Embedding negative structures to model holes and cut-outs

S. Ilanko

Department of Engineering
The University of Waikato
Te Whare Wananga o Waikato
Gate 1 Knighton Road
Private Bag 3105, Hamilton 3240
New Zealand
ilanko@waikato.ac.nz

It has now been established that geometric boundary conditions and continuity conditions can be modelled by using either positive or negative stiffness or inertia type penalty term [1-5]. The experience of working with negative stiffness and inertial parameters has led to the question: what if both stiffness and mass were to be taken as negative? Changing the sign of all stiffness and inertial terms of a structure is simply equivalent to multiplying both sides of an eigenvalue equation by minus one, which does not change its frequencies or modes. Basically, a negative structure in such a sense has the same vibratory properties as that of its positive counterpart, although the structure itself may not have a physical meaning.

However, an interesting question emerges about the behaviour of a structure formed by attaching such a negative structure to a larger positive structure: would it be possible to effect a hole or a cut-out in a plate by embedding a "negative plate" to a plate without the hole? The idea is that the negative plate to be attached must have the same shape as that of the hole and the same magnitude of stiffness and mass distribution but with opposite sign (see Figure 1). The negative plate unit would be bonded to the larger (uncut) plate by using distributed penalty stiffness over the area of bonding to prevent any relative motion between the negative part and the positive part.

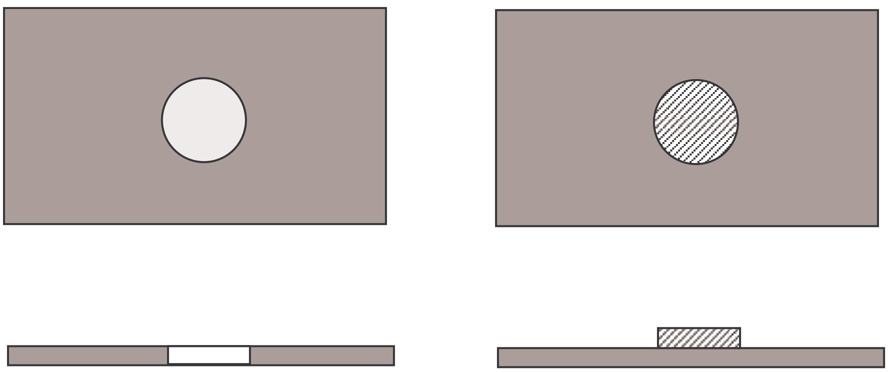

Figure 1

To see if this concept works, a cantilever beam of length L, flexural rigidity EI, and mass per unit length m was made into a free-free beam of length $L/2$, by attaching another cantilever beam of length $L/2$, flexural rigidity $-EI$ and mass per unit length $-m$. The two beams were connected by means of a uniform distribution of elastic springs of high stiffness k per unit length along the entire length of the negative beam, and a Rayleigh-Ritz analysis was carried out to determine the natural frequencies and modes (see Figure 2). Positive and negative values were used for *k*, and the average of the two results was taken to minimise any error due to violation of the constraint condition along the bonded length.

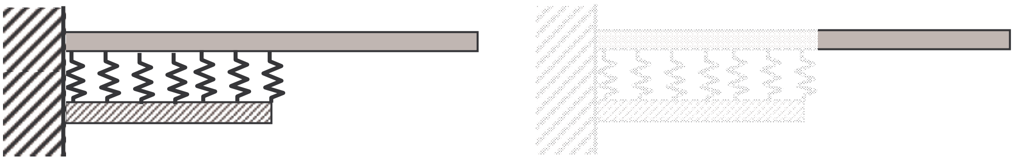

Figure 2

The result was interesting. Using a series of 7 terms each for the two cantilever beams and a non-dimensional stiffness coefficient $\alpha = 10^{11}$, which may be regarded as a penalty parameter, the first three non-zero natural frequencies of the free-free beam were determined. These are given in Table 1 together with those calculated using a straight-forward Rayleigh-Ritz approach with simple polynomials. Interestingly, while the first natural frequency is slightly worse than that obtained using simple polynomials for a free-free beam, the second one is slightly better and the third is worse. It is not clear why the trend changes but, unlike in typical penalty applications where the penalty terms are used to enforce geometric boundary conditions, in this case, they are indirectly being used to relax the conditions at the centre of a clamped beam to make to the free end of a smaller beam. However, the results show that embedding a negative structure to model a cut-out may be possible. While this may seem encouraging, it has to be stated that the use of more terms for the displacement or the use of a negative beam of very short or very long length causes numerical problems. In addition, the first two natural frequencies are not exactly zero but are either small values or complex numbers. This may be due to the fact that the set of functions used are not from a complete set that would allow perfectly free conditions to be modelled. For the final free-free beam, all admissible functions used, when extended to the original clamped support point, have a zero displacement and a zero slope. Another potential contributing factor is that with penalty terms it is not possible to obtain a complete cancellation of the positive structure by the negative structure in practical applications because the penalty values must be finite. Further investigations are being conducted to identify the source(s) of the problem and to see if these could be addressed by using other types of admissible functions.

	7 terms per beam with $\alpha = \pm\ 10^{11}$	Simple polynomial 7 terms	Exact
ω_1	4.732	4.730	4.730
ω_2	7.874	7.971	7.853
ω_3	11.665	11.367	10.996

Table 1 First three non-zero natural frequencies of a free-free beam

References

1. Ilanko, S. & Dickinson, S.M. 1999 Asymptotic modelling of rigid boundaries and connections in the Rayleigh–Ritz Method. *Journal of Sound and Vibration* 219: 370–378.

2. Ilanko, S. 2002 Existence of natural frequencies of systems with imaginary Restraints and their convergence in asymptotic modelling. *Journal of Sound and Vibration*, 255, 5, 883-898.

3. Ilanko, S. 2005 Introducing the use of positive and negative inertial functions in asymptotic modelling. *Proc. R. Soc. A* 461, (2060), 2005: 2545–2562.

4. Williams, F.W. & Ilanko, S. 2006 The use of the reciprocals of positive and negative inertial functions in asymptotic modelling. *Proc. R. Soc. A,* 462 (2070):1909 –1915.

5. Ilanko, S. & Williams, F.W. 2008 Wittrick-Williams algorithm proof of bracketing and convergence theorems for eigenvalues of constrained structures with positive and negative penalty parameters, *International Journal for Numerical Methods in Engineering*, 75 (1), 83 – 102.