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Roots of the Ehrhart Polynomial of Hypersimplices

by

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Abstract. The Ehrhart polynomial of the d-th hypersimplex $\Delta(d,n)$ of order n is studied. By computational experiments and a known result for $d=2$, we conjecture that the real part of every roots of the Ehrhart polynomial of $\Delta(d,n)$ is negative and larger than $-\frac{n}{d}$ if $n \geq 2d$. In this paper, we show that the conjecture is true when $d=3$ and that every root a of the Ehrhart polynomial of $\Delta(d,n)$ satisfies $-\frac{n}{d} < \mathrm{Re}(a) < 1$ if $4 \leq d \ll n$.

Introduction

Let $\mathcal{P} \subset \mathbb{R}^n$ be an integral convex polytope of dimension p. Recall that an *integral* convex polytope is a convex polytope all of whose vertices have integer coordinates. Given an integer $m > 0$, we write $i(\mathcal{P}, m)$ for the number of integer points belonging to $m\mathcal{P} = \{m\alpha \mid \alpha \in \mathcal{P}\}$, that is,

$$i(\mathcal{P}, m) = |m\mathcal{P} \cap \mathbb{Z}^n| \quad m = 1, 2, \ldots .$$

It is known that $i(\mathcal{P}, m)$ is a polynomial in m of degree p. We call $i(\mathcal{P}, m)$ the *Ehrhart polynomial* of $\mathcal{P}$. In general, $i(\mathcal{P}, 0) = 1$ and the leading coefficient of $i(\mathcal{P}, m)$ is equal to the normalized volume of $\mathcal{P}$. In [1], it was conjectured that each root $a \in \mathbb{C}$ of $i(\mathcal{P}, m)$ satisfies $-p \leq \mathrm{Re}(a) \leq p-1$. However, several counterexamples for the conjecture are given in [3, 9] recently. On the other hand, it is known [2] that all the roots of $i(\mathcal{P}, m)$ lie inside the disc with center $-\frac{1}{2}$ and radius $p(p-\frac{1}{2})$.

In this paper, we study roots of the Ehrhart polynomial of a hypersimplex. Let d and n be integers such that $1 \leq d < n$. The *hypersimplex* $\Delta(d,n)$ is a convex polytope in $\mathbb{R}^n$ which is the convex hull of

$$\{\mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_d} \mid 1 \leq i_1 < \cdots < i_d \leq n\},$$

where each $\mathbf{e}_j$ is the unit coordinate vector of $\mathbb{R}^n$. In general, it is known that

- The dimension of $\Delta(d,n)$ is $n-1$;
- $\Delta(d,n)$ is isomorphic to $\Delta(n-d,n)$.

Thus, throughout this paper, we always assume that d and n satisfy the condition

$$2d \leq n. \tag{1}$$

The Ehrhart polynomial $i(\Delta(d,n),m)$ of $\Delta(d,n)$ is given in [4]:

$$i(\Delta(d,n),m)=\sum_{s=0}^{d-1}(-1)^s\binom{n}{s}\binom{(d-s)m+n-1-s}{n-1}.$$

Katzman computed the Hilbert polynomial of corresponding semigroup rings and it is equal to $i(\Delta(d,n),m)$ since the semigroup ring is normal. Exactly speaking, the Hilbert polynomial is equal to the *normalized* Ehrhart polynomial if and only if the semigroup ring is normal. For the sake of completeness, we will later show that the normalized Ehrhart polynomial is equal to the Ehrhart polynomial in this case.

If $d=1$, then $i(\Delta(1,n),m)$ is an $(n-1)$-simplex and

$$i(\Delta(1,n),m)=\binom{m+n-1}{n-1}.$$

Hence, roots of $i(\Delta(1,n),m)$ are $-(n-1),-(n-2),\dots,-2,-1$. If $d=2$, then

$$i(\Delta(2,n),m)=\binom{2m+n-1}{n-1}-n\binom{m+n-2}{n-1}.$$

In [8], it is shown that every root $a\in\mathbb{C}$ of $i(\Delta(2,n),m)$ satisfies

$$-\frac{n}{2}<\mathrm{Re}(a)<0$$

when $2d=4\le n$. Computational experiments[1] suggest the following conjecture:

CONJECTURE 0.1. *Let* $2d\le n$. *Then, every root* $a\in\mathbb{C}$ *of* $i(\Delta(d,n),m)$ *satisfies*

$$-\frac{n}{d}<\mathrm{Re}(a)<0\,.$$

In this paper, we show that

THEOREM 0.2. *Let d and n be positive integers, and let $a\in\mathbb{C}$ be a root of $i(\Delta(d,n),m)$. Then, we have the following*:

(i) *If* $d=3$ *and* $n\ge 6$, *then we have* $-\frac{n}{3}<\mathrm{Re}(a)<0$.

(ii) *If* $4\le d\ll n$, *then we have* $-\frac{n}{d}<\mathrm{Re}(a)<1$.

1. Fundamental facts on $i(\Delta(d,n),m)$

In this section, we present fundamental facts on $i(\Delta(d,n),m)$ which will later play an important role. First we confirm that the Ehrhart polynomial of the hypersimplex $\Delta(d,n)$ is

$$i(\Delta(d,n),m)=\sum_{s=0}^{d-1}(-1)^s\binom{n}{s}\binom{(d-s)m+n-1-s}{n-1}. \tag{2}$$

[1] A rough bound was obtained by Masanori Tajima in his master's thesis (in Japanese).

It is pointed out in [4, Remark 2.3] that the right-hand side of (2) is equal to the *normalized* Ehrhart polynomial of $\Delta(d,n)$. In other words,

$$|m\Delta(d,n)\cap\mathbb{Z}A| = \sum_{s=0}^{d-1}(-1)^s\binom{n}{s}\binom{(d-s)m+n-1-s}{n-1},$$

where $A = \{\mathbf{e}_{i_1}+\cdots+\mathbf{e}_{i_d} \mid 1\le i_1<\cdots<i_d\le n\}$. Note that $\mathbb{Z}A\ne\mathbb{Z}^n$. Since the following fact is not stated in [4], we show it for the sake of completeness:

PROPOSITION 1.1. *The Ehrhart polynomial of $\Delta(d,n)$ is equal to the normalized Ehrhart polynomial of $\Delta(d,n)$.*

Proof. In general, we have $m\Delta(d,n)\cap\mathbb{Z}A\subset m\Delta(d,n)\cap\mathbb{Z}^n$. Hence, it is enough to show that $m\Delta(d,n)\cap\mathbb{Z}A\supset m\Delta(d,n)\cap\mathbb{Z}^n$. Let $\alpha=(\alpha_1,\ldots,\alpha_n)\in m\Delta(d,n)\cap\mathbb{Z}^n$. Since $\alpha\in m\Delta(d,n)$, we have $\alpha_1+\cdots+\alpha_n=dm$. Remark that

$$\mathbf{e}_1-\mathbf{e}_2=(\mathbf{e}_1+\mathbf{e}_3+\cdots+\mathbf{e}_{d+1})-(\mathbf{e}_2+\mathbf{e}_3+\cdots+\mathbf{e}_{d+1})\in\mathbb{Z}A.$$

Similarly, $\mathbf{e}_1-\mathbf{e}_j$ belongs to $\mathbb{Z}A$ for each $2\le j\le n$. Hence,

$$d\mathbf{e}_1=(\mathbf{e}_1+\mathbf{e}_2+\cdots+\mathbf{e}_d)+\sum_{j=2}^{d}(\mathbf{e}_1-\mathbf{e}_j)\in\mathbb{Z}A\,.$$

Thus,

$$\alpha=\left(\sum_{j=1}^{n}\alpha_j\right)\mathbf{e}_1-\sum_{j=2}^{n}\alpha_j(\mathbf{e}_1-\mathbf{e}_j)=dm\,\mathbf{e}_1-\sum_{j=2}^{n}\alpha_j(\mathbf{e}_1-\mathbf{e}_j)$$

belongs to $\mathbb{Z}A$, as desired. □

In order to study roots of $i(\Delta(d,n),m)$, we will use the following fact:

PROPOSITION 1.2 (Rouché). *Let D be a simply connected region and let f and g be holomorphic functions in $\overline{D}$. If $|f(z)|>|g(z)|$ holds for every $z\in\partial D$, then f and $f+g$ have the same number of zeros in D, where each zero is counted as many times as its multiplicity.*

Fix an integer $d>0$. For $s=0,1,\ldots,d-1$, let

$$f_{n,s}(m)=\binom{n}{s}((d-s)m+n-1-s)\cdots((d-s)m+1-s)\,.$$

Then

$$i(\Delta(d,n),m)=\frac{1}{(n-1)!}\sum_{s=0}^{d-1}(-1)^s f_{n,s}(m)\,.$$

We will apply Rouché Theorem by considering the functions $f(z)=f_{n,0}(z)$ and

$$g(z)=\sum_{s=1}^{d-1}(-1)^s f_{n,s}(z)\,.$$

Let $\varphi_{n,d,s}(z)=\frac{|f_{n,s}(z)|}{|f_{n,0}(z)|}$.

LEMMA 1.3. *For every $z = \beta\sqrt{-1}$ with $\beta \in \mathbb{R}$, we have $\varphi_{n+1,d,s}(z) < \varphi_{n,d,s}(z)$.*

Proof. Since

$$\varphi_{n,d,s}(\beta\sqrt{-1}) = \binom{n}{s}\sqrt{\frac{((d-s)^2\beta^2+(n-1-s)^2)\cdots((d-s)^2\beta^2+(1-s)^2)}{(d^2\beta^2+(n-1)^2)\cdots(d^2\beta^2+1)}}$$

holds, we have

$$\frac{\varphi_{n+1,d,s}(z)}{\varphi_{n,d,s}(z)} = \sqrt{\frac{(n+1)^2(d-s)^2\beta^2+(n+1)^2(n-s)^2}{(n+1-s)^2d^2\beta^2+(n+1-s)^2n^2}}.$$

Moreover, since

$$\begin{aligned}
&((n+1-s)^2d^2\beta^2+(n+1-s)^2n^2)-((n+1)^2(d-s)^2\beta^2+(n+1)^2(n-s)^2)\\
&=\beta^2((n+1)(d-s)+(n+1-s)d)(n-d+1)s+((n+1)(n-s)+(n+1-s)n)s\\
&>0
\end{aligned}$$

we have $\frac{\varphi_{n+1,d,s}(z)}{\varphi_{n,d,s}(z)} < 1$, as desired. □

LEMMA 1.4. *Suppose $n \geq d^2-2$. Then, for every $z = -\frac{n}{d}-\beta\sqrt{-1}$ with $\beta \in \mathbb{R}$, we have $\varphi_{n+d,d,s}(z) < \varphi_{n,d,s}(z)$.*

Proof. Since $\varphi_{n,d,s}(z)$ is equal to

$$\binom{n}{s}\sqrt{\frac{((d-s)^2\beta^2+(-1-s+\frac{sn}{d})^2)\cdots((d-s)^2\beta^2+(-n+1-s+\frac{sn}{d})^2)}{(d^2\beta^2+1)\cdots(d^2\beta^2+(n-1)^2)}},$$

we have

$$\begin{aligned}
&\frac{\varphi_{n+d,d,s}(z)}{\varphi_{n,d,s}(z)}\\
&=\frac{\binom{n+d}{s}}{\binom{n}{s}}\sqrt{\frac{((d-s)^2\beta^2+(1-\frac{sn}{d})^2)\cdots((d-s)^2\beta^2+(s-\frac{sn}{d})^2)}{(d^2\beta^2+n^2)\cdots(d^2\beta^2+(n+s-1)^2)}}\\
&\quad\times\sqrt{\frac{((d-s)^2\beta^2+(n+s-\frac{sn}{d})^2)\cdots((d-s)^2\beta^2+(n+d-1-\frac{sn}{d})^2)}{(d^2\beta^2+(n+s)^2)\cdots(d^2\beta^2+(n+d-1)^2)}}\\
&<\frac{(n+d)(n+d-1)\cdots(n+d+1-s)}{n(n-1)\cdots(n+1-s)}\\
&\quad\times\sqrt{\frac{((d-s)^2\beta^2+(1-\frac{sn}{d})^2)\cdots((d-s)^2\beta^2+(s-\frac{sn}{d})^2)}{(d^2\beta^2+n^2)\cdots(d^2\beta^2+(n+s-1)^2)}}.
\end{aligned}$$

For $1 \leq k \leq s$ $(\leq d-1)$,

$$\frac{((d-s)^2\beta^2+(k-\frac{sn}{d})^2)}{(d^2\beta^2+(n+k-1)^2)} = \left(\frac{d-1}{d}\right)^2\frac{((\frac{d-s}{d-1})^2\beta^2+(\frac{k-\frac{sn}{d}}{d-1})^2)}{(\beta^2+(\frac{n+k-1}{d})^2)}$$

and

$$\frac{|n+k-1|}{d} - \frac{|k-\frac{sn}{d}|}{d-1} = \frac{(n+k-1)(d-1)+dk-sn}{d(d-1)}$$
$$= \frac{n(d-1-s)+dk+(d-1)(k-1)}{d(d-1)} > 0\,.$$

Hence, we have

$$\sqrt{\frac{(d-s)^2\beta^2+(k-\frac{sn}{d})^2}{d^2\beta^2+(n+1-k)^2}} < \frac{d-1}{d}.$$

Thus,

$$\frac{\varphi_{n+d,d,s}(z)}{\varphi_{n,d,s}(z)} < \frac{(n+d)(n+d-1)\cdots(n+d+1-s)}{n(n-1)\cdots(n+1-s)}\left(\frac{d-1}{d}\right)^s.$$

Moreover, for $1 \le k \le s$ $(\le d-1)$,

$$(n+1-k)d-(n+d+1-k)(d-1) = n-(d^2-2)+(d-1-k) \ge 0$$

and hence

$$\frac{n+d+1-k}{n+1-k}\cdot\frac{d-1}{d} \le 1\,.$$

Therefore, we have $\frac{\varphi_{n+d,d,s}(z)}{\varphi_{n,d,s}(z)} < 1$, as desired. □

LEMMA 1.5. *For every $z = -\alpha + \lambda n\sqrt{-1}$ with $0 \le \alpha \le \frac{n}{d}$ and $\lambda \in \mathbb{R}$, we have*

$$\varphi_{n,d,s}(z) < \binom{n}{s}\left(\frac{(d-s)^2+\frac{1}{\lambda^2}}{d^2}\right)^{\frac{n-1}{2}}.$$

Proof. Note that

$$\varphi_{n,d,s}(z) = \binom{n}{s}\sqrt{\prod_{k=1}^{n-1}\frac{(d-s)^2\lambda^2n^2+(k-s-(d-s)\alpha)^2}{d^2\lambda^2n^2+(k-d\alpha)^2}}\,.$$

For $1 \le k \le n-1$ and $1 \le s \le d-1$, we have

$$-n < -n+s\left(\frac{n}{d}-1\right)+1 = 1-s-(d-s)\frac{n}{d} < k-s-(d-s)\alpha < n-1-s < n\,.$$

Hence, $(k-s-(d-s)\alpha)^2 < n^2$. Thus,

$$\varphi_{n,d,s}(z) < \binom{n}{s}\left(\frac{(d-s)^2\lambda^2n^2+n^2}{d^2\lambda^2n^2}\right)^{\frac{n-1}{2}} = \binom{n}{s}\left(\frac{(d-s)^2+\frac{1}{\lambda^2}}{d^2}\right)^{\frac{n-1}{2}},$$

as desired. □

2. The case of $d = 3$

In this section, we prove that Conjecture 0.1 is true if $d = 3$.

THEOREM 2.1. *Let* $d = 3$ *and* $n \geq 6$ $(= 2d)$. *Then, every root* $a \in \mathbb{C}$ *of* $i(\Delta(3, n), m)$ *satisfies*

$$-\frac{n}{3} < \mathrm{Re}(a) < 0\,.$$

Proof. If $n = 6$, then the Ehrhart polynomial of $\Delta(3, 6)$ is

$$\begin{aligned} i(\Delta(3,6), z) &= \binom{3z+5}{5} - 6\binom{2z+4}{5} + 15\binom{z+3}{5} \\ &= \frac{1}{20}(z+1)\left(11(z+1)^4 + 5(z+1)^2 + 4\right). \end{aligned}$$

Since $0 < 4 < 5 < 11$ holds, by Eneström–Kakeya Theorem (see, e.g., [5]), it follows that every root $a \in \mathbb{C}$ satisfies $|(a+1)^2| < 1$. Hence, in particular, we have $-2 < \mathrm{Re}(a) < 0$.

Let $n \geq 7$. We apply Rouché Theorem for the functions

$$f(z) = f_{n,0}(z),\ \ g(z) = -f_{n,1}(z) + f_{n,2}(z)$$

and the region

$$D = \left\{ z \in \mathbb{C} \;\middle|\; -\frac{n}{3} < \mathrm{Re}(z) < 0,\ -\sqrt{2}n < \mathrm{Im}(z) < \sqrt{2}n \right\}.$$

Remark that the roots of $f(z)$ are

$$-\frac{n-1}{3}, -\frac{n-2}{3}, \dots, -\frac{2}{3}, -\frac{1}{3}$$

and all of them belong to D. Thus, it is enough to show that

$$|f_{n,1}(z)| + |f_{n,2}(z)| < |f_{n,0}(z)|$$

for all $z \in \partial D$.

CASE 1. $z = \beta\sqrt{-1}$ where $\beta \in \mathbb{R}$.

If $n = 7$ and $s = 1$, then

$$\begin{aligned} \frac{|f_{7,1}(\beta\sqrt{-1})|}{|f_{7,0}(\beta\sqrt{-1})|} &= 7\sqrt{\frac{(4\beta^2+5^2)(4\beta^2+4^2)(4\beta^2+3^2)(4\beta^2+2^2)(4\beta^2+1^2)(4\beta^2)}{(9\beta^2+6^2)(9\beta^2+5^2)(9\beta^2+4^2)(9\beta^2+3^2)(9\beta^2+2^2)(9\beta^2+1)}} \\ &= \frac{1792}{2187}\sqrt{\frac{9(\beta^2+\frac{25}{4})(\beta^2+\frac{9}{4})(\beta^2+\frac{1}{4})\beta^2}{16(\beta^2+\frac{25}{9})(\beta^2+\frac{16}{9})(\beta^2+\frac{4}{9})(\beta^2+\frac{1}{9})}}\,. \end{aligned}$$

We now show

$$\frac{9(\beta^2+\frac{25}{4})(\beta^2+\frac{9}{4})(\beta^2+\frac{1}{4})\beta^2}{16(\beta^2+\frac{25}{9})(\beta^2+\frac{16}{9})(\beta^2+\frac{4}{9})(\beta^2+\frac{1}{9})} < 1\,.$$

Let

$$f(x) = 16\left(x+\frac{25}{9}\right)\left(x+\frac{16}{9}\right)\left(x+\frac{4}{9}\right)\left(x+\frac{1}{9}\right)$$

$$-9\left(x+\frac{25}{4}\right)\left(x+\frac{9}{4}\right)\left(x+\frac{1}{4}\right)x$$
$$=7x^4+\frac{109}{36}x^3-\frac{10969}{432}x^2+\frac{739711}{46656}x+\frac{25600}{6561}.$$

Then, since

$$f(y+1)=7y^4+\frac{1117}{36}y^3+\frac{11099}{432}y^2+\frac{100567}{46656}y+\frac{1844635}{419904}>0$$

for all $y\geq 0$, it follows that $f(x)>0$ for all $x\geq 1$. Moreover, if $0\leq x<1$, then

$$\begin{aligned} f(x) &> 6x^4+3x^3-27x^2+15x+3 \\ &= 3(1-x)(1+x+x(2x+5)(1-x)) \\ &> 0. \end{aligned}$$

Thus, $f(x)>0$ for all $x\geq 0$.

If $n=7$ and $s=2$,

$$\begin{aligned} &\frac{|f_{7,2}(\beta\sqrt{-1})|}{|f_{7,0}(\beta\sqrt{-1})|} \\ &= 21\sqrt{\frac{(\beta^2+4^2)(\beta^2+3^2)(\beta^2+2^2)(\beta^2+1^2)(\beta^2+0^2)(\beta^2+(-1)^2)}{(9\beta^2+6^2)(9\beta^2+5^2)(9\beta^2+4^2)(9\beta^2+3^2)(9\beta^2+2^2)(9\beta^2+1)}} \\ &= \frac{14}{81}\sqrt{\frac{(\beta^2+16)(\beta^2+9)(\beta^2+1)\beta^2}{36(\beta^2+\frac{25}{9})(\beta^2+\frac{16}{9})(\beta^2+\frac{4}{9})(\beta^2+\frac{1}{9})}}. \end{aligned}$$

It then follows that

$$\frac{(\beta^2+16)(\beta^2+9)(\beta^2+1)\beta^2}{36(\beta^2+\frac{25}{9})(\beta^2+\frac{16}{9})(\beta^2+\frac{4}{9})(\beta^2+\frac{1}{9})}<1$$

since

$$\begin{aligned} &36\left(x+\frac{25}{9}\right)\left(x+\frac{16}{9}\right)\left(x+\frac{4}{9}\right)\left(x+\frac{1}{9}\right)-(x+16)(x+9)(x+1)x \\ &= 35x^4+158x^3+\frac{5}{3}x^2+\frac{232}{81}x+\frac{3484}{729}+(10x-2)^2 \\ &> 0 \end{aligned}$$

for all $x\geq 0$.

Thus, by Lemma 1.3, if $n\geq 7$, then

$$\begin{aligned} \frac{|f_{n,1}(\beta\sqrt{-1})|+|f_{n,2}(\beta\sqrt{-1})|}{|f_{n,0}(\beta\sqrt{-1})|} &\leq \frac{|f_{7,1}(\beta\sqrt{-1})|}{|f_{7,0}(\beta\sqrt{-1})|}+\frac{|f_{7,2}(\beta\sqrt{-1})|}{|f_{7,0}(\beta\sqrt{-1})|} \\ &< \frac{1792}{2187}+\frac{14}{81} \\ &< 1. \end{aligned}$$

Hence, we have $|f_{n,1}(\beta\sqrt{-1})|+|f_{n,2}(\beta\sqrt{-1})|<|f_{n,0}(\beta\sqrt{-1})|$.

CASE 2. $z=-\frac{n}{3}+\beta\sqrt{-1}$ with $\beta\in\mathbb{R}$.

First, we study the case when $n = 7, 8, 9$. If $n = 7$, then

$$\begin{aligned}
&\frac{|f_{7,1}(z)|}{|f_{7,0}(z)|}\\
&=7\sqrt{\frac{(4\beta^2+(\frac{1}{3})^2)(4\beta^2+(\frac{2}{3})^2)(4\beta^2+(\frac{5}{3})^2)(4\beta^2+(\frac{8}{3})^2)(4\beta^2+(\frac{11}{3})^2)(4\beta^2+(\frac{14}{3})^2)}{(9\beta^2+1)(9\beta^2+2^2)(9\beta^2+3^2)(9\beta^2+4^2)(9\beta^2+5^2)(9\beta^2+6^2)}}\\
&=7\left(\frac{2}{3}\right)^4\sqrt{\frac{(4\beta^2+(\frac{1}{3})^2)(4\beta^2+(\frac{2}{3})^2)(4\beta^2+(\frac{5}{3})^2)(4\beta^2+(\frac{8}{3})^2)}{(4\beta^2+(\frac{2}{3})^2)(4\beta^2+(\frac{2}{3}\cdot 2)^2)(4\beta^2+(\frac{2}{3}\cdot 3)^2)(4\beta^2+(\frac{2}{3}\cdot 4)^2)}}\\
&\quad\times\frac{\frac{11}{3}}{5}\cdot\frac{\frac{14}{3}}{6}\sqrt{\frac{((\frac{6}{11})^2\beta^2+1)((\frac{3}{7})^2\beta^2+1)}{((\frac{3}{5})^2\beta^2+1)((\frac{1}{2})^2\beta^2+1)}}\\
&\le\frac{8624}{10935},
\end{aligned}$$

and

$$\begin{aligned}
&\frac{|f_{7,2}(z)|}{|f_{7,0}(z)|}\\
&=21\sqrt{\frac{(\beta^2+(\frac{5}{3})^2)(\beta^2+(\frac{2}{3})^2)(\beta^2+(\frac{1}{3})^2)(\beta^2+(\frac{4}{3})^2)(\beta^2+(\frac{7}{3})^2)(\beta^2+(\frac{10}{3})^2)}{(9\beta^2+1)(9\beta^2+2^2)(9\beta^2+3^2)(9\beta^2+4^2)(9\beta^2+5^2)(9\beta^2+6^2)}}\\
&=21\cdot\frac{1}{2}\cdot\frac{1}{3^6}\cdot\frac{7}{3}\cdot\frac{10}{3}\sqrt{\frac{((\frac{3}{7})^2\beta^2+1)((\frac{3}{10})^2\beta^2+1)}{(\beta^2+1)((\frac{1}{2})^2\beta^2+1)}}\\
&\le\frac{245}{2187}.
\end{aligned}$$

Then, $\frac{8624}{10935}+\frac{245}{2187}<1$. Moreover, if $n = 8$, then

$$\frac{|f_{8,1}(z)|}{|f_{8,0}(z)|}<\frac{8}{7}\sqrt{\frac{4\beta^2+(\frac{16}{3})^2}{9\beta^2+7^2}}\frac{|f_{7,1}(z)|}{|f_{7,0}(z)|}<\frac{|f_{7,1}(z)|}{|f_{7,0}(z)|},$$

and

$$\frac{|f_{8,2}(z)|}{|f_{8,0}(z)|}<\frac{4}{3}\sqrt{\frac{\beta^2+(\frac{11}{3})^2}{9\beta^2+7^2}}\frac{|f_{7,2}(z)|}{|f_{7,0}(z)|}<\frac{|f_{7,2}(z)|}{|f_{7,0}(z)|}.$$

On the other hand, if $n = 9$, then we have

$$\begin{aligned}
&\frac{|f_{9,1}(z)|}{|f_{9,0}(z)|}\\
&=9\cdot\frac{2^6}{3^6}\sqrt{\frac{9\beta^2(9\beta^2+\left(\frac{3}{2}\right)^2)(9\beta^2+\left(\frac{3}{2}\right)^2)(9\beta^2+\left(\frac{9}{2}\right)^2)(4\beta^2+5^2)(4\beta^2+6^2)}{(9\beta^2+1)(9\beta^2+2^2)(9\beta^2+4^2)(9\beta^2+5^2)(9\beta^2+7^2)(9\beta^2+8^2)}}\\
&<\frac{64}{81},
\end{aligned}$$

and

$$\frac{|f_{9,2}(z)|}{|f_{9,0}(z)|}$$

$$= 36 \cdot \frac{1}{2^4 \cdot 3^3} \sqrt{\frac{9\beta^2(4\beta^2+2^2)(4\beta^2+4^2)(\beta^2+3^2)(4\beta^2+6^2)(4\beta^2+8^2)}{(9\beta^2+1)(9\beta^2+2^2)(9\beta^2+4^2)(9\beta^2+5^2)(9\beta^2+7^2)(9\beta^2+8^2)}}$$

$$< \frac{1}{12}.$$

Then, $\frac{64}{81} + \frac{1}{12} < 1$.

Note that $d^2 - 2 = 7 \leq n$. By Lemma 1.4, it follows that

$$|f_{n,1}(z)| + |f_{n,2}(z)| < |f_{n,0}(z)|.$$

CASE 3. $z = -\alpha \pm \sqrt{2n}\sqrt{-1}$ with $0 \leq \alpha \leq \frac{n}{3}$.

By Lemma 1.5,

$$\frac{|f_{n,1}(z)|}{|f_{n,0}(z)|} < n \left(\frac{4+\frac{1}{2}}{9}\right)^{\frac{n-1}{2}} = n \left(\frac{1}{2}\right)^{\frac{n-1}{2}},$$

and

$$\frac{|f_{n,2}(z)|}{|f_{n,0}(z)|} < \frac{n(n-1)}{2} \left(\frac{1+\frac{1}{2}}{9}\right)^{\frac{n-1}{2}} = \frac{n(n-1)}{2} \left(\frac{1}{6}\right)^{\frac{n-1}{2}}.$$

Since

$$\frac{(n+1)\left(\frac{1}{2}\right)^{\frac{n}{2}}}{n\left(\frac{1}{2}\right)^{\frac{n-1}{2}}} = \frac{n+1}{\sqrt{2}n} < 1$$

and

$$\frac{\frac{n(n+1)}{2}\left(\frac{1}{6}\right)^{\frac{n}{2}}}{\frac{n(n-1)}{2}\left(\frac{1}{6}\right)^{\frac{n-1}{2}}} = \frac{n+1}{\sqrt{6}(n-1)} < 1$$

hold for $n \geq 7$, we have

$$\frac{|f_{n,1}(z)|}{|f_{n,0}(z)|} < n \left(\frac{1}{2}\right)^{\frac{n-1}{2}} \leq 7 \left(\frac{1}{2}\right)^3 = \frac{7}{8}$$

and

$$\frac{|f_{n,2}(z)|}{|f_{n,0}(z)|} < \frac{n(n-1)}{2} \left(\frac{1}{6}\right)^{\frac{n-1}{2}} < 21 \left(\frac{1}{6}\right)^3 = \frac{7}{72}.$$

Thus, $|f_{n,1}(z)| + |f_{n,2}(z)| < |f_{n,0}(z)|$ follows from $\frac{7}{8} + \frac{7}{72} < 1$. □

3. The case of $d \geq 4$

In this section, we study the case of $d \geq 4$. Although, we could not prove that Conjecture 0.1 is true for $d \geq 4$, we prove inequalities which are close to those in Conjecture 0.1 when $d \ll n$.

THEOREM 3.1. *Suppose that integers d and n satisfy $d \geq 4$ and $n \geq 6d^2-16d+13$. Then, every root $a \in \mathbb{C}$ of $i(\Delta(d,n),m)$ satisfies*

$$\mathrm{Re}(a) < 1.$$

Proof. First, we prove that, every $z \in \mathbb{C}$ with $1 \leq \mathrm{Re}(z)$ satisfies

$$\frac{|f_{n,s}(z)|}{|f_{n,0}(z)|} < \frac{2^s}{3^s\, s!}$$

for $s = 1, 2, \ldots, d-1$. Let $z = \alpha+1+\beta\sqrt{-1}$ with $\alpha \geq 0$ and $\beta \in \mathbb{R}$ and let $m = z-1 \in \mathbb{C}$. Then,

$$f_{n,s}(z) = \binom{n}{s}((d-s)m + d + n - 1 - 2s)\cdots((d-s)m + d + 1 - 2s).$$

For $i = 1, 2, \ldots, n-d$, we have

$$d(d+n-i-2s) - (d-s)(d+n-i) = s(n-d-i) \geq 0,$$

and hence,

$$0 < \frac{d-s}{d+n-i-2s} \leq \frac{d}{d+n-i}.$$

Thus,

$$\left|\frac{(d-s)m+d+n-i-2s}{dm+d+n-i}\right| = \frac{d+n-i-2s}{d+n-i}\sqrt{\frac{(\frac{d-s}{d+n-i-2s}\beta)^2 + (\frac{d-s}{d+n-i-2s}\alpha+1)^2}{(\frac{d}{d+n-i}\beta)^2 + (\frac{d}{d+n-i}\alpha+1)^2}}$$
$$\leq \frac{d+n-i-2s}{d+n-i}.$$

On the other hand, for $j = 1, 2, \ldots, s$, since

$$(d-1)(2d-j) - d(2d-j-2s) = 2d(s-1) + j > 0$$

and

$$(d-1)(2d-j) + d(2d-j-2s) = 2d(2(d-1-s) + (s-j) + 1) + j > 0$$

hold, we have

$$\left|\frac{2d-j-2s}{d-1}\right| < \frac{2d-j}{d}.$$

Thus,

$$\left|\frac{(d-s)m+2d-j-2s}{dm+2d-j}\right| = \frac{d-1}{d}\left|\frac{\frac{d-s}{d-1}m + \frac{2d-j-2s}{d-1}}{m + \frac{2d-j}{d}}\right|$$

$$= \frac{d-1}{d}\sqrt{\frac{(\frac{d-s}{d-1}\beta)^2 + (\frac{d-s}{d-1}\alpha + \frac{2d-j-2s}{d-1})^2}{\beta^2 + (\alpha + \frac{2d-j}{d})^2}}$$
$$< \frac{d-1}{d}.$$

In addition, for $k = 1, 2, \ldots, s$,

$$\begin{aligned}&(2d-1)(2d-2)(d+n-k) - (2d-2k+1)(2d-2k)(d+n-1)\\ &= 2(k-1)(2(d+n)(d-1-k) + 2dn + n + 2k) \geq 0.\end{aligned}$$

Hence,

$$\frac{(2d-2k+1)(2d-2k)}{d+n-k} \leq \frac{(2d-1)(2d-2)}{d+n-1}.$$

Therefore,

$$\begin{aligned}
&\frac{|f_{n,s}(z)|}{|f_{n,0}(z)|}\\
&= \binom{n}{s}\frac{|(d-s)m+d+n-1-2s|\cdots|(d-s)m+d+1-2s|}{|dm+d+n-1|\cdots|dm+d+1|}\\
&= \binom{n}{s}\prod_{i=1}^{n-d}\left|\frac{(d-s)m+d+n-i-2s}{dm+d+n-i}\right| \prod_{j=1}^{s}\left|\frac{(d-s)m+2d-j-2s}{dm+2d-j}\right|\\
&\quad\times \prod_{j=s+1}^{d-1}\left|\frac{(d-s)m+2d-j-2s}{dm+2d-j}\right|\\
&< \binom{n}{s}\prod_{i=1}^{n-d}\frac{d+n-i-2s}{d+n-i}\ \left(\frac{d-1}{d}\right)^s\\
&= \binom{n}{s}\frac{(2d-1)\cdots(2d-2s)}{(d+n-1)\cdots(d+n-2s)}\left(\frac{d-1}{d}\right)^s\\
&= \frac{1}{s!}\left(\prod_{k=1}^{s}\frac{n+1-k}{d+n-k-s}\right)\left(\prod_{k=1}^{s}\frac{(2d-2k+1)(2d-2k)}{d+n-k}\right)\left(\frac{d-1}{d}\right)^s\\
&\leq \frac{1}{s!}\left(\frac{(2d-1)(2d-2)}{d+n-1}\cdot\frac{d-1}{d}\right)^s\\
&\leq \frac{2^s}{3^s s!}\left(\frac{(2d-1)(d-1)^2}{d(2d^2-5d+4)}\right)^s.
\end{aligned}$$

Since $d \geq 4$,

$$1 - \frac{(2d-1)(d-1)^2}{d(2d^2-5d+4)} = \frac{1}{d(2d^2-5d+4)} = \frac{1}{d(2(d-1)(d-2)+d)} > 0\,.$$

Thus,

$$\sum_{s=1}^{d-1} \frac{|f_{n,s}(z)|}{|f_{n,0}(z)|} < \sum_{s=1}^{d-1} \frac{2^s}{3^s s!} < -1 + e^{\frac{2}{3}} < 1\,,$$

and hence

$$\left| \sum_{s=1}^{d-1} (-1)^s f_{n,s}(z) \right| < |f_{n,0}(z)|\,.$$

Therefore, z is not a root of $i(\Delta(d,n),m)$. □

THEOREM 3.2. *Suppose that integers d and n satisfy $d \geq 4$ and $n \geq d^2+2d$. Then, every root $a \in \mathbb{C}$ of $i(\Delta(d,n),m)$ satisfies*

$$-\frac{n}{d} < \mathrm{Re}(a)\,.$$

Proof. We prove that, every $z \in \mathbb{C}$ with $\mathrm{Re}(z) \leq -\frac{n}{d}$ satisfies

$$\frac{|f_{n,s}(z)|}{|f_{n,0}(z)|} < \frac{1}{d-1}$$

for $s = 1, 2, \ldots, d-1$. Let $z = -\alpha - \frac{n}{d} - \beta\sqrt{-1}$ with $\alpha \geq 0$ and $\beta \in \mathbb{R}$ and let $m = -z - \frac{n}{d} = \alpha + \beta\sqrt{-1} \in \mathbb{C}$. Then,

$$f_{n,s}(z) = (-1)^{n-1} \binom{n}{s} ((d-s)m - \frac{ns}{d} + 1 + s) \cdots ((d-s)m - \frac{ns}{d} + n - 1 + s)$$

and

$$\begin{aligned}
&\frac{|f_{n,s}(z)|}{|f_{n,0}(z)|} \\
&= \binom{n}{s} \frac{|(d-s)m - \frac{ns}{d} + 1 + s| \cdots |(d-s)m - \frac{ns}{d} + n - 1 + s|}{|dm+1| \cdots |dm+n-1|} \\
&= \binom{n}{s} \prod_{k=1}^{\lfloor \frac{ns}{d} \rfloor - s} \left| \frac{(d-s)m - (\frac{ns}{d} - \lfloor \frac{ns}{d} \rfloor - 1 + k)}{dm+k} \right| \\
&\quad \times \prod_{k=\lfloor \frac{ns}{d} \rfloor - s + 1}^{n-d} \left| \frac{(d-s)m - \frac{ns}{d} + k + s}{dm+k} \right| \times \prod_{k=n-d+1}^{n-1} \left| \frac{(d-s)m - \frac{ns}{d} + k + s}{dm+k} \right|.
\end{aligned}$$

For $k = 1, 2, \ldots, \lfloor \frac{ns}{d} \rfloor - s$, we have

$$-k < 1 - \left(\frac{ns}{d} - \left\lfloor \frac{ns}{d} \right\rfloor \right) - k \leq 1 - k \ (\leq 0)\,.$$

Hence,

$$\left| \frac{(d-s)m - (\frac{ns}{d} - \lfloor \frac{ns}{d} \rfloor - 1 + k)}{dm+k} \right| < 1\,.$$

For $k = \lfloor \frac{ns}{d} \rfloor - s + 1, \lfloor \frac{ns}{d} \rfloor - s + 2, \ldots, n-d$, we have

$$-\frac{ns}{d} + k + s > 0$$

and

$$\frac{k}{d}-\frac{-\frac{ns}{d}+k+s}{d-s}=\frac{s}{d(d-s)}(n-d-k)\geq 0\,.$$

Hence, for $k=\left\lfloor\frac{ns}{d}\right\rfloor-s+1,\left\lfloor\frac{ns}{d}\right\rfloor-s+2,\ldots,n-d$,

$$\left|\frac{(d-s)m-\frac{ns}{d}+k+s}{dm+k}\right|=\frac{d-s}{d}\left|\frac{m+\frac{-\frac{ns}{d}+k+s}{d-s}}{m+\frac{k}{d}}\right|<\frac{d-s}{d}\,.$$

For $k=n-d+1,n-d+2,\ldots,n-1$, we have

$$\frac{d}{k}-\frac{d-s}{-\frac{ns}{d}+k+s}=\frac{ds(k-(n-d))}{k(kd-(n-d)s)}>0\,.$$

Hence

$$\left|\frac{(d-s)m-\frac{ns}{d}+k+s}{dm+k}\right|=\frac{-\frac{ns}{d}+k+s}{k}\left|\frac{\frac{d-s}{-\frac{ns}{d}+k+s}m+1}{\frac{d}{k}m+1}\right|<\frac{-\frac{ns}{d}+k+s}{k}\,.$$

Therefore,

$$\begin{aligned}&\frac{|f_{n,s}\left(-m-\frac{n}{d}\right)|}{|f_{n,0}\left(-m-\frac{n}{d}\right)|}\\&\leq\binom{n}{s}\left(\frac{d-s}{d}\right)^{n-d-\lfloor\frac{ns}{d}\rfloor+s}\frac{-\frac{ns}{d}+n-d+s+1}{n-d+1}\cdots\frac{-\frac{ns}{d}+n-1+s}{n-1}\\&\leq\frac{n^s}{s!}\left(\frac{d-s}{d}\right)^{(d-s)(\frac{n}{d}-1)}\frac{-\frac{ns}{d}+n-d+s+1}{n-d+1}\cdots\frac{-\frac{ns}{d}+n-1+s}{n-1}\,.\end{aligned}$$

Let

$$g(n,d,s)=\log\left(\frac{n^s}{\Gamma(s+1)}\left(\frac{d-s}{d}\right)^{(d-s)(\frac{n}{d}-1)}\right),$$

where $\Gamma(-)$ is the gamma function. Since $1\leq s\leq d-1$, we have

$$\log\frac{d-s}{d}<-\frac{s}{d}$$

and hence, for any $n\geq d^2+2d$,

$$\begin{aligned}\frac{\partial g}{\partial n}&=\frac{s}{n}+\frac{d-s}{d}\log\left(\frac{d-s}{d}\right)\\&<\frac{s}{d^2+2d}+\frac{d-s}{d}\left(-\frac{s}{d}\right)\\&=\frac{s}{d^2(d+2)}(-2-(d+2)(d-1-s))\\&<0.\end{aligned}$$

Thus, $g(n+1,d,s)<g(n,d,s)$. Moreover, for $k=1,2,\ldots,d-1$ $(<n)$,

$$\frac{\partial}{\partial n}\left(\frac{-\frac{ns}{d}+n-k+s}{n-k}\right)=-\frac{s(d-k)}{d(n-k)^2}<0\,.$$

Therefore, for $n \geq d^2 + 2d$,

$$\begin{aligned}
&\frac{|f_{n,s}\left(-m-\frac{n}{d}\right)|}{|f_{n,0}\left(-m-\frac{n}{d}\right)|}\\
&\leq \frac{(d^2+2d)^s}{s!}\left(\frac{d-s}{d}\right)^{(d-s)(d+1)}\frac{(d-s)(d+1)+1}{d^2+d+1}\cdots\frac{(d-s)(d+1)+d-1}{d^2+2d-1}\,.
\end{aligned}$$

CASE 1. Suppose $s = d-1$.

For each $d \geq 4$,

$$\begin{aligned}
&\frac{(d^2+2d)^{d-1}}{(d-1)!}\left(\frac{1}{d}\right)^{d+1}\frac{d+2}{d^2+d+1}\cdots\frac{2d}{d^2+2d-1}\\
&=\frac{1}{d}\frac{(d+2)^{d-1}}{d!}\frac{d+2}{d^2+d+1}\cdots\frac{2d}{d^2+2d-1}\\
&=\frac{1}{d}\cdot\frac{2^{d-2}\,3}{d!}\frac{2d(d+2)}{3(d^2+2d-1)}\prod_{k=1}^{d-2}\frac{(d+2)(d+1+k)}{2(d^2+d+k)}\,.
\end{aligned}$$

Note that

$$\frac{2^{d-2}\,3}{d!}=\prod_{k=4}^{d}\frac{2}{k}<1\,,$$

$$3(d^2+2d-1)-2d(d+2)=(d-1)(d+3)>0\,,$$

and, for each $1 \leq k \leq d-2$

$$2(d^2+d+k)-(d+2)(d+1+k)=d(d-2-k)+d-2>0\,.$$

Hence,

$$\frac{(d^2+2d)^{d-1}}{(d-1)!}\left(\frac{1}{d}\right)^{d+1}\frac{d+2}{d^2+d+1}\cdots\frac{2d}{d^2+2d-1}<\frac{1}{d-1}\,.$$

CASE 2. Suppose $1 \leq s \leq d-2$.

Let

$$h(d,s)=\log\left((d-1)\frac{(d^2+2d)^s}{\Gamma(s+1)}\left(\frac{d-s}{d}\right)^{(d-s)(d+1)}\right)\,.$$

Then, for $1 \leq s \leq d-2$,

$$\frac{\partial h}{\partial s}=\log(d^2+2d)-\frac{\Gamma(s+1)'}{\Gamma(s+1)}-(d+1)\left(\log\frac{d-s}{d}+1\right)$$

and

$$\frac{\partial^2 h}{\partial s^2}=\frac{d+1}{d-s}-\sum_{\ell=0}^{\infty}\frac{1}{(s+1+\ell)^2}\geq\frac{d+1}{d-s}-\left(\frac{\pi^2}{6}-1\right)=\frac{s+1}{d-s}+\left(2-\frac{\pi^2}{6}\right)>0\,.$$

Thus, for each d, we have $h(d,s) \leq \max(h(d,1), h(d,d-2))$ for all $1 \leq s \leq d-2$.

We now show that $h(d,1) < 0$ and $h(d,d-2) < 0$. Note that, for $d \geq 4$,

$$h(d,1)=\log(d-1)+\log(d^2+2d)+(d-1)(d+1)\log\left(\frac{d-1}{d}\right)$$

and

$$\begin{aligned}\frac{\partial h(d,1)}{\partial d} &= \frac{1}{d-1} + \frac{2d+2}{d^2+2d} + \frac{d+1}{d} + 2d\log\left(\frac{d-1}{d}\right)\\ &< \frac{1}{d-1} + \frac{2d+2}{d^2+2d} + \frac{d+1}{d} - 2\\ &= -\frac{(d-4)(d^2+d-1)}{(d-1)d(d+2)}\\ &\le 0.\end{aligned}$$

Thus, for every $d \ge 4$, we have $h(d,1) \le h(4,1) = \log 72 + 15\log\frac{3}{4} = \log\frac{129140163}{134217728} < 0$.

On the other hand,

$$\begin{aligned}(d-1)\frac{(d^2+2d)^{d-2}}{(d-2)!}\left(\frac{2}{d}\right)^{2(d+1)} &= \frac{2^6\ 6^{d-2}(d-1)}{(d-2)!d^6}\left(\frac{4(d^2+2d)}{6d^2}\right)^{d-2}\\ &= \frac{2^6\ 6^{d-2}(d-1)}{(d-2)!d^6}\left(\frac{3d-(d-4)}{3d}\right)^{d-2}\\ &\le \frac{2^6\ 6^{d-2}(d-1)}{(d-2)!d^6}.\\ &= \frac{2^6\ 6^{d-2}(d-1)^2(d+1)}{(d+1)!d^5}\\ &= \frac{2^6\ 6^{d-2}(d-1)}{(d+1)!d^3}\cdot\frac{d^2-1}{d^2}\\ &< \frac{2^6\ 6^{d-2}(d-1)}{(d+1)!d^3}.\end{aligned}$$

For $d = 4$,

$$\frac{2^6\ 6^{4-2}(4-1)}{(4+1)!\,4^3} = \frac{9}{10} < 1\,,$$

and for $d \ge 5$,

$$\frac{2^6\ 6^{d-2}(d-1)}{(d+1)!d^3} = \frac{2^6\ 6^2}{5!\,5^2}\cdot\frac{5^2}{d^2}\cdot\frac{d-1}{d}\prod_{k=6}^{d+1}\frac{6}{k} < \frac{96}{125} < 1\,.$$

Thus,

$$h(d,d-2) = \log\left((d-1)\frac{(d^2+2d)^{d-2}}{(d-2)!}\left(\frac{2}{d}\right)^{2(d+1)}\right) < 0\,.$$

Therefore, it follows that, for every $1 \le s \le d-1$,

$$\begin{aligned}&\frac{|f_{n,s}\left(-m-\frac{n}{d}\right)|}{|f_{n,0}\left(-m-\frac{n}{d}\right)|}\\ &\le \frac{(d^2+2d)^s}{s!}\left(\frac{d-s}{d}\right)^{(d-s)(d+1)}\frac{(d-s)(d+1)+1}{d^2+d+1}\cdots\frac{(d-s)(d+1)+d-1}{d^2+2d-1}\end{aligned}$$

$$< \frac{1}{d-1} \,.$$

Hence,

$$\sum_{s=1}^{d-1} \frac{|f_{n,s}(z)|}{|f_{n,0}(z)|} < \sum_{s=1}^{d-1} \frac{1}{d-1} = 1 \,,$$

and

$$\left| \sum_{s=1}^{d-1} (-1)^s f_{n,s}(z) \right| < |f_{n,0}(z)| \,.$$

Thus, z is not a root of $i(\Delta(d,n),m)$. □

4. Computational experiments

In this section, some computational experiments are given. First, we computed the approximate roots of the Ehrhart polynomial $i(\Delta(d,n),m)$

- for $4 \leq d \leq 10$ and $2d \leq n \leq d^2 + 2d$ (Figures 1 – 7),
- for $4 \leq d \leq 75$ and $n = 2d$ (Figure 8),

by using the software package `Mathematica` [7] ("`N`" and "`Solve`") and `gnuplot`. Second, by the software `Maple` [6] ("`Hurwitz`"), we checked that every root α of the Ehrhart polynomial $i(\Delta(d,n),m)$ satisfies

$$-\frac{n}{d} < \mathrm{Re}(\alpha) < 0$$

for $4 \leq d \leq 10$ and $2d \leq n \leq d^2 + 2d$.

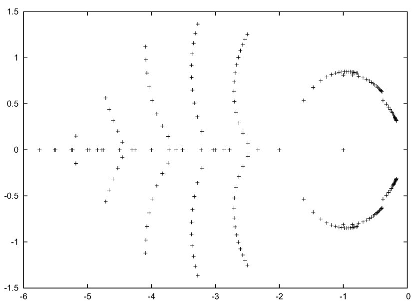

FIGURE 1. $d = 4$

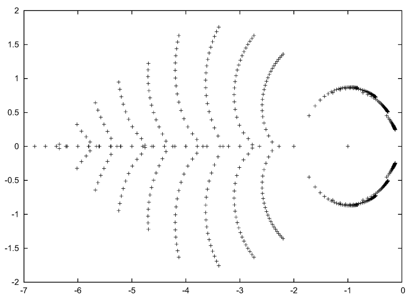

FIGURE 2. $d = 5$

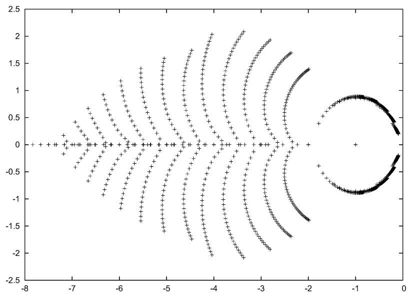

FIGURE 3. $d = 6$

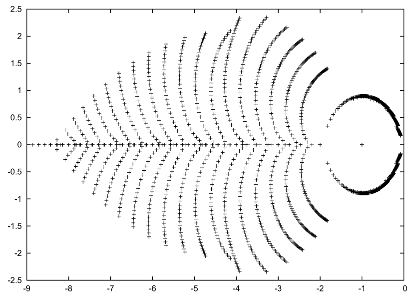

FIGURE 4. $d = 7$

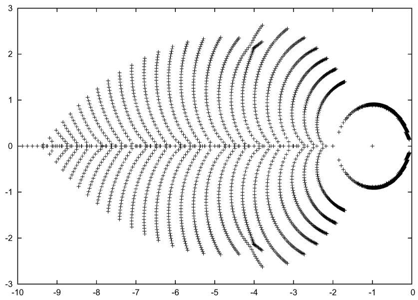

FIGURE 5. $d = 8$

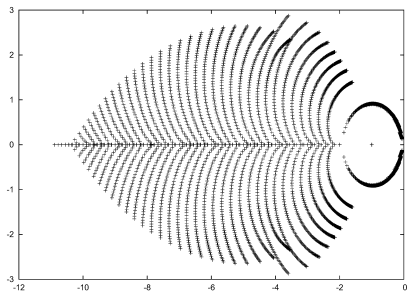

FIGURE 6. $d = 9$

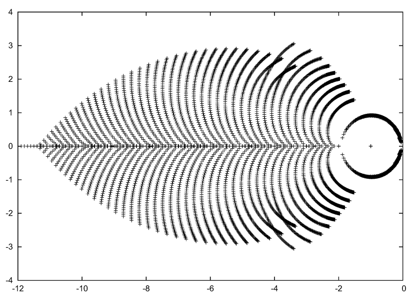

FIGURE 7. $d = 10$

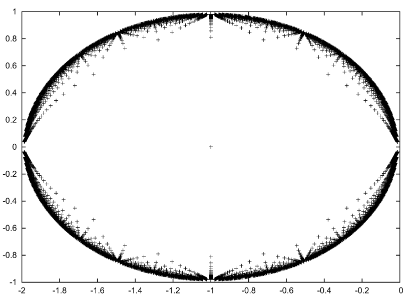

FIGURE 8. $4 \le d \le 75$ and $n = 2d$

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References

[1] M. Beck, J. A. De Loera, M. Develin, J. Pfeifle and R. P. Stanley, Coefficients and roots of Ehrhart polynomials, *in* “Integer Points in Polyhedra—Geometry, Number theory, Algebra, Optimization,” *Contemp. Math.* **374** (2005), 15–36.

[2] B. Braun, Norm bounds for Ehrhart polynomial roots, *Discrete Comput. Geom.* **39** (2008), 191–193.

[3] A. Higashitani, Counterexamples of the Conjecture on Roots of Ehrhart Polynomials, *Discrete Comput. Geom.* **47** (2012), 618–623.
[4] M. Katzman, The Hilbert series of algebras of the Veronese type, *Communications in Algebra* **33** (2005), 1141–1146.
[5] M. Marden, Geometry of Polynomials, Math. Surveys, No. 3; *Amer. Math. Soc.* R.I., Providence (1966).
[6] Maple 15, Maplesoft, a division of Waterloo Maple Inc., Waterloo, Ontario.
[7] Mathematica 9.0, Wolfram Research, Inc., Champaign, IL.
[8] T. Matsui, A. Higashitani, Y. Nagazawa, H. Ohsugi and T. Hibi, Roots of Ehrhart polynomials arising from graphs, *Journal of Algebraic Combinatorics* **34** (2011), 721–749.
[9] H. Ohsugi and K. Shibata, Smooth Fano polytopes whose Ehrhart polynomial has a root with large real part, *Discrete Comput. Geom.* **47** (2012), 624–628.

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