COMMENTARII MATHEMATICI UNIVERSITATIS SANCTI PAULI
Vol. 60, No. 1, 2 2011

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171–8501 JAPAN

Note on Sums of Seven Cubes of Smooth Numbers

by

Koichi KAWADA

(Received April 11, 2011)

Dedicated to Professor Akio Fujii on the occasion of his official retirement

1. Introduction

Extending Vinogradov's celebrated three prime theorem, Akio Fujii [4] proved that for any fixed integer $g \geq 2$, every sufficiently large odd integer can be written as the sum of three numbers, each of which is the product of g prime numbers. More precisely, he [4] showed that for any given positive real numbers $\delta_1, \ldots, \delta_g$ satisfying $\delta_1 + \cdots + \delta_g = 1$, every sufficiently large odd integer n can be written as $n = x_1 + x_2 + x_3$, where for $j = 1$, 2 and 3, the x_j takes the shape

$$x_j = p_{j,1} p_{j,2} \ldots p_{j,g} \quad \text{with primes } p_{j,k} \leq n^{\delta_k} \text{ for } 1 \leq k \leq g\,. \tag{1}$$

In fact, he essentially established an asymptotic formula for the number of the latter representations. As Balog and Sárközy [2] pointed out, this work of Fujii [4] has relation with a problem posed by Erdös on representations of integers by sums of numbers without large prime factors. Such numbers are commonly called *smooth numbers*.

The methods of Fujii [4] may be quite easily altered to show, for example, that every large even integer n can be written as $n = 2x_1 + x_2 + x_3$ with the same constraint on x_j as in (1). Thus it is immediately derived from the above work of Fujii [4] that for any $\varepsilon > 0$, every sufficiently large integer n, regardless of its parity, can be written as the sum of three natural numbers, each of which has no prime factor exceeding n^ε. Later, Balog and Sárközy [2] sharpened this conclusion, replacing the last n^ε by $\exp(3\sqrt{\log n \log\log n})$.

Further, Fujii [5] investigated additive problems of Waring's type for smooth numbers. Amongst others, concerning sums of cubes, on which we concentrate in this short note, he established that every sufficiently large odd integer n can be written as $n = x_1^3 + x_2^3 + \cdots + x_9^3$, where each x_j takes a shape similar to (1), but with the restriction $p_{j,k} \leq n^{\delta_k/3}$ for the prime factors (see Theorem 3 and Corollary 3 of [5]). Again, for large even n, Fujii's argument may lead, with trivial modifications, to the existence of the x_j of the same shape satisfying $n = (2x_1)^3 + x_2^3 + \cdots + x_9^3$, so essentially it follows form Corollary 3 of [5] that for any $\varepsilon > 0$, every sufficiently large n, of either parity, can be written as the sum of nine cubes of natural numbers, each of which has no prime factor exceeding n^ε. This conclusion was refined by Harcos [6], who showed the assertion with $\exp(c\sqrt{\log n \log\log n})$ for some positive absolute constant c, in place of n^ε. Brüdern and Wooley [3] further improved Harcos's result by reducing the number of summands. Namely, they [3] showed that every sufficiently large n is the sum of eight cubes of natural numbers that have no prime factor exceeding $\exp(c\sqrt{\log n \log\log n})$ with some positive absolute constant c.

The purpose of this article is to discuss the corresponding problem for sums of seven cubes. Hereafter, for an integer $m \geq 2$, we denote the largest prime factor of m by $P(m)$, and we seek for an upper bound for $P(x_1x_2\dots x_7)$ in terms of n, with which every sufficiently large integer n admits the representation

$$n = x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 + x_7^3\,. \tag{2}$$

Our result here in this direction is substantially weaker than the aforementioned results, with respect to the smoothness of the variables, and we aim to show that every large n is written in the form (2) with $P(x_1\dots x_7) \leq n^{\sigma/3}$ for a smaller σ. Note that Linnik's seven cube theorem means of course that $\sigma = 1$ is admissible in the latter statement.

If we would be allowed to leave one variable aside in the representation (2), then we may make the remaining six variables satisfactorily smooth. In fact, it may be deduced from a deep theorem of Wooley ([9], Theorem 1.2) that every large n can be written in the form (2) with $P(x_2x_3\dots x_7) \leq \exp(c\sqrt{\log n \log\log n})$ for some $c > 0$. But still considerable technical efforts are required to obtain a significant bound for the largest prime factor of all the seven variables, substantially better than $P(x_1x_2\dots x_7) \leq n^{1/3}$. Such a bound appeared first in the previous work of the author [7] as a by-product of his research on sums of seven cubes of almost primes. On putting

$$\xi = \frac{\sqrt{2833} - 43}{41} \quad \text{and} \quad \sigma_0 = 1 - \frac{4(5 - 16\xi - \xi^2)}{8\xi + 21}\,, \tag{3}$$

Theorem 5 of [7] asserts that for any $\sigma > \sigma_0$, every sufficiently large n can be written in the form (2) with natural numbers x_j $(1 \leq j \leq 7)$ satisfying $P(x_1x_2\dots x_7) \leq n^{\sigma/3}$. We remark that

$$\sigma_0 = (53672\sqrt{2833} - 2784931)/85977$$

is slightly smaller than 0.835239, and also that the above ξ comes from the currently best estimate for the sixth moment of a cubic smooth Weyl sum due to Wooley ([10], Theorem 1.2). In this note we reduce the lower limit for the admissible value of σ by the factor $e^{-1/2}$.

THEOREM. *Let σ_0 be defined by* (3). *Then for any $\sigma > \sigma_0 e^{-1/2}$, every sufficiently large integer n can be written in the form* (2) *with natural numbers x_j $(1 \leq j \leq 7)$ satisfying $P(x_1x_2\dots x_7) \leq n^{\sigma/3}$.*

Note that $\sigma_0 e^{-1/2}$ is slightly smaller than 0.506598.

The Theorem is proved by adding an idea of Balog [1] to the basic work of [7]. This idea of Balog may be regarded as a kind of the switching principle, or the reversal role technique, in the sieve theory.

2. Notation and the strategy

Throughout the paper, we adopt the following notation and conventions.

We write $e(\alpha) = \exp(2\pi i\alpha)$, and the letter p, with a subscript also, denotes prime numbers. The letter ϖ denotes primes congruent to 2 modulo 3.

The constants ξ and σ_0 are defined at (3), and σ denotes any given real number satisfying $\sigma > \sigma_0 e^{-1/2}$, as in the statement of the Theorem. Then there is a positive ε such that

$$\sigma > (\sigma_0 + 9\varepsilon)e^{\varepsilon - 1/2}, \tag{4}$$

and we fix such an ε with $\varepsilon < 10^{-3}$ once for all, within this paper. All the implicit constants, involved in the symbols $\ll$, $\gg$ and O, may depend at most on ε.

With this ε, Theorem 1.2 of Wooley [10] assures the existence of a small positive η such that for any $U > 1$, one has

$$\int_0^1 \Bigg| \sum_{1 \le x \le 2U} e(x^3\alpha) \Bigg|^2 \Bigg| \sum_{\substack{1 \le z \le 2U \\ P(z) \le U^{3\eta}}} e(z^3\alpha) \Bigg|^4 d\alpha \ll U^{3+\xi+\varepsilon}. \tag{5}$$

We fix an $\eta < 10^{-3}$ having the latter property, and to facilitate the later argument, we suppose that $(2\eta)^{-1}$ is an integer, without loss of generality. Note that we may take this η, depending only on ε.

Let n be an integer which shall be supposed to be large in terms of the ε, and put

$$X = \frac{1}{2} n^{1/3}, \quad D = X^{1-\sigma_0-9\varepsilon}, \quad M = X^{(14\xi-1)/(8\xi+21)}.$$

It may be understood from our minor arc estimate (see (10) below) that these choices of D and M are optimal. And it may be helpful to record that $1 - \sigma_0$ and $(14\xi - 1)/(8\xi + 21)$ are approximately 0.16476 and 0.10836, respectively.

Next, for any $U > 1$, we define

$$\mathcal{A}^*(U) = \{l \in \mathbb{N} : U^{1-2\eta} < l \le U^{1-\eta},\ \forall p | l,\ \ 2U^{2\eta} < p \le U^{3\eta}\},$$

imitating the methods of Harcos [6] and Brüdern and Wooley [3], which shows its utility when we evaluate major arc contribution. In accordance with the notation of [7], we also define the sets

$$\begin{aligned} \mathcal{A} &= \{lm : l \in \mathcal{A}^*(X/M),\ 1 \le lm \le 2X/M\}, \\ \mathcal{A}_M &= \{\varpi x : M < \varpi \le 2M,\ x \in \mathcal{A},\ X < \varpi x \le 2X\}, \\ \mathcal{B} &= \{lm : l \in \mathcal{A}^*(X),\ X < lm \le 2X\}. \end{aligned}$$

When $l \in \mathcal{A}^*(X/M)$ and $1 \le lm \le 2X/M$, one has $m \le 2(X/M)^{2\eta}$, while every prime factor of l exceeds $2(X/M)^{2\eta}$. Thus we find that every $x \in \mathcal{A}$ can be written uniquely as $x = lm$ with $l \in \mathcal{A}^*(X/M)$. Similarly, every $z \in \mathcal{B}$ may be written uniquely as $z = lm$ with $l \in \mathcal{A}^*(X)$. Moreover we may notice that

$$\forall x \in \mathcal{A}, \quad P(x) \le (X/M)^{3\eta}, \quad \text{and} \quad \forall z \in \mathcal{B}, \quad P(z) \le X^{3\eta}.$$

We also have $P(y) \le 2M$ for all $y \in \mathcal{A}_M$.

Then we define $R_1(n)$ to be the number of representations of n in the form

$$n = (p_1x_1)^3 + (p_2x_2)^3 + y_1^3 + y_2^3 + y_3^3 + z_1^3 + z_2^3, \tag{6}$$

where

$$D < p_j \le 2D\,, \quad \frac{X}{p_j} < x_j \le \frac{2X}{p_j}\,, \quad y_k \in \mathcal{A}_M\,, \ \ z_j \in \mathcal{B}\,, \tag{7}$$

for $j = 1, 2$, and $k = 1, 2, 3$, and in addition, $P(x_1) \le X^\sigma$. We may provide an asymptotic formula for $R_1(n)$, being based on the minor arc estimate contained essentially in Lemma 5 of [7], as we shall show indeed in the sequel. Then by the fact that $R_1(n) > 0$ for every large n, one obtains a conclusion which is tantamount to Theorem 5 of [7] referred after (3).

To prove the Theorem, we estimate the number, say $R_2(n)$, of the representations of n in the form (6) where the variables satisfy (7) and additional conditions $P(x_1) \le X^\sigma$ and $P(x_2) > X^\sigma$. Further we write $R_3(n)$ for the number of the representations of n in the form (6) where the variables satisfy (7) and $P(x_2) > X^\sigma$.

Now note first that trivially one has $R_2(n) < R_3(n)$. So if we could show that

$$R_1(n) > R_3(n)\,, \tag{8}$$

then we see that $R_1(n) > R_2(n)$, which obviously means that n can be written in the form (6) with the variables satisfying (7), $P(x_1) \le X^\sigma$ and $P(x_2) \le X^\sigma$. Thus the Theorem follows from (8) at once.

We remark that in the above strategy, some additional minor efforts allow us to replace the variables p_j, y_k and z_j for $j = 1, 2$ and $k = 1, 2, 3$, by suitable numbers having no prime factor exceeding $\exp(c\sqrt{\log n \log\log n})$ with some $c > 0$, although such modifications have no impact on the statement of the Theorem. Here we adopt the above setting for the ease of reference to [7].

3. The circle method and minor arc estimates

In order to show (8) by the circle method, we introduce several exponential sums. When d is a natural number and $\mathcal{C}$ is a finite set of integers, we write

$$f(\alpha; d) = \sum_{X/d < x \le 2X/d} e(d^3x^3\alpha)\,, \qquad g(\alpha; \mathcal{C}) = \sum_{x \in \mathcal{C}} e(x^3\alpha)\,,$$

$$f_1(\alpha; d) = \sum_{\substack{X/d < x \le 2X/d \\ P(x) \le X^\sigma}} e(d^3x^3\alpha)\,, \qquad f_3(\alpha; d) = \sum_{\substack{X/d < x \le 2X/d \\ P(x) > X^\sigma}} e(d^3x^3\alpha)\,.$$

For $\nu = 1$ and 3, we also put

$$F_\nu(\alpha) = \sum_{D < p \le 2D} f_\nu(\alpha; p)\,,$$

and then observe, by the orthogonality, that

$$R_\nu(n) = \int_0^1 \sum_{D < p \le 2D} f(\alpha; p) F_\nu(\alpha) g(\alpha; \mathcal{A}_M)^3 g(\alpha; \mathcal{B})^2 e(-n\alpha) d\alpha\,.$$

To evaluate the last integral, we adopt the same Hardy-Littlewood dissection as in [7]. Let $\mathfrak{P}$ be the set of real numbers $\alpha \in [0, 1)$ such that there exist coprime integers q and a satisfying

$$|\alpha - a/q| \le (\log X)^{500} X^{-3}, \quad 1 \le q \le (\log X)^{500}, \quad 0 \le a \le q, \tag{9}$$

and write $\mathfrak{p} = [0, 1) \setminus \mathfrak{P}$. Then the contribution of the minor arcs $\mathfrak{p}$ to the integral representing $R_\nu(n)$ may be swiftly handled by reference to the upper bound for E contained in Lemma 5 of [7]. In fact, on writing

$$I = \int_0^1 |g(\alpha; \mathcal{A})|^6 d\alpha \quad \text{and} \quad J_\nu = \int_0^1 |F_\nu(\alpha)^2 g(\alpha; \mathcal{B})^4| d\alpha,$$

for $\nu = 1$ and 3, and on checking the requirement

$$\max\{X^{1/10}, X^{1/9} D^{-1/3}\} < M \le (X/D)^{1/7},$$

we may obtain the estimate

$$\begin{aligned} &\sum_{D<d\le 2D} \left| \int_{\mathfrak{p}} f(\alpha; d) F_\nu(\alpha) g(\alpha; \mathcal{A}_M)^3 g(\alpha; \mathcal{B})^2 e(-n\alpha) d\alpha \right| \\ &\quad \ll X^{3/4+\varepsilon} M^{3/4} D^{1/4} J_\nu^{1/2} (X^{3/2} + (MI)^{1/2} + (XI)^{1/3} M^{3/2} D^{1/4}) \\ &\quad + X^4 (\log X)^{-50}, \end{aligned} \tag{10}$$

by (4.20) of [7] (see also (4.27)–(4.29), (4.39), (4.40) and (4.42) of [7]). Here only one comment may be in order; on this derivation, we make $F_\nu(\alpha)$ play the role of $g(\alpha; \mathcal{B}_1)$ in [7], and this change has an actual effect only on estimating T_2 which should read

$$T_2 = \int_0^1 |g(\alpha; \mathcal{A}_M)^2 F_\nu(\alpha)^2 g(\alpha; \mathcal{B})^4| d\alpha,$$

in the current context. But, by orthogonality, T_2 is equal to the number of solutions of the equation

$$y_1^3 + (p_1 x_1)^3 + z_1^3 + z_2^3 = y_2^3 + (p_2 x_2)^3 + z_3^3 + z_4^3,$$

subject to $y_j \in \mathcal{A}_M$, $D < p_j \le 2D$, $X/p_j < x_j \le 2X/p_j$ and $z_k \in \mathcal{B}$ for $j = 1, 2$ and $1 \le k \le 4$. Since every natural number less than $2X$ has at most $O(1)$ prime divisors in the interval $(D, 2D]$, we still have

$$T_2 \ll \int_0^1 |g(\alpha; \mathbb{N} \cap [1, 2X])|^8 d\alpha \ll X^5,$$

as is claimed in [7], by virtue of a theorem of Vaughan. Thus we may confirm the estimate (10).

Then, by (5), together with the above remark concerning $F_\nu(\alpha)$, we have $I \ll (X/M)^{3+\xi+\varepsilon}$ and $J_\nu \ll X^{3+\xi+\varepsilon}$, for $\nu = 1$ and 3. Consequently, a modicum of straightforward computation may deduce from (10) that

$$\int_{\mathfrak{p}} \sum_{D<p\le 2D} f(\alpha; p) F_\nu(\alpha) g(\alpha; \mathcal{A}_M)^3 g(\alpha; \mathcal{B})^2 e(-n\alpha) d\alpha$$

$$\ll X^{4+2\varepsilon}\left(X^{-\frac{5-16\xi-\xi^2}{8\xi+21}}D^{1/4}+X^{-\frac{2(5-16\xi-\xi^2)}{8\xi+21}}D^{1/2}\right)+X^4(\log X)^{-50},$$

whence we thus far conclude, for $\nu = 1$ and 3, that

$$R_\nu(n)=\int_{\mathfrak{P}}\sum_{D<p\le 2D} f(\alpha;p)F_\nu(\alpha)g(\alpha;\mathcal{A}_M)^3 g(\alpha;\mathcal{B})^2 e(-n\alpha)d\alpha + O(X^4(\log X)^{-50}). \tag{11}$$

It is essentially a routine to evaluate the contribution of the major arcs $\mathfrak{P}$, and we execute it in the next section.

4. Contribution of the major arcs

We begin with a simple lemma concerning the set $\mathcal{A}^*(U)$.

LEMMA. *Under our convention on* η (*see the comment after* (5)), *we have* $\sum_{l\in\mathcal{A}^*(U)} l^{-1} \gg 1$, *provided that* U *is sufficiently large in terms of* η (*so ultimately*, *in terms of* ε).

Proof. Recall that $(2\eta)^{-1} = h$, say, is an integer. Then, whenever $p_1, \dots, p_{h-1}$ are primes in the interval $(2U^{2\eta}, U^{2\eta+2\eta^2}]$, one has

$$U^{1-2\eta} < (2U^{2\eta})^{h-1} < p_1p_2\dots p_{h-1} \le (U^{2\eta+2\eta^2})^{h-1} < U^{1-\eta},$$

so that $p_1p_2\dots p_{h-1} \in \mathcal{A}^*(U)$. And it is trivial that each natural number can be written as the product of $(h-1)$ primes in at most $(h-1)!$ ways. These facts reveal that

$$\sum_{l\in\mathcal{A}^*(U)} l^{-1} \ge \frac{1}{(h-1)!}\Bigg(\sum_{2U^{2\eta}<p\le U^{2\eta+2\eta^2}} p^{-1}\Bigg)^{h-1},$$

and the sum over p on the right hand side is $\log(1+\eta) + O((\log U)^{-1})$, by Mertens' theorem. Thus the Lemma follows immediately. □

For integers q and a with $q \ge 1$, and a real number β, we define

$$S(q,a)=\sum_{x=1}^{q} e(ax^3/q) \quad \text{and} \quad v(\beta)=\int_X^{2X} e(t^3\beta)dt.$$

For any q and a with $(q,a) = 1$, and for any β, we know that

$$S(q,a) \ll q^{2/3} \quad \text{and} \quad v(\beta) \ll X(1+X^3|\beta|)^{-1}. \tag{12}$$

In fact, the former is Theorem 4.2 of Vaughan [8], and the latter follows by the partial integration with a trivial bound.

Now let $\alpha \in \mathfrak{P}$, and q and a be coprime integers satisfying (9), and write $\beta = \alpha - a/q$. Then, for any integer d, we deduce from Theorem 4.1 of Vaughan [8] that

$$f(\alpha; d) = q^{-1}S(q, ad^3)\int_{X/d}^{2X/d} e(d^3t^3\beta)dt + O((\log X)^{501}) .$$

Noting further that $S(q, ad^3) = S(q, a)$ whenever d is coprime to q, we find in the latter circumstances that

$$f(\alpha; d) = (qd)^{-1}S(q, a)v(\beta) + O((\log X)^{501}) . \tag{13}$$

Accordingly we have the formulae

$$\sum_{D<p\le 2D} f(\alpha; p) = \Delta_1 q^{-1}S(q, a)v(\beta) + O(D(\log X)^{501}) , \tag{14}$$

$$\begin{aligned} g(\alpha; \mathcal{A}_m) &= \sum_{M<\varpi\le 2M}\ \sum_{l\in\mathcal{A}^*(X/M)} f(\alpha; \varpi l) \\ &= \Delta_2 q^{-1}S(q, a)v(\beta) + O\Big(M\Big(\frac{X}{M}\Big)^{1-\eta}(\log X)^{501}\Big) , \end{aligned} \tag{15}$$

$$\begin{aligned} g(\alpha; \mathcal{B}) &= \sum_{l\in\mathcal{A}^*(X)} f(\alpha; l) \\ &= \Delta_3 q^{-1}S(q, a)v(\beta) + O(X^{1-\eta}(\log X)^{501}) , \end{aligned} \tag{16}$$

where

$$\Delta_1 = \sum_{D<p\le 2D}\frac{1}{p} , \quad \Delta_2 = \sum_{M<\varpi\le 2M}\frac{1}{\varpi}\sum_{l\in\mathcal{A}^*(X/M)}\frac{1}{l} , \quad \Delta_3 = \sum_{l\in\mathcal{A}^*(X)}\frac{1}{l} .$$

As for these quantities, the lower bounds

$$\Delta_1 \gg (\log X)^{-1} , \quad \Delta_2 \gg (\log X)^{-1} \quad \text{and} \quad \Delta_3 \gg 1 \tag{17}$$

may be confirmed by our Lemma, Mertens' theorem and its variant for arithmetic progressions. As regards the upper bounds, our argument needs nothing more than the trivial ones like $\Delta_j \ll \log X$ for $j = 1, 2$ and 3.

Since the measure of $\mathfrak{P}$ is $O((\log X)^{1500}X^{-3})$, and all the error terms appearing in the formulae (14)-(16) are $O(X^{1-\eta/2})$, it follows simply from these formulae that

$$\begin{aligned} &\int_{\mathfrak{P}}\sum_{D<p\le 2D} f(\alpha; p)F_\nu(\alpha)g(\alpha; \mathcal{A}_M)^3 g(\alpha; \mathcal{B})^2 e(-n\alpha)d\alpha \\ &= \Delta_1\Delta_2^3\Delta_3^2 \sum_{q\le(\log X)^{500}}\ \sum_{\substack{a=1\\(a,q)=1}}^{q} V_6(q, a, n)W_\nu(q, a, n) + O(X^{4-\eta/3}) , \end{aligned} \tag{18}$$

where, for $s \in \mathbb{N}$ and $\nu = 1$ and 3, we write

$$V_s(q, a, n) = (q^{-1}S(q, a))^s e(-an/q) ,$$

and

$$W_\nu(q,a,n) = \int_{-(\log X)^{500}X^{-3}}^{(\log X)^{500}X^{-3}} F_\nu\left(\frac{a}{q}+\beta\right)v(\beta)^6 e(-n\beta)d\beta\,.$$

We next look into $F_3(\alpha)$. Noting that $X^\sigma > \sqrt{2X/D}$, we have

$$f_3(\alpha; p) = \sum_{p_1 > X^\sigma} f(\alpha; pp_1)\,,$$

for any p with $D < p \le 2D$. The last sum is actually finite, because $f(\alpha; pp_1)$ vanishes for $p_1 > 2X/D$. And, on putting $X_1 = XD^{-1}(\log X)^{-510}$, the contribution of the primes $p_1 > X_1$ is

$$\ll \sum_{X_1 < p_1 \le 2X/D} \frac{X}{pp_1} \ll \frac{X}{p}(\log X)^{-1}\log\log X\,,$$

by Mertens' theorem. Hence we have

$$F_3(\alpha) = \sum_{D<p\le 2D}\ \sum_{X^\sigma < p_1 \le X_1} f(\alpha; pp_1) + O\left(\sum_{D<p\le 2D} \frac{X\log\log X}{p\log X}\right),$$

and therefore, by (13), when q and a are coprime integers satisfying (9), we have

(19) $$F_3(\alpha) = \Delta_1 \Xi q^{-1} S(q,a)v(\beta) + O(\Delta_1 X(\log X)^{-1}\log\log X)\,,$$

where $\beta = \alpha - a/q$ and

$$\Xi = \sum_{X^\sigma < p_1 \le X_1} \frac{1}{p_1} = \log\left(\frac{\log(X/D)}{\sigma\log X}\right) + O\left(\frac{\log\log X}{\log X}\right),$$

by Mertens' theorem again. And, by (4), we see that

(20) $$\Xi = \log\left(\frac{\sigma_0 + 9\varepsilon}{\sigma}\right) + O\left(\frac{\log\log X}{\log X}\right) < \frac{1-\varepsilon}{2}\,.$$

After we get the formula (19) on $F_3(\alpha)$, what remains to execute is completely routine procedure. We have $V_6(q,a,n) \ll q^{-2}$ by the former bound in (12), so applying (19) we see

$$\sum_{q\le(\log X)^{500}}\ \sum_{\substack{a=1\\(a,q)=1}}^{q} V_6(q,a,n)W_3(q,a,n)$$

$$= \Delta_1\Xi \sum_{q\le(\log X)^{500}}\ \sum_{\substack{a=1\\(a,q)=1}}^{q} V_7(q,a,n)\int_{-(\log X)^{500}X^{-3}}^{(\log X)^{500}X^{-3}} v(\beta)^7 e(-n\beta)d\beta$$

$$+ O\left(\sum_{q\le(\log X)^{500}} q^{-1}\Delta_1 X\frac{\log\log X}{\log X}\int_{-\infty}^{\infty}|v(\beta)|^6 d\beta\right),$$

and the last error is $O(\Delta_1 X^4(\log X)^{-1}(\log\log X)^2)$, by the latter bound in (12). Now we introduce the familiar singular series and the singular integral associated with sums of seven cubes, defined respectively by

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} V_7(q,a,n)\,, \quad I(n) = \int_{-\infty}^{\infty} v(\beta)^7 e(-n\beta)d\beta\,.$$

It follows quite easily from (12) that

$$\int_{-(\log X)^{500}X^{-3}}^{(\log X)^{500}X^{-3}} v(\beta)^7 e(-n\beta)d\beta = I(n) + O(X^4(\log X)^{-3000})\,,$$

and also that

$$\sum_{q\le(\log X)^{500}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} V_7(q,a,n) = \mathfrak{S}(n) + O((\log X)^{-500/3})\,.$$

Moreover, one obtains the familiar bounds

$$1 \ll \mathfrak{S}(n) \ll 1\,, \quad X^4 \ll I(n) \ll X^4\,. \tag{21}$$

Actually, the both upper bounds are immmediate by (12). The lower bound for $\mathfrak{S}(n)$ is contained in Theorem 4.6 of Vaughan [8]. As for the lower bound for $I(n)$, one applies Fourier's inversion formula, and gets the expression

$$I(n) = \int\cdots\int \frac{1}{3^7}(u_1u_2\ldots u_6(n-u_1-u_2-\cdots-u_6))^{-2/3}du_1\ldots du_6\,,$$

where the domain of the integral is defined by $X^3 \le u_j \le (2X)^3$ $(1 \le j \le 6)$ and $X^3 \le n-u_1-u_2-\cdots-u_6 \le (2X)^3$. Then, recalling that $(2X)^3 = n$ and considering the contribution of the domain defined by $X^3 \le u_j \le (7/6)X^3$ $(1 \le j \le 6)$, for example, one may confirm the claimed lower bound for $I(n)$.

We gather these results to compute the right hand side of (18) for $\nu = 3$, and then, recalling (11) with (17) and (21), we conclude that

$$R_3(n) = \Delta_1^2\Delta_2^3\Delta_3^2\mathfrak{S}(n)I(n)\big(\Xi + O((\log X)^{-1}(\log\log X)^2)\big)\,. \tag{22}$$

We next turn to $R_1(n)$. From (14), (19) and the trivial formula $f_1(\alpha; p) = f(\alpha; p) - f_3(\alpha; p)$, we may deduce a formula for $F_1(\alpha)$ corresponding to (19), with $1-\Xi$ in place of Ξ. Then, through the same lines as the case of $R_3(n)$ above, we obtain a formula for $R_1(n)$ similar to (22), in which Ξ is replaced by $1-\Xi$. Hence we have

$$R_1(n) - R_3(n) = \Delta_1^2\Delta_2^3\Delta_3^2\mathfrak{S}(n)I(n)\big(1-2\Xi + O((\log X)^{-1}(\log\log X)^2)\big)\,.$$

In view of (20), therefore, the desired inequlity (8) is established, provided that n is sufficiently large in terms of ε. This completes the proof of the Theorem, as we already observed in §2.

References

[1] A. Balog, On additive representation of integers, Acta Math. Hungar. **54** (1989), 297–301.
[2] A. Balog and A. Sárközy, On sums of integers having small prime factors, I, Studia Sci. Math. Hungar. **19** (1984), 35–47.
[3] J. Brüdern and T. D. Wooley, On Waring's problem for cubes and smooth Weyl sums, Proc. London Math. Soc. (3) 82 (2001), 89–109.
[4] A. Fujii, An additive problem in theory of numbers, Acta Arith. **40** (1981), 41–49.
[5] A. Fujii, Some additive problems of numbers, in "Elementary and analytic theory of numbers" (Warsaw, 1982), 121–141, Banach Center Publ., 17, PWN, Warsaw, 1985.
[6] G. Harcos, Waring's problem with small prime factors, Acta Arith. **80** (1997), 165–185.
[7] K. Kawada, On sums of seven cubes of almost primes, Acta Arith. **117** (2005), 213–245.
[8] R. C. Vaughan, The Hardy-Littlewood Method, 2nd ed. Cambridge Univ. Press, 1997.
[9] T. D. Wooley, Breaking classical convexity in Waring's problem: Sums of cubes and quasi-diagonal behaviour, Invent. Math. **122** (1995), 421–451.
[10] T. D. Wooley, Sums of three cubes, Mathematika **47** (2000), 53–61.

Department of Mathematics
Faculty of Education
Iwate University
Morioka, 020–8550 Japan
e-mail: kawada@iwate-u.ac.jp