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On a Mean Value Theorem for the Second Moment of the Riemann Zeta-Function[1]

by

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Dedicated to Professor Akio Fujii on the occasion of his happy retirement

Abstract. Let $E(T)$ be the error term in the mean square formula for the Riemann zeta-function on the critical line. In this paper, a smooth-weighted mean value formula for $E(T)^2$ over the interval $[0, P]$ is obtained in which the error term is $O(P\log^2 P)$. As a corollary, it is proved that the classical mean-value formula for $E(T)^2$ over $[0, P]$ has an error term which is $\Omega_-(P\log^2 P\log\log P)$.

1. Introduction and statement of results

A central problem in classical analytic number theory concerns the $2k$-th moments

$$I_k(T) = \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt$$

of the Riemann zeta-function on the critical line. Evaluation of $I_k(T)$ is a notoriously difficult problem and asymptotic formula for $I_k(T)$ has been obtained only for $k = 1$ and 2. Recent developments on random matrix theory have led to many exciting conjectures on the form of the main term for $I_k(T)$.

Let

$$E(T) = \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt - T\left(\log\frac{T}{2\pi}+2\gamma-1\right)$$

be the error term in the formula for $I_1(T)$. Hardy-Littlewood [2] first proved that $E(T) = o(T\log T)$, and Ingham [6] improved this to $E(T) \ll T^{3/4+\varepsilon}$. This bound has been gradually sharpened by many authors in the last eighty years. However, the best result to-date of Huxley [4], [5] that $E(T) \ll T^{131/416}$ is still a long way from the conjectured best bound $E(T) \ll_\varepsilon T^{\frac{1}{4}+\varepsilon}$.

On the other hand, Heath-Brown [3] applied a classical formula of Atkinson [1] to prove that

$$\int_0^P E(T)^2 dT = cP^{3/2} + O(P^{5/4}\log^2 P) \tag{1.1}$$

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where $c = \frac{2}{3}(2\pi)^{-1/2}\sum_{n=1}^{\infty} d(n)^2 n^{-3/2}$ and $d(n)$ is the divisor function. The rather large O-term in Atkinson's formula does not allow much improvement in the above O-term in (1.1).

Let

$$F(T) = \int_0^T E(t)^2 dt - cT^{3/2}.$$

In [9] Meurman developed a smoothened version of Atkinson's formula with a much sharper error term (see §2 (2.1)) and thereby obtained the improved estimate $F(P) \ll P\log^5 P$ in(1.1). Subsequently Meurman's bound has been further sharpened to $P\log^4 P$ and $P\log^3 P\log\log P$, by Pressimann [11] and by Lau-Tsang [8] respectively. Further improvement on this, to $P\log^3 P$, say, would be difficult and would require some novel techniques.

Estimations for $F(T)$ are also related to bounds for $E(t)$ and $\zeta(1/2+it)$. Indeed one can deduce an upper bound for $E(t)$ from a bound for $F(T)$ as follows.

First we notice that for $z > y > 0$,

$$\begin{aligned} E(z) - E(y) &= I_1(z) - I_1(y) - (z-y)\left(\log\frac{v}{2\pi} + 2\gamma\right) \quad \text{for some } v \in [y, z] \\ &\geq -2(z-y)\log z. \end{aligned}$$

Thus, $E(t)$ can only decrease slowly, at a rate $\ll \log t$.

Now suppose $T > 0$ is large and let

$$M = \max_{T/2<t\leq T} |E(t)| = |E(\tau)| \quad \text{for a } \tau \in [T/2, T].$$

Obviously, we may assume that $M > T^{1/4+\epsilon}$. If $E(\tau)$ is positive, then

$$E(t) \geq \frac{1}{2}E(\tau) = \frac{1}{2}M \quad \text{for } t \in \left[\tau, \tau + \frac{M}{6\log T}\right].$$

Similar argument works in the case that $E(\tau)$ is negative. Thus, writing $w_\pm = \tau \pm \frac{M}{6\log T}$, we have

$$\frac{M^2}{8}(w_+ - w_-) \leq \int_{w_-}^{w_+} E(t)^2\, dt \ll (w_+ - w_-)\sqrt{T} + F(w_+) - F(w_-)$$

and we deduce from this

$$F(T) \ll T^b \log^c T \Longrightarrow M \ll T^{\frac{d}{3}} \log^{\frac{c+1}{3}} T.$$

It is also possible to get an upper bound for $\zeta(\frac{1}{2}+it)$ from $E(t)$. Indeed, by an inequality of Heath-Brown,

$$\begin{aligned} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 &\ll \log t \int_{t-\log^2 t}^{t+\log^2 t} \left|\zeta\left(\frac{1}{2}+iu\right)\right|^2 du + \log t \\ &= \log t\,(I_1(t+\log^2 t) - I_1(t-\log^2 t)) + \log t \end{aligned}$$

$$= (\log t) O(\log^3 t) + (\log t)(E(t + \log^2 t) - E(t - \log^2 t)) + \log t\,.$$

Hence,

$$E(T) \ll T^d \log^e T \Longrightarrow \zeta\left(\frac{1}{2} + it\right) \ll t^{\frac{d}{2}} \log^{\frac{e+1}{2}} t\,.$$

Given the enormous difficulties in reducing the bound for $F(T)$ further, we are prompted to consider adding a smooth weight to the mean square of $E(T)$, in the hope of getting a sharper asymptotic formula. In the paper, we shall prove the following main result.

THEOREM 1. *Suppose the weight function $\omega(t)$ is continuous with piecewise continuous and bounded derivative. Furthermore assume $\omega(t)$ is supported on $\left[\frac{1}{4}, 1\right]$ with $\omega(1/4) = \omega(1) = 0$. Then*

$$\int_0^P \omega\left(\frac{T}{P}\right) E(T)^2 dT = \frac{3c}{2}\left(\int_0^1 \sqrt{t}\,\omega(t)\,dt\right) P^{3/2} - 6\pi^{-2}\left(\int_0^1 \omega(t)dt\right) P \log^2 P \log\log P + O(P \log^2 P)\,. \tag{1.2}$$

As an immediate consequence we deduce fron Theorem 1 the following.

THEOREM 2. *We have*

$$\int_0^P F(T)dT = -3\pi^{-2} P^2 \log^2 P \log\log P + O(P^2 \log^2 P)\,.$$

In particular,

$$F(T) = \Omega_-(T \log^2 T \log\log T)\,.$$

REMARKS 1. The dominance of the main term over the error term in Theorem 2 is very thin, by only $\log\log P$. It is therefore crucial to suppress the error term estimates in our argument to $O(P \log^2 P)$ and the key for the success is that ω is continuous with $\omega(1/4) = \omega(1) = 0$.

2. Let

$$\Delta(x) = \sum_{n \le x} d(n) - x \log x - (2\gamma - 1)x$$

be the error term in the dirichlet divisor problem. This is a well-known companion of $E(T)$ and they share many similar properties. However, Lau-Tsang [7] has proved that

$$\int_0^P \omega\left(\frac{x}{P}\right) \Delta(x)\,dx = c_1\left(\int_0^1 \sqrt{t}\,\omega(t)\,dt\right) P^{3/2} - \left(8\pi^2\right)^{-1}\left(\int_0^1 \omega(t)\,dt\right) P \log^2 P + c_2 P \log P + O(P)$$

for some constanst c_1 and c_2. The second main terms in this and (1.2) are of different orders of magnitude and thus a fundamental difference between $\Delta(x)$ and $E(T)$ is exhibited. This appears to be the first result of such a nature in the literature.

2. Notation and some preliminary results

Throughout this paper $P \geq 100$ is our main parameter and we set $U = P^{1/4}$. The variable x always lies in $\left[\frac{1}{2}\sqrt{2\pi P}, \sqrt{2\pi P}\,\right]$ and hence x is of order $\sqrt{P}$. Integrations with respect to x are over $\left[\frac{1}{2}\sqrt{2\pi P}, \sqrt{2\pi P}\,\right]$ or its sub-intervals. The letters h, m, n denote positive integers $\ll P$ whereas ε denotes an arbitrarily small positive number. We will also invoke freely the well-known upper bound $d(n) \ll_\varepsilon n^\varepsilon$.

The formula we use for $E(T)$ is the following smoothened version of Atkinson's formula in [1] as developed by Meurman [9]:

For $x \in \left[\frac{1}{2}\sqrt{2\pi P}, \sqrt{2\pi P}\right]$ and $a \asymp \sqrt{P}$, we have

$$\sqrt{\frac{\pi}{x}}E\left(\frac{x^2}{2\pi}\right) = \sum\nolimits_1 - \sum\nolimits_2 + \sqrt{\frac{\pi^3}{x}} + O\left(\frac{\log x}{x}\right) \tag{2.1}$$

where

$$\sum\nolimits_1 = \sum_{n \leq (a+U)^2} (-1)^n \eta_n d(n) n^{-3/4} e_n \cos f_n\,, \tag{2.2}$$

$$\sum\nolimits_2 = \sqrt{\frac{\pi}{x}} \sum_{n \leq Z(x,a)} \xi_n d(n) n^{-1/2} \ell_n^{-1} \cos g_n\,,$$

$$\eta_n = \eta_n(a) = 1 - \max\left(0, \ \frac{\sqrt{n} - a}{U}\right),$$

$$e_n = e_n(x) = \left(1 + \frac{\pi^2 n}{x^2}\right)^{-1/4} \left(\frac{x}{\pi\sqrt{n}} \operatorname{arsinh} \frac{\pi\sqrt{n}}{x}\right)^{-1},$$

$$f_n = f_n(x) = \frac{x^2}{\pi} \operatorname{arsinh} \frac{\pi\sqrt{n}}{x} + \sqrt{\pi^2 n^2 + n x^2} - \frac{\pi}{4}\,,$$

$$g_n = g_n(x) = \frac{x^2}{\pi} \log \frac{x}{2\pi\sqrt{n}} - \frac{x^2}{2\pi} + \frac{\pi}{4}\,,$$

$$\ell_n = \ell_n(x) = \log \frac{x}{2\pi\sqrt{n}}\,,$$

$$\operatorname{arsinh} z = \log(z + \sqrt{1 + z^2})\,,$$

$$\xi_n = \xi_n(x, a) = \max\left\{\min\left(1, \frac{1}{U}\left(\left(\frac{x}{2\pi}\right)^2 \frac{1}{\sqrt{n}} - \sqrt{n} - a\right)\right), \ 0\right\} \tag{2.3}$$

$$Z(x, u) = \left(\sqrt{\left(\frac{x}{2\pi}\right)^2 + \frac{u^2}{4}} - \frac{u}{2}\right)^2 . \tag{2.4}$$

Let

$$\mathcal{N}(n,a) = 2\pi\sqrt{n + a\sqrt{n}}\,.$$

Then $n \le Z(x,a)$ if and only if $\mathcal{N}(n,a) \le x$.

We first collect a list of estimates for these functions that will be used in the sequel.

LEMMA 1. *Let $x \in \left[\frac{1}{2}\sqrt{2\pi P}, \sqrt{2\pi P}\,\right]$ and $m < n \ll P$. We have the following.*

(i) $0 \le \eta_n,\ \ \xi_n \le 1,\ 0 < e_n,\ \ \ell_n^{-1} \ll 1.$

(ii) $e_n'(x) \ll nx^{-3},\ \left(\ell_n^{-1}\right)' \ll x^{-1}.$

(iii) $f_n'(x) = \dfrac{2x}{\pi}\operatorname{arsinh}\dfrac{\pi\sqrt{n}}{x},\ \ g_n'(x) = \dfrac{2x}{\pi}\log\dfrac{x}{2\pi\sqrt{n}}.$

(iv) $\xi_n'(x,a) = x\left(2\pi^2 U\sqrt{n}\right)^{-1}$ *for* $\mathcal{N}(n,a) \le x \le \mathcal{N}(n,a+U)$ *and* $\xi_n'(x,a) = 0$ *otherwise.*

(v) $f_n'(x) - f_m'(x) \gg \sqrt{n} - \sqrt{m},\ \ \left(f_n''(x) - f_m''(x)\right)\left(f_n'(x) - f_m'(x)\right)^{-1} \ll nx^{-3};$
$g_n'(x) - g_m'(x) = -\dfrac{x}{\pi}\log\dfrac{n}{m},\ \ \left(g_n''(x) - g_m''(x)\right)\left(g_n'(x) - g_m'(x)\right)^{-1} = x^{-1}.$

(vi) $Z(x,a) \asymp P$ *and* $0 < Z(x,a) - Z(x,a+U) \asymp Ux.$

Proof. These estimates are straightforward from the respective definitions. For instance, for $n \le Z(x,a)$,

$$\frac{x}{2\pi\sqrt{n}} \ge \sqrt{1 + \left(\frac{\pi a}{x}\right)^2} + \frac{\pi a}{x}$$

and hence $\ell_n(x)^{-1} \ll 1$.

LEMMA 2. *Let $y \ge 1$. Then uniformly for $1 \le h \le y^{\frac{15}{14}-\varepsilon}$, we have*

$$\Psi_h(y) = \sum_{m \le y} d(m)d(m+h) = 6\pi^{-2}\int_0^{y/h} m(u;h)\,du + O(y^{3/4})\,.$$

Here

$$\begin{aligned} m(u;h) &= \sigma(h)\log u\log(u+1) \\ &\quad + \left\{\sigma(h)(2\gamma - 2\frac{\zeta'}{\zeta}(2) - \log h) + 2\sigma'(h)\right\}\log(u(u+1)) \\ &\quad + \sigma(h)\left\{(2\gamma - 2\frac{\zeta'}{\zeta}(2) - \log h)^2 - 4\left(\frac{\zeta'}{\zeta}\right)'(2)\right\} \\ &\quad + 4\sigma'(h)\left(2\gamma - 2\frac{\zeta'}{\zeta}(2) - \log h\right) + 4\sigma''(h) \end{aligned} \tag{2.5}$$

and

$$\sigma(h) = \sum_{d|h} d,\ \sigma'(h) = \sum_{d|h} d\log d,\ \sigma''(h) = \sum_{d|h} d\log^2 d. \tag{2.6}$$

Proof. This is adapted from Theorem 1 of [10].

LEMMA 3. *Suppose* $1/2 \le y < z$ *and* $1 \le h \le z^{\frac{15}{14}-\varepsilon}$. *Then*

$$\sum_{y<m\le z} d(m)d(m+h) \ll \frac{\sigma(h)}{h}(z-y)\log^2 z + z^{3/4}\,.$$

Proof. If $y > z/2$, then by Lemma 2

$$\sum_{y<m\le z} d(m)d(m+h) = 6\pi^{-2}\int_{y/h}^{z/h} m(u;h)du + O(z^{3/4})$$

$$\ll \frac{\sigma(h)}{h}(z-y)\log^2 z + z^{3/4}\,.$$

If $y \le z/2$, by Lemma 2 again,

$$\sum_{y<m\le z} d(m)d(m+h) \le \sum_{m\le z} d(m)d(m+h) \ll \frac{\sigma(h)}{h} z\log^2 z + z^{3/4}\,.$$

LEMMA 4. *Let* $y \ge 1/2$. *Then*

$$\Psi_0(y) := \sum_{n\le y} d(n)^2 = \pi^{-2} y\log^3 y + O(y\log^2 y)\,.$$

Proof. This is a well-known result.

LEMMA 5. *For* $y > 1/2$, *we have*

(i) $\displaystyle\sum_{n>y} d(n)^2 n^{-3/2} = 2\pi^{-2} y^{-1/2}\log^3 y + O(y^{-1/2}\log^2 y)$.

(ii) $\displaystyle\sum_{m<n\le y} \frac{d(m)d(n)}{m^\alpha n^\beta (n-m)^2} \ll y^{1-\alpha-\beta}\log^2 y$ *for* $\alpha < 1$ *and* $\alpha+\beta < 1$.

Proof. (i) follows from Lemma 4 by partial summation.

(ii) First, the part of the sum in which $m \le n/2$ is clearly $\ll y^{1-\alpha-\beta}$. For $\frac{n}{2} < m < n \le y$, write $h = n-m$. Then this part of the sum is

$$\ll \sum_{h\le y} h^{-2} \sum_{m\le y} \frac{d(m)d(m+h)}{m^{\alpha+\beta}}\,.$$

By Lemma 3 and partial summation, the inner sum over m is

$$\ll \frac{\sigma(h)}{h} y^{1-\alpha-\beta}\log^2 y.$$

Summation of this over h then leads to the bound in (ii).

LEMMA 6. *Let* $H_1(t), H_2(t), \ldots, H_r(t)$ *be piecewise monotonic functions defined on an interval* I *and let* $F(t)$ *be a real differentiable function such that* $F'(t)$ *is monotonic with* $|F'(t)| \ge m > 0$ *for* $t \in I$. *Then*

$$\left|\int_I H_1(t)H_2(t)\cdots H_r(t)e^{iF(t)}dt\right| \le 4m^{-1}\prod_{i=1}^{r}\max_{t\in I}|H_i(t)|\,.$$

Proof. This is Lemma 2 in [3].

It is technically more convenient to work with $\sqrt{\frac{\pi}{x}}E\big(\frac{x^2}{2\pi}\big)$ instead of $E(T)$. By a simple change of variable

$$\int_0^P \omega\Big(\frac{T}{P}\Big)E(T)^2dT = \frac{P}{\pi^2}\int_0^{\sqrt{2\pi P}} \gamma(x)\left(\sqrt{\frac{\pi}{x}}E\Big(\frac{x^2}{2\pi}\Big)\right)^2 dx$$

where

$$\gamma(x) = \frac{x^2}{P}\omega\Big(\frac{x^2}{2\pi P}\Big), \tag{2.7}$$

which is supported on $[\frac{1}{2}\sqrt{2\pi P}, \sqrt{2\pi P}]$ and $\gamma\big(\frac{1}{2}\sqrt{2\pi P}\big) = \gamma(\sqrt{2\pi P}) = 0$. Furthermore, it is easily verified that

$$\gamma(x) \ll 1,\ \gamma'(x) \ll xP^{-1} \quad \text{and} \quad \int \gamma(x)dx \ll \sqrt{P}\,. \tag{2.8}$$

Then by (2.1),

$$\frac{\pi^2}{P}\int_0^P \omega\Big(\frac{T}{P}\Big)E(T)^2\,dT = I_1 + I_2 - 2I_3 + 2\pi^{3/2}I_4 - 2\pi^{3/2}I_5 + O(\log P)\,, \tag{2.9}$$

where

$$I_1 = \int_0^{\sqrt{2\pi P}} \gamma(x)\sum\nolimits_1^2 dx\,, \tag{2.10}$$

$$I_2 = \int_0^{\sqrt{2\pi P}} \gamma(x)\sum\nolimits_2^2 dx\,, \tag{2.11}$$

$$I_3 = \int_0^{\sqrt{2\pi P}} \gamma(x)\sum\nolimits_1\sum\nolimits_2 dx\,,$$

$$I_4 = \int_0^{\sqrt{2\pi P}} \frac{\gamma(x)}{\sqrt{x}}\sum\nolimits_1 dx\,,$$

$$I_5 = \int_0^{\sqrt{2\pi P}} \frac{\gamma(x)}{\sqrt{x}}\sum\nolimits_2 dx\,.$$

The O-term in (2.9) encompasses five terms, including $\int x^{-1}|\gamma(x)|\log x|\sum_i|dx$ for $i = 1, 2$. We bound these by applying Cauchy-Schwarz's inequality together with (2.8) and the bounds $\int|\gamma(x)|\sum_1^2 dx \ll \sqrt{P}$ and $\int|\gamma(x)|\sum_2^2 dx \ll \log^3 P$, which we shall establish in Lemmas 7 and 8 respectively. We shall estimate I_1 and I_2 asymptotically in §§3, 4 and bound I_3, I_4 and I_5 in §§5, 6. In the course of our estimations, we can allow O-terms only up to the order of $\log^2 P$.

3. The integral I_1

LEMMA 7. *We have*

$$I_1 = \sqrt{\frac{9\pi}{8}c}\int_0^{\sqrt{2\pi P}} \gamma(x)dx - \frac{1}{\pi}\log^3(a+U)^2 \times \int_0^{\sqrt{2\pi P}} \frac{\gamma(x)}{x}\left(\operatorname{arsinh}\frac{\pi(a+U)}{x}\right)^{-1} dx + O(\log^2 P)\,. \tag{3.1}$$

In the following, all integrations with respect to x are over the interval $\left[0, \sqrt{2\pi P}\right]$. But since $\gamma(x)$ is supported on $\left[\frac{1}{2}\sqrt{2\pi P}, \sqrt{2\pi P}\,\right]$, the lower limit of integration is indeed $\frac{1}{2}\sqrt{2\pi P}$ and x is of order $\sqrt{P}$.

From (2.2) and (2.10), by squaring the sum $\sum_1$ and then interchanging the integration and summations, we can write

$$I_1 = \frac{1}{2}S_{11} + S_{12}^{-} + \frac{1}{2}S_{12}^{+} \tag{3.2}$$

where

$$S_{11} = \sum_{n\le(a+U)^2} \eta_n^2 d(n)^2 n^{-3/2}\int e_n^2\gamma(x)dx\,, \tag{3.3}$$

$$S_{12}^{-} = \sum_{m<n\le(a+U)^2} (-1)^{n+m}\eta_m\eta_n d(m)d(n)(mn)^{-3/4}\int e_m e_n \gamma(x)\cos(f_n - f_m)dx \tag{3.4}$$

and

$$S_{12}^{+} = \sum_{m,n\le(a+U)^2} (-1)^{n+m}\eta_m\eta_n d(m)d(n)(mn)^{-3/4}\int e_m e_n \gamma(x)\cos(f_n + f_m)dx\,,$$

corresponding to the diagonal terms and the cross terms. The two main terms in (3.1) come from S_{11}, and $S_{12}^{\pm}$ will be bounded by $\log^2 P$.

The function η_n equals to 1 for $n \le a^2$ and then tapers to 0 at $n = (a+U)^2$. If we change η_n to 1 for $a^2 < n \le (a+U)^2$, the error in S_{11} thus induced is

$$\ll \sum_{a^2<n\le(a+U)^2} d(n)^2 n^{-3/2}\int \gamma(x)dx \ll \sqrt{P}\sum_{a^2<n\le(a+U)^2} n^{\varepsilon-3/2} \ll P^{-1/4+\varepsilon}\,,$$

which is acceptable. So, writing for brevity $A = (a+U)^2$ which is of order P, we have from (3.3)

$$\begin{aligned} S_{11} &= \sum_{n\le A} d(n)^2 n^{-3/2}\int e_n^2\gamma(x)dx + O(P^{-1/4+\varepsilon}) \\ &= \sum_{n\le A} d(n)^2 n^{-3/2}\int \gamma(x)dx - \sum_{n\le A} d(n)^2 n^{-3/2}\int (1-e_n^2)\gamma(x)dx + O(P^{-1/4+\varepsilon})\,. \end{aligned}$$

By Lemma 5,

$$\sum_{n\le A} d(n)^2 n^{-3/2} = \sqrt{\frac{9\pi}{2}}c - \frac{2}{\pi^2}A^{-1/2}\log^3 A + O(P^{-1/2}\log^2 P)\,.$$

Thus, on noting that $\int \gamma(x)dx \ll \sqrt{P}$,

$$S_{11} = \sqrt{\frac{9\pi}{2}}c\int \gamma(x)dx - 2\pi^{-2}A^{-1/2}\log^3 A\int \gamma(x)dx$$
$$- \sum_{n\le A} d(n)^2 n^{-3/2}\int (1-e_n^2)\gamma(x)dx + O(\log^2 P)\,. \tag{3.5}$$

For the last sum, we use Stieltjes integration to obtain

$$\sum_{n\le A} d(n)^2 n^{-3/2}\int (1-e_n^2)\gamma(x)dx = \int\left(\int_0^A u^{-3/2}(1-e_u^2)d\Psi_0(u)\right)\gamma(x)dx\,. \tag{3.6}$$

By Lemma 4, the inner integral is equal to

$$\pi^{-2}\int_0^A u^{-3/2}(1-e_u^2)(\log^3 u + 3\log^2 u)du$$
$$+ u^{-3/2}(1-e_u^2)O(u\log^2 u)\Big|_0^A - \int_0^A O(u\log^2 u)\frac{d}{du}\big(u^{-3/2}(1-e_u^2)\big)du\,.$$

Since $1-e_u^2 \ll ux^{-2}$ and $\frac{d}{du}\big(u^{-3/2}(1-e_u^2)\big) \ll u^{-3/2}x^{-2}$, we find that

$$\int_0^A u^{-3/2}(1-e_u^2)d\Psi_0(u)$$
$$= \pi^{-2}\int_0^A u^{-3/2}(1-e_u^2)\log^3 u\,du + O(A^{1/2}x^{-2}\log^2 A)$$
$$= \frac{2}{\pi x}\int_0^A \log^3 u\frac{d}{du}\left(\frac{-x}{\pi\sqrt{u}} + \left(\operatorname{arsinh}\frac{\pi\sqrt{u}}{x}\right)^{-1}\right)du + O(\sqrt{P}x^{-2}\log^2 P)$$
$$= \left\{\frac{-2}{\pi^2\sqrt{A}} + \frac{2}{\pi x}\left(\operatorname{arsinh}\frac{\pi\sqrt{A}}{x}\right)^{-1}\right\}\log^3 A + O(P^{-1/2}\log^2 P)\,.$$

Putting this back to (3.6) then yields

$$\sum_{n\le A} d(n)^2 n^{-3/2}\int (1-e_n^2)\gamma(x)dx = \frac{-2}{\pi^2\sqrt{A}}\log^3 A\int \gamma(x)dx$$
$$+ \frac{2}{\pi}\log^3 A\int \frac{\gamma(x)}{x}\left(\operatorname{arsinh}\frac{\pi\sqrt{A}}{x}\right)^{-1}dx$$
$$+ O(\log^2 P)\,.$$

Hence from (3.5), we deduce that

$$S_{11} = \sqrt{\frac{9\pi}{2}}c \int \gamma(x)dx - \frac{2}{\pi}\log^3 A \int \frac{\gamma(x)}{x}\left(\operatorname{arsinh}\frac{\pi\sqrt{A}}{x}\right)^{-1} dx + O(\log^2 P)\,. \quad (3.7)$$

We now handle $S_{12}^{\pm}$, the sums involving the cross terms, and proceed to show that $S_{12}^{\pm} \ll \log^2 P$.

Applying integration by parts once and noting that $\gamma(x)$ vanishes at the upper and lower integration limits, we find that the integral inside the double sum in (3.4) is

$$\int e_m e_n \gamma(x)\cos(f_n - f_m)dx = -\int \frac{d}{dx}\left\{\frac{e_m e_n \gamma(x)}{f_n' - f_m'}\right\}\sin(f_n - f_m)dx\,. \quad (3.8)$$

The derivative inside the integral is equal to

$$\left(e_m' e_n \gamma + e_m e_n' \gamma + e_m e_n \gamma' - e_m e_n \gamma \frac{f_n'' - f_m''}{f_n' - f_m'}\right)(f_n' - f_m')^{-1}$$

which is $\ll P^{-1/2}(\sqrt{n} - \sqrt{m})^{-1}$, by Lemma 1 and (2.8). Hence applying Lemma 6, the integral in (3.8) is

$$\ll P^{-1/2}(\sqrt{n} - \sqrt{m})^{-1} \max_x |f_n' - f_m'|^{-1} \ll P^{-1/2}(\sqrt{n} - \sqrt{m})^{-2}\,.$$

Thus,

$$\begin{aligned} S_{12}^{-} &\ll P^{-1/2} \sum_{m<n\le A} d(m)d(n)(mn)^{-3/4}(\sqrt{n} - \sqrt{m})^{-2} \\ &\ll P^{-1/2} \sum_{m<n\le A} d(m)d(n)m^{-3/4}n^{1/4}(n-m)^{-2} \\ &\ll \log^2 P\,, \end{aligned}$$

by Lemma 5 (ii) with $\alpha = 3/4$, $\beta = -1/4$.

The estimation of S_{12}^{+} is easier and has the same bound. Combining these, (3.7) and (3.2), we complete the proof of Lemma 7.

4. The integral I_2

LEMMA 8. *We have*

$$\begin{aligned} I_2 = {} & \pi^{-1}\log^3 P \int_0^{\sqrt{2\pi P}} \frac{\gamma(x)}{x}\left(\operatorname{arsinh}\frac{\pi(a+U)}{x}\right)^{-1} dx \\ & -6\pi^{-1}\left(\int_{\frac{1}{2}\sqrt{2\pi P}}^{\sqrt{2\pi P}} \frac{\gamma(x)}{x}dx\right)\log^2 P \log\log P + O(\log^2 P)\,. \end{aligned}$$

The argument of proof of Lemma 8 is along the same line as Lemma 7. From (2.11) we have

$$I_2 = \frac{1}{2}S_{21} + S_{22}^{-} + \frac{1}{2}S_{22}^{+} \quad (4.1)$$

where

$$S_{21} = \pi \int \Bigg(\sum_{n \le Z(x,a)} d(n)^2 n^{-1} \ell_n^{-2} \xi_n^2 \Bigg) x^{-1} \gamma(x) dx \,, \tag{4.2}$$

$$S_{22}^{-} = \pi \sum_{m<n \le Z(\sqrt{2\pi P},a)} d(m)d(n)(mn)^{-1/2}$$

$$\times \int_{\mathcal{N}(n,a)}^{\sqrt{2\pi P}} (\ell_m \ell_n)^{-1} \xi_m \xi_n x^{-1} \gamma(x) \cos(g_n - g_m) dx \,, \tag{4.3}$$

$$S_{22}^{+} = \pi \sum_{m,n \le Z(\sqrt{2\pi P},a)} d(m)d(n)(mn)^{-1/2}$$

$$\times \int_{\mathcal{N}(\max(n,m),a)}^{\sqrt{2\pi P}} (\ell_m \ell_n)^{-1} \xi_m \xi_n x^{-1} \gamma(x) \cos(g_n + g_m) dx \,.$$

The two main terms in I_2 comes from the diagonal terms in S_{21}, which is quite straightforward to estimate. The bounding of S_{22}^{-}, however, is a lot more difficult than S_{12}^{-} in §3.

First we can shorten the sum inside S_{21} to $\sum\limits_{n \le Z(x,a+U)}$ with an error

$$\ll \int \sum_{Z(x,a+U)<n \le Z(x,a)} d(n)^2 n^{-1} x^{-1} |\gamma(x)| dx$$

$$\ll P^{\varepsilon} U \sqrt{P} P^{-1} \int x^{-1} |\gamma(x)| dx \ll P^{-1/4+\varepsilon} \,,$$

by the observation that $Z(x,a) - Z(x,a+U) \ll U\sqrt{P}$ in Lemma 1 (vi) and $U = P^{1/4}$.

For $n \le Z(x, a+U)$, $\xi_n = 1$ and we are led to evaluating the sum

$$\sum_{n \le Z(x,a+U)} d(n)^2 n^{-1} \ell_n^{-2}$$

inside S_{21}. By Lemma 4 and Stieltjes integration,

$$\begin{aligned}
&\sum_{n \le Z(x,a+U)} d(n)^2 n^{-1} \ell_n^{-2} \\
&= \int_{1^-}^{Z(x,a+U)} u^{-1} \ell_u(x)^{-2} d\Psi_0(u) \\
&= \pi^{-2} \int_1^{Z(x,a+U)} u^{-1} \ell_u(x)^{-2} (\log^3 u + 3\log^2 u) du \\
&\quad - \int_1^{Z(x,a+U)} O(u \log^2 u) \frac{d}{du}\big(u^{-1}\ell_u(x)^{-2}\big) du + O(\log^2 P) \\
&= \frac{16}{\pi^2} \log^3 \frac{x}{2\pi} \Bigg(\log \frac{x}{2\pi\sqrt{Z(x,a+U)}} \Bigg)^{-1} - \frac{48}{\pi^2} \log^2 \frac{x}{2\pi} \log\log \frac{x}{2\pi} + O(\log^2 P) \\
&= \frac{16}{\pi^2} \log^3 \frac{x}{2\pi} \Bigg(\operatorname{arsinh} \frac{\pi(a+U)}{x} \Bigg)^{-1} - \frac{48}{\pi^2} \log^2 \frac{x}{2\pi} \log\log \frac{x}{2\pi} + O(\log^2 P) \,.
\end{aligned}$$

Then from (4.2), we get

$$S_{21} = \frac{16}{\pi}\int \Big(\log^3 \frac{x}{2\pi}\Big)\Big(\operatorname{arsinh}\frac{\pi(a+U)}{x}\Big)^{-1}\frac{\gamma(x)}{x}dx - \frac{48}{\pi}\int \Big(\log^2 \frac{x}{2\pi}\log\log\frac{x}{2\pi}\Big)\frac{\gamma(x)}{x}dx + O(\log^2 P)\,. \tag{4.4}$$

Define

$$\mathcal{H}_0(y) = \int_{\frac{1}{2}\sqrt{2\pi P}}^{y}\Big(\operatorname{arsinh}\frac{\pi(a+U)}{x}\Big)^{-1}\frac{\gamma(x)}{x}dx \ \ for\ \ y \in \Big[\frac{1}{2}\sqrt{2\pi P}, \sqrt{2\pi P}\Big].$$

Then the first term on the right hand side of (4.4) is equal to

$$\frac{16}{\pi}\mathcal{H}_0(\sqrt{2\pi P})\log^3\sqrt{\frac{P}{2\pi}} - 3\int \frac{\mathcal{H}_0(x)}{x}\log^2\frac{x}{2\pi}dx = \frac{2}{\pi}\mathcal{H}_0(\sqrt{2\pi P})\log^3\frac{P}{2\pi} + O(\log^2 P)\,,$$

since $\mathcal{H}_0(y) \ll \log y - \log\frac{1}{2}\sqrt{2\pi P} \ll 1$.

Define

$$\mathcal{H}_1(y) = \int_{\frac{1}{2}\sqrt{2\pi P}}^{y} x^{-1}\gamma(x)dx \quad \text{for} \quad y \in \Big[\frac{1}{2}\sqrt{2\pi P}, \sqrt{2\pi P}\Big].$$

The second term on the right hand side of (4.4) is equal to

$$\begin{aligned}&-\frac{48}{\pi}\int \log^2\frac{x}{2\pi}\log\log\frac{x}{2\pi}d\mathcal{H}_1(x)\\ &= -\frac{48}{\pi}\mathcal{H}_1(\sqrt{2\pi P})\log^2\sqrt{\frac{P}{2\pi}}\log\log\sqrt{\frac{P}{2\pi}} + \frac{48}{\pi}\int \mathcal{H}_1(x)\frac{d}{dx}\Big\{\log^2\frac{x}{2\pi}\log\log\frac{x}{2\pi}\Big\}dx\\ &= -12\pi^{-1}\mathcal{H}_1(\sqrt{2\pi P})\log^2 P\log\log P + O(\log^2 P)\,.\end{aligned}$$

Putting these back to (4.4), we deduce that

$$S_{21} = \frac{2}{\pi}\mathcal{H}_0(\sqrt{2\pi P})\log^3\frac{P}{2\pi} - \frac{12}{\pi}\mathcal{H}_1(\sqrt{2\pi P})\log^2 P\log\log P + O(\log^2 P)\,. \tag{4.5}$$

We come now to the estimation of the cross terms in S_{22}^{-}. If we follow the same argument of S_{12}^{-}, we would obtain the bound $O(\log^3 P)$, which just misses the target by a factor of $\log P$. In order to save a factor of $\log P$, we need to utilize the oscillation of the sine function; more precisely, we shall use

$$\sum_{h=1}^{P}\frac{\sin\alpha h}{h} \ll 1$$

instead of

$$\sum_{h=1}^{P} \frac{|\sin \alpha h|}{h} \ll \log P$$

to obtain the necessary saving. Thus the bounding of S_{22}^{-} is more delicate than that of S_{12}^{-}.

First using an integration by parts for the integral in (4.3), and noting that $\xi_n(x)\gamma(x)$ vanishes at both the upper and lower limits of the integration, we have

$$\int_{\mathcal{N}(n,a)}^{\sqrt{2\pi P}} \xi_m \xi_n (\ell_m \ell_n)^{-1} x^{-1} \gamma(x) \cos(g_n - g_m) dx$$

$$= -\int_{\mathcal{N}(n,a)}^{\sqrt{2\pi P}} \sin(g_n - g_m) \frac{d}{dx} \left\{ \xi_m \xi_n (\ell_m \ell_n)^{-1} x^{-1} \gamma(x) (g_n' - g_m')^{-1} \right\} dx \,. \tag{4.6}$$

The derivative inside the integral is equal to

$$\pi \big(\xi_m \xi_n\big)' \left(x^2 \ell_m \ell_n \log \frac{m}{n} \right)^{-1} \gamma(x) + \pi \xi_m \xi_n \frac{d}{dx} \left\{ \left(x^2 \ell_m \ell_n \log \frac{m}{n} \right)^{-1} \gamma(x) \right\} .$$

By the estimates in Lemma 1 and (2.8), the second term here is $\ll P^{-3/2} \log \frac{n}{m}$. When this is substituted back to the integral in (4.6) and on applying Lemma 6, we get a term $O\big(P^{-2} \big| \log \frac{m}{n} \big|^{-2}\big)$. The contribution of this to S_{22}^{-} is

$$\ll P^{-2} \sum_{m<n\le Z(\sqrt{2\pi P},a)} d(m)d(n)(mn)^{-1/2} \left(\log \frac{m}{n} \right)^{-2}$$

$$\ll P^{-2} \sum_{n\le Z(\sqrt{2\pi P},a)} d(m) n^{-1/2} \sum_{m\le n/2} d(m) m^{-1/2} + P^{-2} \sum_{\substack{n\le Z(\sqrt{2\pi P},a) \\ n/2<m<n}} \frac{d(m)d(n)m}{(n-m)^2}$$

$$\ll \log^2 P \,,$$

by Lemma 5 (ii). Thus

$$S_{22}^{-} = -\pi^2 \int \left\{ \sum_{m<n\le Z(x,a)} \frac{d(m)d(n)}{\sqrt{mn}} \left(\log \frac{n}{m} \right)^{-1} \frac{\big(\xi_m \xi_n\big)'}{\ell_m \ell_n} \sin \left(\frac{x^2}{2\pi} \log \frac{n}{m} \right) \right\} \frac{\gamma(x)}{x^2} dx$$
$$+ O(\log^2 P) \,.$$

Denote the double sum inside the above integral by $\sigma(x)$, that is

$$\sigma(x) = \sum_{m<n\le Z(x,a)} \frac{d(m)d(n)}{\sqrt{mn}} \left(\log \frac{n}{m} \right)^{-1} \frac{\big(\xi_m \xi_n\big)'}{\ell_m \ell_n} \sin \left(\frac{x^2}{2\pi} \log \frac{n}{m} \right)$$

$$= \sum_{\substack{m<n\le Z(x,a) \\ n-m\le U}} + \sum_{\substack{m<n\le Z(x,a) \\ n-m>U}} = \sigma_1(x) + \sigma_2(x) \,, \quad \text{say.}$$

In the estimation of $\sigma_1(x)$, we make use of the fact that $\xi_n'(x) \neq 0$ if and only if $Z(x, a+U) < n < Z(x,a)$. Furthermore, from Lemma 1 (vi), $Z(x,a) - Z(x,a+U) \asymp$

$Ux \asymp U\sqrt{P}$. Hence $\sigma_1(x)$ is really a short sum over a range of length $\asymp U\sqrt{P}$ and m, n are of order P. Write $h = n - m \le U$. Then first order approximations give

$$\left(\sqrt{mn}\log\frac{n}{m}\ell_m\ell_n\right)^{-1}\sin\left(\frac{x^2}{2\pi}\log\frac{n}{m}\right) = h^{-1}\ell_n^{-2}\left(\sin\frac{x^2h}{2\pi n} + O\left(\frac{h^2}{n}\right)\right)$$
$$= h^{-1}\ell_n^{-2}\sin\frac{x^2h}{2\pi n} + O\big(hP^{-1}\big). \qquad (4.7)$$

For $m < n \le Z(z, a)$, careful scrutiny of the definition of ξ_n in (2.3) shows that

(i) $\big(\xi_m\xi_n\big)' = \xi_m'\xi_n + \xi_m\xi_n' = 0$ for $n > Z(x, a)$ or $n < Z(x, a+U)$; (4.8)

(ii) $$\Big(\xi_m\xi_n\Big)' = \frac{d}{dx}\Big\{U^{-2}\Big(n^{-1/2}\Big(\frac{x}{2\pi}\Big)^2 - \sqrt{n} - a\Big)\Big(m^{-1/2}\Big(\frac{x}{2\pi}\Big)^2 - \sqrt{m} - a\Big)\Big\}$$
$$= -(\pi U)^{-2}x\left(1 + \frac{a}{\sqrt{n}} - \frac{x^2}{4\pi^2 n}\right) + O\big(hP^{-1}\big) \qquad (4.9)$$

for $Z(x, a+U) + U < n \le Z(x, a)$;

(iii) $\big(\xi_m\xi_n\big)' \le \big|\xi_m'\big| + \big|\xi_n'\big| \ll \dfrac{x}{U\sqrt{m}}$ for $Z(x, a+U) < n \le Z(x, a)$. (4.10)

In view of (4.8), we now further split the sum $\sigma_1(x)$ into

$$\sigma_1(x) = \sum_{h\le U}\sum_{\substack{Z(x,a+U)+U<n\le Z(x,a)\\ m=n-h}} + \sum_{h\le U}\sum_{\substack{Z(x,a+U)<n\le Z(x,a+U)+U\\ m=n-h}} = \sigma_{11}(x) + \sigma_{12}(x), \quad \text{say.}$$

Estimating crudely by invoking Lemma 1 (i) and (4.10), we have

$$\sigma_{12}(x) \ll \sum_{h\le U}\sum_{Z(x,a+U)<n<Z(x,a+U)+U}\frac{d(n-h)d(n)}{n}\left(\frac{h}{n}\right)^{-1}U^{-1} \ll P^{\varepsilon}. \qquad (4.11)$$

For $\sigma_{11}(x)$, we use (4.7) and (4.9) to deduce that

$$\sigma_{11}(x) = -\frac{x}{\pi^2U^2}\sum_{h\le U}h^{-1}\sum_{Z(x,a+U)+U<n\le Z(x,a)}$$
$$\times d(n-h)d(n)\left(1 + \frac{a}{\sqrt{n}} - \frac{x^2}{4\pi^2 n}\right)\ell_n^{-2}\sin\frac{x^2h}{2\pi n}$$
$$+ O\left(P^{-1}\sum_{h\le U}h\sum_{n\sim P}d(n-h)d(n)U^{-1}\right) + O\left(P^{-1}\sum_{h\le U}\sum_{n\sim P}d(n-h)d(n)\right)$$
$$= -\frac{x}{\pi^2U^2}\sum_{h\le U}h^{-1}\int_{Z(x,a+U)+U}^{Z(x,a)}\left(1 + \frac{a}{\sqrt{u}} - \frac{x^2}{4\pi^2 u}\right)\ell_u^{-2}\sin\frac{x^2h}{2\pi u}\,d\Psi_h(u) + O(UP^{\varepsilon}).$$

By Lemma 2, the integral inside the summation is equal to

$$\frac{6}{\pi^2}h^{-1}\int_{Z(x,a+U)+U}^{Z(x,a)}\left(1 + \frac{a}{\sqrt{u}} - \frac{x^2}{4\pi^2 u}\right)\ell_u^{-2}\sin\frac{x^2h}{2\pi u}m\left(\frac{u}{h}; h\right)du$$

$$+\,O\left(u^{3/4}\left(1+\frac{a}{\sqrt{u}}-\frac{x^2}{4\pi^2 u}\right)\Big|_{Z(x,a+U)+U}^{Z(x,a)}\right)$$
$$-\int_{Z(x,a+U)+U}^{Z(x,a)} O\big(u^{3/4}\big)\frac{d}{du}\left\{\left(1+\frac{a}{\sqrt{u}}-\frac{x^2}{4\pi^2 u}\right)\ell_u^{-2}\sin\frac{x^2 h}{2\pi u}\right\}du\,.$$

The last two terms are $\ll \sqrt{P}$ since $Z(x,a)-Z(x,a+U)\ll U\sqrt{P}=P^{3/4}$, and

$$\left(1+\frac{a}{\sqrt{u}}-\frac{x^2}{4\pi^2 u}\right)=u^{-1}(\sqrt{u}-\sqrt{Z(x,a)})(\sqrt{u}+a+\sqrt{Z(x,a)})$$
$$\ll P^{-1}(u-Z(x,a))\ll P^{-1}\sqrt{P}U=P^{-1/4}\,. \tag{4.12}$$

Thus,

$$\sigma_{11}(x)=\frac{-6x}{\pi^4U^2}\sum_{h\le U}h^{-2}\int_{Z(x,a+U)+U}^{Z(x,a)}\left(1+\frac{a}{\sqrt{u}}-\frac{x^2}{4x^2 u}\right)\ell_u^{-2}\sin\frac{x^2h}{2\pi u}m\left(\frac{u}{h};h\right)du$$
$$+\,O(\sqrt{P}\log P)\,. \tag{4.13}$$

We claim

$$\sum_{h\le U}\sin\left(\frac{x^2}{2\pi u}h\right)h^{-2}m\left(\frac{u}{h};h\right)\ll \log^2 P\,. \tag{4.14}$$

Note that if we disregard the oscillation of the sine function and use the trivial bound $\left|\sin\frac{x^2h}{2\pi u}\right|\le 1$, the above sum would be $\ll \log^3 P$, which would miss our target by a factor of $\log P$.

Write $\beta=\dfrac{x^2}{2\pi u}$ for brevity. In view of (2.5) and (2.6), the sum $\sum\limits_{h\le U}\sin(\beta h)h^{-2}m\big(\frac{u}{h};h\big)$ is a combination of finitely many sums of the form

$$(\log u)^i\sum_{r\le U}\frac{(\log r)^j}{r^2}\sum_{d\le\frac{U}{r}}\frac{\sin(\beta rd)}{d}\log^k d$$

where $i,j,k\ge 0$ and $i+j+k\le 2$. The inner sum over d is $\ll 1$ and hence

$$\sum_{h\le U}\sin(\beta h)h^{-2}m\left(\frac{u}{h};h\right)\ll \log^2 u\,.$$

This proves (4.14). Back substitution into (4.13) and in view of (4.12) then leads to the bound $\sigma_{11}(x)\ll\sqrt{P}\log^2 P$. This together with (4.11) confirms the bound

$$\sigma_1(x)\ll\sqrt{P}\log^2 P\,,$$

and so the contribution of $\sigma_1(x)$ to S_{22}^- is $\ll \sqrt{P}\log^2 P\int x^{-2}\gamma(x)\,dx\ll \log^2 P$.

It remains to handle $\sigma_2(x)$, whose contribution to S_{22}^- is

$$-\pi^2\sum_{\substack{m<n\le Z(\sqrt{2\pi P},a)\\ n-m>U}}\frac{d(m)d(n)}{\sqrt{mn}}\int\big(\xi_m\xi_n\big)'\frac{\gamma(x)}{x^2\ell_m\ell_n}\left(\log\frac{n}{m}\right)^{-1}\sin\left(\frac{x^2}{2\pi}\log\frac{n}{m}\right)dx$$

$$\ll \sum_{\substack{m<n\le Z(\sqrt{2\pi P},a)\\ n-m>U}} \frac{d(m)d(n)}{\sqrt{mn}} \max_{x\in[\frac{1}{2}\sqrt{2\pi P},\sqrt{2\pi P}]} \left| (\xi_m\xi_n)' \left(x^3 \ell_m \ell_n \log^2 \frac{n}{m}\right)^{-1} \gamma(x) \right|$$

$$\ll P^{-1}U^{-1} \sum_{\substack{m<n\le Z(\sqrt{2\pi P},a)\\ n-m>U}} \frac{d(m)d(n)}{m\sqrt{n}} \left(\log \frac{n}{m}\right)^{-2} , \tag{4.15}$$

by applying Lemma 6 together with the estimates Lemma 1 (i), (2.8) and (4.10). The part of this sum for which $m < n/2$ is easily seen to be $\ll P^{1/2+\varepsilon}$. For $n/2 < m < n$, the sum is

$$\ll \sum_{\substack{n/2<m<n-U\\ n\le Z(\sqrt{2\pi P},a)}} \frac{d(m)d(n)}{m\sqrt{n}} \left(\frac{n}{n-m}\right)^2$$

$$\ll \sum_{h\ge U} h^{-2} \sum_{m\le Z(\sqrt{2\pi P},a)} \sqrt{m}\, d(m)d(m+h)$$

$$\ll \sum_{h\ge U} h^{-3}\sigma(h) P^{3/2} \log^2 P + \sum_{h\ge U} h^{-2} P^{5/4} ,$$

by invoking Lemma 3. Plainly $\sum_{h\ge U} h^{-3}\sigma(h)$ is $\ll U^{-1}$. All in all, the expression in (4.15) is $\ll \log^2 P$, and whence

$$S_{22}^- \ll \log^2 P .$$

In a similar but much easier manner, one shows that $S_{22}^+ \ll \log^2 P$. In view of (4.1) and (4.5), this completes the proof of Lemma 8.

5. The integral I_3

We come now to the estimation of I_3, for the cross term $\sum_1 \sum_2$. The oscillating factors $\cos f_n(x)$ in $\sum_1$ and $\cos g_n(x)$ in $\sum_2$ are almost in phase when m and n are close to $(a+U)^2$ and $Z(x,a)$ respectively. But the totality of all such cases is small and we shall use a special trick to utilize this observation.

LEMMA 9. *We have*

$$I_3 \ll \log^2 P .$$

The trick is that we transform a small tail section of $\sum_2$ into a small tail section of $\sum_1$ by noting that the parameter a in (2.1), apart from having order of $\sqrt{P}$, still has some degree of freedom. More precisely, we use the formula (2.1) twice, first with value a and then with a replaced by $a+2U$ (but keeping the same U). Thus, by writing

$$r_n(x) = (-1)^n d(n) n^{-3/4} e_n(x) \cos f_n(x) ,$$
$$s_n(x) = d(n) n^{-1/2} \ell_n(x)^{-1} \cos g_n(x) ,$$

for brevity and noting that they are independent of a, we have, by (2.1)

$$\sum_{n\le(a+U)^2}\eta_n(a)r_n(x)-\sqrt{\frac{\pi}{x}}\sum_{n\le Z(x,a)}\xi_n(a)s_n(x)+O(x^{-1}\log x)$$
$$=\sqrt{\frac{\pi}{x}}E\left(\frac{x^2}{2\pi}\right)=\sum_{n\le(a+3U)^2}\eta_n(a+2U)r_n(x)$$
$$-\sqrt{\frac{\pi}{x}}\sum_{n\le Z(x,a+2U)}\xi_n(a+2U)s_n(x)+O(x^{-1}\log x)\,.$$

Here, instead of η_n and ξ_n we have to write $\eta_n(a)$ and $\xi_n(a)$, to indicate their dependence on the parameter a. Then

$$\sqrt{\frac{\pi}{x}}\left\{\sum_{n\le Z(x,a)}\xi_n(a)s_n(x)-\sum_{n\le Z(x,a+2U)}\xi_n(a+2U)s_n(x)\right\}$$
$$=-\sum_{n\le(a+3U)^2}\eta_n(a+2U)r_n(x)+\sum_{n\le(a+U)^2}\eta_n(a)r_n(x)+O(x^{-1}\log x)\,.$$

We may therefore express

$$\sum\nolimits_1\sum\nolimits_2=\sum\nolimits_1\sqrt{\frac{\pi}{x}}\sum_{n\le Z(x,a+2U)}\xi_n(a+2U)s_n(x)$$
$$-\sum\nolimits_1\left\{\sum_{n\le(a+3U)^2}\eta_n(a+2U)r_n(x)-\sum_{n\le(a+U)^2}\eta_n(a)r_n(x)+O(x^{-1}\log x)\right\}$$
$$=\sum\nolimits_1\sum\nolimits_2^*-\sum\nolimits_1\sum\nolimits_1^*+O\left(x^{-1}\log x\left|\sum\nolimits_1\right|\right),\quad say,$$

where

$$\sum\nolimits_2^*=\sqrt{\frac{\pi}{x}}\sum_{n\le Z(x,a+2U)}\xi_n(a+2U)s_n(x)$$

is similar to the original $\sum_2$, but with a tail section removed and $\xi_n(a)$ changed to $\xi_n(a+2U)$, and

$$\sum\nolimits_1^*=\sum_{n\le(a+3U)^2}(\eta_n(a+2U)-\eta_n(a))r_n(x).$$

We see that $\sum_1^*$ has the same shape as $\sum_1$, but with the new smoothening factor $\eta_n(a+2U)-\eta_n(a)$ which has support on the very short interval $[a^2,(a+3U)^2]$. So $\sum_1^*$ is a short sum of length $\asymp aU$, having the same oscillating factor as $\sum_1$. Similar to the estimation of I_1 in Lemma 7, we see that the contribution of the cross terms in

$$\int\gamma(x)\sum\nolimits_1\sum\nolimits_1^*dx$$

is $\ll \log^2 P$, while the diagonal terms yield the contribution

$$\sum_{a^2<n\le(a+2U)^2} (\eta_n(a+2U)-\eta_n(a))^2 d(n)^2 n^{-3/2} \int e_n^2 \gamma(x)dx$$
$$\ll \sqrt{P} \sum_{a^2<n\le(a+3U)^2} d(n)^2 n^{-3/2} \ll P^{-1/4+\varepsilon} .$$

Furthermore, by Cauchy-Schwarz's inequality and the bound for $\int \sum_1^2 \gamma(x)dx$ in Lemma 7, we find that

$$\int x^{-1} \log x \left| \sum\nolimits_1 \right| \gamma(x)dx \ll \log P$$

which is again sufficient. To finish the proof of Lemma 9, it remains to establish that

$$I_{31} = \int \sum\nolimits_1 \sum\nolimits_2^* \gamma(x)dx \ll \log^2 P .$$

The estimation of this follows the same argument of $S_{22}^{\pm}$ in §4. More precisely, after interchanging the integration and summation, we have

$$I_{31} = S_{31}^{+} + S_{31}^{-} ,$$

where

$$S_{31}^{\pm} = \frac{\sqrt{\pi}}{2} \sum_{m\le(a+U)^2} \sum_{n\le Z(\sqrt{2\pi P},a+2U)} \frac{(-1)^m d(m)d(n)}{m^{3/4}\sqrt{n}} \eta_m(a)$$
$$\int_{\mathcal{N}(n,a+2U)}^{\sqrt{2\pi P}} e_m \xi_n(a+2U)\ell_n^{-1} x^{-1/2} \gamma(x) \cos(g_n \pm f_m)dx . \tag{5.1}$$

As before, S_{31}^{-} is the more difficult one and we shall prove

$$S_{31}^{\pm} \ll P^{-1/4+\varepsilon} . \tag{5.2}$$

By the same argument of S_{22}^{-} and noting that $\xi_n(a+2U)\gamma(x)$ vanishes at the upper and lower limits of the integration, we obtain, after an integration by parts,

$$\int_{\mathcal{N}(n,a+2U)}^{\sqrt{2\pi P}} = \int_{\mathcal{N}(n,a+2U)}^{\sqrt{2\pi P}} \sin(g_n - f_m) \frac{d}{dx}\Big\{ e_m \xi_n(a+2U)\ell_n^{-1} x^{-1/2} \gamma(x) \big(g_n' - f_m'\big)^{-1} \Big\} dx .$$

By Lemma 6, this is

$$\ll \max \left| g_n' - f_m' \right|^{-1} \left| \frac{d}{dx} \Big\{ e_m \xi_n(a+2U)\ell_n^{-1} x^{-1/2} \gamma(x) \big(g_n' - f_m'\big)^{-1} \Big\} \right|$$

where the maximum is over $x \in \big[\max(\mathcal{N}(n,a+2U), \frac{1}{2}\sqrt{2\pi P}), \sqrt{2\pi P} \,\big]$. We will show in a moment that

$$\left| g_n' - f_m' \right|^{-1} \ll U^{-1} \quad \text{and} \quad \left| g_n'' - f_m'' \right| \ll 1 \tag{5.3}$$

for x in the above range. Then, in view of Lemma 1 (i)–(iv) and (2.8) the integral in (5.1) is

$$\ll U^{-1}\sqrt{\frac{x}{n}}\big|g_n' - f_m'\big|^{-2} + x^{-1/2}\big|g_n'' - f_m''\big|\,\big|g_n' - f_m'\big|^{-3}$$
$$\ll U^{-3}\sqrt{\frac{x}{n}} + x^{-1/2}U^{-3} \ll P^{-1/2}n^{-1/2} + P^{-1} \ll P^{-1/2}n^{-1/2}.$$

Therefore

$$S_{31}^{-} \ll P^{-1/2} \sum_{m\le(a+U)^2}\ \sum_{n\le Z(\sqrt{2\pi P},a+2U)} \frac{d(m)d(n)}{m^{3/4}n} \ll P^{-1/4+\varepsilon},$$

and the same bound holds for S_{31}^{+}. Hence (5.2) is proved.

Finally, we establish the bounds in (5.3). Direct from their definitions (c.f. Lemma 1 (iii)), we find that

$$\begin{aligned} g_n' - f_m' &= \frac{2x}{\pi}\left(\log\frac{x}{2\pi\sqrt{n}} - \operatorname{arsinh}\frac{\pi\sqrt{m}}{x}\right)\\ &= \frac{2x}{\pi}\log\left(\frac{x}{2\pi\sqrt{n}}\left\{\frac{\pi\sqrt{m}}{x} + \left(1+\frac{\pi^2 m}{x^2}\right)^{1/2}\right\}^{-1}\right) \end{aligned} \tag{5.4}$$

and

$$g_n'' - f_m'' = \frac{2}{\pi}\left(\log\frac{x}{2\pi\sqrt{n}} - \operatorname{arsinh}\frac{\pi\sqrt{m}}{x} + 1\right) + 2\frac{\sqrt{m}}{x}\left(1+\frac{\pi^2 m}{x^2}\right)^{-1/2}.$$

Plainly, for $m \le (a+U)^2 \asymp P \asymp x^2$ and $n \le Z(x, a+2U)$, we have

$$g_n'' - f_m'' \ll 1.$$

From (2.4), one verifies directly

$$\frac{x}{2\pi}\left\{\frac{\pi\sqrt{m}}{x} + \left(1+\frac{\pi^2 m}{x^2}\right)^{1/2}\right\}^{-1} = \sqrt{Z(x,\sqrt{m})}.$$

Hence for $n \le Z(x, a+2U)$,

$$\begin{aligned} g_n' - f_m' &= \frac{2x}{\pi}\left(\log\sqrt{Z(x,\sqrt{m})} - \log\sqrt{n}\right)\\ &\ge \frac{2x}{\pi}\Big(\log\sqrt{Z(x,\sqrt{m})} - \log\sqrt{Z(x,a+2U)}\Big)\\ &= \frac{2x}{\pi}\big\{m-(a+2U)^2\big\}\frac{d}{du}\log\sqrt{Z(x,\sqrt{u})}\Big|_{u=u_0} \quad \text{for some } u_0 \in \big(m,(a+2U)^2\big)\\ &= \frac{x}{2\pi}\left(u_0\left(\left(\frac{x}{2\pi}\right)^2 + \frac{u_0}{4}\right)\right)^{-1/2}\big((a+2U)^2 - m\big)\\ &\gg xP^{-1}\big((a+2U)^2 - (a+U)^2\big) \gg U. \end{aligned}$$

This proves (5.3) and our Lemma 9 hence follows.

6. Proofs of Theorem 1 and 2.

The treatments of I_4 and I_5 are quite straightforward, by integrating term by term of $\sum_1$ and $\sum_2$ and then applying Lemma 6. We have

$$I_4 \ll P^{-1/4}\,, \quad I_5 \ll P^{-1/4}\log P\,.$$

Putting these and the estimates for I_1, I_2, I_3 from Lemmas 7, 8, 9 into (2.9), we conclude that

$$\frac{\pi^2}{P}\int_0^P \omega\left(\frac{T}{P}\right)E(T)^2\,dT = \sqrt{\frac{9\pi}{8}}c\int_0^{\sqrt{2\pi P}}\gamma(x)\,dx \\ -6\pi^{-1}\left(\int x^{-1}\gamma(x)dx\right)\log^2 P\log\log P + O(\log^2 P)\,.$$

In view of (2.7), Theorem 1 follows.

To deduce Theorem 2, we notice that for any function $\phi(T)$

$$\int_0^P \phi(T)F(T)\,dT = \int_0^P\left(\int_T^P \phi(t)\,dt\right)\left(E(T)^2 - \frac{3}{2}cT^{1/2}\right)dT\,.$$

Using the function

$$\phi(T) = \begin{cases} -2\,, & \dfrac{P}{4} \le T \le \dfrac{P}{2}\,, \\ 1\,, & \dfrac{P}{2} < T \le P\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

and find that

$$Q(P) - 2Q(P/2) = P\int_0^P \omega\left(\frac{T}{P}\right)\left(E(T)^2 - \frac{3}{2}cT^{1/2}\right)dT$$

where

$$Q(Y) = \int_{\frac{Y}{2}}^Y F(T)dT$$

and

$$\omega(x) = \begin{cases} 2x - \dfrac{1}{2}\,, & \dfrac{1}{4} \le x \le \dfrac{1}{2}\,, \\ 1 - x\,, & \dfrac{1}{2} \le x \le 1\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Then by Theorem 1, we have

$$Q(P) - 2Q(P/2) = -\frac{9}{8\pi^2}P^2\log^2 P\log\log P + O(P^2\log^2 P)\,.$$

Replacing P by $P2^{-j}$, then multiplying throughout by 2^j and then sum j from 0 to J where $J = [3 \log\log P]$, we obtain (by noting the bound $F(T) \ll T \log^4 T$)

$$Q(P) = -\frac{9}{4\pi^2} P^2 \log^2 P \log\log P + O(P^2 \log^2 P)\,.$$

Whence

$$\int_0^P F(T)dT = \sum_{j=0}^{\infty} Q\left(\frac{P}{2^j}\right) = -\frac{3}{\pi^2} P^2 \log^2 P \log\log P + O(P^2 \log^2 P)$$

and Theorem 2 follows.

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