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A representation of the natural numbers by means of cycle-numbers, with consequences in number theory

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Abstract

In this paper we give rules for creating a number triangle **T** in a manner analogous to that for producing Pascal's arithmetic triangle; but all of its elements belong to $\{0, 1\}$, and cycling of its rows is involved in the creation. The method of construction of any one row of **T** from its preceding rows will be defined, and that, together with starting and boundary conditions, will suffice to define the whole triangle, by sequential continuation.

We shall use this triangle in order to define the so-called *cycle-numbers*, which can be mapped to the natural numbers. **T** will be called the '*cycle-number triangle*'.

First we shall give some theorems about relationships between the cycle-numbers and the natural numbers, and discuss the cycling of patterns within the triangle's rows and diagonals. We then begin a study of figures (i.e. (0,1)-patterns, found on lines, triangles and squares, etc.) within **T**. In particular, we shall seek relationships which tell us something about the prime numbers. For our later studies, we turn the triangle onto its side and work with a doubly-infinite matrix **C**.

We shall find that a great deal of cycling of figures occurs within **T** and **C**, and we exploit this fact whenever we can. The phenomenon of cycling patterns leads us to muse upon a 'music of the integers', indeed a 'symphony of the integers', being played out on the cycle-number triangle or on **C**. Like Pythagoras and his 'music of the spheres', we may well be the only persons capable of hearing it!

Keywords: cycle-number triangle, cycle-number, prime cycle-numbers

MSC: 11R99

Music is the pleasure the human mind experiences from
counting without being aware that it is counting.
G. W. Leibnitz (1646–1714)

1. Introduction

In his scholarly book on Pascal's '*Arithmetical Triangle*', the author AWF Edwards [2] says of the *Triangle*: "It reveals patterns which delight the eye, raises questions which tax the number-theorists, and ..." (he adds, quoting D. Knuth) "amongst the coefficients there are so many relations present that, when someone finds a new identity, there aren't many people who get excited about it any more, except the discoverer!"

Pascal, in his own publication on the famous triangle, in 1654, said that he was fascinated by the mathematical richness of the patterns that he had discovered in it, and that: "He had had to leave out more than he could put in!"

In the triangle that we are about to define, none of its elements rise above 1 (its alphabet is $\{0, 1\}$), and yet (echoing Pascal) we have found a great richness in the geometric patterns of 0s and 1s that arise, many of which carry with them secrets about the prime numbers. Further (again echoing Pascal), we have had to leave out much more than we could put in.

We have called our triangle '*the cycle-number triangle*' because of the many cyclic phenomena involved in the $(0, 1)$-patterns of most interest, and because we derive from it a new representation of the natural numbers, each one exhibiting cyclic behaviour. We have given our triangle the general label $\mathbf{T}$.

We declare that we have not seen this triangle defined before in the literature, but, of course, it may well have been described several times in the past and produced nothing of sufficient interest to keep mathematicians using and mentioning it. If we may quote Pascal yet again, he wrote in his autobiography, apropos his triangle: "Let no one say I have said nothing new. The arrangement of the subject is new. When we play tennis, we both play with the same ball, but one of us places it better!"

Whatever is the case, we hope that our methods and studies of the cycle-number triangle contain something new and worthy of their presentation.

2. Example, definition and construction of T

We begin by showing the cycle-number triangle $\mathbf{T}$, down to row 6, in Figure 1 below. The apex triangle and directions for the central axis and i,j reference axes are shown.

Note in Figure 1 the left- and right-boundaries of $\mathbf{T}$, two sloping lines, each containing the sequence $\mathbf{0}$, $\mathbf{1}$, $\mathbf{0}$, $\mathbf{0}$, $\mathbf{0}$. These are the given elements, to start construction of the triangle.

To reference an element in **T** we may use two coordinates $(\mathbf{i}, \mathbf{j})$, with the $\mathrm{i}, \mathrm{j}^{\mathrm{th}}$ element occurring at the intersection of the sloping lines $\mathbf{R_i}$ (the ith diagonal parallel to the right-boundary) and $\mathbf{L_j}$ (the jth diagonal parallel to the left-boundary). The two directed reference lines are indicated in Figure 1 on either side of the triangle $\mathbf{T(6)}$.

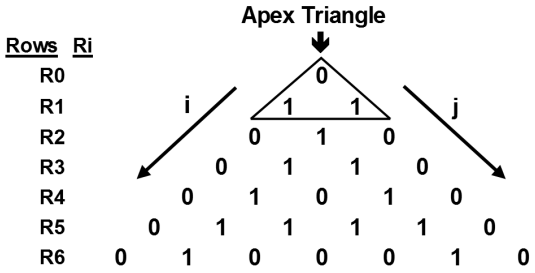

Figure 1: The Cycle-number Triangle $\mathbf{T(6)}$ (with apex triangle defined)

The general rules for generating **T** now follow.

2.1. Constructing the Triangle T

(1) The apex and triangle boundaries

The cycle-number triangle **T** is constructed according to the following rules:

(i) The elements have alphabet $\{\mathbf{0,\ 1}\}$;

(ii) The apex element is $\mathbf{0}$;

(iii) The left-boundary **L** is (from the apex downwards to the left) $\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \ldots$;

(iv) The right-boundary **R** runs from the apex downwards to the right, with the same $(\mathbf{0,1})$-sequence as that of **L**.

The rows following the apex triangle are then constructed, row by row, by making a sequence of 'neck-tie' applications, as explained next.

(2) The 'neck-tie' figure and its uses

The figure which we use repeatedly to generate the elements of **T**, row-by-row, is called a *neck-tie* in view of its shape. It consists of an equi-sided triangle, ∇, supported by two long, sloping legs which are potentially infinite.

When applied to row **Ri** of **T**, the top side of the triangle is marked with the $(0, 1)$ elements of **Ri**, and the other two sides take the same markings, in order, cycling around the triangle (see Figure 2 for an example applied to **R3**).

The side-length, defined to be the number of elements on each side, increases by one with each row application.

The neck-tie (designated **ni**) is completed by adding the two spreading legs from the lowest vertex of the neck triangle. The left leg slopes down to the left, and has the (0,1)-pattern from the right side of ∇ appearing in it, cycling ever downwards.

Similarly, the right leg slopes down to the right, with the (0,1)-pattern from the left side of ∇, cycling ever downwards, appearing in it.

Figure 2 below shows how the Cycle-Number Triangle **T** is constructed, row by row, down to row **R6**. It also includes an expanded view of the neck-tie which is applied to row **R3**.

Notation. We designate by **T(i)** or **Ti** the sub-triangle of **T** which extends from the apex down to row **Ri**. And **ni** will designate the neck-tie which is applied to row **Ri** of **T**. It must be noted that the constructed neck-tie lines are notional. They do not appear normally in diagrams of **T**.

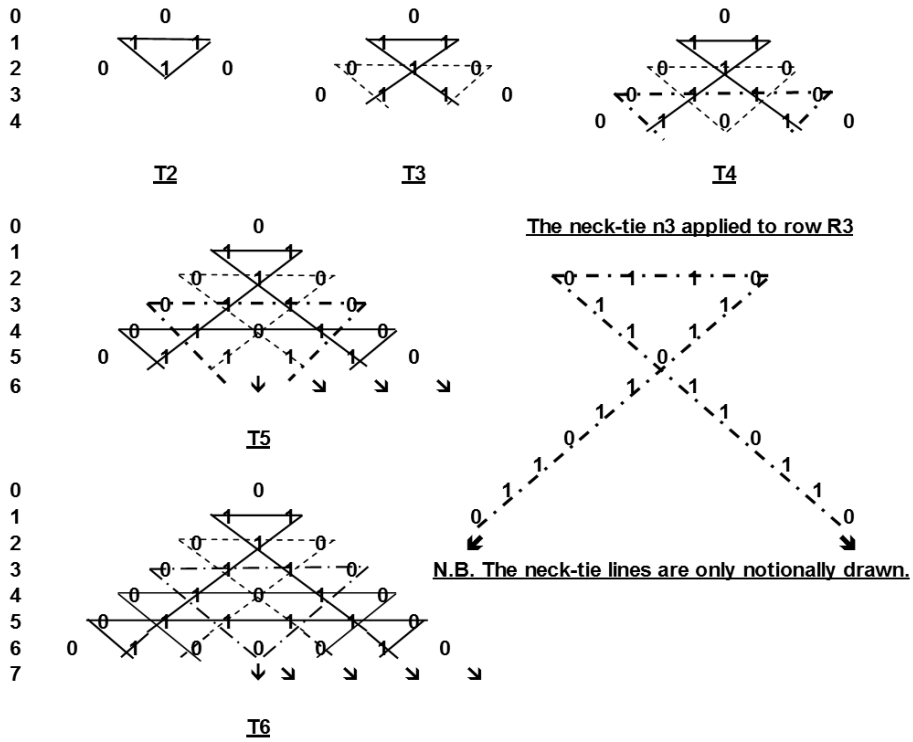

Figure 2: Constructing triangle **T** row-by-row **T(2)** $\rightarrow$ **T(6)**

(3) The cycle-numbers and their fundamental cycles (f.c.s)

Observe how the string (110) from the top of the neck-tie **n3** cycles around the neck (in both directions). It also cycles down the left and right legs. This string is defined to be *the fundamental cycle (f.c.)* of the cycle-number $\underline{\mathbf{3}}$. Some general properties of cycle-numbers will now be developed.

The nth cycle-number is designated by $\underline{\mathbf{n}}$, and its fundamental cycle by $\underline{\mathbf{n}}$**'**.

Definition 2.1. The cycle-number $\underline{\mathbf{n}}$ is the infinite string obtained by cycling its fundamental cycle $\underline{\mathbf{n}}$**'** indefinitely: e.g. $\underline{\mathbf{3}} = \overline{110}$.

Evidently, for each value of $n > 0$, two pictorial representations of $\underline{\mathbf{n}}$ occur in triangle **T**, cycling down the left and right legs respectively of the corresponding neck-tie.

Definition 2.2. The infinite sequence of cycle-numbers with $n > 0$ will be denoted by $\underline{\mathbf{N}}$. (It corresponds one-to-one with the natural number sequence **N**.)

Later we shall display the cycle-numbers as rows of a doubly infinite matrix **C**.

Since we cannot apply a neck-tie to $\underline{\mathbf{0}}$ in row **R0**, we have to define its cycle-number specially.

Definition 2.3. The zero cycle-number is $\underline{\mathbf{0}} \equiv 01000\cdots = (01)\overline{0}$. (This is the string on the R and L diagonals of **T**. It is special in that its cycling does not begin until after (01) occurs. With all the other cycle-numbers ($n > 0$), the cycling begins with the first digit of the string for $\underline{\mathbf{n}}$.)

With Figures 1 and 2 to guide us, we can make and prove the following general observations, as our first theorems about the cycle-number triangle, and the cycle-numbers derived from it.

Theorem 2.4. *The first six fundamental cycles (f.c.s), taken from the rows **R1** to **R6** of **T6**, are (1), (10), (110), (1010), (11110) and (100010). Generally:*

*(i) Every f.c. after **R0** is of length n (it has n letters);*

*(ii) Every f.c. after **R1** begins with a 1 and ends with a 10.*

Proof. The proofs of each item follow immediately from the neck-tie construction and applications to the sequence of rows, or from each other, so they will not be spelled out. □

Note that, like $\underline{\mathbf{0}}$, the cycle-number $\underline{\mathbf{1}} = \overline{1}$ is 'special', arising from the second row of the apex triangle. It cycles from its f.c. $\underline{\mathbf{1}}$**'** indefinitely, never acquiring a 0 (c.f. Peano's first two axioms, which are needed to establish 0, and its successor S(0) which is later labelled 1: both 'special numbers').

Theorem 2.5 (Palindrome Principles).

(i) The $(0,1)$*-string on the upper side of a neck-tie* ∇ *is a palindrome; In general, row* $\boldsymbol{Ri}$ *of* $\boldsymbol{T}$ *is a* $(0,1)$*-palindrome of length* $i+1$*, for* $\boldsymbol{i} = 1, 2, 3, \ldots$.

(ii) The first $n-1$ *elements of a fundamental cycle (after* $\boldsymbol{R0}$*) form a palindrome.*

Proof. The proofs of each item follow immediately from the neck-tie construction and applications to the sequence of rows of $\mathbf{T}$, or from each other, so they will not be spelled out. □

As an example, the complete (0,1)-string from the upper side of neck-tie **n6** is (0100010), which is a palindrome. And the f.c. of cycle-number $\underline{6}$ is $\underline{\mathbf{6'}}$ = (100010), whose first five digits form a palindrome.

N.B. The Palindrome Principles, simple though they are, turn out to be powerful tools in enabling us to look ahead in $\mathbf{T}$ to discern patterns in number sequences.

3. Some theorems on lines in T

We have shown how to define the cycle-numbers, and given a few results about their $(0,1)$-patterns, and their fundamental cycles. We now begin a study of the $(0,1)$-patterns which occur on lines in $\mathbf{T}$.

Theorem 3.1. *We already defined above (see Figure 1) how to reference elements* $(\boldsymbol{i}, \boldsymbol{j})$ *in* $\boldsymbol{T}$ *using the sloping reference diagonals for* $\boldsymbol{i}$ *and* $\boldsymbol{j}$ *coordinates. As explained above,* $\boldsymbol{L_i}$ *is the* $\boldsymbol{i}^{th}$ *sloping line in* $\boldsymbol{T}$ *parallel to the* $\boldsymbol{L}$*-boundary, and* $\boldsymbol{R_j}$ *is the* $\boldsymbol{j}^{th}$ *sloping line in* $\boldsymbol{T}$ *parallel to the* $\boldsymbol{R}$*-boundary. (Thus* $\boldsymbol{R_i}//\boldsymbol{R}$ *and* $\boldsymbol{L_i}//\boldsymbol{L}$*.) Then:*

(i) $\boldsymbol{L_1}$ *and* $\boldsymbol{R_1}$ *are both sequences of cycled 1s, which we designate as unit-cycle lines;*

(ii) $\boldsymbol{L_2}$ *and* $\boldsymbol{R_2}$ *are both sequences of cycles of* $(1,0)$ *(starting after the boundary element* $\boldsymbol{0}$*), which we designate as* $\boldsymbol{2}$*-cycle lines, having pattern* $\overline{10}$*;*

(iii) $\boldsymbol{L_3}$ *and* $\boldsymbol{R_3}$ *are both sequences of cycles of* $(1,1,0)$ *(starting after the boundary element* $\boldsymbol{0}$*), which we designate as* $\boldsymbol{3}$*-cycle lines, having pattern* $\overline{110}$*;*

(iv) $\boldsymbol{L_4}$ *and* $\boldsymbol{R_4}$ *are both sequences of cycles of* $(1,0,1,0)$ *(starting after the boundary element* $\boldsymbol{0}$*), which we designate as* $\boldsymbol{4}$*-cycle lines* $\overline{1010}$*;*

(v) This sequence of pairs $(\boldsymbol{L_i}, \boldsymbol{R_i})$ *of i-cycle lines, with* $\boldsymbol{L_i} = \boldsymbol{R_i}$*, continues indefinitely as* $\boldsymbol{i}$ *increases by 1 at each row-step in* $\boldsymbol{T}$*.*

Proof. The proofs of each item follow immediately from the neck-tie construction and applications to the sequence of rows of $\mathbf{T}$, or from each other, so they will not be spelled out. □

The above theorem has shown that there is an ordered sequence of cycling $(0,1)$-strings down the left- and right- diagonals, equal in pairs. The next Theorem 3.2 will prove that the same sequences occur in vertical columns, the pairs being equidistant from the central axis of **T**. Before presenting this theorem, let us define the Cartesian axis frame, to which we can refer elements in horizontal and vertical directions.

3(2) The (x, y) Cartesian axes

The origin of the frame is at the apex of **T** so the apex is at point $(\mathbf{0,0})$. The Cartesian y-axis is oriented vertically, with direction downwards. The Cartesian x-axis is the horizontal through the apex, with positive direction to the right. Its scale unit is that distance which separates the columns of **T** (equally spaced).

We shall use **Cn** to denote the nth column, which contains the (**0,1**)-string which appears down the vertical line $x = n$, for $n = 1, 2, 3, \ldots$.

3(3) The axis line and its (0,1) pattern

The axis line is $x = 0$. Thus the column **C0** is the **T-**triangle axis, and each digit appearing on it below the apex triangle is the lower vertex of a neck-triangle, which in turn is a cycling of a boundary digit **0** (after **n1**). Hence the axis bears the same $(0,1)$ pattern as do the boundaries, viz. 01000....

To the left of the axis, x will take corresponding negative values.

It follows from Theorem 2.5(i) (Palindrome Principle), that we need only deal with the columns **Cn** when **n** is positive. The corresponding columns in the negative direction will carry the same cycle-number sequence. On the few occasions which we refer to columns to the left of the axis we shall write **C(-n).**

We have shown that the sets of **L**-diagonals and **R**-diagonals are equal in pairs, w.r.t. their (**0,1**)-patterns. The next Figure and theorem shows that this same phenomenon occurs in the vertical columns, when taken in pairs equidistant from the axis of **T**.

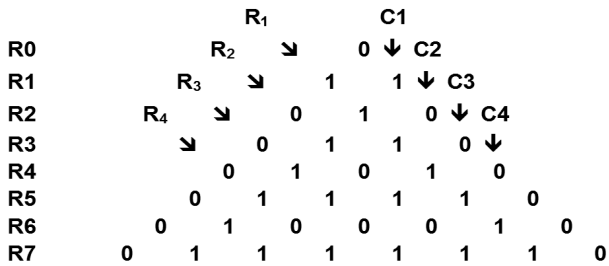

Figure 3: Triangle **T(7)**, indicating the rows **Rj** and cols. **Cj** for $\mathbf{j} = 1, 2, 3, 4$

Theorem 3.2 (The vertical column patterns)**.**
Diagram. *Refer to Figure 3. Recall that* $\boldsymbol{R_i}//\boldsymbol{R}$*, where* $\boldsymbol{R}$ *is the right-boundary of* $\boldsymbol{T}$*.*

Let us abbreviate the phrase 'the (0,1)-pattern in column $\boldsymbol{Cn}$*' to 'the* $\boldsymbol{Cn}$*-pattern'.*

Subsection 3(3) above proved the case $\boldsymbol{C0} = \boldsymbol{R_0}$*.*

Now we assert that (refer to Figure 3):

(i) The $\boldsymbol{Cj}$*-pattern is equal to the* $\boldsymbol{R_j}$*-pattern, for* $\boldsymbol{j} = 1, 2, 3, \ldots$
And by axis-symmetry of $\boldsymbol{T}$ *the* $\boldsymbol{C(-j)}$*-pattern is also equal to the* $\boldsymbol{R_j}$*-pattern.*

(ii) The $\boldsymbol{Cj}$*-pattern lies along both the lines* $x = j$ *and* $x = -j$*.*

Proof. (i) Referring to the diagram of **T7** above, we construct an inductive proof, using properties of the neck-tie triangles (see Figure 2 and 3): we begin by showing the theorem to be true for j = 1.

We note that the (0,1) elements of **C1** occur in rows **R1, R3, R5**, etc. (i.e. in the odd rows). The first element of **C1** is 1, by definition of the diagonal pattern in $\mathbf{R_0}$.

Then the following statements are evidently true:

The second element of **C1** is in row **R3**, which is the third element of diagonal $\mathbf{R_1}$, which is equal to 1 (by Theorem 3.1(i); $\mathbf{R_1}$ is a unit-line).

The third element of **C1** is in row **R5**, which is the fifth element of $\mathbf{R_1}$, which equals 1 since $\mathbf{R_1}$ is a unit-line.

In general, the ith element of **C1** is in row $\mathbf{R(i+2)}$, which is the $(\mathbf{i+2})^{\mathrm{th}}$ element of $\mathbf{R_1}$ and hence is equal to 1.

The proof that the **C1**-pattern is a string of 1s is now easily completed by induction, using the neck-tie construction rules. Then, by the Palindrome Principle, we can assert that the **C(-1)** pattern is also a string of 1s.

We can apply the same arguments to determine that the **C2**-pattern is the same as the $\mathbf{R_2}$-pattern, and equals the **C(-2)**-pattern.

Induction can now be used to generalize this overall argument, to prove the statement that for all j > 0 the **Cj**-pattern is the same as the $\mathbf{R_j}$-pattern, and equals the **C(-j)**-pattern. This will complete the proof of Theorem 3.2, for the (0,1)-patterns in the columns of **T**. □

Before going on to study (0,1)-patterns in **T** other than those occurring in straight lines, as treated above, we shall now present an alternative method for generating the elements of **T**. The starting triangle for this method bears comparison with the Pascal triangle in 'binomial coefficient form'. Moreover, it immediately shows how the rows of **T** relate to the natural numbers in **N** (in two directions) and their 'coprimeness properties'.

4. A second method for generating the cycle-number triangle T

4.1. The enteger triangle E

The cycle-number triangle **T(6)** was shown in Figure 1, Section 2, and then it was shown how to generate **T** generally by the neck-tie algorithm. Now we shall obtain the triangle **T(6)** by writing down a triangle **E** of ordered pairs of integers, called *entegers*, and then operating on each enteger by the so-called "coprime-function" named *kappa* ($\boldsymbol{\kappa}$). (N.B. We introduced the notion of 'enteger' in [4] and [5]. We write $\mathrm{n_m}$ for an enteger. Two entegers are added as with vectors.) Before defining kappa, below we give the enteger triangle **E** (on the left) of entegers down to **R6**, and (on the right) the triangle after the transformations by kappa have taken place.

	The enteger triangle E(6)		The cycle-number triangle T(6)
R0	0_0		0
R1	1_0 1_1		1 1
R2	2_0 2_1 2_2	T = κE	0 1 0
R3	3_0 3_1 3_2 3_3	⟶	0 1 1 0
R4	4_0 4_1 4_2 4_3 4_4		0 1 0 1 0
R5	5_0 5_1 5_2 5_3 5_4 5_5		0 1 1 1 1 0
R6	6_0 6_1 6_2 6_3 6_4 6_5 6_6		0 1 0 0 0 1 0

Figure 4: The triangle ***E*** of entegers (ordered pairs), transformed by $\boldsymbol{\kappa}$ to **T**

We shall define **E** by giving its $\mathrm{n^{th}}$ line, then define the function kappa, and then establish the validity of the general transformation $\boldsymbol{\kappa}\mathbf{E} = \mathbf{T}$

Definition 4.1. The general row **Rn** for the enteger triangle **E** is:

$$n_0,\ n_1,\ n_2,\ \ldots\ ,\ n_{n-1},\ n_n.$$

(For comparison, in Pascal's triangle the row is $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n-1}, \binom{n}{n}$.)

Lemma 4.2. *Given the triangle **T(n)**, we can extend it to **T(n+1)** thus: add 1_0 ('vectorial' enteger addition) to the last enteger in each left diagonal $\boldsymbol{L_i}$ for $i = 0, 1, 2, \ldots, n$, and add 1_1 to the last enteger n_n of **Rn**.*

Definition 4.3. Let $\mathrm{e} = \mathrm{s_t}$, with s,t $\in$ **N**. Then the *'coprime operator (kappa)'* is defined as follows:

$$\kappa(e) \equiv \begin{cases} 1, & \text{if } s \text{ and } t \text{ are coprime;} \\ 0, & \text{otherwise (see also special cases below).} \end{cases}$$

Special cases: $\kappa(0_0) \equiv 0$; $\kappa(1_0) \equiv 1$; $\kappa(t_0) \equiv 0$ for all $t > 1$, where $e = s_t$ is the general notation for an enteger, with s and t written diagonally.

Without using the notion of divisibility, we can determine when a number pair is coprime by means of the 'repeated-pair-subtraction method' used in the simplest version of Euclid's Algorithm (call it EA) (some examples of this are given in Figure 5).

Lemma 4.4. *Let kappa be applied (element-wise) to the nth row of* $\boldsymbol{E}$*. Then the result is a palindromic* $(0,1)$*-string.*

Proof. Consider the elements of **Rn** of **E**, taken in pairs symmetrically placed relative to the axis of **E**. The ith pair, after applying kappa to each, is $\kappa(n_i)$ and $\kappa(n_{n\text{-}i})$. Applying only the first subtraction in Euclid's Algorithm, we find from the second of our pairs that after this one subtraction $\kappa(n_{n\text{-}i}) = \kappa(i_n) = \kappa(n_i)$. This is true for all pairs in the row (if there is a single central enteger, as in the even-numbered rows, then it is immaterial what kappa-value it takes) so a palindromic (0,1)-string results from the row **Rn**. □

4.2. Relationships between E and T

We claimed in Figure 4 that $\boldsymbol{\kappa}(\mathbf{E}) = \mathbf{T}$, the cycle-number triangle. This sub-section is concerned with proving this claim.

We have already shown that the nth rows of both triangles $\boldsymbol{\kappa}(\mathbf{E})$ and **T** have the same lengths n+1, and that these rows each consist of a (0,1)-string which is palindromic. So both triangles are symmetric w.r.t. their axis-lines.

It is immediate that they both have the same **R** and **L** diagonals, and that in both, the $\mathbf{R_1}$ and $\mathbf{L_1}$ diagonals are unit-lines. The two both have the same axis lines, since in **E** the axis $x = 0$ carries the entegers 0_0, 2_1, 4_2, ..., which is an A.P. of entegers having common difference 2_1. (We can extend Lemma 4.2 to show that this sequence extends indefinitely.) Applying kappa to the sequence, we get 0, 1, 0, 0, ..., which is the axis pattern in **T**.

Recall that **T** was constructed by applying a neck-tie construction, say **nn,** to each row **Rn**, We shall show how the same type of construction can be used to build **E**, and moreover that the same type of cycling around and down the neck-ties, as in **T**, then occurs in $\boldsymbol{\kappa}\mathbf{E}$. The construction of neck-tie $\mathbf{nn} = \nabla_{\mathbf{n}}$ for **E** requires the following steps:

(i) the top side of ∇_n is n_0, n_1, n_2, ..., $n_{n\text{-}1}$, n_n (see Definition 4.1)

(ii) the left side of ∇_n is n_0, $(n+1)_1$, $(n+2)_2$, ..., $(2n\text{-}1)_{n\text{-}1}$, $(2n)_n$

(iii) the right side of ∇_n is n_n, $(n+1)_n$, $(n+2)_n$, ..., $(2n\text{-}1)_n$, $(2n)_n$

(iv) the left leg is a continuation of the sequence in (iii)

(v) the right leg is a continuation of the sequence in (ii)

To exemplify this definition of ∇_{n} we will show the neck-tie ∇_3, and on its right, show what happens to it when certain of its elements are reduced by applying the 'pair subtraction' method of Euclid's Algorithm to them (recall that such applications do not change the kappa values from those of the original pairs). We show only the neck, and one cycle of the left leg and the right leg. It will be evident how the leg cycles must continue.

$$\begin{array}{ccccccc}
3_0 & & 3_1 & & 3_2 & & 3_3 \\
 & 4_1 & & & & 4_3 & \\
 & & 5_2 & & 5_3 & & \\
 & & & 6_3 & & & \\
 & & 7_3 & & 7_4 & & \\
 & 8_3 & & & & 8_5 & \\
9_3 & & & & & & 9_6
\end{array}
\qquad
\begin{array}{ccccccc}
3_0 & & 3_1 & & 3_2 & & 0_3 \\
 & 3_1 & & & & 1_3 & \\
 & & 3_2 & & 2_3 & & \\
 & & & 3_0 & & & \\
 & & 1_3 & & 3_1 & & \\
 & 2_3 & & & & 3_2 & \\
0_3 & & & & & & 3_0
\end{array}
\qquad
\begin{array}{ccccccc}
0 & & 1 & & 1 & & 0 \\
 & 1 & & & & 1 & \\
 & & 1 & & 1 & & \\
 & & & 0 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & & & 1 & \\
0 & & & & & & 0
\end{array}$$

Figure 5: The neck-tie $\nabla_{\boldsymbol{3}}$; then same with appropriate EA changes; then $\kappa(\nabla_{\boldsymbol{3}})$

Remarks. The left diagram shows the neck-tie as defined for row **R3** of **E.** The centre diagram is attractive, found by applying EA (Euclid's Algorithm) subtractions appropriately, to elements of $\nabla_{\mathbf{3}}$. It shows how the kappa values of elements of **n3** must cycle in the desired way and become equal to corresponding values in **T**.

Its generalization to the neck-tie $\nabla_{\mathbf{n}}$ and to **T** is automatic.

The claim that $\boldsymbol{\kappa}(\mathbf{E}) = \mathbf{T}$ now follows by induction on $\nabla_{\mathbf{n}}$. Thus, if we verify it for $n = 1$ and 2, we can extend those to verify it for $n = 3$, and so on.

4.3. On 'Coprimeness' and 'Primeness' in the Cycle-Numbers

This is an appropriate moment for us to link the cycle-numbers to the natural numbers with regard to the concepts of coprimeness and primeness, in such a way that the two number systems can be said to represent one another exactly in those regards.

Consider the two triangles **T** and **E** in Figure 4, where $\mathbf{T} = \boldsymbol{\kappa}\mathbf{E}$. It is seen that each row of **T** (e.g. **Rn**) carries a complete record of what we shall call the coprimeness relation of **n** with each of the integers $i = 1, 2, 3, \ldots, n$. This gives rise to the following lemma, which directly relates fundamental cycles and the totient function of Euler:

Lemma 4.5. *Let $\omega(\underline{\mathrm{n}})$, the weight of cycle-number $\underline{\mathrm{n}}$, be the sum of the elements of **n'** in row **Rn** of **T**. Then $\omega(\underline{n}) = \varphi(n)$, where φ (i.e. phi) is Euler's totient function.*

Proof. This follows immediately from Definition 4.1, $\mathbf{T} = \boldsymbol{\kappa}\mathbf{E}$ and the definition of φ. □

We can with reason speak of the *coprimeness* of a cycle-number $\underline{\mathrm{n}}$ and assign a measure of it by the ratio (or index) $\omega(\underline{\mathrm{n}})/\mathrm{n} = \varphi(\mathrm{n})/\mathrm{n}$.

We continue these notions by coining the slogan that "coprimeness begets primeness", and presenting the following definitions to add precision to it.

Definition 4.6 (Coprimeness index, and primeness of $\underline{\mathbf{n}}$)**.** We define a *cycle-number* $\underline{\mathrm{n}}$ to be *prime* if its coprimeness index has value $(n-1)/n = 1 - 1/n$. The value of the index can never be 1, since $\kappa(\mathrm{n_n}) = 0$.

Note that with this definition, it can be shown that $\underline{\mathrm{n}}$ is a prime cycle-number iff n is a prime integer. For if n is not prime, there exists some integer $m < n$ such that $\kappa(\mathrm{n_m}) = 0$, and the coprimeness index of $\underline{\mathrm{n}}$ is less than $1 - 1/n$, so $\underline{\mathrm{n}}$ is not prime.

Conversely, if n is prime, then $\varphi(n) = n - 1 = \omega(\underline{\mathrm{n}})$, the coprimeness index is (n-1)/n , and so $\underline{\mathrm{n}}$ is prime.

Examples may be seen in rows 2, 3, 5, and 7 of **T** in Figure 3.

5. Three operations on cycle-numbers

5.1. Definitions of the operators

The following three operations and their symbols are defined on the cycle-numbers, which allow us to discover and develop various algebraic relationships between the cycle-numbers. We shall not report on the subsequent algebra further than we need to, in order to study (0,1)-patterns in the triangle **T** and a later-derived matrix C.

The three operators and their symbols are:

(1) 'star' ($*$) , **(2)** 'add' ('+'), and **(3)** 'multiply' ($\wedge$).

Definition 5.1. The 'star' (or 'conjoin') operation is one of conjoinment of two given (0,1)-vectors or strings. Thus if $\underline{\mathbf{m}}$ and $\underline{\mathbf{n}}$ are two (0,1)-strings, then $\underline{\mathbf{m}} * \underline{\mathbf{n}}$ is the string obtained by writing first the $\underline{\mathbf{m}}$-string, and then continuing with the $\underline{\mathbf{n}}$-string, thus creating a string of length m+n. Clearly this operation does not generally commute.
It can be extended in the obvious way to deal with three or more strings.

Care must be taken when interpreting the conjoin of two f.c.s of cycle-numbers. The result is *not necessarily* another cycle-number f.c.; in fact, *it usually isn't*.

Definition 5.2. Two cycle-numbers $\underline{\mathrm{m}}$ and $\underline{\mathrm{n}}$ are added in a natural way as follows (letting their 'sum' be $\underline{\mathrm{m}}$ '+' $\underline{\mathrm{n}} \equiv \underline{\mathrm{s}}$).
Let s = m+n (sum of the two cycle-number f.c. lengths), and find from row $\mathbf{R}$(m+n) of the enteger triangle **E** the enteger string which, on applying κ to its elements, yields **s'**. Using Def. 4.1 we find the required string to be $(\mathrm{m+n})_1$, $(\mathrm{m+n})_2$, ... , $(\mathrm{m+n})_{\mathrm{s-1}}$, $(\mathrm{m+n})_{\mathrm{s}}$. Applying κ to this string yields $\underline{\mathrm{s}}$', and hence $\underline{\mathrm{s}}$.

Note: To make full sense of this operation, we must think of it as taking place between rows **Rm** and **Rn** of the enteger triangle **E**. We are extending the triangle in the 'natural way', by 'adding' (strictly 'appending') n further rows to it (down from **Rm**) according to theorems previously given for patterns on the diagonal lines.

Finally, having reached $\mathbf{R_{m+n}} = \mathbf{R_s}$, we drop the first element s_0 and we are left with $\underline{s}$' as required. (As an example, see Figure 3 and add $\underline{2}$ to $\underline{3}$ to get $\underline{5}$.)

Definition 5.3 (The cap product)**.** Two cycle-numbers $\underline{m}$ and $\underline{n}$ in **N** (not including $\underline{0}$) are 'multiplied' (in a not-so-natural way), as defined below. Again we carry out the initial operations upon the two respective f.c.s, $\underline{m}$' and $\underline{n}$' and arrive at the fundamental cycle of a new cycle-number which we shall call the Boolean Product (B.P.), or the 'cap product', of the two cycle-numbers. This 'product' is a powerful tool for us in our study of cycle-number patterns. Its definition is as follows:
Let mn = k. Then the Boolean Product of $\underline{m}$ and $\underline{n}$ is a cycle-number whose f.c. is of length k, and is found from the following formula: $\underline{k}' = (n * \underline{m}') \wedge (m * \underline{n}')$.
The left-hand bracket contains the (0,1)-string of n conjoined cycles of the f.c. of $\underline{m}$, and the right-hand bracket contains the (0,1)-string of m conjoined cycles of the f.c. of $\underline{n}$. The cap symbol between the two bracketed terms indicates that an element-wise product has to be computed, according to the following binary multiplication table (Boolean): $0\wedge0 = 0\wedge1 = 1\wedge0 = 0$, and $1\wedge1 = 1$.

It is easy to see that the two multiplication sets from **N(x)** and $\underline{\mathbf{N}}(\wedge)$ are isomorphic.

A simple example will illustrate the use of the operation $\wedge$**.**

Example. Let $\underline{m} = \underline{2}$ and $\underline{n} = \underline{3}$. Then

$$(n * \underline{m}')\wedge(m * \underline{n}') = (\mathbf{101010})\wedge(\mathbf{110110}).$$

Note that each string is of length 2x3 =6. Applying $\wedge$ element-wise gives the result (**100010**), which is the f.c. of $\underline{6}$.

The reader should check this result in Fig. 3, and observe how the $\underline{2}$-cycles and $\underline{3}$-cycles arrive at **R6** in their respective neck-ties, with their end 0s filling three places in $\underline{6}$'. If one lays either $\underline{2}$' or $\underline{3}$' along the length of $\underline{6}$', as with two moving rulers, one finds that each ruler cycles $\underline{6}$' exactly, with regard to their end 0s. We say (using Euclid's language) that both $\underline{2}$' and $\underline{3}$' *measure* $\underline{6}$' because of this.

It is helpful to place $n * \underline{m}$' above $m * \underline{n}$', and apply the cap products vertically, in the k resulting 2x1 columns. Thus, with the example:

$$\begin{aligned} 3 * \underline{2}' &= 101010 \\ 2 * \underline{3}' &= 101110 \\ \wedge \quad &= 100010 \end{aligned}$$

Note that for a 1 to occur in the result, there must be two 1s above it. We shall exploit this fact later.

Before leaving our discussion of the cycle-number triangle **T**, we remark that all manner of patterns can be discovered in **T**, based on the arrangements of the 0s and 1s, and how they are related to the cycle-numbers.

We have already discussed many of the most obvious patterns, and built definitions and theorems about them. We shall end this Section by presenting an interesting theorem that demonstrates a fractal property, namely that **T** can properly include a copy of itself ... indeed an infinite sequence of such copies. Our proof will be 'pictorial', extending to two inclusions only.

Theorem 5.4. $\mathbf{T} \supset \mathbf{T} \supset \mathbf{T} \supset \ldots$ *(proper inclusions).*

Proof. Pictorial Proof (See Figure 6). □

The Figure 6 first shows **T** to row 10 (i.e. **T(10)**), with its first two neckties, coloured black and blue respectively. The second and third **T**s of the theorem are shown below the first one.

Clearly the second two triangles have elements which map directly to themselves and to those of the original **T**. They are mapped from the original neck-ties, but with changed Euclidean shapes. They remain similar in congruent triples. In terms of cycle-numbers their necks are still equi-sided triangles, and their legs still carry the same (0,1)-patterns. The neck-ties undergo anti-clockwise, Euclidean rotations.

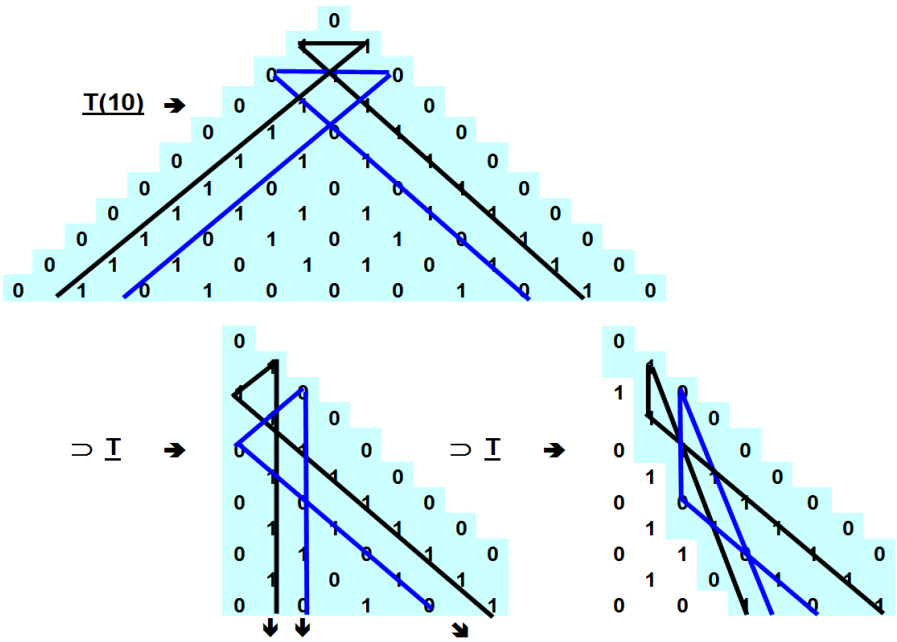

Figure 6: A fractal property of **T**

This sequence of included **T**s and their corresponding triangles and neck-ties

can be extended, and other Ts and sequences of Ts can be found elsewhere in the original triangle.

6. Definition of the cycle-number matrix C

Our construction of the cycle-number triangle **T** enabled us to introduce the notion of cycle-numbers and define various of their properties and operations on them. We now wish to display all the cycle-numbers in a doubly-infinite matrix called **C,** which provides a more convenient view-point of their domain for us to proceed with their study. It is easy to produce **C**, for it is just a matter of turning triangle **T** 'on its side' and 'dropping' the boundary diagonals **R** and **L**. Sub-section 6.1 clarifies this.

6.1. Producing C by using the f.c.s from T

To be more precise, we place the fundamental cycles of the cycle-numbers in the rows of **C**, with n' (from 1' onwards) occupying the first n elements of row **Rn**. Then we allow each number to cycle indefinitely in its row, from the leading diagonal (l.d.) towards the right, potentially filling all the rows of **C**.

To reference elements in the matrix, we shall envisage perpendicular Cartesian axes y (vertically down) and x (horizontally across) both taking all values in positive **N**. The following diagram exemplifies all these arrangements up to $n = 13$.

Row	y/x	1	2	3	4	5	6	7	8	9	10	11	12	13
R1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
R2	2	1	0	1	0	1	0	1	0	1	0	1	0	1
R3	3	1	1	0	1	1	0	1	1	0	1	1	0	1
R4	4	1	0	1	0	1	0	1	0	1	0	1	0	1
R5	5	1	1	1	1	0	1	1	1	1	0	1	1	1
R6	6	1	0	0	0	1	0	1	0	0	0	1	0	1
R7	7	1	1	1	1	1	1	0	1	1	1	1	1	1
R8	8	1	0	1	0	1	0	1	0	1	0	1	0	1
R9	9	1	1	0	1	1	0	1	1	0	1	1	0	1
R10	10	1	0	1	0	0	0	1	0	1	0	1	0	1
R11	11	1	1	1	1	1	1	1	1	1	1	0	1	1
R12	12	1	0	0	0	1	0	1	0	0	0	1	0	1
R13	13	1	1	1	1	1	1	1	1	1	1	1	1	0 (l. d.)

Figure 7: The Cycle-Number matrix **C(13)**

Observe that the triangle beneath (and including) the l.d., is the cycle-number

triangle 'left-justified' and with the zero-lines removed. Its rows are the fundamental cycles (f.c.s) of the cycle-numbers.

Remarks. (On the column elements in C(13).) We are now going to observe what happens in columns C1, C2, ... as the rows are introduced sequentially, R1 to R2 to R3 etc. In order to describe what we are doing we need to add to our vocabulary several graphic terms and notations, such as 'potential prime (pP)', 'potential twin prime (pT or pTP)', 'stalactite in Cj (j-stal)' and 'n-sieve' or 'p-sieve'. Each of these will be defined when introduced.

Observations. (Many have already been noted earlier, from **T**.)

(i) **C** is symmetric about the leading diagonal, so **Rn** = **Cn**.

(ii) The leading diagonal (l.d.) is 1, 0, 0, 0, ...

(iii) The f.c. $\underline{n}$' of cycle-number $\underline{n}$, in **Rn**, runs across from **C1** to the l.d. Its transpose, in **Cn**, is equal to it and runs from **R1** down to the l.d.

(iv) All elements in **R1** are 1, being placed there by $\underline{1}$' as it cycles along to the right.

(v) We say that all elements in the columns of **R1** are *potentially prime* (pP), and that each begins '*growing a stalactite of* 1s' in its column (c.f. a real stalactite, starting to grow down from the roof of a cave).

(vi) All elements in **R2** are produced by $\underline{2}$' cycling to the right; thus $1, 0, 1, 0, \ldots$ are the elements placed in **R2** of columns **C1, C2**, etc.

(vii) We now think of the process in (vi) as being a 'sieving' action, thus: the 0s are placed in the even cols., and each one 'stops' the stalactite above it from growing its column of 1s any further. Thus all stalactites in the even columns are now 'stopped' at **R2**.

(viii) The stalactite in **C2** 'has reached' the l.d. of **C**, and the f.c. composition (10) satisfies our definition of primeness. So we say that the 2-stal is prime; sometimes we say that the stalactite in col. 2 is prime, and even that **C2** is prime (if n is prime, then **Cn** contains a prime stalactite). Thus '$\underline{2}$ is P'.

(ix) In all even cols. after **C2**, the pP stalactites are stopped when the **2**-sieve cycles by; and their stalactites become nonP (or nP, or not-prime). When this happens, we say that the stalactite has reached its final length in its column. This length is the number of 1s acquired, plus a 1 for the final 0.

(x) Stalactites which reach the l.d. are 'prime stals'. Their columns are 'prime cols'.

(xi) In all odd columns the stalactites are not 'stopped' by the **2**-sieve. Their lengths all 'grow' by 1, and they remain as potential primes (pPs). We say they have passed through the **2**-sieve. Now we imagine the **3**-sieve beginning its cycling, and we ask how many of the remaining pP stalactites will survive its passage.

(xii) We can carry on this process for ever, letting the row sieves pass along to the right, here and there stopping a stalactite from growing further.

(xiii) Observe that some pairs of rows have identical (0,1)-patterns. Examples are: **R2, R4, R8**; and **R3, R9**; and **R6, R12**. Conditions for this are given by:

Theorem 6.1. *Two cycle-numbers* $\underline{\mathrm{m}}$ *and* $\underline{\mathrm{n}}$ *have the same* $(0,1)$*-pattern in their rows if and only if* m *and* n *have the same radical; that is, iff* $r(m) = r(n)$*. In case* $m < n$*, we have* m *measures* n*, and* $\underline{\mathrm{m}}$*' cycles in* $\underline{\mathrm{n}}$*'.*

Proof. The proof is left to the reader. □

Example.

(i) $\underline{2}$, $\underline{4}$, and $\underline{8}$ have the same $(0,1)$ pattern, since $r(2) = 2 = r(4) = r(8)$. The f.c.s are respectively 10, 1010, and 10101010; $\underline{2}$' cycles in $\underline{4}$' and $\underline{8}$'; and $\underline{4}$' cycles in $\underline{8}$'.

(ii) $\underline{6}$ and $\underline{12}$ have the same $(0,1)$ pattern, since $r(6) = 6 = 2 \cdot 3$ and $r(12) = r(2^2 \cdot 3) = 2 \cdot 3$. We have $\underline{6}$'= 100010, which cycles in $\underline{12}$'= 100010100010.

Definition 6.2. The relation 'has the same (0,1)-pattern' is denoted by ρ(rho). It is easy to show that ρ is an equivalence relation on the rows of **C**. Hence the set of rows of **C** is partitioned by ρ.

We now introduce a matrix derived from **C**, denoted by **PBPS(C),** and obtained by sequentially computing its rows.

6.2. The PBPS matrix

A useful pictorial device is obtained by transforming the matrix **C** as we go along, row by row, and placing the modified rows in a new matrix, say $\mathbf{S} \equiv$ **PBPS(C).** The acronym stands for **P**artial **B**oolean **P**roduct (row)-**S**equence. The rows of **C** form the sequence $\underline{\mathbf{N}}$, of the cycle-numbers $\underline{\mathrm{n}}$. The ith partial BP of this sequence, denoted by $\mathbf{s_i}$, is the ith row of **S**. Thus the rules of the computations are as follows:

Row computation Rules:
Let $\underline{\mathbf{n}}$ denote the nth row of **C**, and $\underline{\mathbf{s_n}}$ denote the corresponding row in **S**. Then

(i) $\mathbf{s_1} = \underline{1} = \overline{1}$; and

(ii) $\mathbf{s_n}$ = factorial $\underline{\mathrm{n}}$ (using BP multiplication) = primorial $\underline{\mathrm{i}}$ (using BP multiplication, and lemma 6.3 below).

Example.

$$\begin{aligned}\mathbf{s}_6 &= \underline{1}\wedge\underline{2}\wedge\underline{3}\wedge\underline{4}\wedge\underline{5}\wedge\underline{6} \text{ which reduces to}\\ &= \underline{2}\wedge\underline{3}\wedge\underline{5},\\ &= \text{primorial } \underline{5}.\end{aligned}$$

(The notation for this (see [6]) is $\mathbf{5}\#$ where cap or BP multiplication is understood. We occasionally use a personal notation $\mathbf{X_i}$ for primorial p_i, where $\mathbf{X}$ is capital 'chi'. Thus for example, $\mathbf{5}\# = \mathbf{X_3}$.)

The sequence of primorials rises rapidly in lengths, since $\mathbf{p_{n+1}\#} = \mathbf{p_{n+1}} \wedge \mathbf{p_n\#}$.

Lemma 6.3. *factorial $\underline{n}$ = primorial $\boldsymbol{p_n}$, where $\boldsymbol{p_n}$ is the greatest prime cycle-number less than or equal to $\underline{n}$.*

Proof. Any **n**-sieve which is not a prime sieve cannot supply a 0 to a column which has an unstopped stalactite in it, and hence can be ignored. For if **n** were not prime, it would be measurable by one or more primes, **p** say, with $\mathbf{p}<\mathbf{n}$, and one of the **p**-sieves arising from them would already have stopped the stalactite. □

Example. In $\mathbf{s_6}$, the $\underline{4}$ and $\underline{6}$ cycle-numbers are not prime.

(i) Now $\underline{2} \wedge \underline{4} \equiv \underline{2}$ (in its whole (0,1)-string) so a stalactite which has passed through the **2**-sieve must also pass through the **4**-sieve. Thus the $\underline{4}$ may be ignored.

(ii) For $\underline{6}$, the other non-prime, we have $\underline{6} = \underline{2} \wedge \underline{3}$. Therefore any 0 presented to a column in $\underline{6}!$ by the 4-sieve or the 6-sieve will find that the stalactite has already been stopped by either the 2-sieve or the 3-sieve. The computation of $\underline{6}$' shows how this must happen:

$$\begin{array}{cc} & 101010\\ \wedge & \underline{110110}\\ & \underline{100010}\end{array}$$

The resulting **PBPS** matrix need show only the **C1** column of 1s, and the completed stalactites (with their final 0s) in the other columns. In the leading diagonal we place a P in each prime column, for ease of locating prime rows. All other entries in the matrix are 0s, and these are not shown (i.e. their cells are left blank).

Below is the reduced matrix **PBPS(C13)**. (See Lemma 6.3 for explanation of why we can write factorial $\underline{n}$ for obtaining each row **Rn** . For each prime row, we could write $\underline{\mathbf{p}}\#$, and for each non-prime row we would have $\underline{\mathbf{n}}\mathbf{!} \equiv \underline{\mathbf{p}}\#$ where p is the greatest prime $<$ n; and all multiplication is $\wedge$.)

	y/x	1	2	3	4	5	6	7	8	9	10	11	12	13
1!	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
2!	**2**	**1**	**P**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**
3!	**3**	**1**		**P**		**1**		**1**		**0**		**1**		**1**
4!	**4**	**1**			**0**	**1**		**1**				**1**		**1**
5!	**5**	**1**				**P**		**1**				**1**		**1**
6!	**6**	**1**					**0**	**1**				**1**		**1**
7!	**7**	**1**						**P**				**1**		**1**
8!	**8**	**1**							**0**			**1**		**1**
9!	**9**	**1**								**0**		**1**		**1**
10!	**10**	**1**									**0**	**1**		**1**
11!	**11**	**1**										**P**		**1**
12!	**12**	**1**											**0**	**1**
13!	**13**	**1**												**P**

Figure 8: The matrix **PBPS(C13)** (a prime rib diagram)

The reader will appreciate why we have added the bracketed phrase to the figure's caption. The prime and twin prime stalactites stand out like ribs in a rib-cage.

Note in particular that when a growing stalactite acquires a 0 from a passing sieve, it 'stops growing'. More precisely, its pattern now ends in (1,0), and the next $\wedge$ operation in its column is $1 \wedge 0 = 0$. This happens to all stalactites eventually (except the stal. in **C1**). Those which reach the l.d. become primes; whilst those pPs which are not destined to become primes are stopped by a 0 above the l.d.

Much more can be said, and deduced from, the **C** and **PBPS** matrices. This must all be left for a segue paper. To end this one, we shall include a Section 7 which muses upon the 'music' made by cycling (0,1)-patterns formed within **T** and **C**.

7. Musings on the 'music of the cycle-numbers'

At the head of this paper (p. 2), we gave a quote by Leibnitz which expresses very beautifully a relationship he claims between music, man and mathematics, one with which we whole-heartedly empathize. Here we take up his theme with some of our own feelings (well, Turner's anyway!) about musical images arising from studies of the cycle-numbers in **T**. The author du Sautoy, in his book *Music of the Primes* [1], traces many connections between mathematics and music, stemming from work due to Pythagoras, Euler, and so on up to the present day, where his focus of attention is on the distribution of the primes and their relations to the zeros of the zeta function and Riemann's Hypothesis, and on related musical ideas.

With our cycle-numbers, we have shown that each number $\underline{\mathbf{n}}$ has an interior pattern, or structure, with a fundamental cycle (a (0,1)-string of length n) which

cycles around its **n**-necktie and down the two legs indefinitely, within the cycle-number triangle **T**. Similarly, in matrix **C**, the cycling takes place linearly in two (and more) directions. It is easy to compare these 'movements' with the vibrations of tuned strings on a musical instrument, or on bars of a xylophone. One can even turn the cycled patterns into music, by clapping or drumming the (0,1)-strings using Morse-code rhythms, and accenting beats on the starts of each cycle. For example, the number $\underline{\mathbf{2}}$ has f.c. (10), which can be clapped in 2/4 time as it cycles, thus: **da-di, da-di, da-di,** ... with stresses on each **da** . Similarly $\underline{\mathbf{3}}$, with f.c. (110), can be clapped in 3/4 or 3/8 time as **da-da-di, da-da-di,** etc. with stresses on each first **da**. The notion of polyphony is easily introduced via the cap product. For example, $\underline{\mathbf{2}}$ and $\underline{\mathbf{3}}$ can oscillate together, as the joint vector $\underline{\mathbf{2}} \wedge \underline{\mathbf{3}} = \underline{\mathbf{6}}$. This has f.c. (100010) (clapped as **da-di-di-di-da-di**),which can be stressed in various ways to produce differing rhythms and 'sounds'.

In this manner, one can think of each twin prime having its own distinctive rhythms and sounds; e.g. ($\underline{\mathbf{3}}$, $\underline{\mathbf{5}}$) resonates with $\underline{\mathbf{15}}$, and so on. These patterns, or pieces of linear patterns, occur and recur in different ways and places throughout the matrix **C**, causing 'overtones' or 'harmonics' in the 'music'.

One interesting comment, about the entrance of each successive prime, will suffice to end this musing. When a new prime arises in **T**, it breaks various previous symmetries, and introduces its own distinctive rhythm into the music which is sounding within and about its new linear 'melodies', on its own grid in **T** or **C**.

Perhaps, like Pythagoras and his 'music of the spheres', we (i.e. Turner) may well be the only person capable of hearing the 'music of the cycle-numbers'.

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