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# Some Mean Value Theorems for the Riemann Zeta-Function and Dirichlet $L$-Functions

by

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*Dedicated to Professor Akio Fujii on the occasion of his retirement*

## 1. Introduction

The theory of the Riemann zeta-function $\zeta(s)$ and Dirichlet $L$-functions $L(s,\chi)$ abounds with unsolved problems. Chronologically the first of these, now known as the Riemann Hypothesis (RH), originated from Riemann's remark that it is very probable that all non-trivial zeros of $\zeta(s)$ lie on the line $\Re s = \frac{1}{2}$. Later on Piltz conjectured the same for all of the functions $L(s,\chi)$ (GRH). The vertical distribution of the zeta zeros is the subject of Montgomery's pair correlation conjecture, which can also be generalized for $L(s,\chi)$. In this theory there are many other questions, most of them still open, about value distribution, non-existence of linear relations among zeros of a function, non-existence of common zeros of these functions etc. The results mentioned in this article may be seen as a first step in addressing some of these matters.

We adopt the usual notation, $s = \sigma + it \in \mathbb{C}$ with $\sigma, t \in \mathbb{R}$. The non-trivial zeros of $\zeta(s)$ will be denoted as $\rho = \beta + i\gamma$, and those of $L(s,\chi)$ as $\rho(\chi) = \beta(\chi) + i\gamma(\chi)$. The parameter $T$ tends to $\infty$. It is well-known that the number of zeros of $\zeta(s)$ in $0 < t \le T$ is $N(T) = \dfrac{T}{2\pi}\log\dfrac{T}{2\pi e} + O(\log T)$, and for a primitive character $\chi$ to the modulus $q$ this number is $N(T,\chi) = \dfrac{T}{2\pi}\log\dfrac{qT}{2\pi e} + O(\log qT)$. In this paper the Dirichlet $L$-functions are always those associated with primitive characters. The Gaussian sum associated with a character $\chi \pmod q$ is $\tau(\chi) = \displaystyle\sum_{m=1}^{q} \chi(m) e^{\frac{2\pi i m}{q}}$. For basic facts about the Riemann zeta-function and Dirichlet $L$-functions we refer the reader to the books of Davenport [1], Montgomery and Vaughan [9], and Titchmarsh [10], and for related material to Gonek [6].

Among the contributions of A. Fujii to this subject are the following results from [2]–[5]:

$$\sum_{0<\gamma\le T} \zeta'(\rho) \sim \frac{T}{4\pi}\log^2\frac{T}{2\pi},$$

$$\lim_{T\to\infty}\frac{2\pi}{T}\sum_{0<\gamma\le T}\Big(L\Big(\frac{1}{2}+i\gamma,\chi\Big)-1\Big) = -L(1,\overline{\chi})\chi(-1)\tau(\chi)\frac{\mu(q)}{\phi(q)}+\frac{L'}{L}(1,\chi),$$

$$\lim_{T\to\infty}\frac{2\pi}{T}\sum_{0<\gamma(\chi)\le T}\Big(L\Big(\frac{1}{2}+i\gamma(\chi),\psi\Big)-1\Big) = -\delta(q_\chi,q_\psi)L(1,\overline{\psi}\chi)\psi(-1)\tau(\psi)\frac{\tau(\overline{\chi}\psi_0)}{\phi(q_\psi)} + \frac{L'}{L}(1,\psi\overline{\chi}),\ \text{(on GRH for } L(s,\chi)),$$

where $\psi$ is a primitive character to the modulus $q_\psi \ge 3$, and $\chi$ is a primitive character to the modulus $q_\chi \ge 1$, $\chi \ne \psi$, $\psi_0$ is the principal character mod $q_\psi$, and $\delta(q_\chi, q_\psi) = 1$ if $q_\chi \mid q_\psi$ and $= 0$ otherwise. For the first formula here, Fujii also gave the secondary terms and the error term both unconditionally and conditionally on RH. In fact, Fujii proved more general versions and variations (involving $\frac{1}{2}+i\frac{\gamma}{K}$ where $K$ is an integer $\ge 2$ and with certain weight functions in the summands) of these formulas.

## 2. Statement of results

The results announced here are generalizations of the above quoted results of Fujii, and are all unconditional.

For mean values over the zeros of the Riemann zeta-function we showed

$$\sum_{0<\gamma\le T}\zeta^{(j)}(\rho) = \frac{(-1)^{j+1}}{j+1}\frac{T}{2\pi}\Big(\log\frac{T}{2\pi}\Big)^{j+1} + O_j\big(T\log^j T\big),\quad (j\ge 1),$$

where $\zeta^{(j)}(s)$ is the $j$-th derivative of $\zeta(s)$. Assuming RH, this reveals very clearly that there are points $|s| \asymp T$ on the critical line, which are in particular zeros of $\zeta(s)$, at which the size of $|\zeta^{(j)}(s)|$ is $\gg \log^j T$.

Next, for $\chi$ a primitive character mod $q$, $3 \le q \le (\log T)^A$, where $A$ can be taken to be an arbitrarily large fixed number, we have

$$\sum_{0<\gamma\le T}\big(L(\rho,\chi)-1\big) = \Big(\frac{L'}{L}(1,\chi) - \frac{\mu(q)\chi(-1)\tau(\chi)L(1,\overline{\chi})}{\phi(q)}\Big)\frac{T}{2\pi} + O_A\Big(Te^{-c\sqrt{\log T}}\Big),$$

and for $j \ge 1$,

$$\sum_{0<\gamma\le T}L^{(j)}(\rho,\chi) = -\frac{\mu(q)\chi(-1)\tau(\chi)}{\phi(q)}\frac{T}{2\pi}\sum_{w=0}^{j}\frac{j!}{w!}\sum_{v=w}^{j}\frac{(-1)^v L^{(v-w)}(1,\overline{\chi})}{(v-w)!}\Big(\log\frac{qT}{2\pi}\Big)^w + \frac{T}{2\pi}\Big(\frac{L'}{L}\Big)^{(j)}(1,\chi) + O_{A,j}\Big(Te^{-c\sqrt{\log T}}\Big),$$

where $c$ is a non-effective constant depending on $A$ and $j$. Using the bounds

$$L^{(j)}(1,\chi) \ll_j (\log q)^{j+1},\quad (j=0,1,2,\dots),$$

$$\Big(\frac{L'}{L}\Big)^{(j)}(1,\chi) \ll_{A,j} q^{\frac{j+1}{A}},\quad (j=0,1,2,\dots),$$

$$L(1,\chi) \gg_A \frac{1}{q^{\frac{1}{A}}},$$

these results imply, for the range of $q$ specified as above, the simpler forms

$$\sum_{0<\gamma\le T} L(\rho,\chi) = \frac{T}{2\pi}\log\frac{T}{2\pi}\left(1 + O_A\left(\frac{\log\log T}{\log T}\right)\right),$$

i.e. the average value of $L(\rho,\chi)$ is 1;

$$\lim_{T\to\infty}\frac{1}{N(T)}\sum_{0<\gamma\le T} L'(\rho,\chi) = -\frac{\mu(q)\chi(-1)\tau(\chi)L'(1,\overline{\chi})}{\phi(q)},$$

so that since $\phi(q) \gg \dfrac{q}{\log\log q}$, $|\tau(\chi)| = \sqrt{q}$, $L'(1,\overline{\chi}) \ll \log q$, the average value of $L'(\rho,\chi)$ gets closer to 0 with larger $q$; for $j \ge 1$ we see that

$$\sum_{0<\gamma\le T} L^{(j)}(\rho,\chi) = \frac{(-1)^{j+1}\mu(q)\chi(-1)\tau(\chi)L(1,\overline{\chi})}{\phi(q)}\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^j\left(1+O_{A,j}\left(\frac{\log\log T}{\log T}\right)\right)$$

if $q$ is square-free, and if $q$ is not square-free then

$$\sum_{0<\gamma\le T} L^{(j)}(\rho,\chi) = \frac{T}{2\pi}\left(\frac{L'}{L}\right)^{(j)}(1,\chi) + O_{A,j}\left(Te^{-c\sqrt{\log T}}\right)$$

in which case since we do not know how close $\left(\frac{L'}{L}\right)^{(j)}(1,\chi)$ can get to 0 we can only say

$$\sum_{0<\gamma\le T} L^{(j)}(\rho,\chi) \ll_{A,j} T.$$

For mean values over the zeros with ordinates in $[0,T]$ of a Dirichlet $L$-function $L(s,\chi)$ where $\chi$ is a primitive character to the modulus $q_\chi \le \exp\left(c\sqrt{\log\frac{T}{2\pi}}\right)$, $c$ being an appropriate positive constant, the averages of $\zeta^{(\mu)}(s)$ are

$$\sum_{0<\gamma_\chi\le T}\left(\zeta(\rho_\chi)-1\right) = \frac{L'}{L}(1,\overline{\chi})\frac{T}{2\pi} - \frac{\zeta(\beta)}{\beta^2}\left(\frac{T}{2\pi}\right)^\beta + O\left(T\exp\left(-C\sqrt{\log\frac{T}{2\pi}}\right)\right),$$

and for $\mu \ge 1$

$$\begin{aligned}\sum_{0<\gamma_\chi\le T}\zeta^{(\mu)}(\rho_\chi) &= \left[\left(\frac{d}{ds}\right)^{\mu}\frac{L'}{L}(s,\overline{\chi})\right]_{s=1}\cdot\frac{T}{2\pi}\\ &\quad - \sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\sum_{j=0}^{\mu-\kappa}\frac{(\mu-\kappa)!}{(\mu-\kappa-j)!}\frac{(-1)^j\zeta^{(\kappa)}(\beta)}{\beta^{j+2}}\left(\frac{T}{2\pi}\right)^\beta\left(\log\frac{T}{2\pi}\right)^{\mu-\kappa-j}\\ &\quad + O\left(T\exp\left(-C\sqrt{\log\frac{T}{2\pi}}\right)\right),\end{aligned}$$

where $\beta$ is the possible exceptional zero of $L(s,\chi)$ (and the terms involving it are present only if $\beta$ exists), and $C$ is some positive constant. For $q$ satisfying $q(\log q)^4 \le \log T$, all of the terms involving $\beta$ can be absorbed into the error term.

Next, in the cases $\mu \ge 1$, we have for $q \le (\log T)^A$ with any fixed $A > 0$,

$$\sum_{0<\gamma_\chi \le T} L^{(\mu)}(\rho_\chi, \chi) = \frac{(-1)^{\mu+1}}{(\mu+1)} \frac{T}{2\pi}\left(\log \frac{qT}{2\pi}\right)^{\mu+1} + O\big(T(\log T)^{\mu+\epsilon}\big)$$

for any fixed $\epsilon > 0$.

Now let $\psi$ be a primitive character to the modulus $q_\psi$ and $\psi_0$ denote the principal character modulo $q_\psi$. Assume that $q = [q_\chi, q_\psi]$ (the least common multiple) satisfies $q(\log q)^4 \le \log T$. We have

$$\sum_{0<\gamma_\chi \le T} \big(L(\rho_\chi, \psi) - 1\big) = -\delta(q_\chi, q_\psi)\frac{\psi(-1)\tau(\psi)\tau(\overline{\chi}\psi_0)L(1, \chi\overline{\psi})}{\phi(q_\psi)}\frac{T}{2\pi} + \frac{L'}{L}(1, \overline{\chi}\psi)\frac{T}{2\pi} + O\left(T\exp\left(-C\sqrt{\log \frac{q_\psi T}{2\pi}}\right)\right).$$

For $\mu \ge 1$,

$$\sum_{0<\gamma_\chi \le T} L^{(\mu)}(\rho_\chi, \psi) = -\frac{\psi(-1)\tau(\psi)\tau(\overline{\chi}\psi_0)L(1, \chi\overline{\psi})}{\phi(q_\psi)}\frac{T}{2\pi}\left(\log \frac{q_\psi T}{2\pi}\right)^{\mu} + O\big(T(\log T)^{\mu-1+\epsilon}\big) \quad \text{if } q_\chi \mid q_\psi\,,$$

and

$$\sum_{0<\gamma_\chi \le T} L^{(\mu)}(\rho_\chi, \psi) = (-1)^{\mu}\left[\left(\frac{d}{ds}\right)^{\mu}\bigg|_{s=1} \frac{L'}{L}(s, \overline{\psi}\chi)\right]\frac{T}{2\pi} + O\left(T\exp\left(-C\sqrt{\log \frac{q_\psi T}{2\pi}}\right)\right) \quad \text{if } q_\chi \nmid q_\psi\,.$$

## 3. Remarks about the proofs

The basic idea of the proofs is to see the sums to be evaluated as a sum of residues, so we use

$$\sum_{c<\tilde{\gamma}<T} f(\tilde{\rho}) = \frac{1}{2\pi i}\int_R f(s)\frac{g'}{g}(s)\,ds\,,$$

where $\tilde{\rho}$ with imaginary part $\tilde{\gamma}$ runs through the zeros of $g$, and $R$ is the rectangle having corners at $a+ic,\ a+iT,\ 1-a+iT,\ 1-a+ic$ with an appropriate $c$ (a constant) and $T$ (so as to avoid the poles of the integrand). Using well-known estimates for $\zeta(s),\ L(s,\chi)$ and their derivatives, the horizontal integrals and the integral on the vertical segment with real part $a$ can be bounded easily, and the main term is seen to come from the integral along the vertical segment with real part $1-a$. Classical techniques and results of analytic number theory such as Perron's formula, Dirichlet's hyperbola method, partial summation, the prime number theorem (also for arithmetic progressions), the Pólya-Vinogradov inequality are employed in the calculations.

The detailed version of the proofs are in [7] and [8].

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