4) &= \left(\sum_{k=1}^{a}(c\boldsymbol{e}_2 + (k-1)\boldsymbol{e}_4, 2\wedge 4)\right) + (c\boldsymbol{e}_2 + a\boldsymbol{e}_4 - \boldsymbol{e}_1, 1\wedge 2)\\
E_2(\sigma)(\mathbf{0}, 2\wedge 3) &= (\mathbf{0}, \sigma(2)\wedge\sigma(3)) = (\mathbf{0}, 3\wedge 4)\\
E_2(\sigma)(\mathbf{0}, 2\wedge 4) &= \left(\sum_{k=1}^{-c}(-k\boldsymbol{e}_2, 2\wedge 3)\right) + \left(\sum_{k=1}^{a}(c\boldsymbol{e}_2 + (k-1)\boldsymbol{e}_4, 3\wedge 4)\right)\\
&\quad + (c\boldsymbol{e}_2 + a\boldsymbol{e}_4 - \boldsymbol{e}_1, 1\wedge 3)\\
E_2(\sigma)(\mathbf{0}, 3\wedge 4) &= \left(\sum_{k=1}^{-c}(-k\boldsymbol{e}_2, 2\wedge 4)\right) + (c\boldsymbol{e}_2 + a\boldsymbol{e}_4 - \boldsymbol{e}_1, 1\wedge 4)\ .
\end{aligned}
$$

Therefore, we see that $E_2(\sigma)$ has the POP property.

Now let us consider on P_c, that is, we have $\pi_c\boldsymbol{e}_1 = x_{31}\boldsymbol{v}_3 + x_{41}\boldsymbol{v}_4$ and we know that

$$
\begin{aligned}
\pi_c\boldsymbol{e}_2 &= \pi_c A\boldsymbol{e}_1 = \lambda_3 x_{31}\boldsymbol{v}_3 + \lambda_4 x_{41}\boldsymbol{v}_4\\
\pi_c\boldsymbol{e}_3 &= \pi_c A\boldsymbol{e}_2 = \lambda_3^2 x_{31}\boldsymbol{v}_3 + \lambda_4^2 x_{41}\boldsymbol{v}_4\\
\pi_c\boldsymbol{e}_4 &= \pi_c A\boldsymbol{e}_3 = \lambda_3^3 x_{31}\boldsymbol{v}_3 + \lambda_4^3 x_{41}\boldsymbol{v}_4\ .
\end{aligned}
$$

Therefore $\{\pi_c\boldsymbol{e}_i\}_{1\le i\le 4}$ also satisfies the good star property since $-1 < \lambda_3 < \lambda_4 < 0$ and $x_{31} \le 0$, $x_{41} \ge 0$ by Lemma 4.5. So, we obtain Figure 19 and the proto-tiles set is given by

$$V_c = \{2\wedge 1,\ 1\wedge 3,\ 4\wedge 1,\ 3\wedge 2,\ 2\wedge 4,\ 4\wedge 3\}\ (= V(2))\ .$$

Operating the 2-dimensional extension $E_2(\theta)$ to the proto-tiles of V_c, then we obtain the following:

$$
\begin{aligned}
E_2(\theta)(\mathbf{0}, 2\wedge 1) = (\mathbf{0}, \theta(2)\wedge\theta(1)) &= \left(\sum_{k=1}^{a}((k-1)\boldsymbol{e}_3, 1\wedge 3)\right)\\
&\quad + (a\boldsymbol{e}_3 + c\boldsymbol{e}_1 - \boldsymbol{e}_4, 4\wedge 1)
\end{aligned}
$$

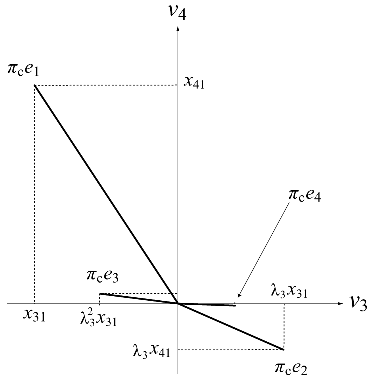


FIGURE 19. The image of $\{\pi_c \boldsymbol{e}_i\}_{1\le i\le 4}$.

$$\begin{aligned}
E_2(\theta)(\mathbf{0}, 1\wedge 3) &= \left(\sum_{k=1}^{a-1}((k-1)\,\boldsymbol{e}_3, 3\wedge 2)\right) + \left(\sum_{k=1}^{-c}(a\boldsymbol{e}_3 - k\boldsymbol{e}_1, 2\wedge 1)\right)\\
&\quad + (a\boldsymbol{e}_3 + c\boldsymbol{e}_1 - \boldsymbol{e}_4, 2\wedge 4)\\
E_2(\theta)(\mathbf{0}, 4\wedge 1) &= \left(\sum_{k=1}^{-c}(a\boldsymbol{e}_3 - k\boldsymbol{e}_1, 1\wedge 3)\right) + (a\boldsymbol{e}_3 + c\boldsymbol{e}_1 - \boldsymbol{e}_4, 4\wedge 3)\\
E_2(\theta)(\mathbf{0}, 3\wedge 2) &= (\mathbf{0}, \theta(3)\wedge\theta(2)) = (\mathbf{0}, 2\wedge 1)\\
E_2(\theta)(\mathbf{0}, 2\wedge 4) &= (\mathbf{0}, \theta(2)\wedge\theta(4)) = (\mathbf{0}, 1\wedge 3)\\
E_2(\theta)(\mathbf{0}, 4\wedge 3) &= (\mathbf{0}, \theta(4)\wedge\theta(3)) = (\mathbf{0}, 3\wedge 2)\,.
\end{aligned}$$

Therefore, we see that $E_2(\theta)$ has the POP property.

For the case of (2) $a \le -5$, we get the conclusion analogously. □

Proof of Theorem 4.2. The first part (1) is obtained by Lemma 4.6 in the case $a \le -5$, $a \ge 5$, and for each a $(-5 \le a \le 5)$, we can check that the proto-tiles set V_e, V_c and the fact that $E_2(\sigma)$ has the POP property explicitly (see the table in the section 4.5). The second part (2), mentioned that the family of compact sets $\{\gamma_{i\wedge j,e}\}$, $i \wedge j \in V_e$ (resp. $\{\gamma_{i\wedge j,c}\}$, $i\wedge j \in V_c$) satisfies (N1), (N2), (N3) of the complex Pisot numeration system property, can be obtained by the analogous proof of Theorem 3.31. □

4.4. Example of the complex Pisot numeration system

EXAMPLE 4.7. Let us consider the minimal polynomial

$$p(x) = x^4 - x^3 + 1\,,$$

then its companion matix A is given by

$$A = \begin{bmatrix} 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1 \end{bmatrix}$$

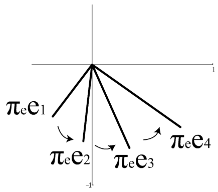


FIGURE 20.

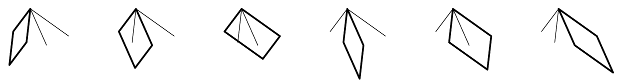
which is corresponding to $(a, c) = (1, 0)$ in Theorem 4.2.

The eigenvalues of A satisfy

$$|\lambda_1| = |\lambda_2| > 1 > |\lambda_3| = |\lambda_4|$$

and the distributions $\{\pi_e \boldsymbol{e}_i\}_{1 \leq i \leq 4}$ is as Figure 20. The proto-tiles set V_e is chosen as

$$V_e = \{1 \wedge 2,\ 1 \wedge 3,\ 1 \wedge 4,\ 2 \wedge 3,\ 2 \wedge 4,\ 3 \wedge 4\}$$

$= \Big\{$  $\Big\}$.

Let us define the automorphism σ of $F\langle 1, 2, 3, 4\rangle$ by

$$\sigma : \begin{cases} 1 \to 2 \\ 2 \to 3 \\ 3 \to 4 \\ 4 \to 41^{-1} \end{cases}$$

and using the automorphism σ, we see that the 2-dimentional extension $E_2(\sigma)$ from the positive orientated face to the "patch" of faces is given by

$$E_2(\sigma)(\mathbf{0}, 1 \wedge 2) = (\mathbf{0}, \sigma(1) \wedge \sigma(2)) = (\mathbf{0}, 2 \wedge 3)$$

$$E_2(\sigma)(\mathbf{0}, 1 \wedge 3) = (\mathbf{0}, 2 \wedge 4)$$

$$E_2(\sigma)(\mathbf{0}, 1 \wedge 4) = \left(\mathbf{0}, 2 \wedge 41^{-1}\right) = (\mathbf{0}, 2 \wedge 4) + \left(f(4), 2 \wedge 1^{-1}\right)$$

$$\overset{(*)}{=} (\mathbf{0}, 2 \wedge 4) + ((\boldsymbol{e}_4 - \boldsymbol{e}_1), 1 \wedge 2)$$

$$E_2(\sigma)(\mathbf{0}, 2 \wedge 3) = (\mathbf{0}, 3 \wedge 4)$$

$$E_2(\sigma)(\mathbf{0}, 2 \wedge 4) = \left(\mathbf{0}, 3 \wedge 41^{-1}\right) \overset{(*)}{=} (\mathbf{0}, 3 \wedge 4) + ((\boldsymbol{e}_4 - \boldsymbol{e}_1), 1 \wedge 3)$$

$$E_2(\sigma)(\mathbf{0}, 3 \wedge 4) = \left(\mathbf{0}, 4 \wedge 41^{-1}\right) = (\mathbf{0}, 4 \wedge 4) + \left(f(4), 4 \wedge 1^{-1}\right) \overset{(*)}{=} ((\boldsymbol{e}_4 - \boldsymbol{e}_1), 1 \wedge 4)$$

where $(*)$ is the rearrangement which is introduced in the section 3. Then, we see that $E_2(\sigma)$ has the POP-property (see Figure 21).

Starting $(-\boldsymbol{e}_3, 3 \wedge 4)$, we see that

$$E_2(\sigma)^3(-\boldsymbol{e}_3, 3 \wedge 4) \succ \pi_e(-\boldsymbol{e}_3, 3 \wedge 4),$$

moreover, let

$$T_e = \left\{\pi_e(\boldsymbol{x}, i \wedge j) \,\middle|\, \pi_e(\boldsymbol{x}, i \wedge j) \in \pi_e E_2(\sigma)^{3n}(-\boldsymbol{e}_3, 3 \wedge 4) \text{ for some } n \in \mathbb{N}\right\},$$

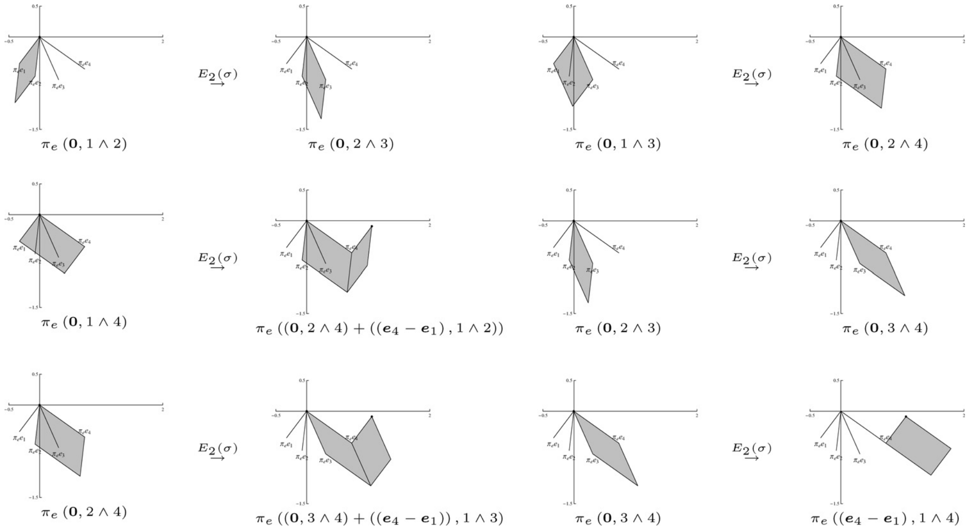


FIGURE 21. $\pi_e(\mathbf{0}, i \wedge j)$ and $\pi_e E_2(\sigma)(\mathbf{0}, i \wedge j)$, $i \wedge j \in V_e$.

then $\mathscr{T}_e = \bigcup_{\pi_e(z, i\wedge j) \in T_e} \pi_e(z, i \wedge j)$ is the quasi-periodic tiling of P_e (see Figure 22).

Now we can find the octagonal patch $\mathcal{U}_e$ satisfying $E_2(\sigma)^3 \mathcal{U}_e \succ \mathcal{U}_e$ (see Figure 23):

$$\mathcal{U}_e = \begin{array}{l} ((-\boldsymbol{e}_3 - \boldsymbol{e}_1 - \boldsymbol{e}_2), 1 \wedge 2) + ((\boldsymbol{e}_4 - \boldsymbol{e}_1 - \boldsymbol{e}_3), 1 \wedge 3) + ((-\boldsymbol{e}_3 - \boldsymbol{e}_1), 1 \wedge 4) \\ + ((\boldsymbol{e}_4 - \boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_3), 2 \wedge 3) + ((-\boldsymbol{e}_3 - \boldsymbol{e}_1 - \boldsymbol{e}_2), 2 \wedge 4) + (-\boldsymbol{e}_3, 3 \wedge 4) \end{array} .$$

Let us define

$$\gamma_{i\wedge j} := \lim_{n\to\infty} A^{-n} \pi_e E_2(\sigma)^n \left(\boldsymbol{x}_{i\wedge j}, i \wedge j\right) \quad \text{for} \quad \left(\boldsymbol{x}_{i\wedge j}, i \wedge j\right) \in \mathcal{U}_e.$$

Then, the family of the compact sets $\left\{\mathrm{cl}\left(\mathrm{int}\left(\gamma_{i\wedge j}\right)\right)\right\}_{i\wedge j \in V_e}$ has the following set equations:

$$\begin{aligned}
A\gamma_{1\wedge 2} &= \gamma_{2\wedge 3} + \pi_e(-2\boldsymbol{e}_4 + \boldsymbol{e}_1) \\
A\gamma_{1\wedge 3} &= \gamma_{2\wedge 4} + \pi_e \boldsymbol{e}_3 \\
A\gamma_{1\wedge 4} &= (\gamma_{2\wedge 4} + \pi_e(\boldsymbol{e}_3 + \boldsymbol{e}_1 - \boldsymbol{e}_4)) \cup (\gamma_{1\wedge 2} + \pi_e \boldsymbol{e}_3) \\
A\gamma_{2\wedge 3} &= \gamma_{3\wedge 4} + \pi_e(\boldsymbol{e}_1 + \boldsymbol{e}_2) \\
A\gamma_{2\wedge 4} &= (\gamma_{3\wedge 4} + \pi_e(-\boldsymbol{e}_2 - \boldsymbol{e}_4)) \cup (\gamma_{1\wedge 3} + \pi_e(-\boldsymbol{e}_2 - \boldsymbol{e}_4)) \\
A\gamma_{3\wedge 4} &= \gamma_{1\wedge 4} + \pi_e \boldsymbol{e}_3
\end{aligned}$$

(see Figure 23).

Moreover, we can see that

$$\mathrm{cl}\left(\mathrm{int}\left(\gamma_{i\wedge j}\right)\right) = \gamma_{i\wedge j}, \quad \mu_e\left(\partial\gamma_{i\wedge j}\right) = 0, \quad \text{and} \quad \gamma = \bigcup_{i\wedge j \in V_e} \gamma_{i\wedge j} \text{ is disjoint},$$

therefore, we see that $(A, \mathcal{P})$, $\mathcal{P} = \left\{\gamma_{i\wedge j}\right\}_{i\wedge j \in V_e}$ is the complex Pisot numeration system.

The labeled graph $(V, E, i, t, \mathcal{L})$ is given by Figure 24.

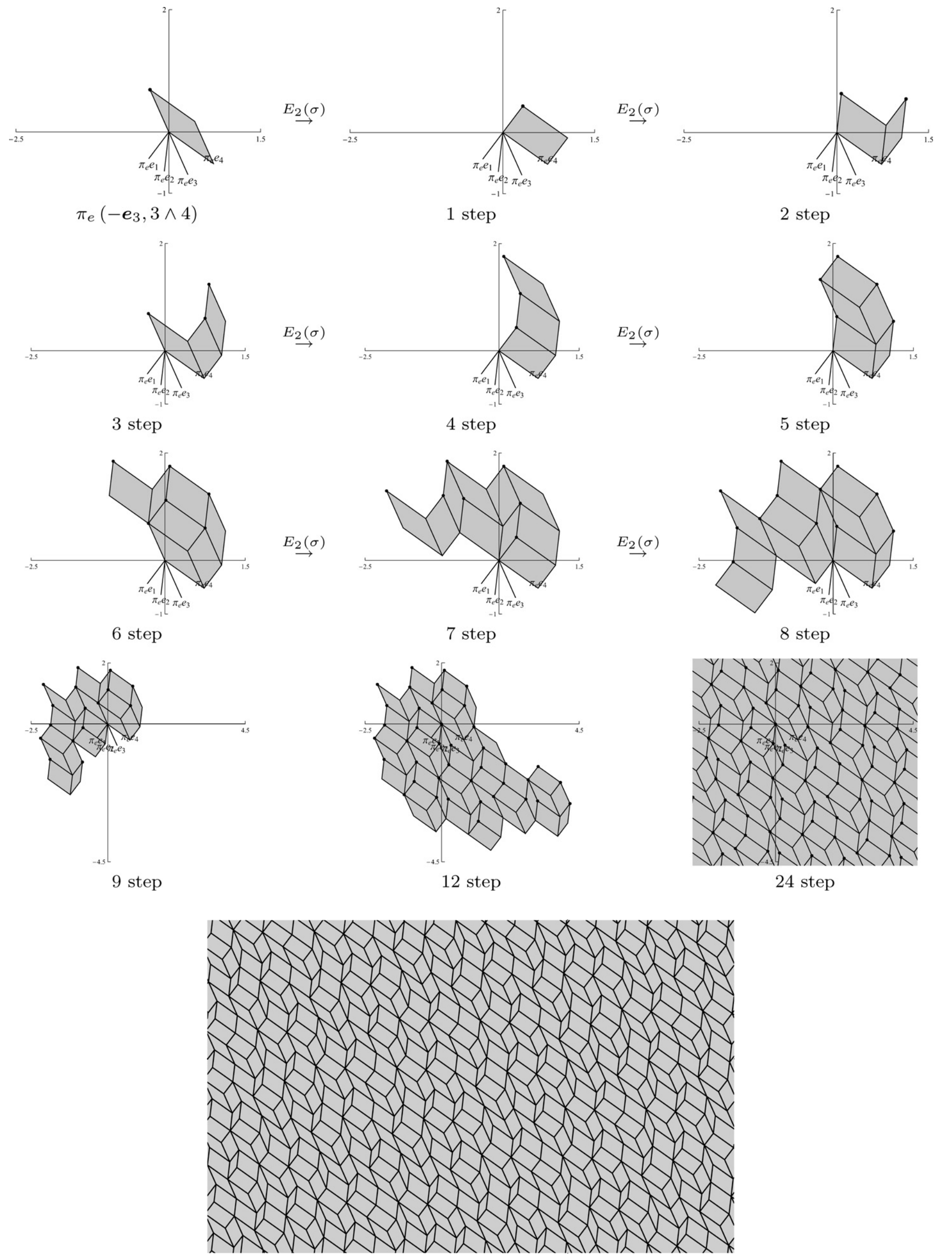
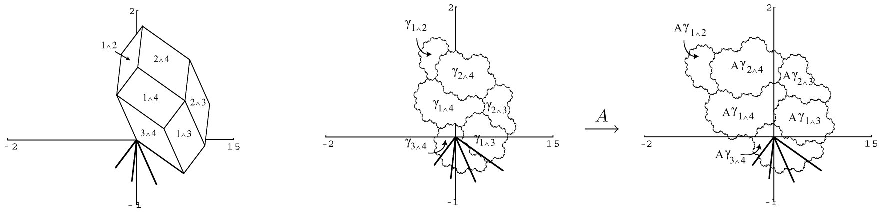


FIGURE 22. The quasi-periodic tiling $\mathscr{T}_e$.



$\mathcal{U}_e$ $\quad\quad$ $\bigcup_{i\wedge j\in V_e}\gamma_{i\wedge j}$ $\quad\quad$ $\bigcup_{i\wedge j\in V_e} A\gamma_{i\wedge j}$

FIGURE 23.

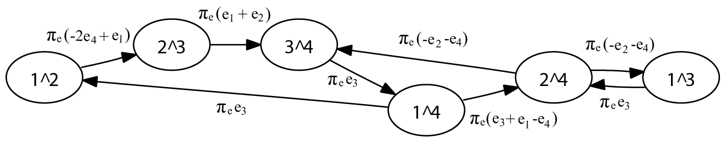


FIGURE 24. The graph $(V, E, i, t, \mathcal{L})$.

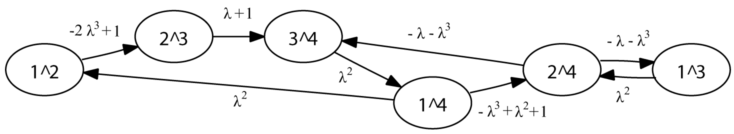


FIGURE 25. The graph $(V, E, i, t, (\phi\mathcal{L}))$.

From the fact that

$$\phi\left(\pi_e \boldsymbol{e}_1\right) = 1\,, \quad \phi\left(\pi_e \boldsymbol{e}_2\right) = \lambda\,, \quad \phi\left(\pi_e \boldsymbol{e}_3\right) = \lambda^2\,, \quad \phi\left(\pi_e \boldsymbol{e}_4\right) = \lambda^3\,,$$

the labeled graph $(V, E, i, t, (\phi\mathcal{L}))$ is given by Figure 25.

Therefore, let $\Omega_{i\wedge j}$ be the labeled admissible sequence space which is starting from the vertex $i \wedge j$ by the labeled graph $(V, E, i, t, (\phi\mathcal{L}))$ and its element be $(a_1, a_2, \ldots)$, $a_i \in \left\{-2\lambda^3+1, \lambda^2, -\lambda^3+\lambda^2+1, \lambda+1, -\lambda^3-\lambda\right\}$, then $z \in \phi\left(\bigcup_{i\wedge j\in V_e}\gamma_{i\wedge j}\right)$ is represented by $z = \sum_{n=1}^{\infty} a_{n-1}\lambda^{-n}$ where $a_n = \phi_e\left(\pi_e \boldsymbol{f}_{k_{n-1}}^{(j_{n-1})}\right)$.

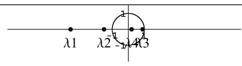
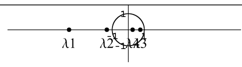
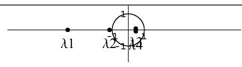
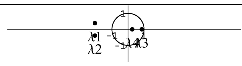
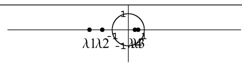
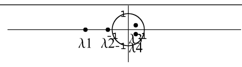
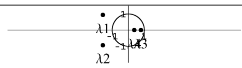
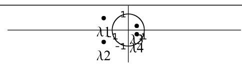
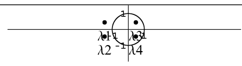
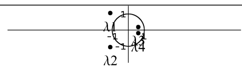
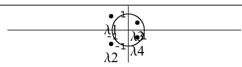
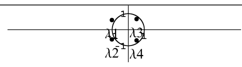
4.5. Appendix: The table

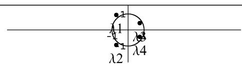
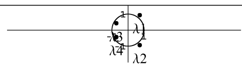
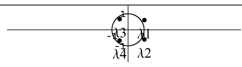
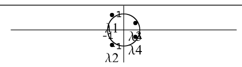
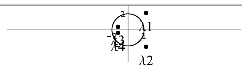
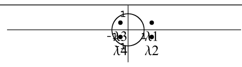
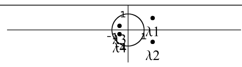
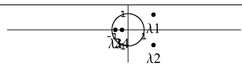
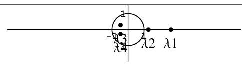
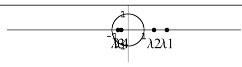
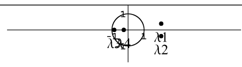
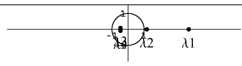
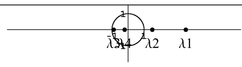
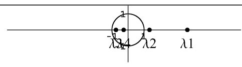
Finally, we will show the table how the eigenvalues λ_i are distributed depending on $a, c \in \mathbb{Z}$. Notation on the table is as follows:

(1) "Comp" ("resp. Real") means the complex (resp. real) number respectively.

(2) $V(i)$, $i = 0, 1, 2$ are the set of the proto-tiles such that

$$V(0) = \{1\wedge 2,\ 1\wedge 3,\ 1\wedge 4,\ 2\wedge 3,\ 2\wedge 4,\ 3\wedge 4\}$$
$$V(1) = \{1\wedge 2,\ 3\wedge 1,\ 1\wedge 4,\ 2\wedge 3,\ 4\wedge 2,\ 3\wedge 4\}$$
$$V(2) = \{2\wedge 1,\ 1\wedge 3,\ 4\wedge 1,\ 3\wedge 2,\ 2\wedge 4,\ 4\wedge 3\}\ .$$

a	c	$x^4 - ax^3 - cx + 1$	$\lambda_1,\ \lambda_2$	$\lambda_3,\ \lambda_4$	Distribution of λ_i	V_e	V_c
$a \leq -5$			Real	Real		$V(2)$	$V(0)$
-4	5	$x^4 + 4x^3 - 5x + 1$	Real	Real		$V(2)$	$V(0)$
-4	4	$x^4 + 4x^3 - 4x + 1$	Real	Real		$V(2)$	$V(0)$
-4	3	$x^4 + 4x^3 - 3x + 1$	Real	Comp		$V(2)$	$V(0)$
-3	4	$x^4 + 3x^3 - 4x + 1$	Comp	Real		$V(1)$	$V(0)$
-3	3	$(x^2 + 2x - 1)$ $(x^2 + x - 1)$	Real	Real		$V(2)$	$V(0)$
-3	2	$x^4 + 3x^3 - 2x + 1$	Real	Comp		$V(2)$	$V(0)$
-2	3	$x^4 + 2x^3 - 3x + 1$	Comp	Real		$V(1)$	$V(0)$
-2	2	$x^4 + 2x^3 - 2x + 1$	Comp	Comp		$V(1)$	$V(0)$
-2	1	$x^4 + 2x^3 - x + 1$	Comp	Comp		$V(1)$	$V(0)$
-1	2	$x^4 + x^3 - 2x + 1$	Comp	Comp		$V(1)$	$V(0)$
-1	1	$x^4 + x^3 - x + 1$	Comp	Comp		$V(1)$	$V(0)$
-1	0	$x^4 + x^3 + 1$	Comp	Comp		$V(1)$	$V(0)$

a	c	$x^4 - ax^3 - cx + 1$	$\lambda_1,\ \lambda_2$	$\lambda_3,\ \lambda_4$	Distribution of λ_i	V_e	V_c
0	1	$x^4 - x + 1$	Comp	Comp		$V(1)$	$V(0)$
0	-1	$x^4 + x + 1$	Comp	Comp		$V(0)$	$V(1)$
1	0	$x^4 - x^3 + 1$	Comp	Comp		$V(0)$	$V(1)$
1	-1	$x^4 - x^3 + x + 1$	Comp	Comp		$V(0)$	$V(1)$
1	-2	$x^4 - x^3 + 2x + 1$	Comp	Comp		$V(0)$	$V(1)$
2	-1	$x^4 - 2x^3 + x + 1$	Comp	Comp		$V(0)$	$V(1)$
2	-2	$x^4 - 2x^3 + 2x + 1$	Comp	Comp		$V(0)$	$V(1)$
2	-3	$x^4 - 2x^3 + 3x + 1$	Comp	Real		$V(0)$	$V(2)$
3	-2	$x^4 - 3x^3 + 2x + 1$	Real	Comp		$V(0)$	$V(1)$
3	-3	$(x^2 - x - 1)$ $(x^2 - 2x - 1)$	Real	Real		$V(0)$	$V(2)$
3	-4	$x^4 - 3x^3 + 4x + 1$	Comp	Real		$V(0)$	$V(2)$
4	-3	$x^4 - 4x^3 + 3x + 1$	Real	Comp		$V(0)$	$V(1)$
4	-5	$x^4 - 4x^3 + 5x + 1$	Real	Real		$V(0)$	$V(2)$
4	-4	$x^4 - 4x^3 + 4x + 1$	Real	Real		$V(0)$	$V(2)$
$a \geq 5$			Real	Real		$V(0)$	$V(2)$

On other pairs (a, c), $c = -a - 1, -a, -a - 1$, $\{\lambda_i\}_{1 \leq i \leq 4}$ are totally real.

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