HIGH FREQUENCY DIFFRACTION OF AN ELECTROMAGNETIC PLANE WAVE BY AN IMPERFECTLY CONDUCTING RECTANGULAR CYLINDER AT GRAZING INCIDENCE.

by A. D. RAWLINS

(*Department of Mathematical Sciences, Brunel University, Uxbridge, UB8 3PH, United Kingdom*)

[Received Revise]

Summary

We shall derive new results for the electromagnetic scattered far wave field produced when a high frequency plane E-polarized wave is at grazing incident on an imperfectly conducting rectangular cylinder. The solution to the problem is obtained by using the geometrical theory of diffraction, multiple diffraction methods, the canonical solution for the problem of the diffraction of a plane wave by a right-angled impedance wedge, in conjunction with a novel analytic approach.

1. Introduction

In dealing with mobile phone propagation in cities the effect of building corners and their surface cladding is of importance for the signal strength of the phones, see references in Rawlins(**1**). A building of rectangular cross-section can be modelled by four of these corners. With appropriate polarization this building can be effectively modelled for high frequency diffraction by a rectangular impedance cylinder in two dimensions. To obtain quantitative and qualitative results for the signal strength far from the building when there are multiple diffraction from such corners an effective approach is to use the Keller's method of the geometrical theory of diffraction(GTD) Keller(**2**), and information about the "diffraction coefficient" which are obtained from the asymptotic solution of canonical impedance wedge problems, Rawlins(**3**). In a previous work Rawlins(**1**), by using some uniform asymptotics developed from the canonical wedge solution Rawlins(**3**), useful asymptotic results were obtained for the oblique incidence case. These asymptotic results were in good agreement with numerical calculations using a development of the Nystrom method, and geometrical modeling of the corner structure by Smith and Rawlins(**4**). An alternative numerical method using a hybrid of physical optics and boundary element approach to deal with impedance structures with sharp corners has been developed by Chandler-Wilde, Langdon and Mokgolele(**5**). However the the asymptotic method used for the oblique case breaks down when the incident wave is at grazing incidence, that is when the incident ray is parallel to a side of the rectangular cylinder. The breakdown of the previous methods occurs because near the grazing incidence shadow boundaries, which lie along the sides of the rectangular cylinder faces, the asymptotic form of the far field changes rapidly for slight angular deviation from these boundaries. In using the plane wave asymptotic expansion formula of Zitron and Karp(**6**) for the far field it was necessary to evaluate angular derivatives of the diffraction coefficient for the far field. At grazing incidence the far field varies rapidly for small angular variation because there is a coalescence of specular and shadow boundaries and Keller's straightforward method is no longer applicable because the diffraction coefficient becomes infinite. The analytical reason for this is that multiple poles of the integrand of the integral solution of the canonical wedge problem move towards a critical saddle point. This requires a more delicate asymptotic evaluation of the "diffraction coefficient", for two different approaches see Rawlins (**7**) for the ideal wedge problem, and Osipov(**8**), Osipov, Hongo and Kobayashi (**9**) for analogous uniform asymptotics for the impedance wedge problem. Initially an attempt was made to apply a method used by Morse(**10**) for the simpler problem of grazing incidence on an ideally conducting cylinder. His method consisted of substituting the known complete uniform asymptotic expansions involving Fresnel integrals of the solution to the canonical ideal conducting wedge problem given by Oberhettinger (**11**) directly into the Green's theorem field representation for the far field. The field on the surface of the cylinder was represented as various derivatives of the canonical right-angled wedge solution. Hence their application required that they differentiate and integrate these asymptotic series which resulted in dealing with arbitrary constants of integration which were functions of the cylinder dimensions, and divergent integrals †. Even though some analogous complete asymptotic expansions were available for the impedance wedge problem Rawlins (**3**), the complete grazing asymptotics required for the present application produced insuperable problems for the present author when trying to deal with the analogous divergent integrals and arbitrary constants of integration. This was because of the complexity of the integrand of the canonical solution of the impedance wedge problem. Hence a simpler alternative method had to be sought in order to derive the grazing incidence far field for the impedance cylinder. The alternative approach was to use Green's theorem and analytic manipulation of the complex integral solution of the impedance wedge diffraction problem and *then* carry out the asymptotic evaluations to any desired order of accuracy. To achieve this the complex integral representation of the canonical problem, Rawlins(**1**)(**12**), for the diffraction field for a right-angled impedance wedge is used; in conjunction with the multiple diffraction that arises from waves traveling from corner to corner of the rectangle.

A time harmonic E-polarized plane wave $\mathbf{E}_i = u_i(r,\theta)\mathrm{e}^{-i\omega t}\hat{\mathbf{z}}$, $u_i(r,\theta) = e^{-ikr\cos(\theta-(\pi/2-\theta_0))} = e^{-ikr\sin(\theta+\theta_0)}$, is incident on the cylinder; for convenience the angle of incidence θ_0 is measured from the vertical y-axis. $\hat{\mathbf{z}}$ is the usual unit vector in the positive z-direction. The resultant total field, which consists of the incident and the field scattered by the cylinder, is given by $\mathbf{E} = u(r,\theta)\mathrm{e}^{-i\omega t}\hat{\mathbf{z}}$. In the rest of the paper the time dependence $\mathrm{e}^{-i\omega t}$ will be suppressed. From the symmetry of the scatterer, Maxwell's equations, and incident polarization the total electromagnetic field can be reduced to a scalar problem involving the determination of the scalar function $u(r,\theta)$. The starting point of this alternative approach is Green's theorem in two dimensions which gives

$$u_s(r,\theta) = \frac{i}{4}\oint\left[u\frac{\partial}{\partial n}H_0^{(1)}(kR) - H_0^{(1)}(kR)\frac{\partial u}{\partial n}\right]ds, \tag{1.1}$$

where $u_s(r,\theta) = u(r,\theta) - u_i(r,\theta)$ is the scattered field at the point (r,θ) for any angle of incidence. The field quantity u which appears in the integrand of (1.1), is the total

† For example in equation(3.44) of Morse(**10**) the integral$\partial I/\partial\rho$ does not exist for $\alpha > \pi, \rho = 0$.

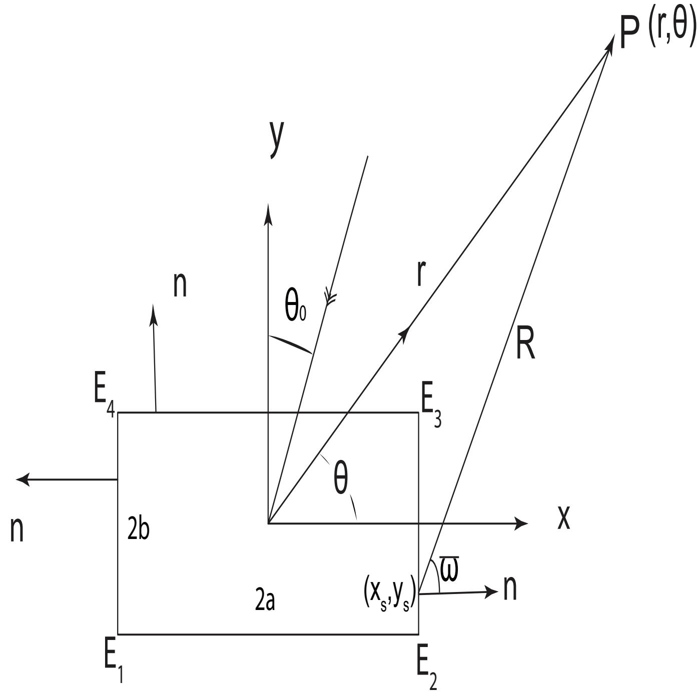

Fig. 1 Co-ordinates $(r, \theta) = (x, y)$ at the centre of the absorbing rectangular cylinder of dimensions $2a \times 2b$. The corners E_1, E_2, E_3, E_4 of the rectangular cylinder have respective cartesian co-ordinates (-a,-b),(a,-b),(a,b),(-a,b). In the far-field $R \gg 1, r \gg 1; R = r - x_s \cos\theta - y_s \sin\theta$ where R is the straight line from the co-ordinates on the cylinder (x_s, y_s) and the far field point $P(r, \theta)$.The angle ϖ between the unit vector $\mathbf{n}$ and $\mathbf{R}$ is given by $\mathbf{n} \cdot \mathbf{R} = R\cos\varpi$

field on the surface of the cylinder; $\mathbf{n}$ is the outer normal from the sides of the cylinder, and $R(s, r)$ the distance from the point s on the cylinder to the observation point P, see Fig. 1. In the work that follows we are going to assume that high frequency means that the sides of the rectangular cylinder ($2a$ and $2b$) are large compared to the incident wavelength, $ka > 1, kb > 1$, but small enough to include curvature of the the fields propagated along the cylinder faces; and that we are calculating the field far away from the scatterer, that is, $r \gg 2\sqrt{a^2 + b^2}, kr \gg 1$. In order to determine the asymptotic representation of the integral (1.1) for r large compared to the lengths $2a$ and $2b$, the following well known asymptotic representation is required,

$$H_0^{(1)}(kR) = \left(\frac{2}{\pi kR}\right)^{\frac{1}{2}} e^{i(kR - \pi/4)} + O((kR)^{-3/2}), \tag{1.2}$$

together with the fact that

$$\frac{\partial H_0^{(1)}(kR)}{\partial n} = \frac{\partial H_0^{(1)}(kR)}{\partial(kR)}\frac{\partial(kR)}{\partial n} = -ik\cos\varpi\left(\frac{2}{\pi kR}\right)^{\frac{1}{2}} e^{i(kR-\pi/4)} + O((kR)^{-3/2}), \quad (1.3)$$

where ϖ is the angle between the unit vector $\mathbf{n}$ and the unit vector $\mathbf{R}/R$. By using the convention that the faces $\overline{E_1E_2}, \overline{E_2E_3}, \overline{E_3E_4}, \overline{E_4E_1}$, are identified by the parameter$j(j = 1, 2, 3, 4$ respectively) the expression (1.3) can be re-written as

$$\frac{\partial H_0^{(1)}(kR)}{\partial n} = ik\sin[\theta + (1-j)\frac{\pi}{2}]\left(\frac{2}{\pi kR}\right)^{\frac{1}{2}} e^{i(kR-\pi/4)} + O((kR)^{-3/2}). \quad (1.4)$$

By substituting (1.2) and (1.4) into (1.1) we get for any angle of incidence

$$\begin{aligned} u_s(r,\theta) &= \frac{e^{i(kr-\frac{\pi}{4})}}{2\sqrt{2\pi kr}}\sum_{j=1}^{4}\exp\left[-ik(x_j\cos\theta + y_j\sin\theta)\right] \\ &\times \int_0^{2d_j}\left[-ku_{j,j+1}\sin[\theta+(1-j)\frac{\pi}{2}] - i\frac{\partial u_{j,j+1}}{\partial n}\right]e^{-ik\rho_j\cos[\theta+(1-j)\frac{\pi}{2}]}d\rho_j \\ &\quad + O[(kr)^{-3/2}], \end{aligned} \quad (1.5)$$

where $\rho_j, (j = 1, 2, 3, 4)$, is the distance measured from the vertex E_j towards the vertex E_{j+1}, $2d_j$ is the length of the side $\overline{E_jE_{j+1}}$, and (x_j, y_j) are the coordinates of E_j; the field quantity $u_{j,j+1}$ is the field on the side $\overline{E_jE_{j+1}}$. The impedance boundary conditions on the cylinder faces is given by

$$\frac{\partial u}{\partial n} = ik\cos\vartheta u, \quad (1.6)$$

which on substituting into (1.5) gives

$$\begin{aligned} u_s(r,\theta) &= -k\frac{e^{i(kr-\frac{\pi}{4})}}{2\sqrt{2\pi kr}}\sum_{j=1}^{4}\exp\left[-ik(x_j\cos\theta + y_j\sin\theta)\right] \\ &\times[\sin[\theta+(1-j)\frac{\pi}{2}] - \cos\vartheta]\int_0^{2d_j} u_{j,j+1}e^{-ik\rho_j\cos[\theta+(1-j)\frac{\pi}{2}]}d\rho_j + O[(kr)^{-3/2}]. \end{aligned} \quad (1.7)$$

We shall now use (1.7) to determine the forward scattered field when the incident field rays are parallel to the sides of the rectangle, that is when $\theta = \frac{\pi}{2}$ $\theta_0 = \pi$. Substituting these values into (1.7) gives

$$\begin{aligned} u_s(r,\frac{\pi}{2}) &= -k\frac{e^{i(kr-\frac{\pi}{4})}}{2\sqrt{2\pi kr}}[e^{ikb}(1-\cos\vartheta)\int_0^{2a} u_{1,2}d\rho_1 - e^{ikb}\cos\vartheta\int_0^{2b} u_{2,3}e^{-ik\rho_2}d\rho_2 \\ &-e^{-ikb}(1+\cos\vartheta)\int_0^{2a} u_{3,4}d\rho_3 - e^{-ikb}\cos\vartheta\int_0^{2b} u_{4,1}e^{-ik\rho_4}d\rho_4] + O[(kr)^{-3/2}], \end{aligned} \quad (1.8)$$

where in terms of the coordinates (x, y) located at the centre of the rectangle the integration variables are given by $\rho_1 = x + a, \rho_2 = y + b, \rho_3 = a - x, \rho_4 = b - y$.

2. Calculation of the field at the corners of the cylinder

It is now necessary to evaluate the field quantities under the integral signs of (1.8). To achieve this the field at each corner of the cylinder must first be calculated, as they will be the incident fields that produce the diffracted fields along the cylinder faces and hence contribute towards computing the integrands of (1.8). We shall use the notation for the vertex field $u_l(E_j)$ where if $l = i$ the contribution is from the diffracted field produced by the incident plane wave directly incident on the corner j; if $l = 1$ the contribution is from the diffracted field from the corner j that has been produced by a diffracted field incident on corner j that resulted from the plane wave directly incident on another corner.Thus a subscript $l = 1$ means that the incident field on the corner$j$ has been produced by diffraction of the incident plane wave by another corner.The field at the corners E_1 and E_2 due to direct illumination by the incident wave is $u_i(E_1) = u_i(E_2) = e^{-ikb}$. It is required to determine the field at the corners E_3 and E_4 after the incident wave has been diffracted by the corners E_2 and E_1 respectively. We shall also need to consider the field at E_1 produced by diffraction of the incident wave by E_2 and vice-versa. As in previous computations carried out in Rawlins(**1**), only terms up to order $O[(kd)^{-3/2}]$, where $d = a$ or $d = b$, will be retained. The canonical solution for the diffraction of a time-harmonic plane wave $U_i(\rho, \theta) = e^{-ik\rho\cos(\theta-\theta_0)}$ by a right angled wedge results in an edge diffracted field $U_d(\rho, \theta)$ given (see(7) of Rawlins(**3**)) by

$$U_d(\rho, \theta) = \frac{1}{2\pi i} \int_{S(\theta)} F(\gamma, \theta_0) e^{ik\rho\cos(\gamma-\theta)} d\gamma, \tag{2.1}$$

where

$$\begin{aligned} F(\gamma, \theta_0) \quad &= \quad \frac{\frac{2}{\sqrt{3}} \sin\frac{2\gamma}{3} \sin\frac{2\theta_0}{3} (\cos\gamma - \cos\vartheta)(\sin\gamma + \cos\vartheta)}{(\cos\theta_0 + \cos\vartheta)(\sin\theta_0 - \cos\vartheta)(\cos\frac{2\pi}{3} - \cos\frac{2(\gamma-\theta_0)}{3})} \\ &\times \quad \frac{(\cos\frac{4\theta_0}{3} - \cos\frac{4(\pi+\vartheta)}{3})(2\cos\frac{2\theta_0}{3}\cos\frac{2\gamma}{3} + \frac{1}{2} - \cos\frac{4(\pi+\vartheta)}{3})}{(\cos\frac{2\pi}{3} - \cos\frac{2(\gamma+\theta_0)}{3})(\cos\frac{4(\gamma-\pi-\vartheta)}{3} + \frac{1}{2})(\cos\frac{4(\gamma+\pi+\vartheta)}{3} + \frac{1}{2})}. \end{aligned} \tag{2.2}$$

The contour of integration $S(\theta), 0 \leqq \theta \leqslant \pi$, is the path of steepest descent through $\gamma = \theta$ which starts at $\theta + \frac{\pi}{2} - i\infty$ and ends at $\theta + \frac{\pi}{2} + i\infty$ and its shape is given by the equation $\Re\gamma = \theta - \arccos(1/\cosh(\Im\gamma))sign(\Im\gamma)$ and is shown schematically in Fig. 2. Upper case letters U_d will denote the canonical solution to the right-angled impedance wedge problem, and lower case letters u_d the application of this solution to the rectangular impedance cylinder problem. The field $u_1(E_3)$ at E_3 produced by the diffraction of the incident plane wave $u_i(\rho, \pi)$ from E_2 can be obtained by referring to Fig. 3 and using the following method. As shown in Fig. 3 a coordinates system (ξ_3, η_3) has been located at the corner E_3 and $P(\xi_3, \eta_3)$ is a point in the neighbourhood of E_3 such that $\eta_3 > 0$. The total field at the corner E_3 is defined by $u_1(E_3) = u_i(P) + u_d(P, E_2)$; and from expression (2.1) with $\rho\cos\theta = 2b + \xi_3, \rho\sin\theta = \eta_3, \theta_0 = \pi, \theta = \delta_3 \to 0^+$ we get

$$u_d(P, E_2) = \frac{e^{-ikb}}{2\pi i} \int_{S(\delta_3)} v(\gamma) F(\gamma, \pi) e^{2ikb\cos\gamma} d\gamma, \tag{2.3}$$

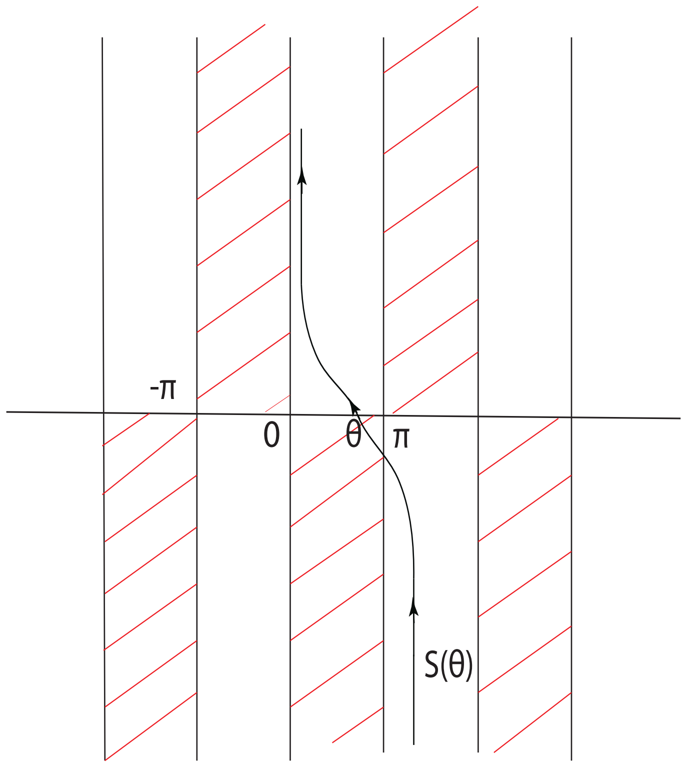

Fig. 2 The path of steepest descent $S(\theta) : \Re\alpha = \theta - \arccos(1/\cosh(\Im\gamma))sign(\Im\gamma)$ in the complex γ-plane which starts at $\theta + \frac{\pi}{2} - i\infty$ and terminates at $\theta - \frac{\pi}{2} + i\infty$ where $0 \leqslant \theta \leqslant \pi$.

where $v(\gamma) = e^{ik(\xi_3 \cos\gamma + \eta_3 \sin\gamma)}$. By using the explicit form (2.1) for $F(\gamma, \pi)$, we expand the expression $v(\gamma)F(\gamma, \pi)$ as a Laurent series , about the saddle point $\gamma = 0$. By using Mathematica Series this gives :

$$v(\gamma)F(\gamma, \pi) = -\frac{2v(0)}{\gamma} - 2\left[v'(0) + \frac{v(0)}{\cos\vartheta}\right] + O[\gamma]. \tag{2.4}$$

Thus as $\delta_3 \to 0$,

$$u_d(P, E_2) = -\frac{v(0)e^{-ikb}}{\pi i}\int_{S(0)} \frac{e^{2ikb\cos\gamma} d\gamma}{\gamma} - \frac{e^{-ikb}}{\pi i}\int_{S(0)} \left[v'(0) + \frac{v(0)}{\cos\vartheta} + O(\gamma)\right] e^{2ikb\cos\gamma} d\gamma. \tag{2.5}$$

The first integral in the above expression can be replaced by half a residue contribution at the origin and a principal value integral over $S(0)$; the second integral can be evaluated by

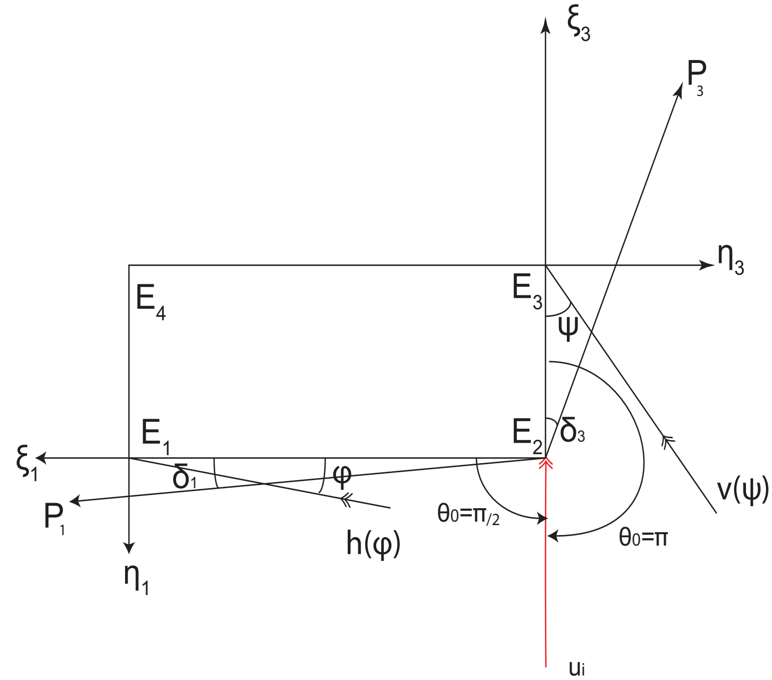

Fig. 3 The diffracted field from corner E_2 represented as plane wave incident fields at corners E_3 and E_1 in terms of plane waves $v(\psi)$ and $h(\varphi)$ respectively.

the method of steepest descent. Because of the oddness of the integrand of the principal value integral and the symmetry of the path of integration $S(0)$ its value is zero. Thus

$$u_d(P, E_2) = -v(0)e^{ikb} - \frac{e^{ikb+i\pi/4}}{\sqrt{\pi kb}}\left[v'(0) + \frac{v(0)}{\cos\vartheta}\right] + O[(kb)^{-3/2}]. \tag{2.6}$$

We obtain after some simplifying and noting that $u_i(P) = e^{ikb}v(0)$, and $u_1(E_3) = u_i(P) + u_d(P, E_2)$ where $P \simeq E_3$:

$$u_1(E_3) = -\frac{e^{ikb+i\pi/4}}{\sqrt{\pi kb}}\left[v'(0) + \frac{v(0)}{\cos\vartheta}\right] + O[(kb)^{-3/2}]. \tag{2.7}$$

We note that we already know the solution to the problem of the diffraction a plane wave. Hence the response to the incident plane wave $v(0), v'(0)$ is given respectively by $u_d(0), -u_d'(0)$. Because symmetry the field at E_4 due to diffraction by E_1 of the incident plane wave, is also given by (2.7), that is, $u_1(E_3) = u_1(E_4)$. Consistent with our definition for $u_1(E_3)$ the only singly diffracted field reaching E_1 comes from E_2. As shown in Fig. 3 a coordinates system (ξ_1, η_1) has been located at the corner E_1 and $P(\xi_1, \eta_1)$ is a point in the neighbourhood of E_1 such that $\eta_1 > 0$. The total field at the corner E_1 is defined by

$u_1(E_1) = u_i(P) + u_d(P, E_2)$; and from (2.1) with $\rho\cos\theta = 2a + \xi_1, \rho\sin\theta = \eta_1, \theta_0 = \pi, \theta = \delta_1 \to 0^+$ we get

$$u_d(P, E_2) = \frac{e^{-ikb}}{2\pi i}\int_{S(\delta_1)} h(\gamma)F(\gamma, \pi/2)e^{2ika\cos\gamma}d\gamma, \tag{2.8}$$

where $h(\gamma) = e^{ik(\xi_1\cos\gamma+\eta_1\sin\gamma)}$; and expanding by using Mathematica we get

$$h(\gamma)F(\gamma, \pi/2) = -2\frac{(3 - 2\cos\frac{4(\pi+\vartheta)}{3})}{(3 + 6\cos\frac{4(\pi+\vartheta)}{3})}h(0)\gamma + O[\gamma^2]. \tag{2.9}$$

An application of the saddle point method then gives

$$u_1(E_1) = O[(ka)^{-3/2}], \tag{2.10}$$

also by symmetry $u_1(E_1) = u_1(E_2)$. Thus the second order diffracted fields at these corners(E_1 and E_2), need not be considered because they are of $O[(kd)^{-3/2}]$.

3. Calculation of the field on the cylinder faces

Having found the vertex fields we can now proceed to calculate the field on any face of the cylinder. The notation $[u_{mn}/u_l(E_j)]$ will denote the component of $u_{m,n}$ contributed by the vertex field $u_l(E_j)$.

Determination of $u_{3,4}$

When the incident plane wave illuminates the corner E_2 the diffracted field can be represented as a series of plane waves; these plane waves will now illuminate the corner E_3. We shall thus require the response of the corner E_3 to a plane wave. The field along $\overline{E_3E_4}$ due to a plane wave incident on corner E_3 at an angle $\theta_0 \simeq 0$ is given by $u_d(\rho_3, 3\pi/2)$, where u_d is given by (2.1). To find the field along the face $\overline{E_3E_4}$ after the incident field has been diffracted by E_2 and then subsequently E_3 we substitute the above field $u_d(\rho_3, 3\pi/2)$ into (2.7) and allow $\theta_0 \to 0$. Hence

$$\begin{aligned}
[u_{34}|u_1(E_3)] &= \frac{e^{i(kb+\frac{\pi}{4})}}{2\sqrt{\pi kb}}[\frac{\partial u_d(\rho_3, 3\pi/2)}{\partial\theta_0} - \frac{u_d(\rho_3, 3\pi/2)}{\cos\vartheta}]_{\theta_0\to 0} + O[(kd)^{-3/2}],\\
&= \frac{e^{i(kb+\frac{\pi}{4})}}{2\sqrt{\pi kb}}[\frac{\partial u_d(\rho_3, 3\pi/2)}{\partial\theta_0}]_{\theta_0\to 0} + O[(kd)^{-3/2}].
\end{aligned} \tag{3.1}$$

Since from the expression (2.1) and (2.2) $u_d(\rho, \theta)_{\theta_0\to 0} = 0$. By symmetry the contribution to the surface field u_{34}, by the incident wave after first being diffracted by corner E_1 and then by the corner E_4, that is $[u_{34}|u_1(E_4)]$ is the same as (3.1) except that ρ_3 is replaced by $2a - \rho_3$.

$$[u_{34}|u_1(E_4)] = \frac{e^{i(kb+\frac{\pi}{4})}}{2\sqrt{\pi kb}}[\frac{\partial u_d(2a - \rho_3, 3\pi/2)}{\partial\theta_0}]_{\theta_0\to 0} + O[(kd)^{-3/2}]. \tag{3.2}$$

The total field at the point P on the face $\overline{E_3E_4}$ is now given by adding (3.1) and (3.2) together. Thus

$$u_{34}(P) = [u_{34}|u_1(E_4)] + [u_{34}|u_1(E_3)].$$

$$u_{3,4} = \frac{e^{i(kb+\frac{\pi}{4})}}{2\sqrt{\pi kb}} \left[\frac{\partial u_d}{\partial \theta_0}(\rho_3, \frac{3\pi}{2}) + \frac{\partial u_d}{\partial \theta_0}(2a - \rho_3, \frac{3\pi}{2})\right]_{\theta_0 \to 0} + O[(kd)^{-3/2}]. \tag{3.3}$$

Determination of $u_{2,3}$

The field at a point P along the face $\overline{E_2E_3}$ contributed by the incident wave being diffracted by corner E_2 is given by the expression

$$[u_{23}|u_i(E_2)] = [u_d(\rho_2, 0)e^{-ikb}]_{\theta_0 \to \pi}. \tag{3.4}$$

The contribution to u_{23} from the incident field diffracted by the corner E_2 and then subsequently by the corner E_3, that is, $[u_{23} \mid u_1(E_3)]$, can be calculated by using (2.7) to give

$$[u_{23}|u_1(E_3)] = \frac{e^{i(kb+\frac{\pi}{4})}}{2\sqrt{\pi kb}}[\frac{\partial u_d(2b - \rho_2, 0)}{\partial \theta_0}]_{\theta_0 \to 0} + O[(kd)^{-3/2}]. \tag{3.5}$$

The total field along $\overline{E_2E_3}$ due to diffraction at corners E_2 and E_3 is the sum of (3.4) and (3.5) and hence

$$u_{2,3} = [u_d(\rho_2, 0)e^{-ikb}]_{\theta_0 \to \pi} + \frac{e^{i(kb+\frac{\pi}{4})}}{2\sqrt{\pi kb}}[\frac{\partial u_d(2b - \rho_2, 0)}{\partial \theta_0}]_{\theta_0 \to 0} + O[(kd)^{-3/2}]. \tag{3.6}$$

Determination of $u_{4,1}$

From symmetry the field on the face $\overline{E_4E_1}$ is exactly the same as $u_{2,3}$ except that the field quantities will be in terms of ρ_4 instead of ρ_2. To obtain the appropriate field we must replace ρ_2 in (3.6) by $2b - \rho_4$, since for any particular value of, $y, \rho_2 + \rho_4 = 2b$. Thus

$$u_{1,4} = [u_d(2b - \rho_4, 0)e^{-ikb}]_{\theta_0 \to \pi} + \frac{e^{i(kb+\frac{\pi}{4})}}{2\sqrt{\pi kb}}[\frac{\partial u_d(\rho_4, 0)}{\partial \theta_0}]_{\theta_0 \to 0} + O[(kd)^{-3/2}]. \tag{3.7}$$

Determination of $u_{1,2}$

The field along the face $\overline{E_1E_2}$ due to diffraction of the incident wave by the corner E_1 is given by

$$[u_{12}|u_i(E_1)] = [u_d(\rho_1, 0)e^{-ikb}]_{\theta_0 \to \pi/2}. \tag{3.8}$$

The contribution to u_{12} from the incident field first being diffracted by E_1, and the resulting field being diffracted by E_2 is given by, using the formula (2.10). Thus

$$[u_{12}|u_1(E_2)] = O[(ka)^{-3/2}]. \tag{3.9}$$

Thus the total contribution caused by the incident wave illuminating corner E_1, is given by

$$[u_d(\rho_1, 0)e^{-ikb}]_{\theta_0 \to \pi/2} + O[(ka)^{-3/2}]. \tag{3.10}$$

From symmetry the contribution to u_{12} of the incident ray illuminating E_2, is given by replacing ρ_1 by $2a - \rho_1$ in (3.10), and is

$$[u_d(2a - \rho_1, 0)e^{-ikb}]_{\theta_0 \to \pi/2} + O[(ka)^{-3/2}]. \tag{3.11}$$

Finally the cylinder face $\overline{E_1E_2}$ will experience direct illumination by the incident plane wave, which, because of the impedance boundary condition, gives rise to the geometrical optics field contribution,

$$[u_{12}|u_1^g(E_2)] = \frac{2e^{-ikb}}{(1-\cos\vartheta)}. \tag{3.12}$$

Thus the total field on the face $\overline{E_1E_2}$ is given by adding (3.10),(3.11) and (3.12) together giving

$$u_{1,2} = [(u_d(\rho_1,0) + u_d(2a-\rho_1,0))e^{-ikb}]_{\theta_0\to\pi/2} + \frac{2e^{-ikb}}{(1-\cos\vartheta)} + O[(kd)^{-3/2}]. \tag{3.13}$$

4. Computing the scattered field

Substituting (3.3), (3.6) (3.7) and (3.13) into (1.8); and then making a change of the variable of integration where appropriate, to bring similar terms under a common range of integration, we obtain eventually

$$\begin{aligned}u_s(r,\frac{\pi}{2}) = -k\frac{e^{i(kr-\frac{\pi}{4})}}{2\sqrt{2\pi kr}}[4a - \frac{\cos\vartheta e^{i\pi/4}}{\sqrt{\pi kb}}\int_0^{2b}[\frac{\partial u_d(\rho_3,0)}{\partial\theta_0}]_{\theta_0\to 0}e^{ik\rho_3}d\rho_3\\ -2\cos\vartheta\int_0^{2b}[u_d(\rho_2,0)e^{-ik\rho_2}]_{\theta_0\to\pi}d\rho_2 + 2(1-\cos\vartheta)\int_0^{2a}[u_d(\rho_2,0)]_{\theta_0\to\frac{\pi}{2}}d\rho_2\\ -\frac{(1+\cos\vartheta)e^{i\pi/4}}{\sqrt{\pi kb}}\int_0^{2a}[\frac{\partial u_d(\rho_3,3\pi/2)}{\partial\theta_0}]_{\theta_0\to 0}d\rho_3 + O[(kd)^{-3/2}]] + O[(kr)^{-3/2}].\end{aligned} \tag{4.1}$$

The integrals appearing in (4.1) are in fact double integrals, because $u_d(\rho,\theta)$ is given by (2.1).

For the evaluation of the double integrals appearing in (4.1), it will be necessary to express the canonical solution $U_d(\rho,\theta)$ given by (2.1) in the alternative form:

$$U_d(\rho,\theta) = \frac{1}{2\pi i}\int_{S(\theta)}[F(\gamma+\pi) - F(\gamma-\pi)]e^{ik\rho\cos(\gamma-\theta)}d\gamma, \tag{4.2}$$

where

$$F(\chi) = \frac{2(\cos\chi+\cos\vartheta)(\cos\vartheta-\sin\chi)\sin\frac{2\theta_0}{3}(\cos\frac{4\theta_0}{3} - \cos\frac{4(\vartheta+\pi)}{3})}{3(\cos\theta_0+\cos\vartheta)(\sin\theta_0-\cos\vartheta)(\cos\frac{4\chi}{3} - \cos\frac{4(\pi+\vartheta)}{3})(\cos\frac{2\chi}{3} - \cos\frac{2\theta_0}{3})}. \tag{4.3}$$

By a straight forward change of the variables of integration it can be shown that for any $G(\gamma)$ which renders the following integral convergent:

$$\frac{1}{2\pi i}\int_{S(\theta)}[G(\gamma+\pi) - G(\gamma-\pi)]e^{ik\rho\cos(\gamma-\theta)}d\gamma = \frac{1}{2\pi i}\left\{\int_{S(\pi)} + \int_{S(-\pi)}\right\}G(\theta+\alpha)e^{-ik\rho\cos\alpha}d\alpha. \tag{4.4}$$

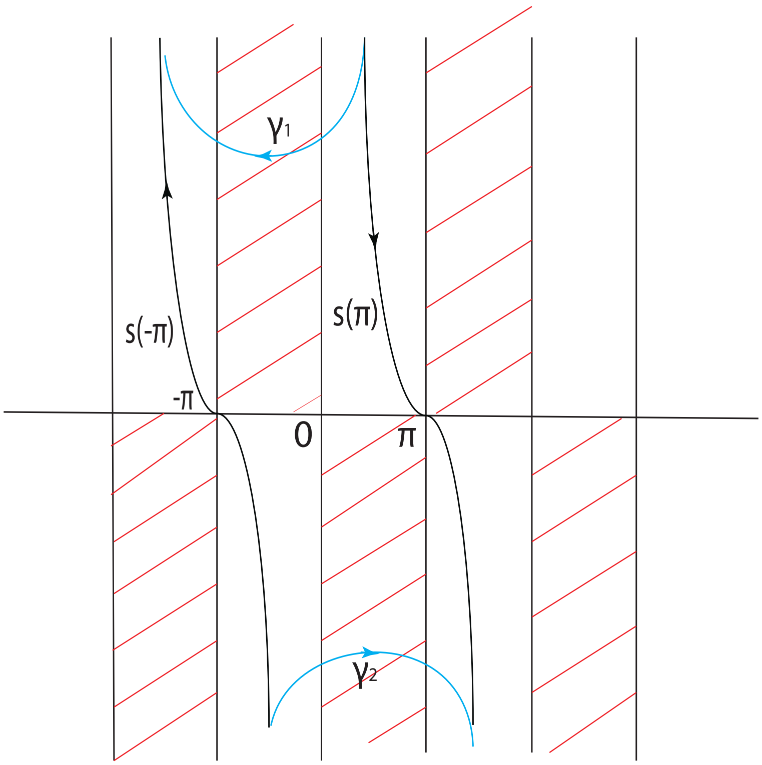

Fig. 4 The contours of integration $S(\pm\pi) : \Re\alpha = \pm\pi - \arccos(1/\cosh(\Im\alpha))sign(\Im\alpha)$; and γ_1, γ_2 in the complex α-plane. The starting point of $S(\pm\pi)$ and $\gamma_{1,2}$ being $\pm(\frac{\pi}{2} + i\infty)$, and the termination point of $S(\pm\pi)$ and $\gamma_{2,1}$ being $\pm(\frac{3\pi}{2} - i\infty)$.

Finally we will require the following result which is proved in Malyuzhinets(**13**). For $\rho = 0$ in (4.4) we have that for arbitrary finite θ

$$\frac{1}{2\pi i}\int_{\gamma_1} G(\alpha+\theta)d\alpha = iG(R), \quad \frac{1}{2\pi i}\int_{\gamma_2} G(\alpha+\theta)d\alpha = -iG(-R), \tag{4.5}$$

where $R = i\infty$, provided $G(\pm R) \to$ Const. as $R \to i\infty$. The contours of integration appropriate to (4.2), (4.4) and (4.5) are shown in Fig. 2 and Fig. 4, where the contours of integration start and terminate within the regions where the exponential term of the integrand ensures uniform convergence of the integral. Any pole singularities of the integrands lie below γ_1 and above γ_2.

Evaluation of the $\int_0^{2a} u_d(\rho_2,\theta)d\rho_2$ *for* $\theta = 0, \theta_0 = \pi/2$

Consider the integral,

$$I = \int_0^{2a} u_d(\rho_2,\theta)d\rho_2, \tag{4.6}$$

which appears in (4.1) for the particular case of $\theta = 0, \theta_0 = \pi/2$. Substituting the contour integral representation for $u_d(\rho_2,\theta)$, by means of (4.2) to (4.4), will give

$$I = \frac{1}{2\pi i}\int_0^{2a}\left\{\int_{S(\pi)} + \int_{S(-\pi)}\right\} F(\theta+\alpha)e^{-ik\rho\cos\alpha}d\alpha d\rho_2. \tag{4.7}$$

It is now possible to interchange the order of integration by virtue of the uniform convergence of the complex contour integral. Thus

$$I = \frac{1}{2\pi i}\left\{\int_{S(\pi)} + \int_{S(-\pi)}\right\} \frac{F(\theta+\alpha)}{(-ik\cos\alpha)}[e^{-2ika\cos\alpha} - 1]d\alpha. \tag{4.8}$$

In the expression (4.8) we now convert the integrals involving $e^{-2ika\cos\alpha}$ back into a single integral along the path $S(\theta)$ by means of (4.4), giving

$$I = \frac{1}{2\pi i}\left\{\int_{S(\pi)} + \int_{S(-\pi)}\right\} \frac{F(\theta+\alpha)}{ik\cos\alpha}d\alpha + \frac{1}{2\pi i\sqrt{3}}\int_{S(\theta)} \frac{D(\gamma,\theta_0)}{ik\cos(\gamma-\theta)}$$
$$\times\left(\frac{1}{\cos\frac{2\pi}{3} - \cos\frac{2(\gamma-\theta_0)}{3}} - \frac{1}{\cos\frac{2\pi}{3} - \cos\frac{2(\gamma+\theta_0)}{3}}\right)e^{ik2a\cos(\gamma-\theta)}d\gamma, \tag{4.9}$$

where

$$D(\gamma,\theta_0) = \frac{(\cos\gamma - \cos\vartheta)(\cos\vartheta + \sin\gamma)(\cos\frac{4\theta_0}{3} - \cos\frac{4(\vartheta+\pi)}{3})(2\cos\frac{2\theta_0}{3}\cos\frac{2\gamma}{3} + \frac{1}{2} - \cos\frac{4(\vartheta+\pi)}{3})}{(\cos\theta_0 + \cos\vartheta)(\sin\theta_0 - \cos\vartheta)(\cos\frac{4(\gamma-\pi-\vartheta)}{3} + \frac{1}{2})(\cos\frac{4(\gamma+\pi+\vartheta)}{3} + \frac{1}{2})}.$$

Now substituting the particular values $\theta = 0, \theta_0 = \pi/2$ gives

$$I = \frac{1}{2\pi i}\left\{\int_{S(\pi)} + \int_{S(-\pi)}\right\} \frac{[F(\alpha)]_{\theta_0=\pi/2}^{\theta=0}}{ik\cos\alpha}d\alpha + \frac{1}{2\pi i\sqrt{3}}\int_{S(0)} \frac{D(\gamma,\pi/2)}{ik\cos\gamma}$$
$$\times\left(\frac{1}{\cos\frac{2\pi}{3} - \cos\frac{2(\gamma-\frac{\pi}{2})}{3}} - \frac{1}{\cos\frac{2\pi}{3} - \cos\frac{2(\gamma+\frac{\pi}{2})}{3}}\right)e^{ik2a\cos\gamma}d\gamma. \tag{4.10}$$

The evaluation of the last integral of (4.10) can be achieved by a direct application of the ordinary saddle point method($ka >> 1$); because no poles of the integrand lie in the vicinity of the saddle point. Thus

$$\frac{1}{2\pi i\sqrt{3}}\int_{S(0)} \frac{D(\gamma,\pi/2)}{ik\cos\gamma}\left(\frac{1}{\cos\frac{2\pi}{3} - \cos\frac{2(\gamma-\frac{\pi}{2})}{3}} - \frac{1}{\cos\frac{2\pi}{3} - \cos\frac{2(\gamma+\frac{\pi}{2})}{3}}\right)e^{ik2a\cos\gamma}d\gamma$$
$$\backsimeq -\frac{2D(0,\pi/2)e^{i2ka+i\pi/4}}{3ik\sqrt{2\pi}(2ka)^{3/2}\cos\vartheta} + O[(ka)^{-5/2}] = O[(ka)^{-3/2}].$$

We now evaluate the remaining integral of (4.10), that is

$$I = \frac{1}{2\pi i}\left\{\int_{S(\pi)} + \int_{S(-\pi)}\right\} \frac{[F(\alpha)]^{\theta=0}_{\theta_0=\pi/2}}{ik\cos\alpha} d\alpha,$$

where

$$\frac{[F(\alpha)]^{\theta=0}_{\theta_0=\pi/2}}{ik\cos\alpha} = -\frac{(\cos\alpha+\cos\vartheta)(\cos\vartheta-\sin\alpha)}{\sqrt{3}ik\cos\alpha(1-\cos\vartheta)\cos\vartheta}$$
$$\times \quad \left\{\frac{(\cos\frac{2\alpha}{3}+\frac{1}{2})}{((\cos\frac{2\alpha}{3})^2-(\cos\frac{2(\pi+\vartheta)}{3})^2)} - \frac{1}{(\cos\frac{2\alpha}{3}-\cos\frac{\pi}{3})}\right\}. \tag{4.11}$$

Clearly,

$$\frac{[F(\pm i\infty)]^{\theta=0}_{\theta_0=\pi/2}}{ik\cos(\pm i\infty)} = 0,$$

so that, by virtue of (4.5), the contour of steepest descent, $S(\pi)$ and $S(-\pi)$ can be joined by the contours γ_1 and γ_2 to form a closed loop. Since no surface wave poles, which would be complex, are enclosed by this closed contour, see Rawlins(**1**)(**3**), we can deform $S(\pi)$ and $S(-\pi)$ to take up the straight line contours shown in Fig. 5. This closed contour will be denoted by C. Hence

$$\frac{1}{2\pi i}\left\{\int_{S(\pi)} + \int_{S(-\pi)}\right\} \frac{[F(\alpha)]^{\theta=0}_{\theta_0=\pi/2}}{ik\cos\alpha} d\alpha = \frac{1}{2\pi i}\int_C \frac{[F(\alpha)]^{\theta=0}_{\theta_0=\pi/2}}{ik\cos\alpha} d\alpha$$
$$= -\Sigma\text{Residues of poles of} \frac{[F(\alpha)]^{\theta=0}_{\theta_0=\pi/2}}{ik\cos\alpha} \text{enclosed by C.} \tag{4.12}$$

The only poles enclosed by C are the poles along the real axis between $(-\pi, \pi)$. These poles occur at the values of α for which

$$\cos\alpha = 0, \text{that is}, \alpha = \pm\pi/2;$$

$$\cos\frac{2\alpha}{3} - \cos\frac{\pi}{3} = 0, \text{that is}, \alpha = \pm\pi/2.$$

Thus single and double poles occur in the expressions (4.11) at $\alpha = \pm\pi/2$. By means of Mathematica the sum of the residues of (4.11) are given by

$$\sum_{\alpha=\pm\frac{\pi}{2}} \text{Res}\frac{[F(\alpha)]^{\theta=0}_{\theta_0=\pi/2}}{ik\cos\alpha} =$$
$$\frac{2}{ik(1-\cos\vartheta)}\left\{\frac{1}{\cos\vartheta} + \frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}(\frac{1}{4}-(\cos\frac{2(\pi+\vartheta)}{3})^2)}\right\}. \tag{4.13}$$

Thus

$$\lim_{\theta_0\to\pi/2}\int_0^{2a} u_d(\rho_2, 0)d\rho_2$$
$$= -\frac{2}{ik(1-\cos\vartheta)}\left\{\frac{1}{\cos\vartheta} + \frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}(\frac{1}{4}-(\cos\frac{2(\pi+\vartheta)}{3})^2)}\right\} + O[(ka)^{-3/2}]. \tag{4.14}$$

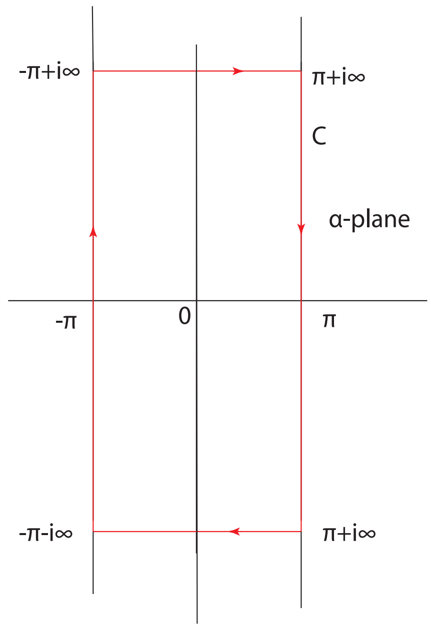

Fig. 5 The closed contour of integration C

Evaluation of the $\lim_{\theta_0 \to 0} \int_0^{2a} \frac{\partial u_d}{\partial \theta_0}(\rho_3, 3\pi/2) d\rho_3$
We shall now consider the integral:

$$I = \lim_{\theta_0 \to 0} \int_0^{2a} \frac{\partial u_d}{\partial \theta_0}(\rho_3, 3\pi/2) d\rho_3.$$

After differentiating under the integral sign of the integral representation for $u_d(\rho_3, \theta)$, this being permissible because the integral converges uniformly before and after the operation, we obtain

$$\lim_{\theta_0 \to 0} \frac{\partial u_d}{\partial \theta_0}(\rho_3, 3\pi/2) = \frac{1}{2\pi i} \int_{S(\frac{3\pi}{2})} \frac{4D(\gamma, 0) \sin \frac{2\gamma}{3} e^{ik\rho_3 \cos(\gamma - \frac{3\pi}{2})}}{3\sqrt{3}(\cos \frac{2\pi}{3} - \cos \frac{2\gamma}{3})^2} d\gamma.$$

Transforming the above contour, as before, to take up the two contours $S(\pi)$ and $S(-\pi)$ and integrating with respect to ρ_3 gives

$$I = \int_0^{2a} \frac{1}{2\pi i} \left\{ \int_{S(\pi)} + \int_{S(-\pi)} \right\} [\frac{\partial F}{\partial \theta_0}(\alpha + \frac{3\pi}{2})]_{\theta_0 \to 0} e^{-ik\rho_3 \cos \alpha} d\alpha d\rho_3,$$

where

$$[\frac{\partial F}{\partial \theta_0}(\alpha+\frac{3\pi}{2})]_{\theta_0\to 0}=\frac{4(\sin\alpha+\cos\vartheta)(\cos\vartheta+\cos\alpha)}{9(1+\cos\vartheta)\cos\vartheta}$$
$$\times\quad\left\{\frac{(1-\cos\frac{2\alpha}{3})}{((\cos\frac{2\alpha}{3})^2-(\cos\frac{2(\pi+\vartheta)}{3})^2)}+\frac{1}{(\cos\frac{2\alpha}{3}+1)}\right\}. \tag{4.15}$$

integrating with respect to ρ_3 gives, after re-arranging the integrals as before

$$I=\frac{1}{2\pi i}\int_C[\frac{\partial F}{\partial\theta_0}(\alpha+\frac{3\pi}{2})]_{\theta_0\to 0}\frac{d\alpha}{ik\cos\alpha}$$
$$+\frac{1}{2\pi i}\int_{S(\frac{3\pi}{2})}\frac{4D(\gamma,0)\sin\frac{2\gamma}{3}e^{ik2a\cos(\gamma-\frac{3\pi}{2})}}{3\sqrt{3}ik\cos(\gamma-\frac{3\pi}{2})(\cos\frac{2\pi}{3}-\cos\frac{2\gamma}{3})^2}d\gamma. \tag{4.16}$$

The second integral in (4.16) can be evaluated directly by the normal saddle point method; it is found to be of order $O[(ka)^{-3/2}]$. The evaluation of the first integral of (4.16) is achieved by summing the residues of $[\frac{\partial F}{\partial\theta_0}(\alpha+\frac{3\pi}{2})/ik\cos\alpha]_{\theta_0\to 0}$ enclosed by C. The only poles that can occur in the interval $|\alpha|\leqslant\pi$ are those for which $\cos\alpha=0$, that is, $\alpha=\pm\pi/2$, since

$$\frac{[\frac{\partial F}{\partial\theta_0}(\alpha+\frac{3\pi}{2})]_{\theta_0\to 0}}{ik\cos\alpha}=\frac{4(\sin\alpha+\cos\vartheta)(\cos\vartheta+\cos\alpha)}{9ik\cos\alpha(1+\cos\vartheta)\cos\vartheta}$$
$$\times\quad\left\{\frac{(1-\cos\frac{2\alpha}{3})}{((\cos\frac{2\alpha}{3})^2-(\cos\frac{2(\pi+\vartheta)}{3})^2)}+\frac{1}{(\cos\frac{2\alpha}{3}+1)}\right\}.$$

The sum of the residues is easily found to be

$$-\frac{8}{9ik(\cos\vartheta+1)}\left\{\frac{1}{2(\frac{1}{4}-(\cos\frac{2(\pi+\vartheta)}{3})^2)}+\frac{2}{3}\right\}.$$

Thus

$$\lim_{\theta_0\to 0}\int_0^{2a}\frac{\partial u_d}{\partial\theta_0}(\rho_3,3\pi/2)d\rho_3=$$
$$\frac{8}{9ik(\cos\vartheta+1)}\left\{\frac{1}{2(\frac{1}{4}-(\cos\frac{2(\pi+\vartheta)}{3})^2)}+\frac{2}{3}\right\}+O[(ka)^{-3/2}]. \tag{4.17}$$

Evaluation of the $\lim_{\theta_0\to 0}\int_0^{2b}\frac{\partial u_d}{\partial\theta_0}(\rho_3,0)e^{ik\rho_3}d\rho_3$

Considering the integral

$$I=\lim_{\theta_0\to 0}\int_0^{2b}\frac{\partial u_d}{\partial\theta_0}(\rho_3,0)e^{ik\rho_3}d\rho_3,$$

we obtain, following a similar procedure as before

$$I=-\frac{1}{2\pi i}\int_C[\frac{\partial F}{\partial\theta_0}(\alpha)]_{\theta_0\to 0}\frac{d\alpha}{ik(1-\cos\alpha)}$$
$$+\frac{1}{2\pi i}\int_{S(0)}\frac{4D(\gamma,0)\sin\frac{2\gamma}{3}e^{ik2b(1+\cos\gamma)}}{3\sqrt{3}ik(1+\cos\gamma)(\cos\frac{2\pi}{3}-\cos\frac{2\gamma}{3})^2}d\gamma.$$

The saddle point of the second integral in the above expression occurs at $\gamma = 0$; and since the poles of the integrand are not near the saddle point, a straight forward application of the saddle point method shows that this integral is of order $O[(kb)^{-3/2}]$. As before the first integral in the above expression is equal to the sum of the residues at the poles of

$$-\frac{[\frac{\partial F}{\partial \theta_0}(\alpha)]_{\theta_0\to 0}}{ik(1-\cos\alpha)} = -\frac{4(\cos\alpha+\cos\vartheta)(\cos\vartheta-\sin\alpha)}{9ik(1-\cos\alpha)(1+\cos\vartheta)\cos\vartheta}$$
$$\times \quad \left\{\frac{(1+\cos\frac{2\alpha}{3})}{((\cos\frac{2\alpha}{3})^2-(\cos\frac{2(\pi+\vartheta)}{3})^2)} - \frac{1}{(\cos\frac{2\alpha}{3}-1)}\right\}, \tag{4.18}$$

which lie within $|\alpha| \leqslant \pi$. These poles occur at the values of α for which

$$\cos\alpha - 1 = -2\sin^2\frac{\alpha}{2} = 0,$$

and

$$\cos\frac{2\alpha}{3} - 1 = -2\sin^2\frac{\alpha}{3} = 0,$$

thus the poles all occur at $\alpha = 0$. Rewriting (4.18) in the form

$$-\frac{4(\cos\alpha+\cos\vartheta)(\cos\vartheta-\sin\alpha)(1+\cos\frac{2\alpha}{3})}{9ik(1-\cos\alpha)(1+\cos\vartheta)\cos\vartheta((\cos\frac{2\alpha}{3})^2-(\cos\frac{2(\pi+\vartheta)}{3})^2)}$$
$$+ \quad \frac{4(\cos\alpha+\cos\vartheta)(\cos\vartheta-\sin\alpha)}{9ik(1-\cos\alpha)(1+\cos\vartheta)\cos\vartheta(\cos\frac{2\alpha}{3}-1)},$$

from which we can see that the first term has a double pole at $\alpha = 0$, and the second term a fourth order pole at $\alpha = 0$. the combined residue is given by

$$\frac{16}{9ik[1-(\cos 2(\pi+\vartheta)/3)^2]} - \frac{5}{27ik\cos\vartheta} - \frac{2}{ik(1+\cos\vartheta)\cos\vartheta}.$$

Thus

$$\lim_{\theta_0\to 0}\int_0^{2b}\frac{\partial u_d}{\partial\theta_0}(\rho_3,0)e^{ik\rho_3}d\rho_3 = -\frac{16}{9ik[1-(\cos 2(\pi+\vartheta)/3)^2]}$$
$$+\frac{5}{27ik\cos\vartheta} + \frac{2}{ik(1+\cos\vartheta)\cos\vartheta} + O[(kb)^{-3/2}]. \tag{4.19}$$

Evaluation of the $\lim_{\theta_0\to\pi}\int_0^{2b} u_d(\rho_2,0)e^{-ik\rho_2}d\rho_2$

Finally we will evaluate the remaining integral in (4.1), that is,

$$I = \lim_{\theta_0\to\pi}\int_0^{2b} u_d(\rho_2,0)e^{-ik\rho_2}d\rho_2. \tag{4.20}$$

After substituting into this last expression the integral representation for $u_d(\rho_2, 0)$ and carrying out the ρ_2 integration the result can be put in the form

$$I = \frac{1}{2\pi i}\left\{\int_{S(\pi)} + \int_{S(-\pi)}\right\}\frac{[F(\alpha)]_{\theta_0=\pi}^{\theta=0}}{(-ik(1+\cos\alpha))}[e^{-2ika(1+\cos\alpha)}-1]d\alpha. \tag{4.21}$$

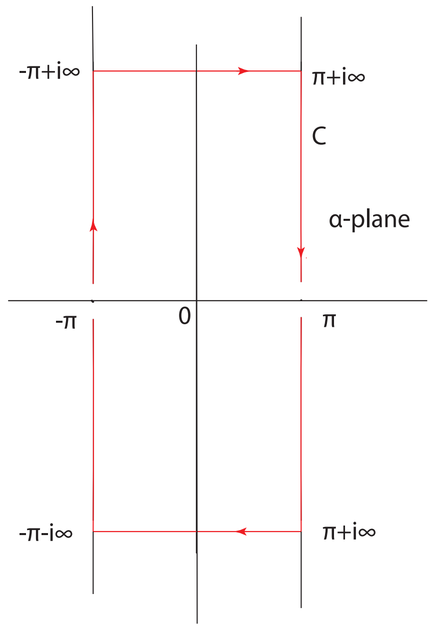

Fig. 6 The broken contour of integration C.

This integral representation exists and is uniformly convergent. However if we split off the integral into the two parts as we have done previously poles will lie on the contours of integration, and in particular at the saddle point. The integrals will therefore not separately exist in the usual sense. It is therefore necessary to interpret the integrals as principal value integrals.

Thus the contours of integration have to have gaps at $\alpha = \pm\pi$, and $\gamma = 0$ giving

$$I = \frac{1}{2\pi i} PV \int_C \frac{[F(\alpha)]^{\theta=0}_{\theta_0=\pi}}{ik(1+\cos\alpha)} d\alpha + \frac{1}{2\pi i\sqrt{3}} PV \int_{S(0)} \frac{D(\gamma,\pi)}{ik(\cos\gamma - 1)}$$
$$\times \left(\frac{1}{\cos\frac{2\pi}{3} - \cos\frac{2(\gamma-\pi)}{3}} - \frac{1}{\cos\frac{2\pi}{3} - \cos\frac{2(\gamma+\pi)}{3}} \right) e^{ik2b(\cos\gamma-1)} d\gamma, \qquad (4.22)$$

where the broken contours C and $S(0)$ are as shown in Fig. 6 and Fig. 7 respectively, and the integrals are principal value integrals. To evaluate the second integral of (4.22) it is required to expand the integrand in terms of powers of γ, around the saddle-point $\gamma = 0$,

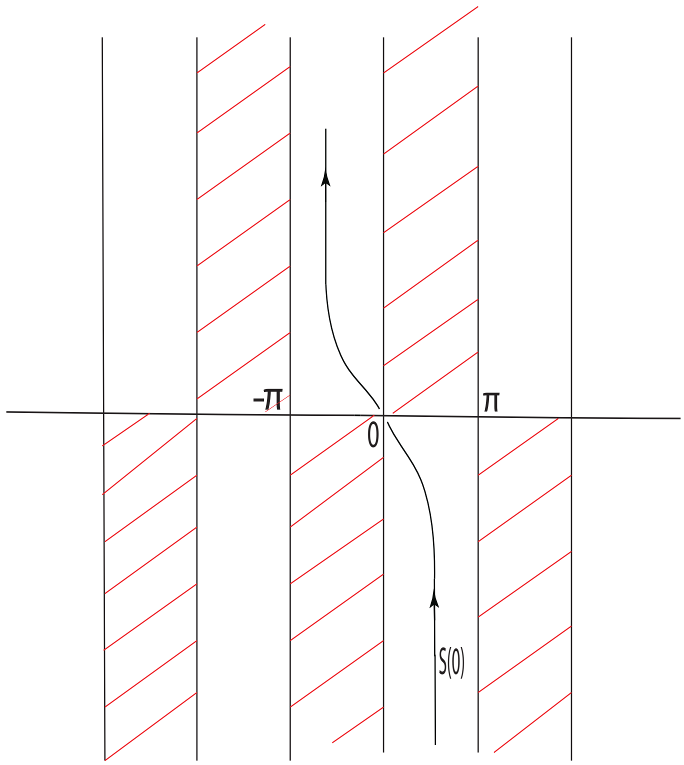

Fig. 7 The broken contour of integration $S(0)$ in the complex γ-plane.

which gives

$$J = \frac{1}{2\pi i\sqrt{3}} PV \int_{S(0)} \frac{D(\gamma,\pi)}{ik(\cos\gamma - 1)} \left(\frac{1}{\cos\frac{2\pi}{3} - \cos\frac{2(\gamma-\pi)}{3}} - \frac{1}{\cos\frac{2\pi}{3} - \cos\frac{2(\gamma+\pi)}{3}} \right) e^{i2kb(\cos\gamma-1)} d\gamma,$$

$$= \frac{1}{2\pi i} PV \int_{S(0)} e^{i2kb(\cos\gamma-1)} \sum_{n=-3}^{\infty} \alpha_n \gamma^n d\gamma, \tag{4.23}$$

where by using Mathematica we get

$$\alpha_{-2} = \frac{-4i}{k\cos\vartheta},$$

$$\alpha_0 = \frac{i}{3k\cos\vartheta} \left(-\frac{1}{3} + \frac{6}{(1-\cos\vartheta)} - \frac{16}{(1+2\cos\frac{4(\pi+\vartheta)}{3})} - \frac{64}{(1+2\cos\frac{4(\pi+\vartheta)}{3})^2} \right).$$

We shall see that it is not necessary to know the explicit forms of the odd coefficients, or to consider terms higher than $n = 0$ for the accuracy we are interested in. The integral (4.23) can be written as

$$J = \frac{\alpha_{-2}}{2\pi i} PV \int_{S(0)} e^{i2kb(\cos\gamma - 1)} \frac{d\gamma}{\gamma^2} + \frac{\alpha_0}{2\pi i} PV \int_{S(0)} e^{i2kb(\cos\gamma-1)} d\gamma$$
$$+\frac{1}{2\pi i}\sum_{n=1}^{\infty} \alpha_{2n} PV \int_{S(0)} e^{i2kb(\cos\gamma-1)}\gamma^{2n} d\gamma + \frac{1}{2\pi i}\sum_{n=0}^{\infty}\alpha_{2n-3} PV \int_{S(0)} e^{i2kb(\cos\gamma-1)}\gamma^{2n-3} d\gamma.$$

From the symmetry of the path of integration $S(0)$ and the oddness of the integrand it can be shown that

$$PV \int_{S(0)} e^{i2kb(\cos\gamma-1)}\gamma^{2n-3} d\gamma = 0.$$

We can now write J as

$$J = -\frac{\alpha_{-2}e^{-i2kb}}{2\pi i} PV \int_{S(0)} e^{i2kb\cos\gamma} d(\gamma^{-1}) + \frac{\alpha_0}{2\pi i} PV \int_{S(0)} e^{i2kb(\cos\gamma-1)} d\gamma$$
$$+\frac{e^{-i2kb}}{2\pi i(-2ikb)}\sum_{n=1}^{\infty}\alpha_{2n} PV \int_{S(0)} (\frac{\gamma^{2n}}{\sin\gamma}) d(e^{i2kb\cos\gamma}).$$

The first and third integral can be evaluated by carrying out integration by parts; the integrated parts vanishes exponentially. We can then apply the method of steepest descent to the remaining integrals so that that (4.23) is asymptotic to

$$J = \sqrt{\frac{kb}{\pi}} e^{-i\frac{\pi}{4}}\alpha_{-2} + \frac{e^{i\frac{\pi}{4}}}{2\sqrt{\pi kb}}(\alpha_0 + \frac{\alpha_{-2}}{24}) + O[(kb)^{-3/2}]. \tag{4.24}$$

We now evaluate the remaining integral of (4.22),

$$\frac{1}{2\pi i} PV \int_C \frac{[F(\alpha)]_{\theta_0=\pi}^{\theta=0} d\alpha}{ik(1+\cos\alpha)}, \tag{4.25}$$

where

$$\frac{[F(\alpha)]_{\theta_0=\pi}^{\theta=0}}{ik(1+\cos\alpha)} =$$
$$-\frac{(\cos\alpha+\cos\vartheta)(\cos\vartheta-\sin\alpha)(\cos\frac{2\alpha}{3}-\frac{1}{2})}{\sqrt{3}ik(1+\cos\alpha)(\cos\vartheta-1)\cos\vartheta((\cos\frac{2\alpha}{3})^2-(\cos\frac{2(\pi+\vartheta)}{3})^2)}$$
$$+ \quad \frac{(\cos\alpha+\cos\vartheta)(\cos\vartheta-\sin\alpha)}{\sqrt{3}ik(1+\cos\alpha)(\cos\vartheta-1)\cos\vartheta(\cos\frac{2\alpha}{3}+\frac{1}{2})}. \tag{4.26}$$

The poles of $\frac{F(\alpha)}{ik(1+\cos\alpha)}$, for $|\alpha| \leqslant \pi$ occur at $\alpha = \pm\pi$. We now close the contour C by adding two small semicircular indentations and by adding or subtracting the appropriate residue

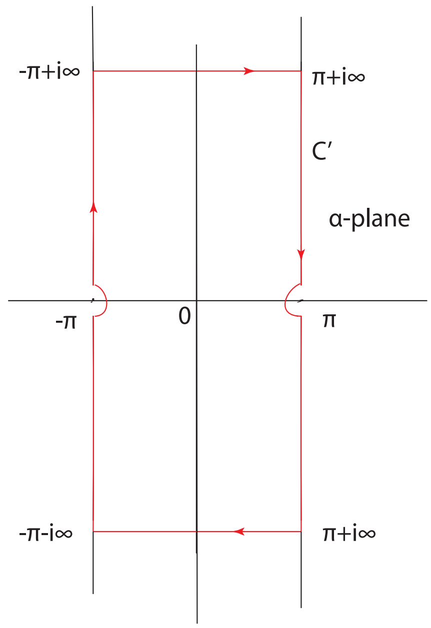

Fig. 8 The closed and indented contour of integration C'.

contributions at $\alpha = \pm\pi$, so that the value of the integral (4.25) remains unchanged. How we indent is arbitrary provided we add or subtract the appropriate residue contribution. We shall indent in such a way that the new indented closed contour C', see Fig. 8, does not enclose any poles of the integrand and so by Cauchy's theorem its value is zero. Thus we have

$$\frac{1}{2\pi i} PV \int_C \frac{[F(\alpha)]_{\theta_0=\pi}^{\theta=0} d\alpha}{ik(1+\cos\alpha)} = -\left\{ \frac{Res[\frac{F(\alpha)}{ik(1+\cos\alpha)}]_{\alpha=\pi} + Res[\frac{F(\alpha)}{ik(1+\cos\alpha)}]_{\alpha=-\pi}}{2} \right\}, \tag{4.27}$$

and after some computation with the help of Mathematica for the evaluation of the residues of (4.26) at the second and third order poles which appear at $\alpha = \pm\pi$ we obtain

$$\frac{1}{2\pi i} PV \int_C \frac{[F(\alpha)]_{\theta_0=\pi}^{\theta=0} d\alpha}{ik(1+\cos\alpha)} = \frac{-2}{\sqrt{3} ik \cos\vartheta} \left(\frac{1}{3} + \frac{1}{[\frac{1}{4} - (\cos\frac{2(\pi+\vartheta)}{3})^2]} \right). \tag{4.28}$$

Hence

$$\lim_{\theta_0\to\pi}\int_0^{2b} u_d(\rho_2,0)e^{-ik\rho_2}d\rho_2 = \frac{1}{ik\cos\vartheta}[\frac{4\sqrt{kb}e^{-\frac{\pi}{4}}}{\sqrt{\pi}} - \frac{2}{3\sqrt{3}}$$
$$-\frac{2}{\sqrt{3}[\frac{1}{4}-(\cos\frac{2(\pi+\vartheta)}{3})^2]} + \frac{e^{\frac{i\pi}{4}}}{2\sqrt{\pi kb}}(\frac{5}{18} - \frac{2}{1-\cos\vartheta}$$
$$+\frac{16}{3(1+2\cos\frac{4(\pi+\vartheta)}{3})} + \frac{64}{3(1+2\cos\frac{4(\pi+\vartheta)}{3})^2}) + O[(kb)^{-3/2}]]. \quad (4.29)$$

The total scattered field, the scattering cross section, and graphical results

Thus the expression for the total scattered field is given by substituting the values of the integrals (4.14),(4.17), (4.19), and (4.29), into (4.1), which gives

$$u_s(r,\frac{\pi}{2}) = -\frac{e^{i(kr-\pi/4)}}{2\sqrt{2\pi kr}}\Bigg(4ka - \frac{\cos\vartheta e^{-i\pi/4}}{\sqrt{\pi kb}}\times$$
$$\left\{\frac{-16}{9(1-(\cos\frac{2(\pi+\vartheta)}{3})^2)} + \frac{5}{27\cos\vartheta} + \frac{2}{(1+\cos\vartheta)\cos\vartheta}\right\} + 2i\left\{\frac{4\sqrt{kb}e^{-\frac{\pi}{4}}}{\sqrt{\pi}}\right.$$
$$-\frac{2}{3\sqrt{3}} - \frac{2}{\sqrt{3}[\frac{1}{4}-(\cos\frac{2(\pi+\vartheta)}{3})^2]} + \frac{e^{\frac{i\pi}{4}}}{2\sqrt{\pi kb}}\left[\frac{5}{18} - \frac{2}{1-\cos\vartheta} + \frac{16}{3(1+2\cos\frac{4(\pi+\vartheta)}{3})}\right.$$
$$\left.\left.+\frac{64}{3(1+2\cos\frac{4(\pi+\vartheta)}{3})^2}\right]\right\} + 4i\left\{\frac{1}{\cos\vartheta} + \frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}(\frac{1}{4}-(\cos\frac{2(\pi+\vartheta)}{3})^2)}\right\}$$
$$-\frac{8e^{-i\pi/4}}{9\sqrt{\pi kb}}\left\{\frac{1}{2(\frac{1}{4}-(\cos\frac{2(\pi+\vartheta)}{3})^2)} + \frac{2}{3}\right\} + O[\frac{1}{(kb)^{3/2}}] + O[\frac{1}{(ka)^{3/2}}]\Bigg) + O[(kr)^{-3/2}].$$

The above expression is quite complicated, however it can be put in a more useful form by expanding everything out in terms of inverse powers of $\cos\vartheta$,(where in the limit for a perfectly conducting cylinder $|\cos\vartheta|\to\infty$), we obtain,

$$u_s(r,\frac{\pi}{2}) = -\frac{e^{i(kr-\pi/4)}}{2\sqrt{2\pi kr}}\Bigg(4ka + \frac{8\sqrt{kb}e^{\frac{i\pi}{4}}}{\sqrt{\pi}} - \frac{19e^{-\frac{i\pi}{4}}}{18\sqrt{\pi kb}} - \frac{1}{\cos\vartheta}\left\{\frac{4}{i} + \frac{4e^{-\frac{i\pi}{4}}}{\sqrt{\pi kb}}\right\}$$
$$+O[\frac{1}{(\cos\vartheta)^2}] + O[\frac{1}{(kb)^{3/2}}] + O[\frac{1}{(ka)^{3/2}}]\Bigg) + O[(kr)^{-3/2}]. \quad (4.30)$$

The total scattering cross section σ for the cylinder can be found by using the the cross section theorem Jones(**14**), that states that if

$$u_s = u + O\left[(kr)^{-\frac{3}{2}}\right],$$

then the total cross section σ is given by

$$\sigma = -\frac{2}{k}\Re\left[\left(\frac{e^{i(kr-i\frac{\pi}{4})}}{\sqrt{2\pi kr}}\right)^{-1} u\right].$$

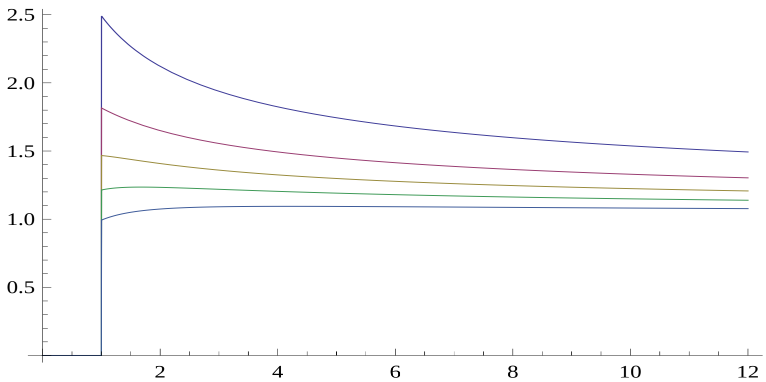

Fig. 9 Scattering cross section σ against k for various rectangles with $a = 1, n = 2; \mu = 1, \mu_0 = 1, \kappa = 1$. The top most graph corresponds to $b = 5$ and the lower graphs to $b = 2, 1, 1/2, 1/5$,respectively in that consecutive order.

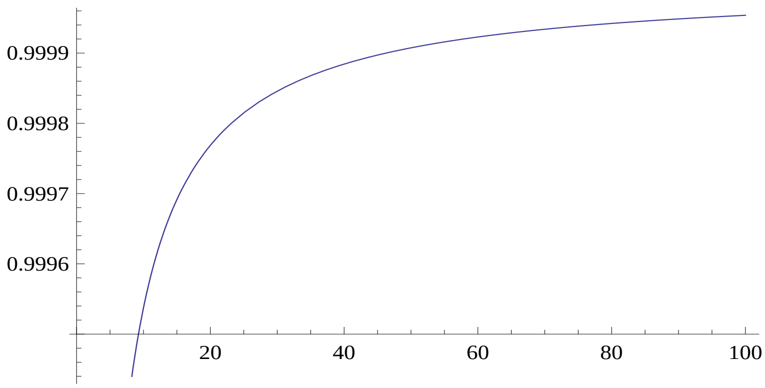

Fig. 10 The ratio σ/σ_∞ against n for $a = 1, b = 1, k = 100; \mu = 1, \mu_0 = 1, \kappa = 1$

The leading term of the above expression corresponds with the Kirchhoff approximation. By substituting the complex refractive index $N = n(1 + i\kappa), n > 0$ for $\cos\vartheta = -\frac{\mu_0}{\mu}N$ and applying the cross section theorem we get

$$\sigma = 4a + \frac{8\sqrt{kb}}{\sqrt{2\pi}k} - \frac{19}{18k\sqrt{2\pi kb}} + \frac{\mu}{\mu_0(1+\kappa^2)n}\left[-\frac{4\kappa}{k} + \frac{4(1-\kappa)}{\sqrt{2\pi kb}k}\right] + O[n^{-2}] + O[\frac{(kd)^{-3/2}}{k}].$$

In the limit as $\mid n \mid \to \infty$ the above expression reduces to the scattering cross section for a perfectly conducting rectangular cylinder σ_∞; which agrees with the expression obtained by Morse(**10**).

5. Conclusions

We have derived new high frequency approximate expressions for grazing incidence of a plane wave E-polarized electromagnetic field scattered by an imperfectly conducting rectangle.The present work compliment the oblique incidence results already published in Rawlins (**3**). The techniques used here can also be applied to any polygonal structure with more complicated boundary conditions provided the canonical wedge problem with these boundary conditions is known in the form of a contour integral. The approach could also be used to deal with the double impedance wedge problem where radiation from the aperture between the wedges is required.A intriguing idea is the modeling of a smooth structure, say an impedance elliptical cylinder, by an impedance polygonal cylinder; the corners are wedges whose canonical solutions is known. In the limit as the number of sides increases this polygonal cylinder structure will approach the smooth cylinder problem. The only caveat to this approach is that as the number of sides increase the polygonal segments become small compared to the wavelength and hence the high frequency approach breaks down so a hybrid method would need to be used. The method of this work is a considerable improvement on the existing approach used by Morse, on a simpler problem, in that it is more direct and less complicated and avoids differentiating complete asymptotic expansions, and using divergent integrals which is not strictly rigorous [†]. These results will be of use in the practical situations described in the author's earlier publication which dealt with oblique incident waves. We have also obtained the scattering cross section for grazing incidence for an impedance cylinder which has high conductivity. The graphs show that the effect of conductivity does result in a reduction of the scattering cross section. The methods used in this paper can be extended to deal with the H-Polarized situation by including the effect of surface waves. This will involve more asymptotic analysis on the known canonical wedge diffraction problem. We remark that the approach used here can be used for any angle of incidence and any observation point. It is also a simple straight forward approach in that it can be used in conjunction with Mathematica and the steepest descent method to derive asymptotic results to any desired order at singular transition points without the necessity of introducing the asymptotics of higher transcendental functions like Fresnel integrals. Finally we remark that the approach used here from (2.3) to derive the result (2.6) is in fact a simpler generalization of the results of Karp and Zitron (**15**) who derived a complete complicated plane wave expansion for diffraction integrals whose integrands are regular. Their results showed that it is always possible to represent the radiated field by a linear combination of plane waves and their derivatives. By using the Series function in Mathematica we have obtained the same sort of plane wave expansion for integrands that have multiple pole singularities that can occur at the saddle points of the integrand. In the appendix we show that it is always possible to represent radiated fields with arbitrary pole singular integrands in the same manner.

6. Acknowledgement

The author would like to thank the referees for drawing the authors attention to relevant references, and for making useful suggestions to improved the presentation of this work.

References

1. A. D.Rawlins, High frequency diffraction of an electromagnetic plane wave by an imperfectly conducting rectangular cylinder. *J. Eng. Math.* **76**(2012) 157– 180.

[†] Despite this lack of rigour an independent calculation was carried out by the present method on Morse's problems to check his results rigorously and they were found non the less to be correct.

2. J. B.Keller, A geometrical theory of diffraction. *J. Opt. Soc. America.***52**(1962) 116–130.
3. A. D.Rawlins, Asymptotics of a right-angled impedance wedge. *J. Eng. Math.* **65**(2009) 355–366.
4. P.D.Smith, and A.D.Rawlins, Diffraction from structures with an impedance boundary. *ICEAA Conference on electromagnetics in advanced applications, Turin.*(2013), ISBN: 978-1-4673-5705-0, DOI: 10.1109/ICEAA.2013.6632459.
5. S.N.Chandler-Wilde,S.Langdon, and M.Mokgolele, A high frequency boundary element method for scattering by convex polygons with impedance boundary conditions. *Comms. Compt. Phys.***11**(2012) 573–593.
6. N.Zitron, and S. N.Karp, Higher order approximations in multiple scattering. I Two-dimentional scalar case. *J. Math. Phys.***2**(1961) 394–402.
7. A. D.Rawlins, A note on uniform asymptotic wave diffraction by a wedge. *Q. Jl. Mech. Appl. Math.***67**(2014) 43-56.
8. A. V.Osipov, Uniform asymptotic formula for calculating an acoustic field near a face of an impedance wedge. *Sov. Phys Acoust.* **36**(1990) 180–183.
9. A. V.Osipov, K.Hongo, and H.Kobayashi, High-frequency approximations for the electromagnetic field near a face of an impedance wedge. *IEEE Trans. Antn. Prop.* **AP-50**(2002) 930–940.
10. B. J.Morse, Diffraction by polygonal cylinders. *J. Math. Phys.***5**(1964) 199–214.
11. F.Oberhettinger, On asymptotic series for functions occuring in the theory of diffraction of waves by wedges. *J. Math. Phys.***34**(1955) 245–255.
12. A. D.Rawlins, Diffraction of an E-or H-polarized electromagnetic plane wave by a right-angle wedge with imperfectly conducting faces. *Q. Jl. Mech. Appl. Math.* **43**(1990) 161–172.
13. G. D.Malyuzhinets, The radiation of sound from vibrating faces of an arbitrary wedge. *Sov. Phys. Acoust.* **1**(1955) 152–174.
14. D. S.Jones, The theory of electromagnetism.(Pergamon Press, Oxford,1964) 608-612.
15. S. N.Karp. and N.Zitron, A plane wave expansion theorem for cylindrically radiated fields. *J. Eng. Math.***8**(1974) 291–296.

APPENDIX A

A generalization of Karp and Zitron's result.

Following the terminology of Karp and Zitron (**15**) the two dimentional radiated field is represented by

$$u(r,\theta) = \int_{C_1} f(\beta) e^{ikr\cos(\theta-\beta)} d\beta.$$

The function in the integrand $f(\beta)$ is now assumed to have a Laurent series such that:

$$f(\beta) = \sum_{m=-M}^{\infty} f_m \beta^m, \quad where \quad f_m = \frac{1}{2\pi i} \oint \frac{f(\beta)}{\beta^{m+1}} d\beta;$$

The circular contour of integration encloses the origin but excludes the nearest singularity of $f(\beta)$ to the origin. We now follow (**15**) and write the integrand in terms of new coordinates of the second scatterer giving

$$u(r,\theta) = \int_{C_1} v(\beta) f(\beta) e^{ikd\cos\beta} d\beta = \int_{C_1} g(\beta) e^{ikd\cos\beta} d\beta,$$

where $v(\beta) = e^{ik(x\cos\beta + y\sin\beta)}$ and $g(\beta) = v(\beta)f(\beta)$. The plane wave $v(\beta)$ and the function $f(\beta)$ can be expanded as a power series and a Laurent series respectively in β giving

$$g(\beta) = \sum_{n=0}^{\infty} v_n \beta^n \sum_{m=-M}^{\infty} f_m \beta^m,$$

where $v_n = v^{(n)}(0)/n!$ and f_m are the known coefficients of the Laurent series for $f(\beta)$. Assuming both series are absolutely convergent we can rearrange the double series representation for $g(\beta)$ as

$$g(\beta) = \sum_{s=-M}^{\infty} (\sum_{k=0}^{s+M} v_k f_{s-k})\beta^s = \sum_{s=-M}^{\infty} g_s \beta^s.$$

Clearly $g_s = \sum_{k=0}^{s+M} v_k f_{s-k}, \quad s \geqslant -M$, is a linear combination of the plane wave $v(\beta)$ and its higher derivatives at $\beta = 0$. We now shift the contour C_1 to take up the new indented path of steepest descent $S(0)$ through $\beta = 0$. Interchanging the order of integration and summation gives

$$u(r,\theta) = \sum_{s=-M}^{\infty} g_s \int_{S(0)} \beta^s e^{ikd\cos\beta} d\beta = (\sum_{s=-M}^{-1} + \sum_{s=0}^{\infty}) g_s \int_{S(0)} \beta^s e^{ikd\cos\beta} d\beta.$$

The integrals that occur for $-M \leqslant s \leqslant -1$ are replaced by residue contribution at the origin and a principal value integral. The principal value integrals for odd s vanish giving

$$u(r,\theta) =$$

$$\sum_{s=-M[sEven]}^{-2} g_s PV \int_{S(0)} \beta^s e^{ikd\cos\beta} d\beta + i\pi \sum_{s=M}^{1} \frac{g_s}{(s-1)!} \frac{d^{s-1}}{d\beta^{s-1}} (e^{ikd\cos\beta}) \mid_{\beta=0} + \sum_{s=0}^{\infty} g_s \int_{S(0)} \beta^s e^{ikd\cos\beta} d\beta.$$

Although explicit expressions can be given on applying the method of steepest descent and higher order differentiation, in the context of modern computer algebra the main result that is important is that the radiated field in the vicinity of the second scatterer is a linear combination of plane waves and its derivatives.