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Uniqueness Theorems for L-Functions

by

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Dedicated to Professor Akio Fujii on the occasion of his retirement

Abstract. We prove uniqueness theorems for L-functions from the (extended) Selberg class. Moreover, we generalize asymptotic formulae for certain discrete moments of Dirichlet L-functions at the zeros of another Dirichlet L-function $L(s,\chi)$ due to Fujii to roots of $L(s,\chi)=c$, where c is an arbitrary complex number.

CONTENTS

1. Introduction and Statement of the Main Results

The famous five value theorem of Rolf Nevanlinna states that any two non-constant meromorphic functions which share five distinct values are identical. Here two meromorphic functions f and g are said to share a value $c \in \mathbb{C} \cup \{\infty\}$ if the sets of preimages of c under f and under g are equal, for short $f^{-1}(c) := \{s \in \mathbb{C} : f(s) = c\} = g^{-1}(c)$. Furthermore, f and g are said to share the value c counting multiplicities (CM) if the latter identity of sets holds and if the roots of the equations $f(s) = c$ and $g(s) = c$ have the same multiplicities; if there is no restriction on the multiplicities, f and g are said to share the value c ignoring muliplicities (IM). Since the functions $f(s) = \exp(s)$ and $g(s) = \exp(-s)$ share the four values $0, \pm 1, \infty$, the number five in Nevanlinna's statement is best possible. If multiplicities are taken into account, Nevanlinna proved that any two meromorphic functions f and g that share four distinct values $c_1, \dots, c_4$ CM are identical or can be transformed into one another by a Moebius transformation M in such a way that $g \equiv M \circ f$ and M fixes two of the points c_j while the other two are interchanged. Also the

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number four of values shared CM is best possible. For more results of this type we refer to [9, 19, 31].

In [27] (resp. Chapter 7 in [28]), Steuding investigated how many values L-functions can share. In this special case better estimates are possible than those which Nevanlinna's theorems provide (since there is additional information about the functions available). Using Nevanlinna theory it was shown that distinct L-functions do not share any finite complex value CM; here the notion of an L-function $\mathcal{L}(s)$ is according to the extended Selberg class $\mathcal{S}^\sharp$ which consists, roughly speaking, of all Dirichlet series $\mathcal{L}(s) = \sum_{n=1}^\infty a(n)n^{-s}$ with coefficients $a(n) \ll n^\epsilon$ which possess a meromorphic continuation of finite order to the whole complex plane with a possible single pole at $s = 1$, and satisfy a functional equation of Riemann type; the precise definition of $\mathcal{S}^\sharp$ is given in Section 2. Examples of such L-functions are the Riemann zeta-function $\zeta(s)$, Dedekind zeta-functions to number fields, Dirichlet and Hecke L-functions attached to characters, L-functions associated with modular forms, as well as certain Epstein zeta-functions built from quadratic forms. The precise formulation of the uniqueness theorem mentioned above is as follows:

THEOREM A. *If two elements of the extended Selberg class $\mathcal{S}^\sharp$ share a complex value $c \neq \infty$ CM, then they are identical.*

It should be noticed that the normalization assumption $a(1) = 1$ in Theorem 7.11 from [28] can be removed by the method explained in Section 7.2 of [28].

Concerning values which are shared IM, recently Bao Qin Li [17] has shown the following result.

THEOREM B. *If two elements of the extended Selberg class $\mathcal{S}^\sharp$ with constant coefficient $a(1) = 1$ satisfy the same functional equation and share two complex values IM, then they are identical.*

It is an interesting question to which extent an L-function can share values with an arbitrary meromorphic function. In this direction, Bao Qin Li [16] has shown the following uniqueness result.

THEOREM C. *Let $a, b \in \mathbb{C}$ be two distinct values and let f be a meromorphic function in $\mathbb{C}$ with finitely many poles. If f and a non-constant L-function $\mathcal{L} \in \mathcal{S}^\sharp$ share a CM and b IM, then $f \equiv \mathcal{L}$.*

In particular, this implies that if f is meromorphic in $\mathbb{C}$ and shares one value CM, a second value IM and the value ∞ IM with a non-constant L-function $\mathcal{L}$, then $f \equiv \mathcal{L}$ [16, Corollary 2]. (Note that $s = 1$ is the only possible pole for an L-function; see Section 2.) This gives a partial answer to the following question by C.C. Yang mentioned in [16]:

QUESTION. *If f is meromorphic in $\mathbb{C}$ and f shares two distinct values $a, b \in \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ CM and a value $c \in \mathbb{C} \setminus \{a, b, 0\}$ IM with the Riemann zeta-function ζ, can we conclude that $f \equiv \zeta$?*

If any one of the values a, b, c is ∞, Theorem C gives an affirmative answer to this question. The following result deals with the case $a, b, c \neq \infty$.

THEOREM 1. *Let f be meromorphic in $\mathbb{C}$ and $\mathcal{L} \in \mathcal{S}^\sharp$ be a non-constant L-function such that f and $\mathcal{L}$ share the values $a, b \in \mathbb{C}$ CM and the value $c \in \mathbb{C}$ IM. Then $f \equiv \mathcal{L}$.*

This theorem will be proved in Section 3 by means of Nevanlinna theory. It turns out that the proof (which is similar to Bao Qin Li's proof of Theorem C) does not fully exploit the various properties of L-functions (at least not explicitly); it just uses the knowledge about the growth of the characteristic function of L-functions (Lemma 6). Therefore, the proof remains valid for a larger class of meromorphic functions, leading to the following result.

THEOREM 2. *Let $a, b, c, d \in \overline{\mathbb{C}}$ be distinct. Let f and g be meromorphic non-constant functions in $\mathbb{C}$ which share the values a, b CM and the value c IM and such that f or g assumes the value d only finitely many times. Assume that f has finite non-zero order and that one of the following two conditions is satisfied.*

(1) *The order of f is not an integer.*

(2) *The order of f is an integer and f has maximal type*[1].

Then $f \equiv g$.

This is a variation on another result of R. Nevanlinna [20, Satz 6] who had shown that two meromorphic functions of finite non-integer order which share three values CM have to coincide. For extensions of Nevanlinna's result see also [31, Section 2.3]; in particular, by [31, Theorem 2.25] the condition that three values are shared CM can be replaced by the condition that two values are shared CM and that for a third value c which is not a Borel exceptional value of f each zero of $f - c$ of multiplicity m is a zero of $g - c$ of multiplicity at least m. The proof of Theorems 1 and 2 will be given in Section 3.

Next we investigate the distribution of values of pairs of L-functions. It is expected that *primitive* L-functions of the Selberg class $\mathcal{S}$ cannot share any complex value. Here $\mathcal{S} \subset \mathcal{S}^\sharp$ is, roughly speaking, the subset of all $\mathcal{L}$ which possess an Euler product; for the precise definition of $\mathcal{S}$ we refer again to Section 2. A function $\mathcal{L} \in \mathcal{S}$ is said to be *primitive* if any factorization within $\mathcal{S}$ is trivial. Examples of primitive functions are the Riemann zeta-function, Dirichlet L-functions associated with a primitive character, and L-functions attached to elliptic curves. We do not want to recall basic results about the Selberg class nor explain its arithmetical relevance but stress that any L-function in $\mathcal{S}$ has a factorization into primitive elements and that this factorization is unique if the deep Selberg orthogonality conjecture is true; another consequence of the latter conjecture is the unsolved Artin conjecture that all Artin L-functions have an analytic continuation to $\mathbb{C} \setminus \{1\}$. A good source for further reading is the monograph of M. R. Murty & V. K. Murty [18] from which we quote that if it could be shown that for any family of primitive L-functions $\mathcal{L}_1, \dots, \mathcal{L}_k$ in $\mathcal{S}$ there exist complex numbers $s_1, \dots, s_k$ such that $\mathcal{L}_j(s_\ell) = 0$ if and only if $j = \ell$, then the factorization into primitive elements would be unique. We note that $\zeta(s)$ and $\zeta(s)^2$ have the same zeros and the same pole. On the contrary, elements of the extended Selberg class $\mathcal{S}^\sharp$ may share a complex value: for instance, if $\mathcal{L} \in \mathcal{S}^\sharp$, then any constant multiple $\lambda\mathcal{L}$ with $\lambda \in \mathbb{C}$ also belongs to the extended Selberg class $\mathcal{S}^\sharp$, hence $\mathcal{L}$ and $\lambda\mathcal{L}$ share their zeros and poles CM provided that $\lambda \neq 0$.

[1] For the definition of functions of maximal type see Section 3.

Concerning primitive L-functions sharing a complex value, we shall prove the following uniqueness theorem for the subset of degree one elements of the Selberg class:

THEOREM 3. *If two elements of the Selberg class $\mathcal{S}$, both of degree one, share a complex value $c \in \mathbb{C} \setminus M$, then they are identical.*

The proof of this theorem relies on a joint universality property of Dirichlet L-functions associated with non-equivalent characters and will be given in Section 4. As Kaczorowski & Perelli [12] proved, the degree one elements of the Selberg class $\mathcal{S}$ are Riemann's zeta-function $\zeta(s)$, Dirichlet L-functions associated with primitive characters and shifts thereof, i.e., $L(s + i\theta, \chi)$, where θ is an arbitrary real number and χ is a primitive character. All degree one functions are primitive elements in $\mathcal{S}$.

Recall the definition of Dirichlet L-function. Given a Dirichlet character $\chi \bmod q$ (i.e., a group homomorphism from the group of prime residue classes modulo q to $\mathbb{C}^*$, extended to $\mathbb{Z}$ by setting $\chi(n) = 0$ for all n which are not coprime with q), the associated Dirchlet L-function is defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where the product is taken over all prime numbers p. Both, the Dirichlet series and the Euler product converge in the half-plane $\operatorname{Re} s > 1$ and define an analytic function. The Riemann zeta-function may be considered as the Dirichlet L-function to the unique character $\chi_0 \bmod 1$. By analytic continuation, $L(s, \chi)$ extends to a meromorphic function in the complex plane with a single pole at $s = 1$ if χ is a principle character (i.e., $\chi(n) = 1$ for all n coprime with some q). A character that is not induced by a character of smaller modulus is said to be primitive; principle characters are not considered as primitive. Of special interest are Dirichlet L-functions associated with primitive characters χ, since any non-principle character $\chi \bmod q$ is induced by a uniquely determined primitive character and the corresponding L-functions differ from one another by a finite Euler product with non-trivial factors only for the prime divisors of q which extends to an entire function with a very regular value-distribution. In a similar way the L-function attached to a principle character $\chi \bmod q$ equals $\zeta(s)$ times a finite Euler product.

Specializing on Dirichlet L-functions, we may also follow a different approach which seems to be of independent interest. Here we obtain more subtle information on the value distribution of Dirichlet L-functions by considering certain discrete moments:

THEOREM 4. *Let $\chi \bmod q$ and $\psi \bmod Q$ be distinct primitive characters and, for fixed $c \in \mathbb{C}$, denote the solutions of $L(s, \chi) = c$ in the right half-plane by $\rho_\chi = \beta_\chi + i\gamma_\chi$. Then, as $T \to \infty$,*

$$\sum_{0<\gamma_\chi<T} L(\rho_\chi, \psi) = \alpha_1 \frac{T}{2\pi} \log T + \alpha_0 \frac{T}{2\pi} + O\left(T \exp(-b_1 (\log T)^{\frac{1}{4}-\epsilon})\right), \tag{1}$$

where $b_1 > 0$ is an absolute constant and the constants $\alpha_j = \alpha_j(c, \chi, \psi)$ for $j = 0, 1$ depend on c as well as on the characters $\chi \bmod q$ and $\psi \bmod Q$, and are given by the formulae (30) *and* (31) *below. The statement is also true if either χ or ψ is equal to* *the principal character* $\chi_0 \bmod 1$ (*in which case the corresponding L-function equals the Riemann zeta-function*).

Here the summation is taken only over those c-points ρ_χ which have non-negative real-part. This is due to the fact that there are, for example, infinitely many zeros of $L(s, \chi)$ on the negative real axis; however, since each of which is related to a pole of the Gamma-factor in the functional equation (12), these zeros are of minor interest than those inside the critical strip $0 \leq \operatorname{Re} s \leq 1$; the situation for general $c \neq 0$ is not much different.

It should be mentioned that Theorem 4 extends previous results due to Fujii [2, 3, 5] who proved under assumption of the respective Riemann hypothesis the following asymptotic formulae:

$$\sum_{|\gamma| \leq T} L(\rho, \chi) = \frac{T}{\pi} \log T + O(T),$$

where the summation is taken over the nontrivial zeros $\rho = \beta + i\gamma$ of the Riemann zeta-function $\zeta(s)$ (which is covered as case $c = 0$ and $\chi = \chi_0 \bmod 1$), as well as

$$\sum_{|\gamma_\chi| \leq T} \zeta(\rho_\chi) = \frac{T}{\pi} \log T + O(T),$$

where the summation is over the zeros $\rho_\chi = \beta_\chi + i\gamma_\chi$ of a Dirichlet L-function $L(s, \chi)$ to a primitive character χ (the case $c = 0$ and $\psi = \chi_0 \bmod 1$ above), and, finally,

$$\sum_{|\gamma_\chi| \leq T} L(\rho_\chi, \psi) = \left(1 - \frac{1}{\varphi([q, Q])} \sum_{\substack{a \bmod [q,Q] \\ (a,[q,Q])=1}} (\overline{\chi}\psi)(a)\right) \frac{T}{\pi} \log T + O(T), \tag{2}$$

where $\chi \bmod q$ and $\psi \bmod Q$ are (not necessarily distinct) primitive characters, the summation is over the nontrivial zeros $\rho_\chi = \beta_\chi + i\gamma_\chi$ of $L(s, \chi)$ (the case $c = 0$ with $\chi, \psi \neq \chi_0 \bmod 1$), and $[\ ,\ ]$ and $(\ ,\ )$ denote the least common multiple and the greatest common divisor, respectively. Actually, Fujii obtained much stronger error terms than those given above, however, applying a slightly different method Steuding [26] succeeded in establishing the above formulae unconditionally at the expense of the weaker error terms stated above. There are some more results of this flavour in the literature; for example, Fujii's work on the Hurwitz zeta-function [4] and Steuding's variation [25], as well as further work [7] of Garunkštis et al. to mention only a few.

We may use Theorem 4 for another proof of the fact that distinct Dirichlet L-functions do not share a complex value c; however, there is a certain technical obstacle to deduce Theorem 3 completely which is related to the arithmetical nature of the coefficients α_j. The details and the proof of Theorem 4 are given in Section 5. We start, however, with the precise definition of the Selberg class and some preliminaries.

2. The (extended) Selberg Class

In 1989, Selberg introduced a rather general class $\mathcal{S}$ consisting of all functions which have a representation as Dirichlet series

$$\mathcal{L}(s) = \sum_n \frac{a(n)}{n^s}$$

and satisfy

(i) **Ramanujan hypothesis**: $a(n) \ll n^\epsilon$ for any $\epsilon > 0$, where the implicit constant may depend on ϵ.

(ii) **Analytic continuation**: there exists a non-negative integer k such that $(s-1)^k\mathcal{L}(s)$ is an entire function of finite order.

(iii) **Functional equation**: $\mathcal{L}(s)$ satisfies a functional equation of type

$$\Lambda_{\mathcal{L}}(s) = \omega\overline{\Lambda_{\mathcal{L}}(1-\overline{s})},$$

where

$$\Lambda_{\mathcal{L}}(s) := \mathcal{L}(s)Q^s\prod_{j=1}^{f}\Gamma(\lambda_j s + \mu_j)$$

with positive real numbers Q, λ_j, and complex numbers μ_j, ω with $\mathrm{Re}\,\mu_j \geq 0$ and $|\omega| = 1$.

(iv) **Euler product**: $\mathcal{L}(s)$ satisfies

$$\mathcal{L}(s) = \prod_p \mathcal{L}_p(s)\,, \quad \text{where} \quad \mathcal{L}_p(s) = \exp\left(\sum_{k=1}^{\infty}\frac{b(p^k)}{p^{ks}}\right)$$

with suitable coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < \frac{1}{2}$.

The extended Selberg class $\mathcal{S}^\sharp$ is defined as the set of all functions $\mathcal{L}$ satisfying **(i)–(iii)**. Usually, a function is said to be an L-function if it possesses an Euler product representation; however, it appears that $\mathcal{S}^s harp$ contains interesting examples of functions which do not have an Euler product, and in some aspects it is worthwhile to study the extended Selberg class.

The degree of $\mathcal{L} \in \mathcal{S}^\sharp$ is defined by $\mathrm{d}_{\mathcal{L}} = 2\sum_{j=1}^{f}\lambda_j$; although the data of the functional equation is not unique, the degree is well-defined as follows from an asymptotic formula for the number $N_c(T)$ of c-points $\rho = \beta + i\gamma$ of $\mathcal{L} \in \mathcal{S}^\sharp$ with $\beta \geq 0$ and $|\gamma| \leq T$, namely,

$$N_c(T) = \frac{\mathrm{d}_{\mathcal{L}}}{\pi}T\log\frac{T}{e} + \frac{T}{\pi}\log(\lambda Q^2) + O(\log T)\,. \tag{3}$$

For $c \neq 1$ this is Theorem 7.7 from [28]; by technical refinement one can prove the above formula also for the case $c = 1$ (as indicated in Section 7.2 from [28]). The quantity λ is defined by $\lambda = \prod_{j=1}^{f}\lambda_j^{2\lambda_j}$. It is worth noticing that the main term in the asymptotic formula is independent of c.

The Selberg class is of particular interest with respect to its structure. Both, $\mathcal{S}$ and $\mathcal{S}^\sharp$ are multiplicatively closed which gives rise to the notions of divisibility and primitivity as mentioned in the introduction. The only element of degree zero in $\mathcal{S}$ is the function constant 1 whereas the set of degree zero elements in the extended class $\mathcal{S}^\sharp$ consists of all constant functions and certain Dirichlet polynomials. There are no elements in $\mathcal{S}^\sharp$ of degree $0 < \mathrm{d} < 1$. As already mentioned above, the degree one elements of the Selberg class $\mathcal{S}$ are, as shown by Kaczorowski & Perelli [12], given by $\zeta(s)$ and $L(s+i\theta, \chi)$, where θ is an arbitrary real number and χ is a primitive character. The case of degree one in the extended class $\mathcal{S}^\sharp$ is more delicate. Here Kaczorowski & Perelli [12] proved that any element in $\mathcal{L} \in \mathcal{S}^\sharp$ of degree one has a unique representation of the form

$$\mathcal{L}(s) = \sum_{\chi \bmod q} P_\chi(s+i\theta) L(s+i\theta, \chi^*)\,,$$

where θ is a real number and the summation is over all characters $\chi \bmod q$ and χ^* denotes the unique primitive or principal character which induces $\chi \bmod q$, and P_χ is a Dirichlet polynomial of degree zero in $\mathcal{S}^\sharp$. Recently, Kaczorowski & Perelli [13] succeeded in proving that $\mathcal{S}^\sharp$ contains no element of degree $1 < \mathrm{d} < 2$. Unfortunately, the characterization of degree two elements in $\mathcal{S}$ or even in $\mathcal{S}^\sharp$ is incomplete.

Excellent references for the theory of the (extended) Selberg class are [10, 21]. Besides, Kaczorowski et al. [11] also considered the modified class $\tilde{\mathcal{S}}^\sharp$ where the conjugation in the axiom on the functional equation is dropped; this class is relevant with respect to Hecke's theory of modular forms and associated Dirichlet series. Most of our results can easily be applied to this class too. We want to remark that there is no proper definition of an L-function. There have been a few attempts to define an axiomatic setting for arithmetically relevant L-functions; the (extended) Selberg class seems to be the most promising approach so far.

3. Nevanlinna Theory–Proof of Theorem 1 & 2

Our proof of Theorems 1 and 2 is based on Nevanlinna theory. For the convenience of the reader, we recall the standard notations and main results as far as they are relevant for our purposes. (For more details we refer to [9].) We denote the proximity function of a meromorphic function f by $m(r, f)$ and its counting functions by $N(r, f)$ (counting multiplicities) and $\overline{N}(r, f)$ (ignoring multiplicities). Then for its Nevanlinna characteristic $T(r, f) := m(r, f) + N(r, f)$ the First Fundamental Theorem

$$T(r, f) = T\left(r, \frac{1}{f}\right) + O(1)$$

holds, and for the sums and products of meromorphic functions $f_1, \dots, f_p$ without poles at the origin we have the estimates

$$U\left(r, \prod_{j=1}^{p} f_j\right) \le \sum_{j=1}^{p} U(r, f_j), \qquad U\left(r, \sum_{j=1}^{p} f_j\right) \le \sum_{j=1}^{p} U(r, f_j) + \log p$$

for $U(r, .) = m(r, .),\ N(r, .),\ \overline{N}(r, .),\ T(r, .)$. Furthermore, $T(r, .)$ is monotonically increasing.

The *order* $\varrho(f)$ of a function f meromorphic in $\mathbb{C}$ is defined as

$$\varrho(f) := \limsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log^+ r}\,.$$

(Here, $\log^+ x := \max\{\log x;\ 0\}$ for $x > 0$ and $\log^+ 0 := 0$.) If f is a non-vanishing entire function of finite order, then f has the form $f = e^p$ with a polynomial p, and we have $\varrho(f) = \deg(p)$ and $T(r, f) = O(r^{\varrho(f)})$; in particular, $\varrho(f)$ is an integer.

For a meromorphic function f in $\mathbb{C}$ of finite order $\varrho(f)$, the *type* $\tau(f)$ is defined as

$$\tau(f) := \limsup_{r\to\infty} \frac{T(r, f)}{r^{\rho(f)}}\,.$$

If $\tau(f) = \infty$, we say that f has *maximal type*.

One of the main insights in Nevanlinna theory is the Second Fundamental Theorem. It has the following extension which is the main tool of our proof. Here, by $S(r, f)$ we denote an arbitrary term which is $o(T(r, f))$ as $r \to \infty$, $r \notin E$ where $E \subseteq [0, \infty[$ is a set of finite Lebesgue measure; if f has finite order, one can choose $E = \emptyset$.

LEMMA 5. *If f is meromorphic in $\mathbb{C}$ and non-constant and $\varphi_1, \varphi_2, \varphi_3$ are three distinct functions meromorphic in $\mathbb{C}$, then the estimate*

$$T(r, f) \le \sum_{j=1}^{3} \overline{N}\left(r, \frac{1}{f-\varphi_j}\right) + \sum_{j=1}^{3}\Big\{O(T(r, \varphi_j)) + S(r, \varphi_j)\Big\} + S(r, f)$$

holds.

Proof. This is a slight variation of the Second Fundamental Theorem for small functions [9, Theorem 2.5] whose proof can be easily adopted. We omit the details. •

Another crucial observation is the following asymptotic estimate for the Nevanlinna characteristic of L-functions [28, Theorem 7.9].

LEMMA 6. *Every L-function $\mathcal{L}$ of degree d satisfies*

$$T(r, \mathcal{L}) = \frac{d}{\pi} \cdot r \log r + O(r)\,.$$

The proof of Theorem 2 makes use of several ideas of Bao-Qin Li's proof of Theorem C, and in fact it would have been possible to stick even closer to his reasoning. However, we prefer to replace the crucial part of his proof by a different argument based on Lemma 5.

Proof of Theorem 2. It is an easy consequence from the First Fundamental Theorem that Moebius transformations leave the Nevanlinna characteristic invariant (up to some additive constant); in particular, they do not affect the order and type of a meromorphic function. Furthermore, if f and g share a value α CM (IM) and if M is a Moebius transformation, then $M \circ f$ and $M \circ g$ share the value $M(\alpha)$ CM (IM). Therefore, after post-composition with an appropriate Moebius transformation we may assume $b = \infty$, hence $a, c, d \in \mathbb{C}$.

The Second Fundamental Theorem and the value sharing properties of g and f yield

$$\begin{aligned}
T(r,g) &\le \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{g-a}\right) + \overline{N}\left(r,\frac{1}{g-c}\right) + S(r,g)\\
&= \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f-a}\right) + \overline{N}\left(r,\frac{1}{f-c}\right) + S(r,g)\\
&\le 3T(r,f) + S(r,g)\,.
\end{aligned}$$

From this estimate it is almost clear that $\varrho(g) \le \varrho(f)$. To be precise, however, an estimate for $T(r,g)$ is required which holds for all $r > 0$ without the exceptional set involved in the term $S(r,g)$. As in the proof of Theorem 1 in [16], this is done by the following standard argument: Let some $\varepsilon > 0$ be fixed. By the last estimate and the definitions of $S(r,g)$ and $\varrho(f)$, there exists a set $E \subset [0,\infty[$ of finite Lebesgue measure such that

$$T(r,g) \le 4T(r,f) \le r^{\varrho(f)+\varepsilon} \quad \text{for all } r > 0 \text{ with } r \notin E\,. \tag{4}$$

Let some $r > 0$ be given. Then there exists some $r_0 \in [r, r+|E|+1]$ with $r_0 \notin E$, so by the monotonicity of $T(r,g)$ we deduce that

$$T(r,g) \le T(r_0,g) \le r_0^{\varrho(f)+\varepsilon} \le (r+|E|+1)^{\varrho(f)+\varepsilon}\,.$$

Since this holds for all $r > 0$, we conclude that $\varrho(g) \le \varrho(f) + \varepsilon$. This holds for all $\varepsilon > 0$ which gives $\varrho(g) \le \varrho(f) \ (< \infty)$. By interchanging the role of f and g we obtain $\varrho(g) = \varrho(f)$.

Since f has finite order, the exceptional sets in the definition of $S(r,f)$ can be chosen to be empty. Therefore the same reasoning leading to (4) (with interchanged roles of f and g) shows the existence of some $r_0 > 0$ such that

$$T(r,f) \le 4T(r,g) \quad \text{for all } r > r_0\,.$$

This shows that if f has maximal type (as in case (2)), also g has maximal type.

Now w.l.o.g. we may assume that g is the function which assumes the value d only finitely many times.

Since f and g share the values a and ∞ CM, the function

$$F := \frac{g-a}{f-a}$$

is entire and non-vanishing. Furthermore, $\varrho(F) \le \max\{\varrho(f), \varrho(g)\} = \varrho(f)$. So F must have the form $F = e^P$ where P is a polynomial of degree at most $q := \lfloor \varrho(f) \rfloor$ (the largest integer not exceeding $\rho(f)$). Therefore

$$T(r,F) = O(r^q)\,.$$

In view of the assumptions on the order and type of f, this means that the growth of F is slower (in the sense of Nevanlinna theory) than the growth of f and g. This is the crucial argument in the proof.

Now we assume $f \not\equiv g$. Then $F \not\equiv 1$. Since g and f share c IM, for each $s \in \mathbb{C}$ with $g(s) = c$ we have $F(s) = 1$. This means

$$\overline{N}\left(r,\frac{1}{g-c}\right) \le \overline{N}\left(r,\frac{1}{F-1}\right) \le T(r,F) + O(1) = O(r^q)\,. \tag{5}$$

(Here, of course, it is crucial that $F \not\equiv 1$.) Now we consider the function

$$h := a + (c - a) \cdot F\,.$$

Let some $s \in \mathbb{C}$ with $g(s) = h(s)$ be given. Then we have

$$f(s) = a + \frac{g(s) - a}{F(s)} = a + \frac{h(s) - a}{F(s)} = a + c - a = c\,,$$

hence $g(s) = c$. This shows

$$\overline{N}\left(r, \frac{1}{g - h}\right) \leq \overline{N}\left(r, \frac{1}{g - c}\right) = O(r^q)\,. \tag{6}$$

Furthermore, since g assumes the value d only finitely many times, we also know

$$\overline{N}\left(r, \frac{1}{g - d}\right) = O(\log r). \tag{7}$$

We want to apply the Second Fundamental Theorem for small functions (Lemma 5) with $\varphi_1 = c$, $\varphi_2 = h$ and $\varphi_3 = d$. For this, we have to make sure that $\varphi_1, \varphi_2, \varphi_3$ are distinct. By assumption we have $c \neq d$, and $F \not\equiv 1$ implies $h \not\equiv c$.

Let us assume that $h \equiv d$. Then F is constant. If there would be some $s_0 \in \mathbb{C}$ with $f(s_0) = c$, then we would have $f(s_0) = c = g(s_0)$, hence $F(s_0) = 1$, and we would obtain $F \equiv 1$, a contradiction. So f and hence g omit the value c. Keeping in mind that $g - d$ has only finitely many zeros, we conclude that $G := \frac{g-d}{g-c}$ is an entire function of finite order with finitely many zeros, hence that it has a representation $G = Q_1 e^{Q_2}$ where Q_1, Q_2 are polynomials and $\deg(Q_2) = \varrho(G) \leq \varrho(g)$, i.e. $\deg(Q_2) \leq q$. This yields

$$T(r, g) = T(r, G) \leq C \cdot r^q + O(\log r) \tag{8}$$

for some constant $C > 0$. In particular, we have $\varrho(g) \leq q$. If $\varrho(f)$ is not an integer, i.e. $q < \varrho(f) = \varrho(g)$, this is an immediate contradiction. If $\varrho(f)$ is an integer, i.e. $q = \varrho(f) = \varrho(g) > 0$, we deduce $\tau(g) \leq C$. But in this case, by assumption we have $\tau(f) = \infty$, hence $\tau(g) = \infty$ as shown above, again a contradiction.

Therefore, $h \not\equiv d$. So $\varphi_1, \varphi_2, \varphi_3$ are distinct, indeed. Furthermore,

$$T(r, h) = T(r, F) + O(1) = O(r^q)\,.$$

Now Lemma 5 and (5), (6) and (7) yield

$$\begin{aligned} T(r, g) &\leq \overline{N}\left(r, \frac{1}{g - c}\right) + \overline{N}\left(r, \frac{1}{g - h}\right) + \overline{N}\left(r, \frac{1}{g - d}\right) \\ &\quad + O(T(r, h)) + S(r, h) + S(r, g) \\ &\leq O(r^q) + O(\log r) + S(r, g)\,. \end{aligned}$$

Here we have used that h has finite order, so $S(r, h) = o(T(r, h))$ for all $r > 0$ without an exceptional set. The same argument holds for g, so we conclude that $T(r, g) = O(r^q) + O(\log r)$. With the same reasoning as in (8), this gives a contradiction. Therefore, $f \equiv \mathcal{L}$. •

Proof of Theorem 2. By Lemma 6, $\mathcal{L}$ has order 1 and maximal type. Furthermore, $\mathcal{L}$ has at most one pole. So Theorem 1 is a special case of Theorem 2. •

4. Joint Universality–Proof of Theorem 3

Now we shall show how to distinguish L-functions by using a remarkable simultaneous approximation property. In 1975 Voronin [29] proved, roughly speaking, that any non-vanishing analytic function can be uniformly approximated by certain shifts of the Riemann zeta-function, and in a sequel [30] he extended this result to a simultaneous approximation theorem for a family of Dirichlet L-functions associated with non-equivalent characters. Here two characters are said to be non-equivalent if they are not induced by the same character. The precise formulation of the latter phenomenon, also called joint universality, is as follows:

LEMMA 7. *Let $\chi_1, \dots, \chi_\ell$ be pairwise non-equivalent Dirichlet characters, $\mathcal{K}_1, \dots, \mathcal{K}_\ell$ be compact subsets of the strip $\frac{1}{2} < \operatorname{Re} s < 1$ with connected complements. Further, for each $1 \le j \le \ell$, let $f_j(s)$ be a continuous non-vanishing function on $\mathcal{K}_j$ which is analytic in the interior of $\mathcal{K}_j$. Then, for any $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] \,:\, \max_{1\le j\le \ell} \max_{s\in\mathcal{K}_j} |L(s+i\tau, \chi_j) - f_j(s)| < \epsilon \right\} > 0 .$$

A proof of this theorem can also be found in [14], Chapter 7, and [28], Chapters 1 and 12, respectively.

Proof of Theorem 3. In view of the characterization of degree one elements of the Selberg class $\mathcal{S}$ (see Section 2) we need to consider two shifted Dirichlet L-functions $L(s+i\theta_j, \chi_j)$ associated with either primitive characters or the principal character mod 1, where the θ_j are real numbers and $\theta_j = 0$ if $\chi_j \equiv \chi_0 \bmod 1$. Now assume that $L(s+i\theta_1, \chi_1)$ and $L(s+i\theta_2, \chi_2)$ share a complex value c. If $\chi_1 = \chi_2$, then $L(s+i\theta_1, \chi_1) = c$ whenever $L(s+i\theta_2, \chi_1) = c$, and it follows that either $\theta_1 = \theta_2$ or the c-points of $L(s, \chi_1)$ are periodically distributed with period $i(\theta_1 - \theta_2)$ which is absurd. Therefore, we may assume that $\chi_1 \ne \chi_2$; hence, being primitive, they are non-equivalent.

Now suppose $c \ne 0$ and that c' is another non-zero complex number different from c. We shall show the existence of some complex number s' such that

$$L(s'+i\theta_1, \chi_1) = c \ne L(s'+i\theta_2, \chi_2) .$$

For this purpose let $\mathcal{K}$ be the closed disk centered at $\frac{3}{4}$ of radius $r \in (0, \frac{1}{4})$. Moreover, define target functions by setting $f_1(s) = c + \lambda(s - i\theta_1 - \frac{3}{4})$ and $f_2(s) = c'$ on sets $\mathcal{K}_j$, where

$$\mathcal{K}_j = \mathcal{K} + i\theta_j := \{s + i\theta_j \,:\, s \in \mathcal{K}\} \tag{9}$$

and λ is a positive real number for which $\lambda r < |c|$. By the latter condition $f_1(s)$ does not vanish on $\mathcal{K}_1$. Thus, an application of Lemma 7 with $0 < \epsilon < \min\{\lambda r, |c - c'|\}$ yields a real number τ such that

$$\max_{1\le j\le 2} \max_{s\in\mathcal{K}_j} |L(s+i\tau, \chi_j) - f_j(s)| < \epsilon . \tag{10}$$

We first deduce that

$$\max_{s\in\mathcal{K}} |L(s+i\theta_1+i\tau, \chi_1) - f_1(s+i\theta_1)| < \epsilon .$$

Since the absolute value of $f_1(s+i\theta_1)-c=\lambda(s-\frac{3}{4})$ on the boundary of $\mathcal{K}$ equals λr which is strictly larger than ϵ, it follows that

$$\max_{s\in\partial\mathcal{K}} |L(s+i\theta_1+i\tau,\chi_1)-c-\{f_1(s+i\theta_1)-c\}| < \epsilon < \min_{s\in\partial\mathcal{K}} |f_1(s+i\theta_1)-c| ,$$

and an application of Rouché's theorem gives the existence of a c-point of $L(s+i\theta_1,\chi_1)$ inside $\mathcal{K}+i\tau := \{s+i\tau \,:\, s\in\mathcal{K}\}$. Secondly, we deduce from (10) that

$$\max_{s\in\mathcal{K}} |L(s+i\theta_2+i\tau,\chi_2)-c'| < \epsilon .$$

Consequently, $L(s+i\theta_2,\chi_2)$ does not assume the value c in $\mathcal{K}+i\tau$ since $\epsilon < |c-c'|$. This already shows that $L(s+i\theta_1,\chi_1)$ and $L(s+i\theta_2,\chi_2)$ do not share any complex value $c\neq 0$.

Since Dirichlet L-functions are expected to have no zeros to the right of the critical line $\frac{1}{2}+i\mathbb{R}$, universality is not an appropriate tool to discuss the remaining case of a shared value $c=0$.

Let us assume that $L(s+i\theta_1,\chi_1)$ and $L(s+i\theta_2,\chi_2)$ share the value $c=0$. In view of the trivial zeros of Dirichlet L-functions on the negative real axis it follows that $\theta_1=\theta_2$. Next we may use Fujii's formula (2) (resp. its unconditional version [26]) to conclude that

$$\lim_{T\to\infty}\frac{\pi}{T\log T}\sum_{|\gamma_{\chi_1}|\leq T} L(\rho_{\chi_1},\chi_2) = 1-\frac{1}{\varphi([q_1,q_2])}\sum_{\substack{a \bmod [q_1,q_2]\\ (a,[q_1,q_2])=1}} (\overline{\chi_1}\chi_2)(a) ,$$

where q_i is the modulus of the character χ_i. It follows from the orthogonality relation for characters that the right-hand side does not vanish for $\chi_1\neq\chi_2$. Hence $L(s+i\theta_1,\chi_1)$ and $L(s+i\theta_2,\chi_2)$ do not share the value 0. This proves Theorem 3. •

As already mentioned in the introduction, it is expected that *independent* L-functions cannot share any complex value. However, besides the notion of an L-function it is not clear what the correct meaning of *independence* should be. Nevertheless, joint universality seems to be an interesting approach to this question. We refer to [28], Chapters 12 and 13, for an overview on joint universality theorems for L-functions. We could have alternatively used a joint universality theorem due to Sander & Steuding [23] which applies to a family of Dirichlet series with periodic coefficients and analytic continuation beyond the abscissa of absolute convergence. This theorem covers indeed the case of the extended Selberg class; however, since elements in $\mathcal{S}^\sharp$ may be linearly dependent, they cannot be jointly universal in general without any restriction. In fact, the joint universality theorem of Sander & Steuding is conditional subject to a linear independence condition on the target functions.

5. Moments of Dirichlet L-Functions–Proof of Theorem 4

This third approach is related to the use of Fujii's formula (2) for the case of zeros in the proof of the Uniqueness Theorem 3 from the previous section.

We start with some preparation for the proof of Theorem 4. The functional equation for the Riemann zeta-function in asymmetric form is given by

$$\zeta(s) = \Delta(s)\zeta(1-s)\,, \tag{11}$$

where

$$\Delta(s) := 2(2\pi)^{s-1}\Gamma(1-s)\sin\tfrac{\pi s}{2}\,.$$

If $\chi \bmod q$ is a primitive character, then $L(s,\chi)$ satisfies the functional equation

$$L(s,\chi) = \Delta(s,\chi)L(1-s,\overline{\chi})\,, \tag{12}$$

where

$$\Delta(s,\chi) := \tau(\chi)(2\pi)^{s-1}q^{-s}\Gamma(1-s)\left(\exp\left(\tfrac{\pi i(s-1)}{2}\right) + \chi(-1)\exp\left(-\tfrac{\pi i(s-1)}{2}\right)\right)\,,$$

and

$$\tau(\chi) := \sum_{a \bmod q} \chi(a)\exp\left(\frac{2\pi i a}{q}\right)$$

is the Gaussian sum associated with χ. If $\chi \bmod q$ is primitive, then $|\tau(\chi)| = \sqrt{q}$, whereas $\tau(\chi_0) = \mu(q)$ for the principal character $\chi_0 \bmod q$, where $\mu(q)$ is the Möbius μ-function. It is easy to verify that

$$\Delta(s,\chi) = \chi(-1)\tau(\chi)q^{-s}\Delta(s)(\cot\tfrac{\pi s}{2})^{\frac{1}{2}(1-\chi(-1))}\,. \tag{13}$$

Note that (12) reduces to (11) for $\chi \bmod 1$. By Stirling's formula, for $t > 1$,

$$\frac{\Delta'}{\Delta}(s,\chi) = \frac{\Delta'}{\Delta}(1-s,\overline{\chi}) = -\log\frac{tq}{2\pi} + O\left(\frac{1}{t}\right)\,. \tag{14}$$

In order to prove Theorem 4 we shall use the following

LEMMA 8. *For any Dirichlet character χ there exist positive constants c_1 and c_2 such that, for $\sigma \le 0$ and $|t| \ge 2$,*

$$|L(\sigma+it,\chi)| > \frac{c_1|t|^{\frac{1}{2}-\sigma}}{(\log t)^7}\,.$$

and

$$|L(\sigma+it,\chi)| < c_2|t|^{\frac{1}{2}-\sigma}\log t\,.$$

Proof. Let χ_0 be a principle character. We start from the well known inequality

$$L^3(\sigma,\chi_0)|L^4(\sigma+it,\chi)L(\sigma+2it,\chi^2)| \ge 1$$

for $\sigma > 1$. Let $t \ge 2$. For $\sigma > 1 - c_3(\log t)^{-1}$ with a suitable constant $c_3 > 0$, we have the estimates

$$L^3(\sigma,\chi) \ll (\sigma-1)^{-3}$$

as $\sigma \to 1$, and

$$L(\sigma+it,\chi) \ll \log t \tag{15}$$

(see formulae (3.5) and (5.13) in Chapter IV in [22]). Thus, for $\sigma > 1$,

$$\left|\frac{1}{L(\sigma+it,\chi)}\right| \leq (L(\sigma,\chi_0))^{\frac{3}{4}}\left|L(\sigma+2it,\chi^2)\right|^{\frac{1}{4}} \ll \frac{(\log t)^{\frac{1}{4}}}{(\sigma-1)^{\frac{3}{4}}}. \tag{16}$$

By Cauchy's integral formula from the bound (15) we deduce, for $\sigma \geq 1$,

$$L'(\sigma+it,\chi) \ll (\log t)^2 .$$

Then, for $\sigma > 1$,

$$L(1+it,\chi) - L(\sigma+it,\chi) = -\int_1^\sigma L'(u+it,\chi)\,\mathrm{d}u \ll (\sigma-1)(\log t)^2 .$$

This in combination with (16) leads with the choice $\sigma - 1 = c_3(\log t)^{-9}$ to

$$|L(1+it,\chi)| \gg (\log t)^{-7} .$$

The assertion of the lemma follows from the functional equation (12) in combination with Stirling's formula. •

Now we are in the position to give the

Proof of Theorem 4. By the calculus of residues,

$$\sum_{0<\gamma_\chi\leq T} L(\rho_\chi,\psi) = \frac{1}{2\pi i}\oint \frac{L'(s,\chi)}{L(s,\chi)-c} L(s,\psi)\,\mathrm{d}s, \tag{17}$$

where the integration is taken over a rectangular contour in counterclockwise direction according to the location of the nontrivial c-points of $L(s,\psi)$, to be specified below. In view of the Riemann-von Mangoldt-type formula (3) the ordinates of the c-points cannot lie too dense. For any large T_0 we can find a $T \in [T_0, T_0+1)$ such that

$$\min_{\rho_\chi} |T-\gamma_\chi| \gg \frac{1}{\log T}, \tag{18}$$

where the minimum is taken over all nontrivial c-points $\rho_\chi = \beta_\chi + i\gamma_\chi$. It follows from the partial fraction decomposition of $L(s,\chi)$ that

$$\frac{L'}{L}(\sigma+iT,\chi) \ll (\log T)^2 \quad \text{for} \quad -1 \leq \sigma \leq 2 \tag{19}$$

(see [1], Chapter 19). Next we shall consider regions free of c-points.

For $\sigma \to +\infty$,

$$L(\sigma+it,\chi) = 1 + o(1)$$

uniformly in t. Hence, there are no c-points for sufficiently large σ provided $c \neq 1$. For the case $c = 1$ define

$$m = \min\{n \geq 2 : \chi(n) \neq 0\}. \tag{20}$$

We observe, for $\sigma \to +\infty$,

$$L(\sigma+it,\chi) - 1 = \frac{\chi(m)}{m^{\sigma+it}}(1+o(1)). \tag{21}$$

Hence, in both cases, $c \neq 1$ and $c = 1$, there are no c-points of $L(s,\chi)$ in the half-plane $\operatorname{Re} s > B - 1$, where $B := \log T$ and T is sufficiently large. Further, define $b = 1 + \frac{1}{\log T}$.

Then we may suppose that there are no c-points on the line segments $[B+i, B+iT]$ and $[1-b+i, 1-b+iT]$ (by varying b slightly if necessary). We also suppose that there are no c-points on the line $[1-b+i, B+i]$ (if there is a c-point on this line we always can slightly shift this line). Moreover, there are only finitely many trivial c-points to the left of $\mathrm{Re}\, s = 1-b$ (in analogy to Lemma 4 from [6]).

Hence, in (17) we may choose the counterclockwise oriented rectangular contour $\mathcal{R}$ with vertices $B+i$, $B+iT$, $1-b+iT$,$1-b+i$, at the expense of a small error for disregarding the at most finitely many nontrivial c-points below $\mathrm{Im}\, s = 1$ and for counting finitely many trivial c-points to the left of $\mathrm{Re}\, s = 1-b$:

$$\sum_{0<\gamma_\chi<T} L(\rho_\chi, \psi) = \frac{1}{2\pi i}\int_{\mathcal{R}} \frac{L'(s,\chi)}{L(s,\chi)-c} L(s,\psi)\, \mathrm{d}s + O(1)\,.$$

We may rewrite the integral on the right hand side as

$$\frac{1}{2\pi i}\left\{\int_{B+i}^{B+iT} + \int_{B+iT}^{1-b+iT} + \int_{1-b+iT}^{1-b+i} + \int_{1-b+i}^{B+i}\right\} \frac{L'(s,\chi)}{L(s,\chi)-c} L(s,\psi)\, \mathrm{d}s$$
$$= \sum_{j=1}^{4} \mathcal{I}_j\,,$$

say. The integrals $\mathcal{I}_1$ and $\mathcal{I}_3$ are producing the main term whereas the other integrals contribute to the error term only. By a similar reasoning as in [6], we find $\mathcal{I}_2, \mathcal{I}_4 \ll T^{\frac{1}{2}+\varepsilon}$.

We start with the integral $\mathcal{I}_1$. First assume $c = 1$. Rewriting the integrand as a product of Dirichlet series shows that

$$\begin{aligned}\frac{L'(B+it,\chi)}{L(B+it,\chi)-c} L(B+it,\psi) &= -\log m + O\left(\left(\frac{m}{m+1}\right)^B\right)\\ &= -\log m + O\left(T^{-\log\frac{m+1}{m}}\right)\end{aligned}$$

uniformly in t, where m is defined by (20). Thus, by interchanging integration and summation,

$$\begin{aligned}\mathcal{I}_1 &= \frac{1}{2\pi}\int_1^T \left(-\log m + O\left(T^{-\log\frac{m+1}{m}}\right)\right) \mathrm{d}t\\ &= -\log m \cdot \frac{T}{2\pi} + O\left(T^{1-\log\frac{m+1}{m}}\right).\end{aligned}$$

For $c \neq 1$, we similarly obtain $\mathcal{I}_1 = O\left(T^{1-\log m}\right)$. We may rewrite the latter two formulae as

$$\mathcal{I}_1 = -\delta_{[c=1]} \log m \cdot \frac{T}{2\pi} + O\left(T^{1-\log\frac{m+1}{m}}\right), \tag{22}$$

where

$$\delta_{[\mathcal{A}]} := \begin{cases} 1 & \text{if } \mathcal{A} \text{ is true}\,,\\ 0 & \text{otherwise}\,.\end{cases} \tag{23}$$

In order to evaluate $\mathcal{I}_3$ we apply Lemma 8 which provides the existence of a positive constant C, depending only on c, such that

$$\left|\frac{c}{L(s,\chi)}\right| < \tfrac{1}{2} \quad \text{for } s = 1-b+it\,, \quad |t| \geq C\,.$$

Hence, the geometric series expansion

$$\frac{L'(s,\chi)}{L(s,\chi)-c} = \frac{L'}{L}(s,\chi)\left\{1 + \frac{c}{L(s,\chi)} + \sum_{k=2}^{\infty}\left(\frac{c}{L(s,\chi)}\right)^k\right\}$$

is valid for s from $[1-b+iC, 1-b+iT]$. Since integration over $[1-b+i, 1-b+iC]$ produces a bounded error, we arrive at

$$\begin{aligned}\mathcal{I}_3 &= \frac{1}{2\pi i}\int_{1-b+iT}^{1-b+i}\frac{L'(s,\chi)}{L(s,\chi)-c}L(s,\psi)\,\mathrm{d}s\\ &= \frac{1}{2\pi i}\int_{1-b+iT}^{1-b+iC}\left\{\frac{L'}{L}(s,\chi)L(s,\psi) + c\frac{L'}{L}(s,\chi)\frac{L(s,\psi)}{L(s,\chi)}\right.\\ &\quad\left. + \frac{L'}{L}(s,\chi)L(s,\psi)\sum_{k=2}^{\infty}\left(\frac{c}{L(s,\chi)}\right)^k\right\}\mathrm{d}s + O(1)\\ &= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + O(1)\,,\end{aligned}$$

say. The bound (19) and Lemma 8 give $\mathcal{J}_3 \ll T^{\frac{1}{2}+\epsilon}$. By a similar reasoning as in [26], the integral

$$\mathcal{J}_1 = \frac{1}{2\pi i}\int_{1-b+iT}^{1-b+iC}\frac{L'}{L}(s,\chi)L(s,\psi)\,\mathrm{d}s$$

can be considered, firstly, up to an error term as a contour integral,

$$\mathcal{J}_1 = \frac{1}{2\pi i}\int_{\mathcal{R}}\frac{L'}{L}(s,\chi)L(s,\psi)\,\mathrm{d}s + O(T^{\frac{1}{2}+\varepsilon})\,,$$

and, secondly, as a sum of residues,

$$\mathcal{J}_1 = \sum_{0<\gamma_\chi^0<T} L(\rho_\chi^0,\psi) + O(T^{\frac{1}{2}+\varepsilon})\,,$$

where $\rho_\chi^0 = \beta_\chi^0 + i\gamma_\chi^0$ denotes the nontrivial zeros of $L(s,\chi)$. In [7] this sum of residues was asymptotically evaluated as

$$\mathcal{J}_1 = \frac{T}{2\pi}\log\frac{Tq}{2\pi e} + a_1\frac{T}{2\pi} + O\left(T\exp(-b_2(\log T)^{\frac{1}{4}-\epsilon})\right)\,, \tag{24}$$

where $b_2 > 0$,

$$a_1 := a_1(\chi,\psi) := \frac{L'}{L}(1,\psi\overline{\chi}) - \delta_{[q|Q]}L(1,\chi\overline{\psi})\psi(-1)\tau(\psi)\frac{\tau(\overline{\chi}\psi_0)}{\phi(Q)}\,, \tag{25}$$

and ψ_0 is the principal Dirichlet character mod Q. Note that the second term on the right does not appear if Q is not divisible by q.

It remains to consider $\mathcal{J}_2$. Using the functional equation (12), we find

$$\begin{aligned}\mathcal{J}_2 &= \frac{c}{2\pi i}\int_{1-b+iT}^{1-b+iC}\frac{L'}{L}(s,\chi)\frac{L(s,\psi)}{L(s,\chi)}\,\mathrm{d}s\\ &= -\frac{c}{2\pi i}\int_{b-iT}^{b-iC}\left(\frac{\Delta'}{\Delta}(1-s,\chi)-\frac{L'}{L}(s,\overline{\chi})\right)\frac{\Delta(1-s,\psi)}{\Delta(1-s,\chi)}\frac{L(s,\overline{\psi})}{L(s,\overline{\chi})}\,\mathrm{d}s\,.\end{aligned}$$

Hence, by complex conjugation,

$$\begin{aligned}\overline{\mathcal{J}_2} &= -\frac{\overline{c}}{2\pi}\int_C^T\left(\frac{\Delta'}{\Delta}(1-b-i\tau,\overline{\chi})-\frac{L'}{L}(b+i\tau,\chi)\right)\\ &\quad\times\frac{\Delta(1-b-i\tau,\overline{\psi})}{\Delta(1-b-i\tau,\overline{\chi})}\frac{L(b+i\tau,\psi)}{L(b+i\tau,\chi)}\,\mathrm{d}\tau\\ &= \mathcal{K}_1+\mathcal{K}_2\,,\end{aligned}$$

say. Since

$$(\cot\tfrac{\pi}{2}(1-b-i\tau))^{(\chi(-1)-\psi(-1))/2} = i^{\frac{1}{2}(\chi(-1)-\psi(-1))}+O\left(e^{-2\tau}\right),$$

as $\tau\to\infty$, we get in view of (13)

$$\begin{aligned}\frac{\Delta(1-b-i\tau,\overline{\psi})}{\Delta(1-b-i\tau,\overline{\chi})} &= (\psi\chi)(-1)\frac{\tau(\overline{\psi})}{\tau(\overline{\chi})}\left(\frac{q}{Q}\right)^{1-b-i\tau} i^{(\chi(-1)-\psi(-1))/2}+O\left(e^{-2\tau}\right)\\ &= \overline{a_2}\left(\frac{q}{Q}\right)^{1-b-i\tau}+O\left(e^{-2\tau}\right),\end{aligned}$$

as $\tau\to\infty$, where

$$a_2 := a_2(\chi,\psi) := \frac{\tau(\psi)}{\tau(\chi)}i^{\frac{1}{2}(\psi(-1)-\chi(-1))} \tag{26}$$

is constant; here we have used the fact that $\tau(\overline{\chi}) = \chi(-1)\overline{\tau(\chi)}$ and the corresponding formula for ψ in place of χ. By this, formula (14), and expressing $L(b+i\tau,\overline{\psi})$ and $L(b+i\tau,\overline{\chi})$ as Dirichlet series, we obtain

$$\mathcal{K}_1 = \frac{\overline{ca_2}}{2\pi}\left(\frac{q}{Q}\right)^{1-b}\sum_{m,n=1}^{\infty}\frac{\mu(m)\chi(m)\psi(n)}{(mn)^b}\int_C^T\left(\frac{Q}{mnq}\right)^{i\tau}\log\frac{\tau q}{2\pi}\,\mathrm{d}\tau+O(1)\,.$$

For $\frac{Q}{mnq}\neq 1$ the integral can be estimated by integrating by parts; these terms contribute an error term $O((\log T)^3)$. Computing the integral for $mn=\frac{Q}{q}$ yields

$$\mathcal{K}_1 = \overline{ca_2a_3}\cdot\frac{T}{2\pi}\log\frac{Tq}{2\pi e}+O((\log T)^3)\,, \tag{27}$$

where

$$a_3 := a_3(\chi,\psi) := \frac{q}{Q}\sum_{d\mid\frac{Q}{q}}\mu(d)\overline{\chi}(d)\overline{\psi}\left(\frac{Q}{dq}\right). \tag{28}$$

Similarly

$$\mathcal{K}_2 = -\frac{\overline{ca_2}}{2\pi}\left(\frac{q}{Q}\right)^{1-b}\sum_{m,n,k=1}^{\infty}\frac{\mu(m)\chi(m)\psi(n)\Lambda(k)\psi(k)}{(mnk)^b}$$
$$\times \int_C^T (mnk)^{-i\tau}\ \mathrm{d}\tau + O(1)$$
$$= O((\log T)^3)\,.$$

Taking into account (22), (24), and (27), we now obtain

$$\sum_{0<\gamma_\chi<T} L(\rho_\chi,\psi) = (1+ca_2a_3)\frac{T}{2\pi}\log\frac{Tq}{2\pi e} \tag{29}$$
$$+ (a_1 - \delta_{[c=1]}\cdot\log m)\frac{T}{2\pi} + O\left(T\exp(-b_1(\log T)^{\frac14-\epsilon})\right)\,.$$

Hence, we may replace (29) by (1) with the constants

$$\alpha_1 = \alpha_1(c,\chi,\psi) := 1 + ca_2a_3, \tag{30}$$
$$\alpha_0 = \alpha_0(c,\chi,\psi) := a_1 - \delta_{[c=1]}\cdot\log m + (1+ca_2a_3)\log\frac{q}{2\pi e} \tag{31}$$

where all relevant information is given by (20), (23), (25), (26), and (28). Theorem 4 is proved. •

Finally, we shall discuss how Theorem 4 can be used to show that distinct Dirichlet L-functions cannot share a complex value c with at most one exception for c. Specializing (3) shows for the number $N_c(T)$ of nontrivial c-points of $L(s,\chi)$ with a primitive character χ mod q with imaginary part γ_c satisfying $|\gamma_c| < T$ that

$$N_c(T) = \frac{T}{\pi}\log\frac{qT}{2\pi e b_c} + O(\log T) \tag{32}$$

with $b_c = 1$ if $c \neq 1$, and $b_1 = m$, where m is defined by (20). We have to take into account that the summation in the formula of Theorem 4 is over all c-points $\rho_\chi = \beta_\chi + i\gamma_\chi$ with imaginary part $\gamma_\chi \in (0,T)$. We observe

$$L(\beta_\chi + i\gamma_\chi,\chi) = c \quad\Longleftrightarrow\quad L(\beta_\chi - i\gamma_\chi,\overline{\chi}) = \overline{c}\,.$$

Hence, we have to conjugate c and χ for c-points in the lower half-plane. In view of Theorem 4 we write

$$\Sigma(c,T,\chi,\psi) := \sum_{0<\gamma_\chi<T} L(\rho_\chi,\psi)$$
$$= \alpha_1(c,\chi,\psi)\frac{T}{2\pi}\log T + \alpha_0(c,\chi,\psi)\frac{T}{2\pi} + \mathcal{E}\,,$$

where $\mathcal{E}$ denotes here and in the sequel an error term of size $O\left(T\exp(-b_1(\log T)^{\frac14-\epsilon})\right)$. We observe

$$\sum_{-T<\gamma_\chi<0} L(\rho_\chi,\psi) = \Sigma(\overline{c},T,\overline{\chi},\psi)$$

and

$$\sum_{|\gamma_\chi|<T} L(\rho_\chi, \psi) = \Sigma(c, T, \chi, \psi) + \Sigma(\overline{c}, T, \overline{\chi}, \psi) + O(1)\,,$$

where the error term is with respect to possible c-points on the real axis. Hence, by our previous observations,

$$\sum_{|\gamma_\chi|<T} L(\rho_\chi, \psi) = (\alpha_1(c, \chi, \psi) + \alpha_1(\overline{c}, \overline{\chi}, \psi))\frac{T}{2\pi}\log T$$
$$+(\alpha_0(c, \chi, \psi) + \alpha_0(\overline{c}, \overline{\chi}, \psi))\frac{T}{2\pi} + \mathcal{E}.$$

Subtracting (32) yields

$$\sum_{|\gamma_\chi|<T} (L(\rho_\chi, \psi) - c)$$
$$= (2 + ca_2(\chi, \psi)a_3(\chi, \psi) + \overline{c}a_2(\overline{\chi}, \psi)a_3(\overline{\chi}, \psi) - 2c)\frac{T}{2\pi}\log T$$
$$+ (\alpha_0(c, \chi, \psi) + \alpha_0(\overline{c}, \overline{\chi}, \psi) - 2c\log\frac{q}{2\pi eb_c})\frac{T}{2\pi} + \mathcal{E}\,.$$

If not all coefficients at $T\log T$ and T vanish, it obviously follows that $L(s, \chi)$ and $L(s, \psi)$ do not share the value c. Since the coefficient at $T\log T$ depends linearly on c, there is at most one value of c for which the series on the left-hand side vanishes. It would be interesting to exclude the existence of such an exceptional value c by a different argument than in §4, however, the coefficients depend in a rather sophisticated way on c, χ and ψ and include additionally algebraic and transcendental constants. Since we could not exclude such an exceptional value of c, we decided to leave this problem for the interested reader.

REMARK. Recently, Ki [15] succeeded in showing that if two elements of $\mathcal{S}^\sharp$ with constant coefficient $a(1) = 1$ satisfy the same functional equation with positive degree and share a non-zero complex value, then they are identical; moreover, he showed that this is not true for $c = 0$ or for zero degree.

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