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On Eisenstein Series of Half Integral Weight and Theta Series Over Imaginary Quadratic Fields

by

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Abstract. In this article we give an answer to an analogy of Kubota's problem, that is, we construct the theta series over the hyperbolic 3-space of the Hamilton quaternion algebra attached to the imaginary quadratic field by using the residue of an analytic Eisenstein series.

1. Introduction

T. Kubota [3], [5] studied about relations of reciprocity laws of quadratic residue symbols over algebraic number fields and the transformation formulas of theta functions. To be precise, he realized certain theta series associated with the zeta functions of algebraic number fields F as automorphic forms with respect to the special linear group of degree 2 over F, and derived explicit factors of automorphy from some properties of Gauss sums. Further he [3], [4], [5] introduced the Eisenstein series $E(u, s_1, s_2, \ldots, s_{r_2})$ defined by quadratic residue symbols of totally imaginary number fields of degree $2r_2$, and proposed to determine a relation between this Eisenstein series $E(u, s_1, s_2, \ldots, s_{r_2})$ and the theta series associated with the zeta function of F. In particular, he has shown that the function determined by this Eisenstein series by means of analytic continuation at $s_1 = s_2 = \cdots = s_{r_2} = 1/2$ has the same analytic properties as them of the theta series in [4].

In [7], G.Shimura investigated modular forms appeared in the value or the residue of Eisenstein series of half-integral weight over totally real number fields K. There he proved that certain theta series associated with K is given as the residue of certain Eisenstein series of half-integral weight over K.

In this paper we shall define the Eisenstein series of half-integral weight over imaginary quadratic fields F according to the method of [7], and shall get the theta series over the hyperbolic 3-space of the Hamilton quaternion algebra $\mathbf{H}$ attached to F in [3] by using the residue of the Eisenstein series in this case. From these results, we get an answer to an analogy of the above problem in the case of imaginary quadratic fields. Though the proof of this fact is basically the same as that of [7], we need to treat technical difficulties at the archimedean prime. There we confront a calculation of an integral formula of the product of special functions. Fortunately, we can calculate explicitly them. Analyzing the archimedean part closely, we deduce our results.

In section 2, we shall introduce the metaplectic group and factors of automorphy of half-integral weight over imaginary quadratic fields F. In section 3, we study Eisenstein series E and E' of half-integral weight over F. There we deduce that the residue of E' at $s = 1/2$ is equal to the theta series attached to imaginary quadratic fields F.

NOTATION. We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For an associative ring R with identity element we denote by $R^\times$ the group of all its invertible elements and by $M_2(R)$ all 2×2 -matrices with entries in R. For $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$ with a, b, c and d in R, we put $a_x = a,\ b_x = b,\ c_x = c$ and $d_x = d$. We fix an imaginary quadratic field F and denote by ∞ and $\mathbf{h}$ the imaginary archimedean prime and the set of nonarchimedean primes of F respectively. Further we denote by $\mathfrak{g}$ and $\mathfrak{d}$ the maximal order of F and the different of F relative to $\mathbb{Q}$.

If $v = \infty$, we view v as an injection of F into $\mathbb{C}$, and identify $F_v = \mathbb{C}$; this means that we fix one of the two injections for $v(= \infty)$. For a fractional ideal $\mathfrak{a}$ in F and $t \in F_{\mathbb{A}}^\times$ we denote by $N(\mathfrak{a})$ the norm of $\mathfrak{a}$ and by $t\mathfrak{a}$ the fractional ideal in F with $(t\mathfrak{a})_v = t_v \mathfrak{a}_v$ for every $v \in \mathbf{h}$, and we put $|t|_v = |t_v|_v$ with the normalized valuation $|\ |_v$ at v and $|t|_{\mathbb{A}} = \prod_{v \in \{\infty\} \cup \mathbf{h}} |t|_v$.

Given an algebraic group $\mathcal{G}$ defined over F, we define the localization $\mathcal{G}_v$ for each $v \in \{\infty\} \cup \mathbf{h}$ and the adelization $\mathcal{G}_{\mathbb{A}}$ as usual, and view $\mathcal{G}$ as a subgroup of $\mathcal{G}_{\mathbb{A}}$. We denote by $\mathcal{G}_\infty$ and $\mathcal{G}_{\mathbf{h}}$ the archimedean and nonarchimedean factors of $\mathcal{G}_{\mathbb{A}}$ respectively. For an element $x \in \mathcal{G}_{\mathbb{A}}$ we denote by $x_v,\ x_\infty$ and $x_{\mathbf{h}}$ its projections to $\mathcal{G}_v,\ \mathcal{G}_\infty$ and $\mathcal{G}_{\mathbf{h}}$.

We put $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. For $z \in \mathbb{C}$, we put $\mathbf{e}(z) = \exp(2\pi\sqrt{-1}z)$ and $\Re(z) = \mathrm{Re}(z)$. We define a character $\mathbf{e}_v : F_v \to \mathbb{T}$ for each v by

$$\mathbf{e}_v(x) = \begin{cases} \mathbf{e}(x + \overline{x}) & \text{if } v = \infty \\ \mathbf{e}\left(-\left\{\text{fractional part of } \mathrm{Tr}_{F_v/\mathbb{Q}_p}(x)\right\}\right) & \text{if } v \in \mathbf{h}\,, \end{cases}$$

where p is the rational prime divisible by v. Further we put $\mathbf{e}_{\mathbb{A}}(x) = \prod_{v \in \{\infty\} \cup \mathbf{h}} e_v(x_v)$ for $x = (x_v) \in F_{\mathbb{A}}$.

2. Metaplectic groups and the factor of automorphy of weight $\frac{1}{2}$

We put

$$\begin{aligned} G &= \left\{\alpha \in \mathrm{GL}_2(F) \mid {}^t\alpha \mathbf{i} \alpha = \mathbf{i}\ \right\} \quad \left(\mathbf{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right), \\ P &= \{\alpha \in G \mid c_\alpha = 0\}, \\ \Omega_{\mathbb{A}} &= \left\{x \in G_{\mathbb{A}} \mid c_x \in F_{\mathbb{A}}^\times\right\}, \quad \text{and} \quad \Omega_v = \{x \in G_v \mid c_x \neq 0\}\,. \end{aligned}$$

We let G acts on $F^2 = F \times F$ by right multiplication. Then we can define the metaplectic groups $\mathrm{Mp}(F_{\mathbb{A}})$ and $\mathrm{Mp}(F_v)$ in the sense of [9] for each $v \in \{\infty\} \cup \mathbf{h}$ with respect to the alternating form $(x, y) \mapsto x\mathbf{i}^t y$ on $F^2 \times F^2$. Recall that these groups, written $\mathrm{M}_{\mathbb{A}}$ and M_v, for simplicity, are groups of unitary transformation on $L^2(F_{\mathbb{A}})$ and on $L^2(F_v)$.

There is an exact sequence

$$1 \to \mathbb{T} \to \mathrm{M}_{\mathbb{A}} \to G_{\mathbb{A}} \to 1$$

and also a similar exact sequence for each v with M_v and G_v in place of $\mathrm{M}_{\mathbb{A}}$ and $G_{\mathbb{A}}$. We denote by **pr** the projection maps of $\mathrm{M}_{\mathbb{A}}$ and M_v to $G_{\mathbb{A}}$ and G_v. There is a natural lift $\gamma : G \to \mathrm{M}_{\mathbb{A}}$ by which we can consider G a subgroup of $\mathrm{M}_{\mathbb{A}}$. There are also two types of splitting lifts

$$\gamma_P : P_{\mathbb{A}} \to \mathrm{M}_{\mathbb{A}}, \quad \gamma_\Omega : \Omega_{\mathbb{A}} \to \mathrm{M}_{\mathbb{A}}, \tag{2.1}$$

$$[\gamma_P(\alpha)f](x) = |\det(a_\alpha)|_{\mathbb{A}}^{\frac{1}{2}} \mathbf{e}_{\mathbb{A}}(xa_\alpha{}^t b_\alpha{}^t x/2) f(xa_\alpha) \quad \text{if } \alpha \in P_{\mathbb{A}}, \tag{2.2a}$$

$$[\gamma_\Omega(\beta)f](x) = \left|\det(c_\beta)\right|_{\mathbb{A}}^{\frac{1}{2}} \int_{F_{\mathbb{A}}} f(xa_\beta + yc_\beta)\mathbf{e}_{\mathbb{A}}(q_\beta(x,y))dy \tag{2.2b}$$

$$\text{if } \beta \in \Omega_{\mathbb{A}}, \quad q_\beta(x,y) = \frac{1}{2}xa_\beta{}^t b_\beta{}^t x + \frac{1}{2}yc_\beta{}^t d_\beta{}^t y + xb_\beta{}^t c_\beta{}^t y.$$

These maps are consistent in the sense that $\gamma = \gamma_P$ on P, $\gamma = \gamma_\Omega$ on $G \cap \Omega_{\mathbb{A}}$ and

$$\gamma_\Omega(\alpha\beta\gamma) = \gamma_P(\alpha)\gamma_\Omega(\beta)\gamma_P(\gamma) \quad \text{if } \alpha, \gamma \in P_{\mathbb{A}}, \quad \beta \in \Omega_{\mathbb{A}}. \tag{2.2c}$$

Moreover there are similar lifts of P_v and Ω_v into M_v given by the same formulas with the subscript $\mathbb{A}$ replaced by v. Here the measure on $F_{\mathbb{A}}$ is the product of the measure $\prod_{v\in\{\infty\}\cup\mathbf{h}} d_v x$ with

$$d_v(x+iy) = 2dxdy \quad \text{for } v = \infty \quad \text{and} \quad \int_{\mathfrak{g}_v} d_v x = N(\mathfrak{d}_v)^{-\frac{1}{2}} \quad \text{for } v \in \mathbf{h}.$$

Let $\mathbf{H}$ be the Hamilton quaternion algebra $\mathbf{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ with the quaternion units i, j and k. We put

$$H = \{\mathfrak{z} = z + jy \in \mathbf{H} \mid z \in \mathbb{C}, y > 0\}. \tag{2.3}$$

We define an action of $\alpha \in G$ on H by

$$\alpha(\mathfrak{z}) = (a_\alpha\mathfrak{z} + b_\alpha)(c_\alpha\mathfrak{z} + d_\alpha)^{-1} \quad \text{for all } \mathfrak{z} \in H. \tag{2.4}$$

For $\mathfrak{z} = z + jy \in H$ and $\alpha \in G$, we put $\mu_0(\alpha,\mathfrak{z}) = c_\alpha\mathfrak{z} + d_\alpha$. Further we put $Y(\mathfrak{z}) = y$ for $\mathfrak{z} = z + jy \in H$. Then we have

$$Y(\alpha(\mathfrak{z})) = m(\alpha,\mathfrak{z})^{-1}Y(\mathfrak{z}) \quad \text{with} \quad m(\alpha,\mathfrak{z}) = |\mu_0(\alpha,\mathfrak{z})|^2. \tag{2.5}$$

We define a compact subgroup C of $G_{\mathbb{A}}$ by

$$C = \prod_{v\in\{\infty\}\cup\mathbf{h}} C_v,$$

$$C_v = \begin{cases} \{\alpha \in G_v \mid \alpha(j) = j\} & \text{if } v = \infty \\ \{\alpha \in G_v \mid (\mathfrak{g} \times \mathfrak{d}^{-1})_v\alpha = (\mathfrak{g} \times \mathfrak{d}^{-1})_v\} & \text{if } v \in \mathbf{h}. \end{cases}$$

We have $G_{\mathbb{A}} = P_{\mathbb{A}}C$. We define a map $\lambda : G_{\mathbb{A}} \to F_{\mathbb{A}}$ by

$$\lambda(\alpha) = d_\alpha \quad (\alpha \in G_{\mathbb{A}}).$$

For $\alpha \in G_{\mathbb{A}}$, we assign a fractional ideal $\mathrm{il}(\alpha)$ of F and a positive real number $\epsilon(\alpha)$ by

$$\mathrm{il}(yw) = \lambda(y)\mathfrak{g}\,, \quad \epsilon(yw) = |\lambda(y)|_{\mathbb{A}} \quad \text{for } y \in P_{\mathbb{A}} \text{ and } w \in C\,, \tag{2.6}$$

where $x\mathfrak{g}$ for $x \in F_{\mathbb{A}}^{\times}$ denotes the ideal with $(x\mathfrak{g})_v = x_v\mathfrak{g}_v$ for every $v \in \mathbf{h}$. Then we have

$$\begin{gathered}\epsilon(\alpha) = \epsilon(\alpha_{\mathbf{h}})\epsilon(\alpha_\infty)\,, \\ \epsilon(\alpha_{\mathbf{h}}) = N(\mathrm{il}(\alpha))^{-1}\,, \quad \epsilon(\alpha_\infty) = m(\alpha_\infty, j)^{\frac{1}{2}} \quad \text{with } j \in H\,.\end{gathered} \tag{2.7}$$

We put $c = 4\mathfrak{g}$ and define a subgroup C' of C by $C' = \prod_v C'_v$ with

$$C'_v = \begin{cases} C_v \text{ if } v = \infty \\ \{\alpha \in C_v \mid b_\alpha \in 2\mathfrak{d}_v^{-1},\ c_\alpha \in 2\mathfrak{d}_v\} \quad \text{if } v \in \mathbf{h}\,. \end{cases} \tag{2.8}$$

We put $\lambda(\tau) = \lambda(\mathbf{pr}(\tau))$, $\epsilon(\tau) = \epsilon(\mathbf{pr}(\tau))$ and $\mathrm{il}(\tau) = \mathrm{il}(\mathbf{pr}(\tau))$ for $\tau \in \mathrm{M}_{\mathbb{A}}$. We fix an element δ of $F_{\mathbb{A}}^{\times}$ with $\delta_v\mathfrak{g}_v = \mathfrak{d}_v$ if $v \in \mathbf{h}$ and $\delta_v = 1$ if $v = \infty$, and define an element η of $G_{\mathbb{A}}$ by

$$\eta_v = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } v = \infty \\[2ex] \begin{pmatrix} 0 & -\delta_v^{-1} \\ \delta_v & 0 \end{pmatrix} & \text{if } v \in \mathbf{h}\,. \end{cases} \tag{2.9}$$

We put $C'' = C' \cup C'\eta$. We define a function $\Phi_{\mathfrak{z},0}(x)$ on $F_{\mathbb{A}}$ by

$$\Phi_{\mathfrak{z},0}(x) = \mathbf{e}(\Re(x_a^2 z) + i\Re(\overline{x}_a y x_a)) \prod_{v \in \mathbf{h}} \varphi_v(x_v) \tag{2.10}$$

for every $x = (x_\infty, x_v) \in F_{\mathbb{A}}$ $(v \in \mathbf{h})$ and $\mathfrak{z} = z + jy \in H$, where φ_v is the characteristic function of $\mathfrak{g}_v$ for $v \in \mathbf{h}$.

For $\mathbf{pr}(\tau) \in P_{\mathbb{A}}C''$ and $\mathfrak{z} \in H$, we denote by $h(\tau, \mathfrak{z})$ the quasi factor of automorphy of weight $\frac{1}{2}$ defined in [8, Theorem 1.2].

PROPOSITION 1. *For every $\xi \in \mathrm{M}_{\mathbb{A}}$ with $\mathbf{pr}(\xi) \in P_{\mathbb{A}}C''$, there is a function $h(\xi, \mathfrak{z})$ determined by*

$$(\xi\Phi_{\mathfrak{z},0})(0) = N(\mathrm{il}(\xi))^{\frac{1}{2}} h(\xi, \mathfrak{z})^{-1}\,. \tag{2.11}$$

Moreover h has the following properties:

$$h(\xi, \mathfrak{z}) m(\mathbf{pr}(\xi), \mathfrak{z})^{-\frac{1}{2}} \quad \text{for a fixed } \xi \text{ does not depend on } \mathfrak{z}\,, \tag{2.12a}$$

$$h(\xi, \mathfrak{z})^2 = \zeta m(\mathbf{pr}(\xi), \mathfrak{z}) \quad \text{with} \quad \zeta \in \mathbb{T}\,, \tag{2.12b}$$

$$\begin{gathered} h(\beta\xi\tau, \mathfrak{z}) = h(\beta, \mathfrak{z}) h(\xi, \mathbf{pr}(\tau)\mathfrak{z}) h(\tau, \mathfrak{z}) \quad \text{if } \mathbf{pr}(\beta) \in P_{\mathbb{A}}\,, \\ \mathbf{pr}(\xi) \in P_{\mathbb{A}}C'' \quad \text{and} \quad \mathbf{pr}(\tau) \in C''G_\infty\,, \end{gathered} \tag{2.12c}$$

$$h(t\gamma_p(\gamma), \mathfrak{z}) = t^{-1} |(d_\gamma)_\infty|^{\frac{1}{2}} \quad \text{if } t \in \mathbb{T} \text{ and } \gamma \in P_{\mathbb{A}}\,. \tag{2.12d}$$

3. Eisenstein series

Now an automorphic form with respect to $h(\xi, \mathfrak{z})$ is a function $g_0 : \mathrm{M}_{\mathbb{A}} \to \mathbb{C}$ satisfying

$$g_0(\alpha\xi u) = g_0(\xi)h(u, j)^{-1} \quad \text{for } j \in H\,,\ \alpha \in G\,,\ \xi \in \mathrm{M}_{\mathbb{A}} \text{ and } u \in \mathrm{M}_{\mathbb{A}} \text{ with } \mathbf{pr}(u) \in B\,, \text{ where } B \text{ is an open subgroup of } C''\,. \tag{3.1}$$

Given such a g_0, we can define a function g on H by

$$g(\xi(j)) = g_0(\xi)h(\xi, j) \quad \text{for } \xi \in \mathrm{M}_{\mathbb{A}} \text{ and } \mathbf{pr}(\xi) \in BG_\infty\,. \tag{3.2}$$

We put $\Gamma = G \cap BG_\infty$. Then we have

$$g(\alpha(\mathfrak{z})) = g(\mathfrak{z})h(\alpha, \mathfrak{z}) \quad \text{for } \alpha \in \Gamma \text{ and } \mathfrak{z} \in H\,, \tag{3.3}$$

$$g_0(\alpha\xi) = g(\xi(j))h(\xi, j)^{-1} \quad \text{for } \alpha \in G \text{ and } \mathbf{pr}(\xi) \in BG_\infty\,. \tag{3.4}$$

Conversely, if g is a function on H satisfying (3.3), we can define a function g_0 on $\mathrm{M}_{\mathbb{A}}$ by (3.4), which satisfies (3.1) and (3.2). We define a function f on $\mathrm{M}_{\mathbb{A}}$ by

$$f(\xi) = \begin{cases} 0 & \text{if } \mathbf{pr}(\xi) \notin P_{\mathbb{A}}C'\,, \\ |h(\xi, j)|h(\xi, j)^{-1} & \text{if } \mathbf{pr}(\xi) \in P_{\mathbb{A}}C'\,. \end{cases} \tag{3.5}$$

By virtue of (2.12.c) and (2.12.d), we have

$$f(\beta\xi u) = f(\xi)h(u, j)^{-1} \quad \text{if } \beta \in P \text{ and } \mathbf{pr}(u) \in C'\,. \tag{3.6}$$

We now define an Eisenstein series $E_{\mathbb{A}}(\xi, s)$ on $\mathrm{M}_{\mathbb{A}}$ by

$$E_{\mathbb{A}}(\xi, s) = \sum_{\alpha \in P\backslash G} f(\alpha\xi)\epsilon(\alpha\xi)^{-2s-\frac{1}{2}}\,. \tag{3.7}$$

We put $E(\mathfrak{z}, s) = E_{\mathbb{A}}(\xi, s)h(\xi, j)$ for $\mathfrak{z} = \xi(j)$ with $\xi \in M_{\mathbb{A}}$ and $\mathbf{pr}(\xi) \in G_\infty$. Put $\Gamma_0(4) = G \cap C'G_\infty$. Then we can easily verify that

$$E(\gamma(\mathfrak{z}), s) = h(\gamma, \mathfrak{z})E(\mathfrak{z}, s) \quad \text{for every } \gamma \in \Gamma_0(4)\,. \tag{3.8}$$

We denote by $\mathcal{S}(F_{\mathbb{A}})$ the Schwartz-Bruhat space of $F_{\mathbb{A}}$. With η defined by (2.9), we choose the element $\tilde{\eta} \in M_{\mathbb{A}}$ with $\mathbf{pr}(\tilde{\eta}) = \eta$ and $\tilde{\eta}\psi = \psi_\infty \prod_{v\in\mathbf{h}} \gamma_\Omega(\eta_v)\psi_v$ for every ψ in $\mathcal{S}(F_{\mathbb{A}})$ of the form $\psi(x) = \psi_\infty(x_\infty)\prod_{v\in\mathbf{h}} \psi_v(x_v)$. Then we define a function $E'_{\mathbb{A}}(\xi, s)$ on $\mathrm{M}_{\mathbb{A}}$ by

$$E'_{\mathbb{A}}(\xi, s) = E_{\mathbb{A}}(\xi\tilde{\eta}, s) \quad \text{for every } \xi \in \mathrm{M}_{\mathbb{A}}\,. \tag{3.9}$$

Both $E_{\mathbb{A}}$ and $E'_{\mathbb{A}}$ satisfy (3.1) with a suitable B, hence we can define a function $E'(\mathfrak{z}, s) = E'_{\mathbb{A}}(\xi, s)h(\xi, j)$ for $\mathfrak{z} = \xi(j)$ with $\xi \in M_{\mathbb{A}}$ and $\mathbf{pr}(\xi) \in BG_\infty$.

We put

$$t(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad \tau(u) = \gamma_p(t(u)) \quad (u \in F_{\mathbb{A}})\,.$$

We fix an element ξ of $\mathrm{M}_{\mathbb{A}}$ with $\mathbf{pr}(\xi) \in G_\infty$. Then $E'_{\mathbb{A}}(\tau(u)\xi, s)$ has a Fourier expansion of the form

$$E'_{\mathbb{A}}(\tau(u)\xi, s) = \sum_{\sigma \in F} b(\sigma, \xi, s)\mathbf{e}_{\mathbb{A}}(\sigma u) \quad (u \in F_{\mathbb{A}})\,, \tag{3.10}$$

where $\Re(s)$ is sufficiently large, and $b(\sigma, \xi, s)$ is the following

$$b(\sigma, \xi, s) = \int_{F_{\mathbb{A}}/F} E'_{\mathbb{A}}(\tau(u)\xi, s)\mathbf{e}_{\mathbb{A}}(-\sigma u)du\,.$$

Using the same argument as that of [7, p. 300], we deduce

$$b(\sigma, \xi, s) = \int_{F_{\mathbb{A}}} f(\mathbf{i}\tau(u)\xi\tilde{\eta})\epsilon(\mathbf{i}\tau(u)\xi\tilde{\eta})^{-2s-\frac{1}{2}}\mathbf{e}_{\mathbb{A}}(-\sigma u)du\,. \tag{3.11}$$

Here we express this in terms of $E'(\mathfrak{z}, s)$. For $\mathfrak{z} \in H$ with $\mathfrak{z} = z + jy$, we define $w \in G_{\mathbb{A}}$ and $u^0 \in F_{\mathbb{A}}$ by

$$w_v = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } v \in \mathbf{h}\,, \\ \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix} & \text{if } v = \infty\,, \end{cases} \qquad u_v^0 = \begin{cases} 0 & \text{if } v \in \mathbf{h}\,, \\ z & \text{if } v = \infty\,. \end{cases} \tag{3.12}$$

Then we have

$$\begin{aligned} E'(\mathfrak{z}, s) &= h(\tau(u^0)\gamma_P(w), j)E'_{\mathbb{A}}(\tau(u^0)\gamma_P(w), s) \\ &= \sum_{\sigma \in \frac{1}{2}\mathfrak{g}} c(\sigma, y, s)\mathbf{e}(\sigma z + \overline{\sigma}\,\overline{z})\,. \end{aligned} \tag{3.13}$$

By the same method as that of [7, 4.10], we obtain $E'(\mathfrak{z}, s) = E(\eta_0(\mathfrak{z}), s)h(\eta_0, \mathfrak{z})^{-1}$ for some $\eta_0 \in G$. The arguments in [7, p. 301] implies that

$$c(\sigma, y, s) = y^{-\frac{1}{4}}b(\sigma, \gamma_P(w), s)\,. \tag{3.14}$$

By $\mathbf{i}\tau(u)\gamma_P(w) = \gamma_\Omega(it(u)w)$ and (2.2b), we obtain

$$\begin{aligned} [\mathbf{i}\tau(u)\gamma_P(w)\Phi_{j,0}](0) = &\prod_{v \in \mathbf{h}} \int_{F_v} \varphi_v(x)\mathbf{e}_v\left(\frac{xu_v{}^t x}{2}\right)dx_v \\ &\times y^{\frac{1}{4}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-2\pi y|x|^2)\mathbf{e}\left(\frac{ux^2 + \overline{u}\,\overline{x}^2}{2}\right)dx_1dx_2 \end{aligned}$$

with $x = x_1 + ix_2$. By virtue of [1, p. 49], we see the following formula

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-2\pi y|x|^2)\mathbf{e}\left(\frac{ux^2 + \overline{u}\,\overline{x}^2}{2}\right)dx_1dx_2 = \frac{1}{2}(|u|^2 + y^2)^{-\frac{1}{2}}\,. \tag{3.15}$$

Since we have that

$$(\mathbf{i}t(u)w\eta)_v = \begin{cases} \begin{pmatrix} -\delta_v & 0 \\ \delta_v u_v & -\delta_v^{-1} \end{pmatrix} & \text{if } v \in \mathbf{h}\,, \\ \begin{pmatrix} 0 & -\sqrt{y}^{-1} \\ \sqrt{y} & u\sqrt{y}^{-1} \end{pmatrix} & \text{if } v = \infty\,, \end{cases}$$

and (2.7), we obtain

$$\epsilon((\mathbf{i}t(u)w\eta)_v) = y + |u|^2 y^{-1} \quad (v = \infty)\,. \tag{3.16}$$

Therefore the arguments in [7, p. 302] yields that

$$h(\mathbf{i}\tau(u)\gamma_P(w)\tilde{\eta}, j)) = \prod_{v \nmid 4} \omega_v(u_v) \times 2(|u|^2 + y^2)^{-\frac{1}{2}} \tag{3.17}$$

with

$$\omega_v(u_v) = \left(\epsilon\left(\begin{pmatrix} 1 & 0 \\ \delta_v^2 u_v & 1 \end{pmatrix}\right)^{\frac{1}{2}} N(\delta_v)^{\frac{1}{2}} \int_{\mathfrak{g}_v} \mathbf{e}_v\left(\frac{x^2 u_v}{2}\right) dx\right)^{-1}.$$

The proof of the following proposition proceeds like the arguments given in [7, Lemma 4.2].

PROPOSITION 2. *$f(\mathbf{i}\tau(u)\gamma_P(w)\tilde{\eta}) \neq 0$ if and only if $u_v \in 2\mathfrak{d}_v^{-1}$ for every v dividing* 4, *in which case we have*

$$\begin{aligned} &f(\mathbf{i}\tau(u)\gamma_P(w)\tilde{\eta})\epsilon\,(\mathbf{i}\tau(u)\gamma_P(w)\tilde{\eta})^{-2s-\frac{1}{2}} \\ &\quad = N(\mathfrak{d})^{-2s-\frac{1}{2}}(y^{-1}(y^2 + |u|^2))^{-2s-\frac{1}{2}} \prod_{\substack{v \in \mathbf{h} \\ v \nmid 4}} \omega_v(u_v)\epsilon\left(\begin{pmatrix} 1 & 0 \\ \delta_v^2 u_v & 1 \end{pmatrix}\right)^{-2s-\frac{1}{2}}. \end{aligned}$$

By Proposition 2, (3.11) and (3.14), we have, for $\sigma \in \frac{1}{2}\mathfrak{g}$,

$$c(\sigma, y, s) = N(\mathfrak{d})^{-2s} \prod_{v \in \mathbf{h}} c_v(\sigma, s) c_\infty(\sigma, y, s)\,,$$

where

$$c_v(\sigma, s) = \begin{cases} N\left(\left(\frac{1}{2}\mathfrak{g}\right)_v\right) & \text{if } v \mid 4\,, \\ \displaystyle\sum_{u \in F_v/\delta_v^{-1}\mathfrak{g}_v} \omega_v(u)\mathbf{e}_v(-\sigma u)\epsilon\left(\begin{pmatrix} 1 & 0 \\ \delta_v^2 u_v & 1 \end{pmatrix}\right)^{-2s-\frac{1}{2}} & \text{if } v \nmid 4\,, \end{cases}$$

$$c_\infty(\sigma, y, s) = y^{2s+\frac{1}{2}} \int_{\mathbb{C}} \frac{\mathbf{e}(-(\sigma u + \overline{\sigma}\,\overline{u}))}{(y^2 + |u|^2)^{2s+\frac{1}{2}}} du\,. \tag{3.18}$$

The integral in (3.18) is equal to

$$y^{1-4s} \frac{K_{\frac{1}{2}-2s}(4\pi y|\sigma|)(4\pi y|\sigma|)^{2s-\frac{1}{2}}}{\Gamma(2s + \frac{1}{2})2^{2s-\frac{1}{2}}}\,, \tag{3.19}$$

where $K_{\frac{1}{2}-2s}(4\pi y|\sigma|)$ is the modified Bessel function of the first kind (cf. [6, p. 271]). Applying the methods of [7, p. 307–308], we deduce

$$E'(\mathfrak{z}, s) = \sum_{\sigma \in \frac{1}{2}\mathfrak{g}} \frac{c_f(\sigma, s)K_{\frac{1}{2}-2s}(4\pi y|\sigma|)(4\pi y|\sigma|)^{2s-\frac{1}{2}}}{N(\mathfrak{d})^{2s}\Gamma(2s + \frac{1}{2})y^{2s-\frac{3}{2}}2^{2s+\frac{3}{2}}} \mathbf{e}(\sigma z + \overline{\sigma}\,\overline{z})\,, \tag{3.20}$$

where $c_f(\sigma, s)$ is defined by

$$c_f(\sigma, s) = \begin{cases} L_4(4s-1, 1)/L_4(4s, 1) & \text{if } \sigma = 0\,, \\ \displaystyle\sum_{\mathfrak{a}, \mathfrak{b}} \frac{\mu(\mathfrak{a})\omega_\sigma(\mathfrak{a})L_4(2s, \omega_\sigma)}{N(\mathfrak{a})^{2s}N(\mathfrak{b})^{4s-1}L_4(4s, 1)} & \text{if } \sigma \neq 0\,, \end{cases} \tag{3.21}$$

with $L_4(s, \chi) = \prod_{\wp \nmid 4}(1 - \chi(\wp)N(\wp)^{-s})^{-1}$. Here ω_σ ($\sigma \in \frac{1}{2}\mathfrak{g}$) is an ideal character defined by $\omega_\sigma(\wp) = \left(\frac{F(\sqrt{2\sigma})/F}{\wp}\right)$ for every prime ideal $\wp$ coprime to 4, μ is the Möbius function and $\mathfrak{a}$, $\mathfrak{b}$ runs over all ordered pair of integral ideals of F coprime to 4 with $\mathfrak{a}^2\mathfrak{b}^2 \supset 4\sigma$.

Further we have the following proposition (cf. [7, Corollary 6.2]).

PROPOSITION 3. *$\Gamma(s)\Gamma(s + \frac{1}{2})L_4(4s, 1)E(\mathfrak{z}, s)$ is holomorphic on the whole s-plane except for a possible simple pole at $s = 1/2$.*

By virtue of (3.20), we deduce the following theorem.

THEOREM 1. *The residue of $E'(\mathfrak{z}, s)$ at $s = 1/2$ is equal to*

$$\frac{1}{8N(\mathfrak{d})}\prod_{\wp \nmid 4}(1 + N(\wp)^{-1})^{-1}\zeta_F(2)^{-1}\mathrm{Residue}_{s=1}\zeta_F(s)\vartheta_F(\mathfrak{z}) \tag{3.22}$$

for every $\mathfrak{z} = z + jy \in H$, where $\zeta_F(s)$ is the Dedekind zeta function associated with F and $\vartheta_F(\mathfrak{z})$ is the theta series attached to F defined by

$$\vartheta_F(\mathfrak{z}) = y^{\frac{1}{2}}\sum_{\alpha \in \mathfrak{g}} \mathbf{e}\left(\frac{\alpha^2 z + \overline{\alpha}^2\,\overline{z} + 2i|\alpha|^2 y}{2}\right).$$

REMARK 1. $\vartheta_F(\mathfrak{z})$ is the theta series over H associated with the zeta function of F studied in [3].

Proof. We have

$$\mathrm{Residue}_{s=\frac{1}{2}}c(\sigma, y, s) = \frac{\mathrm{Residue}_{s=\frac{1}{2}}c_f(\sigma, s)}{\sqrt{2}N(\mathfrak{d})\Gamma\left(\frac{3}{2}\right)}y^{\frac{1}{2}}K_{-\frac{1}{2}}(4\pi y|\sigma|)(4\pi y|\sigma|)^{\frac{1}{2}}\,.$$

By [2, p. 915], we see that

$$K_{-\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z},$$

which yields that

$$\mathrm{Residue}_{s=\frac{1}{2}}c(\sigma, y, s) = \frac{y^{\frac{1}{2}}\mathrm{Residue}_{s=\frac{1}{2}}c_f(\sigma, s)}{2N(\mathfrak{d})\exp(4\pi|\sigma|y)}\,.$$

From (3.21), on one hand, we have

$$\mathrm{Residue}_{s=\frac{1}{2}}c_f(0, s) = \frac{\mathrm{Residue}_{s=1}\zeta_F(s)}{4\zeta_F(2)}\prod_{\wp | 4}(1 + N(\wp)^{-1})^{-1}.$$

On the other hand, for $\sigma \in \frac{1}{2}\mathfrak{g}$ $(\sigma \neq 0)$, we have

$$\mathrm{Residue}_{s=\frac{1}{2}} c_f(\sigma, s) = \begin{cases} \dfrac{\mathrm{Residue}_{s=1}\zeta_F(s)}{2\zeta_F(2)} \displaystyle\prod_{\wp|4} \left(1+\frac{1}{N(\wp)}\right)^{-1} & \\ & \text{if } \sigma = \frac{1}{2}\sigma'^2 \text{for some } \sigma' \in \mathfrak{g}\,, \\ 0 & \text{if otherwise.} \end{cases}$$

Therefore we derive our theorem. □

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