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*Connecting the Points:  
An investigation into student learning about  
decimal numbers.*

**A dissertation presented in partial fulfillment of the  
requirements of the MEd degree**

**By Bruce Moody  
2008**



THE UNIVERSITY OF  
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*Te Whare Wānanga o Waikato*

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*He honore, he kororia ki Te Atua.*

## **Whakatauki**

This proverb guided me in the teaching process.

Ma te tohutohu, ka wareware;

Ma te kite, ka maumahara;

Ma te whakamahi, ka mātau

Tell me, and I will forget;

Show me, and I will remember;

Involve me, and I will understand.



## *Abstract*

The purpose of this research project was to investigate the effects of a short-term teaching experiment on the learning of decimal numbers by primary students. The literature describes this area of mathematics as highly problematic for students.

The content first covered student understanding of decimal symbols, and how this impacted upon their ability to order decimal numbers and carry out additive operations. It was then extended to cover the density of number property, and the application of multiplicative operations to situations involving decimals. In doing so, three areas of cognitive conflict were encountered by students, the belief that longer decimal numbers are larger than shorter ones (irrespective of the actual digits), that multiplication always makes numbers bigger, and that division always makes numbers smaller.

The use of a microgenetic approach yielded data that was able to be presented that provides details of the environment surrounding the moments where new learning was constructed. The characteristics of this environment include the use of physical artifacts and situational contexts involving measurement that precipitate student discussion and reflection.

The methodology allowed for the collection of evidence regarding the highly complex nature of the learning, with evidence of ‘folding back’ to earlier schema and the co-existence of competing schema. The discussion presents reasons as to why the pedagogical approach that was employed facilitated learning.

One of the main findings was that the use of challenging problems situated in measurement contexts that involved direct student participation promoted the extension and/or re-organization of student schema with regard to decimal numbers.

The study has important implications for teachers at the upper primary level wanting to support student learning about the decimal numbers system.

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## Chapter One: Introduction

Problems that students have with the learning of decimal numbers are well documented in the literature. These include a lack of understanding of the meaning of the symbols used to represent decimal quantities (a semiotic problem), and a lack of facility in using them in mathematical operations. The central epistemological issues for the semiotic problem have been researched, with the findings available to the educational community for over twenty years (e.g. Sackur-Grisvard & Leonard, 1985). These findings describe how understanding of two referent systems (place-value and fractions) has to be combined in a reconstructive process in order to create meaning. Subsequent research has expanded upon (rather than conflicted with) this earlier work (e.g. Stacey & Steinle, 1999; Steinle & Stacey, 1998). For many years, the educational community also has had access to research that has documented the deep-seated difficulties learners of all ages have with situations involving operations with decimal numbers (e.g. Burns, 1990; Graeber & Tirosh, 1990). Despite the widespread knowledge about these difficulties, there is evidence to suggest that little has changed in terms of student achievement with decimals (e.g. Bana & Dolma, 2004, Young-Loveridge, 2007).

The use of decimals is ubiquitous in the wider community and includes contexts such as financial and statistical literacy, measurement, and probability. Therefore, knowledge of decimal numbers is not optional for a society that desires all of its citizens to employ mathematics effectively in everyday life.

Studies that have demonstrated effective pedagogical responses to the needs of students learning decimal place-value have fostered connections in student thinking between decimal symbols, concrete models and prior experiences (e.g. Helme & Stacey, 2000; Irwin, 2001). There are some studies that have investigated how students might be assisted to make sense of decimal numbers in situations that require the application of the four basic arithmetic operations. Examples include Bonotto (2005), and Irwin and Britt (2004). The fact that there are few studies that both present examples of successful interventions and provide models for explaining the process of the learning, has established a need to conduct more research in this area.

My involvement with the teaching of decimal numbers stems from a project I was involved with in 2001-2002. I sought to model to teachers the process of using diagnostic data to plan and deliver lesson sequences. Decimal numbers was a context I could use in classes from Years 6–9, and thus work with Primary, Intermediate, and Secondary Schools. The interventional approach I used was influenced by reading recent research of the time, particularly Stacey, Helme, Archer and Condon (2001) and Irwin (2001). My experiences showed me that without attending to the reasoning behind the practice I was demonstrating, teachers were at best likely to copy what I had done, and at worst consider the approach as too different from ‘proper’ maths teaching, by which they meant didactic instruction.

In 2006, I had the opportunity to research the area of decimal numbers in more depth as I engaged in a Directed Study as part of my Post-Graduate Diploma (Moody, 2007). This served as an end in itself, but also provided data that formed part of the present research.

Recent literature shows that educational researchers in mathematics have produced models that seek to explain the process of learning and to account for instances of non-learning. Many of these models are situated in studies involving rational number (e.g. Simon, Tzur, Heinz & Kinzel, 2004). These studies do not focus on the examination of student misconceptions, but rather on understanding the processes that students are engaged in while learning, and the actions teachers can perform to enhance student cognitive development. In order to describe and examine these processes, student data is gathered *while* learning is taking place. This is in order to capture the conditions and conversations surrounding the moments when new connections were made and new thinking expressed. This positive attitude to students’ prior knowledge, and the desire to help teachers improve their personal pedagogical content knowledge, resonated with my own beliefs about teaching. I wanted to contribute to the educational community in similar fashion. This desire shaped the nature of the data that I wanted to collect and thus determined the type of methodology that I went on to use.

This research was a microgenetic study (Siegler, 2007) to examine how students’ understandings of decimals can be enhanced by a short-term teaching experiment.

The next chapter presents a survey of the literature relevant to this research.

## Chapter Two: Review of Literature

### Chapter Outline

This chapter has been divided into seven parts. Literature concerning the models and mechanisms of learning are discussed in parts 1 to 3. Part 4 examines the literature that pertains to decimal numbers. Part 5 describes the issues surrounding the methodology employed in the study. Part 6 serves to present the place of decimals in the New Zealand mathematics curriculum. The chapter ends with the rationale for the research undertaken in this study.

Each of the parts is prefaced with an introductory quote. These serve both as summary statements of the contents of the sections and also as anchor points for the broader discussion in Chapter Five.

The section headings are listed below:

- Constructivist Theory of Learning
- Constructivist Models of Learning
- Mechanisms of Learning
- Learning of Decimal Numbers
- Research Methodology
- Review of the New Zealand Curricula
- Rationale for this Research

### The Constructivist Theory of Learning

Models or metaphors of learning that imply simple levels or steps to be ascended, or a few obstacles to be overcome; fail to account for the wide variation in learner experience and learner dispositions.  
(Mason, Drury & Bills, 2007, p. 56).

### Constructivist Views of Knowledge

The dominant theory in the learning of mathematics in the last 50 years is that of constructivism. According to Van de Walle (2007, p. 22), “The basic tenet of constructivism is simply this: *Children construct their own knowledge*” (italics in original). The work of the developmental psychologist Jean Piaget is often taken as the reference point from which ideas of constructivism have been discussed and extended. Within the many forms of modern constructivism, all have emphasized the view that learners create meaning in an active way. They create new knowledge for themselves and do not simply absorb the knowledge of others (Cobb, 1994; Confrey & Kazak, 2006; Oxford, 1997). Learning is viewed as the process by which this construction occurs, as learners adjust their previous knowledge in a self-conscious exercise (Anthony, 1996;

Brown, Collins & Duguid, 1989). New knowledge is seen as inseparable from the person acquiring that knowledge (Heinz, Kinzel, Simon, & Tzur, 2000; Hiebert, Carpenter, Fennema, Fuson, Human, Murray, Olivier & Wearne, 1996). Steffe, Cobb and von Glasersfeld (1988) succinctly summarize this with the statement “numbers are *made* by children, not *found*” (italics in original) (p. i).

Moyer (2001) captures key aspects of the constructivist view of learning when she writes, “Current research in mathematics education view students as active participants who construct knowledge by reorganizing their current ways of knowing and extracting coherence and meaning from their experiences” (p. 176).

These views of knowledge imply that the most effective instructional strategies are those where the students are the starting point rather than the discipline itself (Van de Walle, 2007).

Our thinking about knowledge has also evolved. As learning is viewed as an active process of constructing knowledge rather than being passively received, we can also describe knowledge itself using more active terms. For example Gilbert (2003) speaks of viewing knowledge actively, as energy not a thing, as a verb and not a noun.

A static view of knowledge allows for key items to be presented to learners as general truth, then for their applications to be practised by students. As these students have the same presentation of knowledge, the expectation that they will reach equivalent endpoints is reasonable. This (admittedly crude) summary may be termed an epistemological approach. Recognition that ‘adult’ truth is not the same as student truth has forced a re-examination about what we mean by knowledge. Dissatisfaction with a purely epistemological approach is exemplified in the following statement:

“The most serious mistake one could make in a discussion of mathematical processes would be to assume that these processes are universal, that they are the same for the learning child and the professional mathematician. This kind of assumption has led some educators to believe that their own introspection would be sufficient to bring about an understanding of the problems faced by school children”  
(Herscovics, 1989, p. 3).

Knowledge is now described as being dynamic; evolving in response to stimuli. As Brown et al (1989) have stated “All knowledge is, we believe, like language (and so is) always under construction” (p. 33). Sfard (1991) called for more work to be done in

producing a unified theory to simultaneously address the philosophical and psychological dimensions of learning mathematics.

Knowledge is not seen as existing independently, but existing within a community who share similar ideas. Knowledge is organised around the experiences of the learner at least as much as it is organised around abstractions (Lesh, Doerr, Carmona & Hjalmarson, 2003). Epistemological research is seen as forming an important role, but these authors point to the need to consider the abstractions, the experiences of learners, and their inter-linkage in order to better understand the process of learning.

A dynamic view of knowledge sees what is learned as not existing in isolated packets but as linked to create an organic whole, a matrix of significant and relevant information (Herscovics, 1989). Some researchers (e.g. Baroody, Cibulskis, Lai & Li, 2004; Nyikos & Hashimoto, 1997) no longer view learning in 'stages' with metaphors of ladders of knowledge but as 'webs' of continued interaction. Growth in knowledge can be conceptualized as the extension of existing learning in any direction (Carraher & Schliemann, 2002).

### **Internal and Social Aspects of Learning**

Two main theoretical perspectives of constructivism have been explored. The first is now often described with the simple label of 'constructivist' (though the other perspective is also constructivist in terms of its central tenet) and the other label is 'socio-cultural'. The difference between the two can be described as individual and collective accounts of knowledge construction respectively. In the former, knowledge is created internally and then expressed socially; while the latter holds that knowledge is created socially and then internalized individually. As Cobb (1994) suggests, a central question is whether the learning mind is best regarded as primarily in the individual or as belonging to the learning community, which in turn leads to the question of whether communication with self (individual reflection) is an extension of communication with others, or its precursor. The development of the socio-cultural strand to constructivism came as a result of research into the settings within which the epistemological and psychological elements are played out. Moving out of clinical settings and into classroom environments showed

that even the amount of understanding produced by considering cognitive psychology could not explain all of the variation seen in learning. Knowledge of the discipline of mathematics and knowledge of the individual learner were both important aspects of understanding how learning occurs, but were incomplete.

These two branches of constructivist thought, the role of the individual learner, and the role of social interaction, have at times been seen as alternative or even conflicting theories (Brown, 2008; Bruner, 1997; Confrey & Kazak, 2006; Dahl 2003; Simon, 1995). A growing number of researchers are deciding that choosing between constructivist and socio-cultural paradigms is unnecessary as in practice they are complementary (Bruner, 1997; Cobb, 2006; Dahl 2003; Druyan, 2001; Seeger, 2001; Sfard, 1998; Simon, 1995). Sfard (1998) warns of the dangers involved with seeking to promote one perspective over the other. Simon (1995) writes that the main issue is not which of the two perspectives (social or cognitive) is primary, but rather what can be learned by combining analyses from the two. It may be unhelpful to emphasise differences between the constructivist and sociocultural approaches, as the resulting dichotomy is at least partly artificial (Wascgescio, 1998).

Having both perspectives of student learning gives access to cognitive psychology with its attendant vocabulary of 'reflect, mental operations, consciously reviewing experiences'; and to social cognition with its drawing on 'interactions, communication, listening, talking and writing' (Cobb, 1994; Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier & Human, 1997).

### **Environmental Influences**

Bronfenbrenner (1979) proposed a model in which influences upon learning included not only the local classroom environment but wider factors such as school structures and values, and more general factors such as the prevailing social and political climates. Progressively embedded systems were described using the qualifiers 'macro-', 'exo-' 'meso-' and 'micro-'. A macro-system phenomenon would include the society's attitude towards girls being taught mathematics, whereas a micro-system phenomenon would include how many questions in a particular mathematics class were directed towards a female student. He used the term 'ecology' to communicate his belief that these elements

were inter-related. The ecology metaphor transcends dichotomies such as individual/social and epistemological/psychological, it recognizes the impact of both classroom and wider cultural influences, and seeks to describe how these form part of an inter-connected whole. Each of these influences continues to be studied individually, but few researchers would now believe that their particular focus can be totally isolated from the wider setting. These complex interactions continue to be studied (e.g. Kemmis, 2008). One concept that arose from these considerations is that of 'situated learning'. This holds that learning is *always* contextualized and that this context refers to the wide range of influences upon the learner including the classroom climate, situations within which the mathematics problems are embedded, the wider community values, the background experiences of students, and the affective processes in learning (Brown et al, 1989). Each of these individual factors has been the subject of research studies, but with the recognition that they are integrated parts of a whole rather than being separate entities. The variety of other factors that have been studied (e.g. meta-cognition, group dynamics, use of manipulatives, task creation) serve to remind us how complex the process of learning is regarded in constructivist understandings. Recent work (e.g. Barab & Squire, 2004; Cobb, Confrey, diSessa, Lehrer & Schauble, 2003) has also built upon the ideas of Bronfenbrenner as they also employ the ecology metaphor to emphasize the inter-connectedness of the learning environment. Increasingly there are statements that all of these factors not only co-exist but are mutually reflexive and inseparable. One of the dangers involved with having many voices speaking into the learning situation is that of fragmentation (Ball, 2000). This exists when attention to content, pedagogy, and cultural issues are seen as competing for the same resources rather than being integrated. A teacher cannot address issues of content without consideration of pedagogy and culture, nor can they solely attend to cultural aspects without having mathematical content as the context for these aspects to be expressed. While various aspects of learning situations can be considered separately by researchers, they are all part of a single environment as far as the learner is concerned (Bronfenbrenner, 1979). This complexity is seen to be evidence against the existence of a single, simple explanation of learning. As Barab and Squire (2004) write, "learning, cognition, knowing, and context are irreducibly co-constituted and cannot be treated as isolated entities or processes" (p. 1).

### **Recent Developments**

Other paradigms have emerged concerning learning. Variation theory places the object of learning as the centre of analysis. It examines how teachers allow or constrain variation within and between problems for pedagogical ends (Marton & Trigwell, 2000; Mason, 2005; Runesson, 1999, 2005). Runesson (2005) expressly points out that variation theory is not setting out to replace other perspectives but to add to the overall understanding of learning and teaching.

Models theory places the emphasis on the processes of learning – learning how to learn – but again its proponents make clear that they see themselves as adding to the collective knowledge of the educational research community rather than replacing existing paradigms (Lesh et al, 2003). Models theory proposes that primitive understandings of a new concept occur early in the learning process and are refined over long time spans. Progression of learning is described as ‘fuzzy’ to contrast their view with learning that is described as showing clearly identifiable stages. Recursive behaviour such as the regression to earlier understandings and practices is regarded as a normal part of the learning process rather than indicating deficiency. One metaphor employed regarding learning is that of journeying through a complex terrain.

### **Links to Classroom Practice**

Recently, Cobb (2006) has proposed that mathematics educators employ theoretical perspectives as sources of ideas to be appropriated and adapted *as found useful* (italics added). Cobb’s pragmatic approach is based upon his belief that the assumed relationship between theory and instructional practice is problematic. Rather than being channeled into accepting a single view about learning, he seems to be advocating the accessing of all perspectives that help us understand this complex activity.

Seeking a universal descriptor of learning may be chasing a chimera. We do not have evidence that learning is the same for each culture, let alone each individual; nor that learning is the same for any single individual across different learning areas and across



time. Making use of models of learning and reporting upon the outcomes (as per Cobb's suggestion) seems equally sensible as a research approach as the focus upon any specific learning factor.

The translation of constructivism as a theory of learning, to classroom teaching that can be described as constructivism in practice, has been problematic. Bridging theories have been developed and researched to make linkage between theory and practice (Confrey & Kazak, 2006). These have contributed to our knowledge by examining elements of constructivist theory such as problem solving, the place of prior concepts, and cognitive development within learning situations. This 'grounding' of research in classroom practice has served to both shape the theories under examination, and to establish credibility as far as practitioners are concerned. The latter point is important if researchers view the research process as not ending with the publication of a report but including the active responses of those reading it (Burns, 2000; Cohen, Manion & Morrison, 2000; Gravemeijer, 2001; Merriam, 2001; Steffe & Kieren, 1994).

Among the profound implications of constructivism for teachers is the idea that the possession of correct content knowledge and its faithful transmission is only part of the requirements of the successful teacher. As the learner constructs knowledge rather than receiving it, studies of teaching also need to focus upon the learner as a partner in the learning process.

Lesh et al (2003) write that

“The claim that knowledge is actively constructed by the learner is of limited use to teachers and researchers, especially if details are lacking about how the constructs are to be developed” (p. 212).

Teachers need to understand the mathematics itself; its conceptual base, key components and its links to other domains within mathematics. This knowledge has been termed 'specialised knowledge of content' to distinguish it from 'common knowledge of content' (Hill, Ball & Schilling, 2004). The two labels serve to draw attention to the distinction between mathematical knowledge that may be regarded as common to all those who can operate with it, and that which allows the person to create or examine alternative representations and provide explanations of the process. An ability to unpack one's own highly compressed and refined understandings is required in order to find the level of granularity needed to engage with students (Ball, 2000). There have been concerns raised

regarding the lack of mathematical knowledge held by U.S. teachers with the phrase ‘dismayingly thin’ applied (Ball, Hill, & Bass, 2005, p. 14).

In addition, it is now recognised that teachers require ‘pedagogical content knowledge’ (PCK). This term, as used by Shulman (1986), describes ‘the dimension of subject matter knowledge for *teaching*’ (italics in original) (p. 9). This knowledge about how children are likely to interact with the content, their likely preconceptions (which may include misconceptions), and where new knowledge will both graft into and be seen in contrast to their existing schema may help teachers plan for effective instruction. While the lesson is proceeding, teachers may also need to respond to the unexpected mathematical comments from students, be prepared to search for synonyms where language is not understood, and produce alternative representations where original models are not transparent to students (Ball et al, 2005). Further refinement of this concept has expanded the frameworks of reference regarding PCK with sub-categories being defined within the over-arching meaning of the term (Hill & Lubienski, 2007).

## **Constructivist Models of Learning**

The most serious didactical obstacle is a lack of opportunity to learn  
(Harel & Sowder, 2005, p. 46)

### **Zone of Proximal Development (ZPD): Introduction**

If the idea that knowledge can be transferred from an expert to a learner by transmission is rejected, the onus is on the educational theorist to explain how knowledge is constructed by a learner (Cobb, 1988; Hall, 2001).

Two lines of thought are needed; one to deal with the apparent paradox that learners can construct meaning out of what they do not know, and the other to address mechanisms by which this can be achieved. In practice, these two overlap.

In describing how it is possible that learners can create new knowledge, Piaget created the term ‘moderate novelty principle’. Moderate novelty describes material that is sufficiently similar to existing knowledge to be engaged with, but sufficiently distanced so as to provide the need for re-examination of current thought and thus occasioning learning (Baroody et al, 2004). In contrast, material that is too different from existing

understanding cannot be connected to existing knowledge and so cannot be assimilated. A lack of novelty involves ready assimilation but little new learning is possible.

One of the most influential models regarding how learning occurs was created by Vygotsky and is known as the 'Zone of Proximal Development' (ZPD). Vygotsky did not believe that the teacher needed to wait until the student was 'ready' for new learning but that instruction should lead the way into new development (Shayer, 2003; Steffe et al, 1988). This apparent contradiction with constructivist thought is resolved by the consideration of three categories of processes vis-à-vis the learner. The first is what the learner can independently work with, while the third describes activity that the learner cannot currently participate in. In between these two is a learning zone where the learner is able to participate *when* supported by a more able peer or teacher. Vygotsky's work emphasized the importance of social interaction as a vehicle for the accessing and assimilation process. He held that many students were able to solve problems in an interactive social environment before they were able to solve them individually (Bruner, 1997; Leikin & Zaslavsky, 1997). Further than this, he argued that many types of learning could not occur without the mediation of a more able individual.

Vygotsky wrote in Russian, a language that uses one term for both teaching and learning. The important consequence of this regarding the ZPD is that when Vygotsky describes essential features of *learning*, he is simultaneously describing essential features of *teaching* (Clarke, 1995). This is parallel to the use of the word 'ako' in Te Reo Māori where a kaiako (taken as 'teacher') is more literally translated as 'the person who causes learning to occur'. These reflexive terms imply reciprocity in the learning process that is not implicit in English terms such as 'lecturer', 'instructor' and 'teacher'.

### **ZPD: Developments**

Recently, theorists have debated the limiting of the ZPD to the individual, and posited that a learning group collectively possesses a zone whose 'area' is larger than any single individual's zone. For example, Nyikos and Hashimoto (1997) assert that "Most interpretations of the ZPD restrict the zone to each individual thereby discounting the broader social phenomenon of growth as a cohesive thought collective" (p. 507). This collective interpretation of ZPD allows for the possibility for learning to occur in the

absence of an external expert. It admits the potential of group synergy whereby ineffective, individually possessed partial knowledge can be recombined into shared, effective new knowledge when the social environment is conducive to this activity (Martin, Towers & Pirie, 2006). The process by which this occurs relates to the establishment of a shared group goal and the allowing of improvisational discussion and use of artifacts. This group learning possibility accords with cultural views of knowledge that both acknowledge the place of the expert, and also affirm the responsibility of social participation. This is illustrated in the Māori whakatauki (proverb), *Nāu te rourou, nāku te rourou; ka ora te tangata*. (With what you bring in your basket, and what I bring in mine, will suffice for all of us).

Another development in ZPD has been suggested by Albert (2000). She proposed that while ZPD is primarily a model of social learning, there is a need to account for and describe the personal internalization process more fully. This has been termed 'Zone of Proximal Practice' (ZPP), where the student starts to function independently but cannot yet be said to have completely generalized the process being learned. Simon (2006) also raised concerns regarding socio-cultural models of mathematics learning because internalization is not fully explained. He proposed study of 'Key Developmental Understandings' (KDUs) which are regarded as being instigated in social contexts but are then re-worked internally. This seems to parallel the ZPP of Albert (2000).

Another development of ZPD arises from the consideration of what actions students believe are permissible - the Zone of Free Movement - and what actions are encouraged by teachers (and/or peers) – the Zone of Promoted Activity. This concept is associated with Valsiner (Galligan, 2008; Goos, 2008).

### **ZPD: In Practice**

The activity within this zone that introduces the learner to working with new concepts was termed 'scaffolding'. This term describes the tutorial process whereby a more knowledgeable person (teacher or peer) assists the learner to engage with a novel task (Wood, Bruner & Ross, 1976). Scaffolding is an apt metaphor for this view of learning as it calls to mind the temporary structures erected around a building that allow for the creation of a solid object once the constructive process is complete. In education, this scaffolding can take many forms and have many participants. It is the dynamic inter-

relationship of ideas and views within the group setting that enhance or hinder an individual's likelihood of achieving their potential (Nyikos & Hashimoto, 1997). The teacher's role is to facilitate cognitive reconstruction (Cobb, 1988). This may be achieved by directing discussion towards variation that exists between data sets, or between new data and previous constructs. Having to defend propositions, collect and analyze data and actively listen to the arguments of others become pathways to new knowledge (Lampert, 1990). Students will thus 'zigzag' their own way towards the wider community view if the teacher does not impose authoritative knowledge directly. If the teacher reduces the cognitive demand for students then a climate of dependency may be created as students become conditioned to wait for external assistance rather than expect to be able to complete tasks independently (Watson & De Geest, 2005).

Scaffolding is deliberately withdrawn by the teacher as the student begins to internalize the new learning and becomes independent of this external assistance (Bliss, Askew & Macrae, 1996). Learners may also act to remove themselves from scaffolding by choosing to operate more independently.

ZPD can be thought of as a building site; initially a place of raw material, worker power and a plan of operation, ultimately leading to a new, lasting structure. The difference between the construction zone of the building and of the learners is that the learners themselves are also the workforce; they are not only the result of the process but also its primary mechanism. The ZPD can be viewed as the setting within which the moderate novelty principle of Piaget is enacted (Baroody et al, 2004).

Working in the ZPD is no guarantee of learning. The student agenda may simply be the completion of the task at hand, whereas the teacher is seeking for the student to develop mathematically. Both parties have agendas that may overlap but cannot be assumed to be identical. The agency of the learners must be taken seriously. Their goals will particularly affect the reflection aspect of learning. External pressure may lead to conformity in terms of task completion, but if students are disengaged, there is less likelihood that internal cognitive reconstruction will occur (Baroody et al, 2004; Simon et al, 2004).

In practice, ZPD is a constant negotiation around tasks between teacher and learner. Fuzziness and confusion is inherent in this negotiation process and need not be seen as a barrier to learning, but a means to it. There is active engagement by all parties as teachers try and explain their understandings through a range of representations (words, diagrams, actions) and learners do likewise, with both sides struggling to make sense of the other. Ambiguity is the norm. This deep level of activity and engagement in the communication process forces the teacher to continuously evaluate student actions and respond accordingly. Scaffolding cannot be reduced to a formulaic procedure because the exact communications surrounding a concept being learned can be sketched only roughly in advance rather than pre-drawn in detail (Bliss et al, 1996). For this evolving communication to be effective, there needs to be an ongoing teacher commitment to listening to what students are saying, not only when they are following expected lines of thought, but perhaps even more so when they are not. One of the most basic skills ascribed to a teacher operating from a constructivist framework is that of responding to the unexpected views of students in a manner that demonstrates that student views are taken seriously and are worth exploring (Fernandez, 2007).

Another key element is the teacher's use of questioning. In one study, over 60% of teacher questions were described as requiring no more than a pre-determined factual response from students (Myhill & Dunkin, 2002). Questions that probe for understanding, that require articulation of student thought, and challenge current perceptions and concepts are more likely to promote student learning (Fraivillig, Murphy & Fuson, 1999; Martino & Maher, 1999; Sahin & Kulm, 2008).

There are dangers with an over-emphasis on the social aspects of learning though. Teachers may view student and group interaction as a goal in its own right at the expense of any deep consideration of what is being learned (Kirshner, 2002). Students can come to view group participation as a new orthodoxy required by teachers and unrelated to their own learning. The 'right answer' is simply replaced by the 'right conversation' (Williams & Baxter, 1996). This is mostly likely to occur when social scaffolding is seen as paramount and analytical scaffolding is absent. Hence the teacher has to be constantly judging the nature and quality of student discussion against the mathematical agenda (Ball, 2003; McClain, 2002). Ball calls this a 'bifocal perspective', where the object of

learning - the mathematics - and the learners' responses to situations of learning are continually interacting. The concept of assessment being a dynamic process, a continual examination of how independent the learner is becoming, is integral to the application of ZPD in a classroom.

The knowledge gained from dynamic assessment creates targeted occasions to teach and defines the nature and scope of the interventions employed (Shepard, 2000). The nature of these interventions includes the setting of appropriate tasks, the real-time provision of questions to stimulate thinking, and the addressing of affective aspects of learning such as providing motivation and encouragement (Winn, 1994). The interventional aspects of scaffolding are gradually removed so that the learner becomes increasingly self-reliant and self-aware of their new empowered position (Oxford, 1997).

Jaworski (2004) adds the dimension of managing the learning (and the learners), to those of mathematical challenge and sensitivity to student responses, to remind us of the realities of classroom teachers. The affective influences upon student learning permeate all of the other classroom factors. Individual historical attitudes towards mathematics as a learning area or the student's personal view of themselves as a mathematician alter the ways in which potential learning opportunities are perceived and responded to (Op 'T Eynde, De Corte & Verschaffel, 2006).

One of the implications of constructivism is that not all students of the same age are able to learn the same concepts at the same time.

Simon et al, (2004) point out that it is the

“State of the learners' conceptions that determine what they can notice; mathematical relationships are not simply picked up (discovered) from universally accessible situations” (p. 325).

The early constructivists brought to our attention that there are both epistemological and cognitive progressions to consider in learning. Piaget set out a framework of the kinds of thinking that are needed to understand different types of mathematical activity. This was to prove powerfully influential in the organisation of school mathematics curricula. According to Adhami (2002), understanding decimal place-value is held to be at Piagetian level 2B- Mature Concrete Operations – while work with decimals as operators is held to be at 3A – Early Formal Operations. According to Shayer and Adhami (2007),

Piaget's work led to the now discredited idea that chronological age determined when students were ready to advance to new levels of learning and that all people reached Formal Operations by age 16. Both Piaget and Vygotsky's models predict that some material *cannot* be learned with understanding until the learner has the required specific prior knowledge.

Critics of constructivist-based approaches are concerned that minimal instruction can lead to students remaining unaware of the content that the teacher wants them to learn. They will then fail to engage with the underlying structures and thus have little long-term learning (Kirschner, Sweller, & Clark; 2006). In the pedagogical approach described above, the teacher's role is anything but passive. They must diagnose current understanding, design effective tasks, engage students in mathematical discussion and use a range of question types to support and extend thinking. Kirschner et al (2006) were able to refer to a large number of studies where minimal guidance produced minimal learning. Minimal guidance is not consistent with the constructivist approach as described by Ball (2003) and McClain (2002). It may be argued that these studies indicate that poor scaffolding results in poor learning, rather than the overall constructivist approach being inherently flawed. It is the nature and quality of the guidance that is the critical difference between different considerations of teacher activity; whether the 'direct instruction and student repetition method' will better produce long-term understanding than an approach that employs carefully considered scaffolding towards this end.

### **Representation**

Constructivism rejects a view of mathematical meaning that is seen as inherent in external representation and instead focuses upon activity – the internal work of making meaning (Cobb, Yackel & Wood, 1992). This does not mean that representation is unimportant or can be discarded, however. Ball (1992) reinforces the idea that understanding, constructing, and exploiting representational contexts for learning mathematics is crucial in the act of teaching.

According to Seeger (1998),

“In learning, representation without a constructive appeal is as empty as construction that does not represent anything” (p. 329).



In the course of engaging with school mathematics, the fact that students *have* to decipher signs and symbols is seen as the core problem by Steinbring (1998). More positively, when symbols are understood, we have evidence that knowledge has been generalized to the extent that the learner can communicate with the wider mathematical community. Yackel and Cobb (1996) write that mathematical learning is ‘both an active individual construction and a process of acculturation into the mathematical practices of wider society’ (p. 460). Mediation between tools and symbols helps enculturate the individual mind into the collective understanding (Neuman, 2001).

Representation is a concept with two aspects; there is the external sense where written encodings or physical objects are understood to communicate a mathematical idea, and there is the internal representation or cognition of that idea (Seeger, 1998). Rather than viewing symbolic understanding as a have/have not situation, it is better to think about the use of symbols and their meaning as co-developing (Cobb et al, 1992). Falle (2003) argues that conversation is instrumental in the learning of dense symbols; a monologue of teacher introduction will not be enough. It is the meaning that symbols take on in the mind of the reader that is of crucial importance (Godfrey & Thomas, 2003). Flores (2001) found that re-wording through many metaphors is necessary for developing understanding with rational operators. This need for oral language to accompany symbolic language is one of the implications of epistemological studies where the wide compass of meaning of a single symbol has been made explicit. This is exemplified in the work of Kieren (1980, 1992) regarding the fraction symbol. The New Zealand Numeracy Development Project (hereafter referred to as ‘NDP’) publication entitled *Book 6 Teaching Multiplication and Division (Revised Edition 2007)* alerts teachers that the ‘ $\times$ ’ symbol occurs in six distinct problem structures, even in the context of primary school mathematics (Ministry of Education, 2007e, p. 5).

### **Generalization: the Goal of Instruction**

When properly understood, the language of mathematics allows the learner to generalize. Generalization is seen as at the heart of mathematical knowledge (Mason, 2001). Mason et al (2007) go as far as saying that “a lesson without the opportunities for learners to generalize is not a mathematics lesson” (p. 42). It includes the ability to understand the

reasons why a piece of mathematics is applicable to a new situation and to talk about the mathematics as an object in its own right, and therefore capable of existing independently of the context from which it arose (Harel & Tall, 1991; Zazkis & Liljedahl, 2002). A learner possessing this ability has achieved what Yackel and Cobb (1996) describe as intellectual autonomy. This view of mathematical learning has often been positioned in contrast to that which only requires computational proficiency as the learning goal. Skemp (1976) used the terms ‘instrumental’ and ‘relational’ while Hiebert et al (1997) used ‘procedural’ and ‘conceptual’ to express similar distinctions. It is important to note that these authors do not regard mathematical accuracy and understanding as mutually exclusive outcomes. If the student is simply expected to reproduce a standard algorithm but does not understand the process, they may possess ‘instrumental’ or ‘procedural’ knowledge. If however, ‘relational’ or ‘conceptual’ learning occurs, the student will still demonstrate procedural accuracy, but, because they understand *why* the procedure is applicable in the initial setting, they can transfer this proficiency to new situations as well. In other words, it is the understanding of the process that allows for generalization to occur. Teaching for understanding is not antithetical to teaching for calculational competence; rather, it is seen as the most likely way of ensuring that this competence transcends initial classroom settings.

The dynamic view of knowledge challenges the way knowledge growth is ascertained in that it looks for the presence of generalised knowledge in the learner. The distinction that has often been made between the acquisition of knowledge and its application is regarded as inappropriate by some writers (e.g. Heinz et al, 2000; Hiebert et al, 1996). They would suggest that students must be able to do something to demonstrate the presence of knowledge. The validity of this demonstration is typically held to be in solving a problem as opposed to remembering a formula or completing an exercise. The key difference is the degree of agency expected of the learner. In a problem-solving context, the learner is expected to recognise and apply an appropriate piece of knowledge. Where a task simply asks for a skill to be demonstrated in a numerical exercise, the learner does not have to make a decision regarding which mathematics to apply, as it will inevitably be the same as in the preceding examples.

Tzur and Simon (2003) produced the terms ‘participatory’ and ‘anticipatory’ as modifiers of the word ‘knowledge’. In the former, the student can work within a familiar context (participates with the mathematical activity) but does not anticipate the use of this activity in a new setting. It is incorrect to say that students have not learned anything when they are only functioning within initial contexts, but their learning is localized. In Tzur (2007) reasons for students remaining at the participatory stage are examined. The teaching they require is that which helps them focus upon the coding or indexing that is applied to the initial task. This search for meaning becomes the primary instructional goal and not the completion of individual tasks (Confrey & Kazak, 2006). ‘Power’ is achieved by having the context implicit or under-represented so that the underlying mathematical structure can be re-applied. This is generalization, but what has been generalized needs to be further explored. When we use the term generalization with reference to students, it is important to remember that their perspective of what this means is different to that of the adult. With theoretical generalization, essential elements are identified and substituted for by prototypes. This kind of generalization is not accessible to students who are unversed in the formal use of algebraic symbols. We thus need to qualify what we mean by the term more explicitly.

### **Generalization: Distinguishing between Types**

‘Empirical generalization’ is based upon the examples that have been worked with by students where common features have been noticed and described (Zazkis, Liljedahl & Chernoff, 2008). A similar term, ‘factual generalization’ has been used by Radford (2003). While this kind of generalization may ultimately prove insufficient, it serves to lay an important foundation upon which more sophisticated reasoning may later be established. The noticing of the essential structural elements that promote deeper generalization may only become apparent when the discussion of counterexamples forces re-examination of the original empirical data (Harel & Tall, 1991; Radford, 2003; Zazkis et al, 2008; Zazkis & Chernoff, 2008). For example, students may initially reject an obtuse-angled triangle as being a true representation of a triangle if their previous experience has only included acute and right-angled triangles.

‘Expansive generalization’ is when a student extends their existing schema to include new material without engagement in the variation this new data provides. The new material is made to fit the existing understanding and contradictory evidence is suppressed or explained away as special cases (Zazkis et al, 2008).

A third type of generalization termed ‘disjunctive’ describes the situation where little attempt is made to link new material to prior understanding. The new learning is situated solely in the new mathematical context. A new set of procedures are given by the teacher and are shown to be effective for the examples presented. The teacher is required to have specialised content knowledge but pedagogical content knowledge is not regarded as essential. There is a lack of cognitive agency required from the learner. This may prove effective in terms of the immediate application to tasks, but may limit the ability of students to link mathematical ideas into a cohesive whole (Harel & Tall, 1991; Zazkis et al, 2008; Zazkis & Chernoff, 2008). Examples of this kind of teaching include the introduction (without explanation) of the common algorithms for the multiplication and division of fractions (Ebby, 2005). Hiebert (1999) found that students who had worked extensively with algorithmic procedures that were poorly understood found it extremely difficult to return to the basis for their thinking. Other researchers (e.g. Kamii & Dominick, 1997; Pesek & Kirshner, 2000) have also suggested that the teaching of algorithms (without attention to understanding) may later interfere with students’ abilities to generalize mathematical operations.

Another type of generalization has been termed ‘reconstructive’ (Harel & Tall, 1991). Learning is not always cumulative with new knowledge being ‘added’ to existing understanding as the new concept may be incompatible with existing schema. Here the pedagogical aim is for the student to respond to the new data by reconstructing their existing schema (Prediger, 2008). The student is asked to change their current thinking rather than extend it (unlike expansive), but the teacher expects that they will make personal sense of the new material (unlike disjunctive). Empirical generalization may lead to reconstructive generalization if the processes employed are highlighted and discussed. This is cognitively more demanding of students, as they are forced to consider which elements of their existing thinking are transferable and which need to be altered,

and to examine the reasons for doing so. This may result in powerful reflective practice, something regarded by many as the 'real' work of mathematics (Lampert, 1990; Schoenfeld, 1992). There is a different demand upon teachers if reconstructive generalization is the goal. The reconstructive process places more attention on the pedagogical content knowledge of teachers, particularly regarding task design and the skill of using these tasks to help students re-shape their thinking.

In variation theory, learning is seen as attending to those features in novel problems that are either similar to, or vary from, existing knowledge (Marton & Trigwell, 2000; Mason, 2005; Runesson, 1999, 2005). This focus upon the content being learned leads to learning being described as a change in perception in the way something is seen, experienced, or understood (Runesson, 2005). As students observe and describe these variations, they are able to recognize other situations where these new adapted systems would also apply. This endpoint does not appear dissimilar to reconstructive generalization.

In order to facilitate reconstruction, careful consideration of the dimensions of variation within a set of tasks is required. These dimensions include the contexts in which the tasks are set, the methods of recording, and the sizes of the numbers involved in them. If the variation is too restricted, then there is little chance of generalization being made as learners need to experience the same mathematics across a number of contexts. Failure to provide variation results in students being able to function within a defined context but remaining unable to anticipate wider application of their knowledge. If however, the learner's perception of the tasks is that they are too dissimilar, their focus will be on the variation, and the tendency will be to categorize rather than generalize (Runesson, 2005; Watson, 2003). This may explain instances where a 'real' piece of knowledge is kept in a separate category from classroom knowledge. For example, students may know from experience that a 1.5L drink bottle is bigger than a 1.25L bottle; but in classroom applications of decimals they may consider 1.25 to be larger than 1.5. These students do not consider linking their life experiences to classroom maths as they are regarded as existing in separate categories, whereas adults immediately recognise the common use of decimals in both situations. This example demonstrates that it is important to note that the

degree of similarity and variation is from the learner's and not the educator's perspective. The educator already possesses the generalization of the mathematics and so recognizes an underlying relationship that cannot be assumed to exist for the learner (Albert & McAdam, 2007).

### **Generalization: Models of the Process**

If it is accepted that activity and knowledge are closely inter-linked (as per situated learning models), it is unsurprising that many students do not transfer knowledge from one situation to another. Indeed, a model is needed to explain how it *can* be transferred (Sfard, 1998). One mechanism to explain this is that while knowledge is connected to the primary learning context (activity and environment); it is also coded by the learner. This coding process allows for the possibility that the code will be recognised as sufficiently similar in a new setting for the correct application (or modification) of the prior understanding. This coding and encoding process is not automatic though.

The mechanism of coding from the known in order to learn the new is at the heart of many tiered learning models. These models share a perspective that understanding gradually evolves from experience towards abstraction. They serve in contrast to didactic traditions whereby the generalization is presented first.

One of the pioneers of this type of model was Jerome Bruner. Bruner created the terms 'enactive', 'iconic' and 'symbolic' in the 1960s to describe three types of representation. 'Enactive' originally referred to situations where the student manipulated familiar physical objects. Later theorists have extended this to cover anything that the student is familiar with operating with and so may include numbers, diagrams or symbols. 'Iconic' refers to a deliberate re-presentation of the original manipulated objects, typically by way of a diagram. The manipulation is beginning to be focused upon rather than the concrete material. Bruner's third stage was 'symbolic', where the use of symbols made sense as communication tools because of the shared meaning they held between the learner and the wider mathematical community. In modern views, symbols may be re-employed in the enactive stage of more advanced learning, i.e. the now-understood symbols become

objects in their own right and can become the source of new investigations (Mason, 2005).

Other models have since been developed that share the view that students can best learn by initially working with a new concept in some concrete form before the concept could be considered in its own right and therefore potentially be generalized. Sfard (1991) held that abstract notions in mathematics can be conceived structurally as objects, and operationally as processes. It is the complementarity of these conceptions that she viewed as necessary for learning. Whereas the structural elements (what this *is*) had usually been presented to students as the starting point for learning, she argued that the operational conception (what it *does*) is for many people the more effective starting point. Sfard believed that learning was more likely to be effective if students began working with actual problems - often aided by equipment - that would model the mathematics of the problem. This would lead to internal coding of the patterns and processes used, which in turn would lead to the abstraction of the underlying mathematical principles. Her terms of 'interiorization', 'condensation', and 'reification' refer to these three stages of learning as the student moves towards generalization from specific local examples. The models of Herscovics (1989) using the terms 'logico-physical understanding', 'logico-physical abstraction', and 'formalization' also reflect this three-tiered approach.

Biddlecomb (2002) describes a similar process in that students working with 'perceptual units' (items available to the senses) can result in the creation of a 'figurative unit' item. (Figurative applies to a representation of the original operation). Working with figurative units does not require the perceptual unit to be literally present, but working with this unit re-enacts the previous perceptual experiences. An 'abstract unit' is constructed when the original context does not need to be remembered in order for a task to be carried out. Working with abstract units allows for patterns to be recognised and these in turn may become permanent objects, capable of being worked upon in their own right.

Work in the Netherlands centred upon the Freudenthal Institute produced an influential body of work under the heading of 'Realistic Mathematics Education' (RME). Its principles are summarized by Meyer (2001) as: the starting points for instruction are experientially real to students, it is accepted that learning takes place gradually over time and moves through different levels of abstraction, reflection (individually or in a group

situation), and student-generated diagrams, models, and other symbolic forms help in a progressive process of abstraction. It can be seen that this work is also linked to the ideas expressed by Bruner.

### **Generalization: More Complex Models**

These models have been further refined, in particular by the work of Pirie and her colleagues. Part of their contribution relates to the recognition of multiple layering within the broad outlines that echo the enactive, iconic, and symbolic phases. Equally important is the understanding that transitions between levels of understanding are recursive and non-linear. Schoenfeld (1989) expressed distrust of the unidirectional models as the evidence observed in classrooms had not shown ‘inexorable progress from naïve understandings to formal knowledge’ (p. 102). Recursive and non-linear models recognise and allow for empirical data showing that students are likely to re-collect their thoughts and repeat to themselves the conditions and arguments that lead to their initial construct being challenged. They are also likely to temporarily revert to a more primitive behaviour even when they have given evidence of operating at a more sophisticated level (Martin et al, 2006; Pirie & Kieren 1992, Pirie & Kieren, 1994; Pirie & Martin, 2000). Tzur (2007) also built upon these tiered models. A learner abstracts a new mathematical conception by what he termed ‘reflection on activity-effect relationships’. By this is meant that the learner has a goal in mind when asked to solve a problem and produces activity to realise this goal. In doing this activity, a different outcome may be reached, and learners may notice and may reflect upon this difference. In repeated iterations, the new cause-and-effect relationship may itself be noticed, reflected upon, and abstracted. This model does not require physical manipulatives, though it may involve them. Roth and Hwang (2006) assert that much learning is a double ascension of concrete and abstract movement. Repeated engagements occur as learners may start with an abstract concept before they work concretely which in turn adjusts their abstract interpretation which then causes them to view the concrete situation differently. This continuous zigzag of attention between concrete and abstract is believed to more accurately describe how understanding is developed than models that imply that abstraction is only produced upon the completion of other activity.



In the NDP, the use of ‘materials’, ‘imaging’, and ‘properties’ and the use of double-headed arrows in its models for teaching can be seen as an interpretation of the Pirie and Kieren model of learning (Hughes, 2002; Ministry of Education, 2007d, p. 5). Hughes (2002) uses a term from Pirie and Martin (2000) ‘folding back’. Hughes acknowledges that he has changed the use of the term in that in his model it describes an explicit behaviour of the *teacher* rather than a sub-conscious *student* activity. The teacher is directed to produce or shield physical materials. This is a major shift to that in the original context as the focus has moved from the individual’s internal construct to the teacher’s physical construction. Pirie and Martin’s (2000) ‘folding back’ concerned student reflection upon action and outcome, and the need for the learner to keep revisiting prior understanding in order to establish new understanding. This model is similar to the ideas of Tzur (1999, 2007). Martin (2008) provided further elaboration of the process of ‘folding back’. He pointed out that Pirie and Kieren did not regard folding back as regressive behaviour but a necessary aspect of new learning that involved the superimposition of new understandings upon earlier ones. Different types of folding back and different reasons for doing so were described in this paper.

With all of these models however, the most important pedagogical issue is not the differences between them but (with Bruner), understanding how the movement of students into new knowledge could be effected (Mason, 2005).

### **Generalization: Barriers**

The models of learning described above all recognize the potential for students to be operating with a procedure but not operating with the conceptual basis of that procedure (Hiebert et al, 1997). They may successfully operate within specific settings but fail to generalize (Simon & Tzur, 2003). In Bruner’s terms, they are enactive; in the NDP model, they are working with materials. Without familiar scaffolding they are inoperative. That their learning is incomplete may be masked from teachers due to the observed success of these students within the immediate setting. Teachers may over-generalize from this limited data and assume a greater depth of understanding than is warranted.

As Shayer and Adhami (2007) claim,

“The language of mathematics itself is so powerful that it lends itself to the production of procedures which can deliver a result even if the students using the procedure have little, if any, understanding of what they are doing” (p. 270).

Analysis and description of factors inhibiting learning have been documented by Henningsen and Stein (1997). The effect of these inhibitors is to block access to meaningful engagement with the mathematics potentially available to them through interaction with new tasks. They therefore remain outside of the learner’s ZPD. Barriers may be due to the cognitive difficulty of the concept, the student’s perception of the difficulty, or the student’s conception of the mathematics.

The best pedagogical practice cannot immediately overcome the case of the first learning barrier. The material is outside of the ZPD, and will remain inaccessible until other mathematical constructs are established by the students. One can envisage presenting calculus to pre-algebraic students for example.

In the second case, the material is inside the ZPD, but until its novelty is seen as ‘moderate’, students do not perceive the material this way. This inhibitor may be minimized or removed by teacher action, a piece of pedagogical engineering that helps students to alter their perception of the level of difficulty. Their perception may be shaped by altering the external appearance of the new concept. This may be accomplished by making explicit the linkages to their current mathematical knowledge and their social environment, and by concrete experiences such as the manipulation of equipment. The cognitive barrier is addressed by minimizing the difficulties associated with the learning. Small, progressive re-adjustments to existing schema provide students with challenge, but not distress.

Another learning inhibitor may arise from the students’ own current conceptions. If these are over-generalised, they form rules that new data must comply with or be rejected. This can be regarded as independent expansive generalization, and failure to address this is recognized as a key element in the lack of success in teaching students rational number.

As Sackur-Grisvard and Leonard (1985) write:

“We shall see that prior knowledge quite often is helpful in learning new concepts, but that it can also impede learning long after the teacher thinks that acquisition is over” (p. 158).

With decimal numbers, ‘longer is larger’ is an example of how an over-generalized student rule hinders correct ordering and ‘multiplication makes bigger’ serves as an example of how operations that involve decimal numbers may be affected. (*These examples will be expanded upon later in this chapter*).

This learning barrier cannot be removed by appeal to the authority of a perceived expert such as a teacher. The new concept is often not recognized as difficult; it is comprehended in the student’s view, but this perception is faulty. The teacher must expose students to the inherent contradictions in the application of their rules to new situations, in such a manner that the students become aware of the need to resolve them. When the students realize that too many conflicts arise from the use of a previous tool, a new accommodation is required. The challenge is an internal matter – a cognitive civil war – and it is likely that there will be more stress associated with the resolution of this type of conflict.

It is an over-simplification to suggest that removal of a learning inhibitor signifies that the learning issues are now resolved. New ideas and skills do not simply replace existing ones. If the cognitive demand of subsequent tasks remains low, superior skills may not always be employed by learners, as the efficiency of the new skill may not be fully recognised. For example, a student who knows how to use place-value partitioning may still ‘count-on’<sup>1</sup> if given single-digit addition tasks. There is a familiarity and lack of stress associated with the earlier skill. Intuitive beliefs are difficult to alter (Tirosh, Stavy & Cohen, 1998).

New constructs may appear in adult eyes to be mutually exclusive with previous ones; for example, either decimals are ordered on the ‘longer is larger’ belief or according to correct place-value understanding. For students, this is not so clearly defined, perhaps because the constructs themselves have not been sufficiently reflected upon. Models of learning that show recursion (e.g. Martin et al, 2006; Pirie & Kieren 1992, Pirie & Kieren, 1994; Pirie & Martin, 2000) and the simultaneous existence of competing constructs (e.g. Siegler, 2007) help us understand and respond to what would otherwise be inexplicable student behaviour. Diverse modes of thinking co-exist for prolonged

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<sup>1</sup> ‘Count-on’ refers to the practice of solving addition problems by counting in ones from one of the addends, e.g.  $28 + 4$  is solved by reciting 29, 30, 31, 32. Place-value partitioning allows for the problem to be solved as  $20 + 12$  or  $30 + 2$ .

periods, and it is a false expectation that stability of thought must inevitably follow new learning.

### **Decimals: Two Approaches**

Two general positions regarding the learning of rational numbers have been adopted; epistemological and psychological. The former approach seeks to clarify the nature of rational numbers as mathematical constructs and so reduce the first learning inhibitor (the inherent difficulties of the mathematics) by providing information concerning best learning sequences and the inter-linkage of those sequences. The latter focuses upon the existing schema children bring to the learning and so informs educators about the other two learning inhibitors (student perception of difficulty and student conception of the mathematics) (Moss & Case, 1999).

Both have the potential to inform the educational community and both have drawbacks. The epistemological approach may fail because meanings and interpretations of mathematical language may be quite different between teacher and student. There is no effective means of communication if the primary mechanism of intended dialogue is not common to all parties. In particular, rational number notation is not transparent for many students (Owens & Super, 1993). The psychological approach may stay focused upon misconceptions rather than conceptions and seek to change rather than build upon prior understandings. If teachers plan solely from mathematical objectives, students may be disengaged or given mathematically impoverished tasks. If planning is based upon tasks for their own interest sake, then unfocussed learning that is difficult to assess is the result (Simon, 1995).

Research has swung away from epistemological dominance to a situation where the psychological perspective has at least equal weighting, if not being the primary area of focus.

According to Steffe and Kieren (1994),

“First order models were of the sequencing of the mathematics, second order models are of the processes by which this occurs” (p. 722).

### **Decimals: Difficulties**

It is pedagogically naïve to think that it is easy for students to connect work with fractions into their whole-number schema. At least part of this difficulty is because fractions do not form part of the natural counting sequence (Charalambous & Pitta-

Pantazi, 2007). The emphasis on enumeration for early years in educational syllabi and the corresponding lack of emphasis on measurement is seen as a contributory factor to the continuing issues regarding low student achievement in rational number (Hunting & Sharpley, 1988, Sophian, 2008). When students write statements such as  $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$ ,  $1.25 > 1.5$  or  $1.4 + 0.25 = 1.29$ , they are drawing from existing whole-number procedures that have proven effective in the past. Their thinking can be viewed as expansive generalization (Harel & Tall, 1991). The new situation is made to conform to the existing schema.

Teachers are faced with decisions about how to proceed when this occurs. Their responses will reflect their own pedagogical content knowledge and their beliefs concerning the learning they are looking to facilitate (Ernest, 1996; Franke, Carpenter, Levi & Fennema, 2001; Leikin & Levav-Waynberg, 2007; Mewborn & Cross, 2007; Wilkins, 2008). Many of the possible alternatives may not even be recognized as existing due to the personal internal constraints of the teacher. For example, if the belief concerning the teaching of mathematics is fixed upon written work, this constrains the set of options the teacher will choose from. Practical activities and games will not be rejected, they will not even be considered. The environmental constraints, what Kemmis (2008) terms the 'practice architectures' may also limit the choices teachers make.

One possible response is to seek to minimize the disruption to the earlier student generalizations by providing procedures that diminish the differences these new situations have from those existing schema. When students are advised to 'add zeroes' to decimal numbers of unequal length in order to judge their relative size, or to 'line up the decimal points vertically before adding', this can be interpreted as teachers wanting to facilitate the application of expansive generalization. When students are told to 'count the decimal places before multiplying' this can be regarded as promoting disjunctive generalization. While cognitively easier for students to work with in the short term, these methods fail to produce new schema and are not likely to lead to mathematical generalization (Hiebert et al, 1997; Skemp, 1976) or intellectual autonomy (Yackel & Cobb, 1996).

## **Mechanisms of Learning**

A central tenet of Piaget's theory is that an individual, disequibrated by a perceived problem situation in a particular context, will attempt to re-equilibrate by assimilating the situation to existing schemas or, if necessary, reconstruct particular schemas to accommodate the situation (Zazkis & Campbell, 1996, p. 544).

### **Introduction**

Piaget's disequilibria model allows for learning to occur in two ways: where a student is forced to reconstruct their schema, and where a student is able to accommodate new material into an existing construction (Zazkis et al, 2008). The former relies upon the student perceiving a need to change their existing thinking. This awareness is termed cognitive conflict. The latter allows for connections to be established between prior knowledge and novel situations. Each of these mechanisms is discussed in more detail below.

### **Cognitive Conflict: Description**

According to some theorists, learning is impelled through disequilibria (e.g. Bruner, 1997). The equilibration model of Piaget's theory assumes that the mechanism of cognitive conflict is at the core of mental growth. Conflict is defined as being when two or more incompatible responses are aroused simultaneously (Druyan, 2001). It refers to a pedagogical setting and an individual learner's cognitive development; that is, what may be invoked as a contradiction or inconsistency in one learner may not be perceived as a conflict in another (Zazkis & Chernoff, 2008; Zazkis et al, 2008). Conflict may be a result of the individual recognizing that their interpretations of two pieces of data are contradictory (self-conflict), or that their perspective is not shared by their peers (group-conflict), or their view is not that of an authority such as the teacher or a mathematics text (expert-conflict). This disequilibrium promotes compensatory responses which may result in higher levels of understanding. In self conflict situations, the individual may be able to ignore the contradiction, but in interpersonal situations the student must not only satisfy themselves, but also seek to satisfy the other group members, which requires a clearly espoused justification. While the term 'conflict' evokes stress, it does not mean that disequilibrium within a group needs to be seen as a win-lose scenario.

Cognitive conflict has the potential to be powerful in that the learners make conceptual advances relatively autonomously. The individual, whether working independently or as part of a group, has to resolve the conflict rather than passively accepting an expert perspective (Simon, 1995). The novel information need not be literally new, but simply newly appreciated or attended to (Zazkis & Chernoff, 2008).

Cognitive uncertainty is perhaps a better term than cognitive conflict. The active word ‘conflict’ may be too narrowly interpreted to allow for the more passive – but no less real – situations of perplexity, confusion and doubt (Zaslavsky, 2005). All of the terms found in the literature; disequilibrium, dissonance, perturbation conflict, and uncertainty all carry with them stressful and negative overtones. It is unclear if users of these terms allow for the motivation for learning to be framed more positively. The anticipated pleasure of being able to know or do something new is surely also a factor in children’s learning. Williams (2003) found that deep engagement with challenging problems created a drive towards anticipated success with students.

The promotion of situations involving cognitive conflict allows for the possibility of two learning products; the piece of mathematical knowledge itself and also the process of verification of this new knowledge.

As Watson (2003) points out,

“The presentation of a possible case for conflict is much more than an attempt to enculturate the student into conventional understandings ... it is an induction into an internal validity of mathematics by offering the student an opportunity to rethink and restructure existing assumptions and understandings” (p. 31).

### **Cognitive Conflict: Teacher Activity**

Teachers can create environments whereby cognitive conflict may precipitate a restructuring of the students’ thinking (Lambdin & Walcott, 2007). A crucial part of the intervention process involves the writing of problems that students will solve (Irwin, 2001). Hodge, Visnovska, Zhao and Cobb (2007) provide two factors in denoting what makes an instructional task effective. It must hold the potential to develop mathematical interest as well as providing access to important mathematical ideas. If tasks can capture both the attention of the learners and challenge their thinking, then it is more likely that active engagement with the mathematical structure of the task will ensue. The teacher may need to make the conflict explicit by the use of probing questions or by highlighting

contradictory statements. This is particularly true when expansive generalization has occurred, as the learners may not realise that the conflict exists.

General diagnostic assessment gives an indication of likely problem areas. More fine-grained diagnostic assessment provides for the recognition of partially correct constructs (Ron, Herschkowitz & Dreyfus, 2008). This allows the teacher to target the specific area of departure from accepted mathematics whilst building upon the knowledge the students possess that relates to the problem at hand.

Unless children are presented with tasks that include misleading features, then their understanding is not probed beyond their ability to recall what has been presented (Moss & Case, 1999). The teacher may need to deliberately adopt a false position or introduce a contradictory example. The resulting conflict forces the students to reflect upon their previous experiences and to either confirm them (and thus move to a higher level of learning) or betray the weakness of their position. By asking ‘What is wrong?’ as well as ‘What is right?’ a new site for reflection is generated (Siegler, 2000). Errors provide an opportunity to re-conceptualize a problem, to explore the contradictions and to pursue alternative strategies (Kazemi & Stipek, 2001).

Mack (2001) notes that returning to initial understandings is crucial in the learning process in order for restructuring to occur. Reorganization is often facilitated by the deliberate re-examination of previous ideas which may need to be stimulated by the teacher.

Teachers may promote cognitive conflict by using combinations of three items suggested from the literature, the use of manipulatives, drawing upon students’ own knowledge and experiences, and provocative statements. All of these aim to ‘trouble the learner’s thinking’ and so begin the journey of reconstruction.

The use of manipulatives by teachers to create cognitive conflict stands in contrast to other uses of these materials. In many instances the aim of using manipulatives is to help students engage with a new concept by first allowing them to handle a concrete representation of the idea as a precursor to abstraction. The pedagogical aim is to provide a means for accommodation of the new concept to begin. In *this* setting however, the concrete representation serves to highlight the difference between the anticipated and



observed result. Physical materials can be used to create data that conflict with student conceptions (Moskal & Magone, 2000). This data may result from the use of concrete equipment or be produced from a calculator or computer (Tirosh & Graeber, 1990).

A person may be reminded of a piece of prior knowledge or action that conflicts with a current stated belief. For example, estimation is not only a useful mathematical skill; it is a necessary part of the mathematization process, as it provides a link between solutions and their reasonableness. Bonotto (2005) found that when asked to compare what students viewed as sensible estimates against their calculations, their perception of reality would outweigh their mathematical (mis)conceptions, forcing a re-examining of their (mis)calculations. Similarly, Tirosh and Graeber (1990) asked adults to express confidence in a recent calculation and then reminded them of previously expressed convictions that contradict this new evidence. This was often a successful intervention as the highlighting of inconsistencies lead many of the learners to re-evaluate their initial 'truth'.

Provocative statements occur when others make claims that are different to one's own thinking. Exposure to others' 'truth' creates disturbances in the learner's equilibrium. This challenges the status quo in the learner's mind and so allows the possibility of change (Cobb, 1994). The interactions of peers can provide the stimulus to re-examine existing conceptions.

In conversational interactions, people's positions vis-à-vis the discussion topic are not fixed but fluid. If this type of discussion is fostered in the learning situation, then learners may use the degree of agency afforded them to take up the 'teacher' role during these interactions (Barnes, 2003; Trognon, 1993). They may pose questions, challenge assumptions, or offer explanations to each other.

Irwin (2001) found that 11- and 12-year-old students who were given decimal tasks involving realistic problems made more progress than those on non-contextual tasks. Transcripts recorded that the first group drew upon different life experiences to supplement their reasoning about the problems. This additional stimulus translated into improved performance when identical tasks were given to both groups. This was ascribed to a greater amount of reciprocity in peer discussions, and the opportunities afforded in the contextual problems for prior student experience to shape thinking, and to critically

examine results. Schwartz, Neuman and Biezuner (2000) put students who had difficulties ordering decimals in pairs, and then had them work in decimal contexts. They found that pairing students with very different misconceptions was more effective than any other possible pairing. This seemed to be because both students in each of these pairs were forced to explain and defend their existing construct and actively looked for data to justify their position. In doing so, they often encountered a third position, the correct one. This attention to data for an argumentative purpose – in contrast to producing data to solve individual tasks – was regarded as a valuable means of effecting change. The provocative statements can also come from the learner themselves. Lamon (2007) holds that reasoning aloud is the most profound and compelling constructivist activity, as students who engage in this are forced to hear and re-consider their own constructs. This may also occur internally in what has been termed by Valsiner as the ‘dialogic self’ (Galligan, 2008).

### **Cognitive Conflict: Resistance**

However, there is evidence of the durability of intuitive schema, even in the face of contradictory data (Tirosh, 2000). In the study mentioned above (Tirosh & Graeber, 1990), around one-third of the adults simply did not recognise any contradiction unless it was explicitly pointed out to them. It is unclear whether this is a result of the individual unconsciously suppressing part of the data to reduce/eliminate cognitive stress, or whether the relationship between the belief and data was simply unnoticed.

If the importance of students’ prior knowledge is acknowledged, then there logically exist situations where this knowledge can operate to inhibit a valid solution being found by the learner (McNeil & Alibali, 2005). Contradictory data is simply dismissed. This is termed an epistemological obstacle (Harel & Tall, 1991), following earlier work by Brousseau. Epistemological obstacles are not solely rooted in the learner, but may be part of the mathematical content (Prediger, 2008). One can think of the reluctance of the mathematical community to accept negative numbers and imaginary numbers, both of which required considerable conceptual re-adjustment from initial views of number (Greer, 2004). The number of necessary changes in conception between natural and rational numbers embodies a range of epistemological obstacles (Prediger, 2008).

In a step-wise model of learning, students recognise the superiority of a new idea and discard an earlier one. However, in other models of learning, primitive constructs do not simply disappear from a learner's consciousness. New knowledge may help a new construct to be formed, but this does not imply that the earlier construct has been replaced. The new and old ideas can co-exist and may both be drawn upon to solve tasks. If a new way of thinking is perceived by the child to be much more effective than any previous way, it sometimes becomes dominant quite quickly, but this cannot be considered the norm for any particular new learning situation (Siegler, 2007).

The continued existence and influence of primitive schema has been described as an obstinacy factor (Harel & Sowder, 2005). The longevity of these intuitive primary models is now better understood. These have the capacity to influence the decision making process "from behind the scene" (Fishbein, 1985) and can be held tenaciously despite contradictory evidence being presented to the learner (Hunter & Anthony, 2003, McNeil & Alibali, 2005, Tirosh & Graeber, 1990; Tirosh et al, 1998).

McNeil and Alibali (2005) produced what they term a 'change resistance account', which argues that operations as well as knowledge may become entrenched into fixed patterns. When a novel situation is encountered, the operational patterns are activated by the recognition of key encodements. This is seen in the 'multiplication makes bigger' and 'division makes smaller' situations where the code of the  $\times$  and  $\div$  symbols (whether literally present or cognitively placed by the student) overrides any consideration of the sensibleness of the answer.

### **Cognitive Conflict: Counterexamples**

Counterexamples have the potential to create conflict and reorganization of mental schema, but do not of themselves guarantee this. A counterexample may be viewed as sufficient evidence by the teacher, but the teacher's highly developed proof schema allows them to recognise this evidence in a way that the naïve learner cannot. Thus a counterexample can be very easily defined mathematically, but this does not of itself guarantee its efficacy in producing cognitive conflict. Researchers have noted that situations involving cognitive conflict may be dismissed, marginalized as only holding true for that example (and therefore constitute an exception to the 'rule'), or accepted as

true without the accompanying inference that the previous view must therefore be incorrect (Zazkis et al, 2008).

From these observations, Zazkis and her colleagues introduced two terms, ‘pivotal example’ and ‘bridging example’. The first term seeks to describe the counterexample from a pedagogical perspective. If the counterexample creates a turning point in the learner’s perception, it becomes a pivotal example. Thus not all counterexamples are pivotal. If the pivotal example goes further than simply producing cognitive conflict but also goes some way towards its resolution, then it serves to bridge the learner’s initial conception with the more appropriate mathematical conception. It thus becomes a bridging example.

While the teacher defines the counterexample, it is important to note that it is the activity of the learner that defines whether the counterexample becomes pivotal or bridging.

Further work on defining the characteristics of what features of counterexamples are more likely to result in them being viewed as pivotal or bridging has been called for by Zazkis and Chernoff (2008). They point to the need to go beyond the creation of cognitive conflict as this of itself does not guarantee new learning.

According to Zazkis and Chernoff (2008),

“While there is some understanding of how a cognitive conflict can be exposed, once a potential conflict is recognised, there is little knowledge on how to help students in resolving the conflict” (p. 2).

### **Connections with Prior Understanding**

Manipulatives, diagrams and problem contexts can serve many purposes. They may be used to create cognitive conflict to guide a learner towards reconstruction as discussed in the preceding section. They may also be employed to initiate new learning by initially reducing the complexity of the mathematics to a series of achievable tasks which are then analyzed in order to extract the generalization. This latter use features in the tiered models of learning described earlier in this chapter.

### **Connections with Prior Understanding: Manipulatives**

Manipulative materials are designed to represent abstract mathematical ideas explicitly and concretely. The teacher may provide physical stimuli and tasks that can be

accomplished in advance of a student's current level of competence with calculation. This can scaffold the learner to a new level of abstraction (Moyer, 2001). If the term 'metaphor' is taken to mean that the language of one domain is used to communicate thoughts about another, then discussion around the manipulation of concrete referents can be seen as providing physical and linguistic metaphors for the targeted mathematical concept (Bills, 2003).

The use of manipulatives has been shown to improve student learning (Ball, 1992; Cramer & Henry, 2002; Moyer, 2001). The linking of new symbols and systems to concrete referents is seen as an important first step in acquiring the more generalised usage of the new symbols and systems (e.g. Goldin & Shteingold, 2001; Goldin & Janvier, 1998; Jones, Thornton, Putt, Hill, Mogill, Rich & van Zoest, 1996; Lachance & Confrey, 2001; Tzur & Simon, 2003). This is predicted in the tiered models of learning. The *Mathematics in the New Zealand Curriculum* document (MiNZC) (Ministry of Education, 1992) advocates the use of manipulatives and the NDP regards their use as integral to classroom teaching (Higgins, 2005; Ministry of Education, 2007d). The use of manipulatives in countries such as the USA, UK, Australia, and New Zealand is prevalent. This has been due in part to mathematical reform initiatives, whether on a national scale (as the UK and NZ) or more localized (as the USA and Australia).

Equipment may provide children with a sense of the magnitude of numbers and a repertoire of symbolic images to draw upon (Houssart, 2004). Sophian (2008) points out that quantities are properties of the physical world around students whereas numbers are used as symbols to represent these properties. Thus children learning to count require objects to count before abstraction of the process occurs. As rational numbers are not able to be abstracted from simple extensions of whole-number schema without extensive conceptual reconstruction, it is reasonable to expect that physical objects will also have a part to play in this process. Cramer, Post and del Mas (2002) found that students who were taught using multiple physical models and who made translations between different modes of representation made significantly more gains in fractional understanding than children who were given traditional fractions instruction. Steencken and Maher (2003) also found that concrete materials aided student learning of rational number.

Equipment that is used to undertake an actual practical task involving measurement is seen to be especially effective in helping students find connections between whole number and fractions.

As Lamon (2001) points out:

“The need for increasingly accurate measurement has been a driving mechanism in the history of mathematics and science, and it appears to be a strong binding force in children’s mathematical development” (p. 163).

Irwin (2001) found that students could make valid connections and establish sound arguments by having soft drink bottles as referents, while Hunter and Anthony (2003) used volumes of water. Both used these everyday measures when addressing relational knowledge of decimals. The use of the pipe numbers equipment is given as an example of relating the decimal system to a concrete referent (Ministry of Education, 2007f, pp.22-24). Moss and Case (1999) also used volumes of water as static representations of rational numbers. Helme and Stacey (2000) have reported on how a single piece of equipment which was specially designed to model decimal place-value can scaffold many students into new understandings, even with minimal teacher intervention.

### **Connections with Prior Understanding: Diagrams**

The use of diagrams can also help scaffold students into new learning. They capture the mathematical features of concrete experiences and allow learners to depict knowledge before a fully symbolic representation is possible. In this manner they may serve as an intermediate linkage between the concrete and abstract representations of knowledge. They can support visual reasoning, allowing students to form arguments for which they have yet to acquire a complete verbal or symbolic vocabulary. They also may facilitate the conceptualization of structure, allowing students to explore and/or explain the crucial elements of a task and their inter-relationships (Diezmann, 2000, 2005).

The number line is a diagram whereby positions on an axis encode quantitative information (Diezmann & Lowrie, 2006). The use of the number line is of special interest for work with decimal numbers and fractions. A number line helps students to consider numbers as measures of quantity to go alongside their conception of numbers as representing enumeration. This construct is necessary as students make sense of numbers (such as fractions) that are outside of the usual counting schemes (Sophian, 2008). Part of

this interest is also linked to the common practice of using number lines as an assessment tool (Ni, 2000). ‘Unstructured’ or ‘empty’ number lines allow for greater student interaction than the traditional number line where structure is provided for the student. On an empty number line, the student has to use the diagram to model the mathematics they want to describe and so have to choose an appropriate scale and suitable end points (Diezmann & Lowrie, 2006). It is this interactivity that may foster the construction of meaning by students as well as enable them to communicate knowledge. Saxe, Shaughnessy, Shannon, Langer-Osuna, Chinn and Gearhart (2007) found that the use of number lines allowed students to make significant gains in their understanding of the concepts of rational number density and the equivalence of rational numbers written in different forms.

### **Connections with Prior Understanding: Context**

Appropriate tasks have at least three features that encourage active student participation. Firstly, the task is problematic, a term that covers the mixture of challenge and interest that makes students want to participate. Secondly, the task allows connections to be made with prior learning and experience, which allows students to begin working with the task from what they already know. Thirdly, the task must embody important mathematics so that discussion and reflection arise out of the task and may lead to the abstraction of mathematical ideas (Hiebert et al, 1996, Hiebert et al, 1997). Helme, Clarke and Kessel (1996) found that the incorporation of out-of-class experiences were a significant focusing mechanism for student learning. Meaning (in the disciplinary sense) was more likely to occur if meaning (in the experiential sense) was attended to by the lesson design and delivery. Contextually-based introductions to rational numbers have proven to be more successful than traditional instruction (Moss & Case, 1999).

Examples of contextually-rich activities involving decimals include cutting ribbon lengths (Bulgar, 2003), examining receipts for goods sold by weight (Bonotto, 2005), the thickness of paper (Brousseau, Brousseau & Warfield, 2004), and pouring water to gain static measures (Hunter & Anthony, 2003). Brousseau, Brousseau, and Warfield (2007) reported how games could serve as promoters of new mathematical thinking relating to

decimal numbers. The intrinsic motivation of students to win the different games provided the impetus to ask new questions and explore new realms of mathematics.

### **Connections with Prior Understanding: Some Cautions**

The term ‘artifact’ has been employed to describe a particular use of concrete or symbolic representations. The emphasis is on the agency of construction to create or communicate meaning - this meaning is not simply appropriated (Seeger, 1998). From the epistemological position, how well the artifact represents the mathematics - its epistemic fidelity – is critical. This can be determined by the educator alone. However, from the psychological perspective, how well the artifact is interpreted by the learner – its transparency – is equally important (Meira, 1998; Stacey, et al, 2001). Transparency may be different between individuals and between groups of students and can be observed in action, but not determined in advance, by the educator. These researchers agree that as students participate in working with artifacts, meaning is created; meaning cannot be simply transferred by an outside agency or simply ascribed to the artifact by the educator. If students do not interpret the artifact in the same way as the teacher, then communication about its inferred meaning is unclear. All representations, no matter how ‘concrete’ require interpretation (Schoenfeld, 1989). The intended structures are not always those that are perceived by learners (Schliemann, 2002). For example the use of place-value blocks (also called Dienes blocks, base-10 blocks, and MAB) to model decimal numbers has been shown to be problematic for some students. They are confused both by the changes in dimensionality and by the previous use of the blocks to represent whole numbers (Stacey et al, 2001). Ni (2000) found that analogues such as the number line were not always perceived by students as being any less abstract than the symbols they were trying to illustrate.

The use of manipulatives does not guarantee mathematical understanding. All of the tiered models of learning that have been discussed earlier emphasize that the initial ‘working with materials’ phase must be regarded as part of a journey towards abstraction. Though the work of Pirie and Kieren (1994) is often referenced as providing justification for the use of manipulatives, they themselves warned of several dangers. Materials may



be looked upon as providing meaning of themselves, teachers may neglect to attend to their role in helping students make mental constructions (not simply physical ones), all students may be forced down the same pathway to learning, and that symbols and diagrams may be under-utilized (Pirie & Kieren, 1992). More recent statements have echoed those same concerns.

For example, Lamon (2001) writes,

“It is easy for the teacher to attribute magical powers to the representations, thinking that the activity, the manipulative, the picture, or the words that have meaning for them will surely persuade students to adopt the teachers’ adult perspective. As any good teacher knows it just doesn’t work that way!” (p. 156).

Moyer (2001) found that many teachers in the UK regarded physical materials as promoting enjoyment of maths but were not important carriers of meaning. Unless there is reflection upon what the manipulative is representing mathematically, the students may not participate in a challenging learning experience. Kamii, Kirkland and Lewis (2001) found it necessary to remind us that the mathematics we are wanting students to learn does not exist in manipulatives; though they may be useful tools towards that end.

Boaler (1993) discusses how the transferability of the mathematics employed in a realistic task to a wider application is a complex issue. Providing engaging starting points (such as the use of concrete materials) does not guarantee that cognitive reconstruction will inevitably follow.

Another problem is that the materials require decoding as well as the mathematics – a cognitive demand additional to the original task. Attention to the decoding of representations may become a formidable, even an insurmountable, educational barrier. This is especially true where changes in representational format are made before any single representation is properly understood. A wide range of representational and situational contexts is conducive to generalization, in that variation from the initial learning situation can be examined for patterns of continuity, and these patterns form the basis of abstractions being made. If, however, the first encounter with new material is not understood by the learner, then subsequent variation may simply overwhelm them with data.

As Seeger (1998) comments;

“Representational overkill exerts a devastating influence, especially on those students whom it is meant to help” (p. 309).

Students may participate in an activity (whether as a thought-experiment or an enactment) and not transfer the results. For example, Graeber and Tirosh (1990) noted that some children indicated that while  $5 \div 15$  was possible when using 5m of rope, it was impossible to do mathematically with numbers! This would seem to indicate that the number 5 was only viewed as representing a discrete set.

The best that can be said regarding the use of any of the mechanisms described above is that they afford the opportunity for connections to be made between student experience and mathematical generalization. While the connection may be transparent to the teacher, it is the perspective of the learner that will determine its efficacy. If the teacher is actively engaged in helping students attend to the meaning of tasks by the use of probing and reflective questions, then the learning environment is regarded as more favourable to mathematical growth, but there is no guarantee of this (Hiebert et al, 1997).

## **Decimal Numbers**

The traditional approach to fractions is defective in all distinguishable sections of the course of instruction, in the internal connections between the various sections and in the embedding of the course in the greater curriculum for arithmetic and mathematics (Streefland, 1991, p. 7).

### **Problematic**

A superficial inspection of the mathematical content of decimals would fail to recognise the deep-seated difficulties students have in learning them. In studying decimals there is only one new symbolic form – the decimal point - and the initial operations used are the familiar four basic ones. There is not an overload of new semiotic inscriptions or great operational novelty, nor are there a wide variety of representational forms that must be mastered. Streefland's quote above is a response to the world-wide data showing that the learning of decimals is highly problematic for students. The problems are not simply the mis-application of procedures but are more fundamental.

In mathematics education, the domain of rational number has been extensively researched. Part of the reason for this is that the difficulties students have with rational number are so widespread that they have been termed a “weeping sore” (Ellerton & Clements, 1994).

Evidence of the difficulties students have with decimals is provided in many studies. Data from New Zealand schools is available from NDP results and the recent NEMP data. NDP tasks assessed students by asking them to coordinate the language of tenths with a decimal number (*How many tenths are in all of this number? 4.67*)<sup>2</sup>, and to order three decimal numbers (*0.39, 0.478, 0.8*) (Ministry of Education, 2007c, p. 37). The initial Year 6 data showed only 4% proficiency, rising to 13% after one year of NDP teaching intervention. Data for Year 7 students relating to addition and multiplication with decimals had initial figures of 5% rising to 16% and 2% rising to 7% respectively (Young-Loveridge, 2007, pp. 155-157). In the NEMP results, only about 50% of Year 8 students in New Zealand could coordinate tenths and the first decimal place (*Write  $4\frac{2}{10}$  as a decimal*) or identify the place-value of hundredths (*0.07 is the same as 7 ones, 7 tens, 7 tenths or 7 hundredths*) (Flockton, Crooks, Smith, & Smith, 2006, pp. 21, 28).

Overseas studies have produced similar results. Most middle school graduates in the USA assert that “large” numbers such as 0.1814 are bigger than “small” numbers such as 0.3 (Moss & Case, 1999, p. 122). This finding is consistent with Australian research (Steinle & Stacey, 1998, 2002).

Even where some understanding of decimals is detected, this is often not generalized. In a study of US Grade 6 students, students were asked to illustrate 0.6 and 0.06 using four given representations: fractions, money, a 10×10 grid, and a number line. While 94% of students had sufficient understanding to get at least one representation correct, only 14% could transfer their knowledge across all settings (Martinie & Bay-Williams, 2003).

An appreciation of the density of number is evidence of the generalization of the place-value structure to include decimals. Typically students are given two boundary numbers and asked to generate a third number that exists between these two amounts. A study involving 12- and 13-year-olds found that only around 40% could produce answers for boundaries of 3.9 and 4 and that this result halved when examples involving a second decimal place were considered (Greer, 1987). A similar study of 15-year-olds showed that many students extended their notion of density to include specific case examples but

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<sup>2</sup> ‘46’ was the expected acceptable answer with the greater accuracy of ‘46.7’ not being required.

had not developed generalized schemas of number density (Vamvakoussi & Vosniadou, 2004).

When decimals are involved in operations a lack of understanding continues, even with older students. Irwin and Britt (2004) found that Year 8 (12-year-old) students in New Zealand demonstrated little ability in operations involving simple decimals; with around 15% being able to calculate  $4 \times 7.8$ . International studies tell a similar story. Bana and Dolma (2004) found that most Year 7 (12-year-old) students in an Australian study were not able to correctly calculate multiplications when one factor was a single-digit decimal (e.g.  $4 \times 0.6$ ). Of those who were correct, only around one-half were able to judge the reasonableness of their answers. Yang (2005) constructed examples where the computational procedure of counting decimal places would fail. By giving the Grade 6 students the problem  $534.6 \times 0.545 = 291357$  and asking them to write the decimal point in the product, numerical computational error was eliminated as a confounding variable. Only 14% of students could correctly write the answer despite the multiplicand of 0.545 being relatively close to  $\frac{1}{2}$ , and so the product was intended to be easily estimated. That 81% gave 29.1357 was interpreted as showing that the majority of students were blindly following the procedure of counting the decimal places with no regard for the reasonableness of their answer. Evidence of students operating with procedures that are not understood is also given by Burns (1990). He reported on how 13 and 17-year-olds were given a multiple-choice question ( $3.04 \times 5.3$ ) with answers differing by orders of ten. The correct response rates were 21% and 39% respectively. He concluded that the students were not attempting to use any basic fact knowledge ( $3 \times 5$ ) but were trying to apply a system of counting decimal places. That the majority of older students were still making elementary errors is especially disturbing. These students were not reorganizing their thinking as they were exposed to more mathematics each year in conventional classrooms, but were perpetuating the mistakes of much younger students.

It might be expected that studies of students operating with simple decimal division would also show problems. Irwin and Britt (2004) found only 21% of Year 8 students could solve  $31 \div 0.5$ . In another study of Australian Year 7 students, only 30% could correctly solve  $3 \div 5$  (Clarke, Roche & Mitchell, 2007). In a similar study in the USA of

10-11, 12-13 and 14-15 year-olds, the percentage of correct answers for the task  $5 \div 15$  were 20, 24 and 41 respectively (Graeber & Tirosh, 1990).

Studies of pre-service teachers have also shown that operations involving rational numbers are often misunderstood (Stacey, Helme, Steinle, Baturo, Irwin & Bana, 2001; Tirosh & Graeber, 1989; Tirosh, Fishbein, Graeber & Wilson, 2001). A recent New Zealand study has found similar results with practicing teachers where, 82% were unable to provide the key understanding required to solve the problem  $1\frac{1}{2} \div \frac{1}{2}$  (Ward & Thomas, 2007).

### **Examination of Causes: Referent Systems**

The origin of decimal fraction usage is generally credited to Stevin Belgium in 1584/5 and the use of the decimal point to Napier in 1616 (Forno, 1929).

Decimal fractions have a decimal number as the denominator, whether expressly written or not, thus  $\frac{5}{100}$  and 0.05 are both decimals. That decimals draw from both common fractions and place-value has important pedagogical implications if teachers are to draw upon students' prior understanding. Fractions and place-value form referent systems upon which students may draw when making sense of decimals. However, students are usually not conscious of the fact that they have these two systems within their existing number network and so may only utilize the knowledge from one of them (Owens & Super, 1993). The extension of the place-value system to include decimal numbers is more likely to make sense to children once they have an understanding of whole-number place-value and a rudimentary grasp of fractions. Students that link their knowledge of tenths with the place-value convention of using columns, can recognise that using the decimal point does not produce a new type of number, but is a new written form of a familiar number (Baroody et al, 2004).  $2\frac{3}{10}$  is *re-written* as 2.3 thereby circumventing the common student misconception that new numbers are *formed* by writing decimals. Students who hold this misconception typically translate whole-number place-value columns after the decimal point so that 3.45 is perceived as three, then forty-five.

In constructivist thought, information can only become conceptual knowledge when it is integrated into a learner's existing cognitive network. Students who make incorrect expansionist generalizations and/or adopt disjunctive generalizations are unlikely to

demonstrate proficiency in problem-solving tasks. As an example of the former, the ordering of decimal numbers based solely upon the length of the number, (1.12 being perceived as larger than 1.9), indicates that the student is applying a rule that has previously enabled them to quickly assign relative magnitude with whole numbers. With the latter, the action of counting decimal places to provide answers,  $184 \times 0.5 = 9.2$  (as the answer ‘must’ have one decimal place) shows that students are adhering to a procedure and ignoring any form of sensible estimate.

Studies of decimal understanding of students in the 1980s established two important truths; firstly that many of the errors students make with decimals can be attributed to misapplication of one of these two main referent systems, and secondly, that individual use of these referent systems is relatively stable (Hiebert & Wearne, 1985, Nesher & Peled, 1986; Resnick, Nesher, Leonard, Magone, Omanson & Peled 1989, Sackur-Grisvard & Leonard, 1985).

According to Nesher & Peled (1986) there is,

“Almost complete correspondence between the pattern of each child’s answers and the answers expected of him (sic), when one finds he is using a certain rule” (p. 74).

In Sackur-Grisvard & Leonard’s (1985) study, 89% of errors were assignable to an underlying stable construct and were not simply random mistakes. More recent studies have confirmed these findings with decimal numbers (Schwartz et al. 2000; Steinle & Stacey, 1998, 2002), and with other fractions (Stafylidou & Vosniadou, 2004).

The most common referent system employed by students is that of whole-number place-value. A widespread implicit interpretation of the decimal point is that it separates two whole-numbers (Steinbring, 1998). Thus, when examining decimals such as 0.12 and 0.6, the former is internalized as ‘twelve’ and the latter as ‘six’. This logic results in statements such as  $0.12 > 0.6$  and  $0.12 + 0.6 = 0.18$  (Ball, 1992; Irwin & Britt, 2004; Saxe, Edd, Taylor & Gearhart, 2005; Steinle & Stacey, 1998; Stacey & Steinle, 1999). Lamon (2001) notes that students fail to distinguish ‘how *many* from how *much*’. Part of this confusion has been ascribed by Sophian (2008) to children’s introduction to numbers being heavily weighted towards enumeration, at the expense of developing an appreciation of quantity.

This transfer of whole-number thinking is so prevalent that a term ‘N-distracter’ (Streefland, 1991) has been used as a shorthand way of describing situations whereby student responses can be explained by this expansive generalization of whole-number understanding. Whole-number thinking is an example of a conceptual barrier to new learning. Usually, this whole-number influence is described in negative terms. Lukhele, Murray and Olivier (1999) found “limiting constructions originating from whole-number schemes that completely blocked out ... the meaning of fractions” (p. 1). In response to observed student problems, Hunting and Sharpley (1988) suggest that fractions be taught to students before the whole-number knowledge becomes the predominant schema.

Incomplete understanding of the fractional referent scheme also explains some student responses to decimal tasks. There is a barrier for students when asked to make non-unit fractions greater than one as this violates the ‘out-of’ schema that many students have constructed (Lamon, 2002; Tzur, 1999). This hinders the ability to recognise equivalents such as 1.2 and 12/10. An over-generalized understanding of denominators produces statements like  $0.2 > 0.91$  as the ‘2’ and ‘91’ are seen as denominators. Thus the longer a decimal is, the smaller its quantity is thought to be (Hiebert & Wearne, 1985, Neshet & Peled, 1986; Resnick et al, 1989; Sackur-Grisvard & Leonard, 1985; Stacey & Steinle, 1999).

Boufi & Skaftourou (2002) note that the misconceptions described above, based upon either whole-number or fractions systems, have been accounted for in psychological terms and not in relation to students’ instructional experiences. They caution that if educators respond directly to these misconceptions, they will still employ a top-down structuralist teaching model. As this does not value and encourage student contribution, it may fail to produce long-term understanding and will simply be a new set of algorithmic procedures - a disjunctive position.

### **Examination of Causes: Associations and Symbols**

Research has identified two additional key reasons for the difficulties that students have with learning rational numbers: a lack of association and a lack of symbolic understanding.

One cause for lack of association is inherent in the mathematics itself, an epistemological issue. Much of the learning that has taken place with whole number does not transfer to rational number. There are three key points of difference between whole number and early rational number mathematics. There is a change from having a unique symbol for each whole number to multiple representations in a rational number setting. Whole numbers have a transparent system of succession with each number being preceded and followed by a unique successor 'one before' and 'one after'. This does not occur with rational numbers. As a consequence of this last point, the principle of density stands in contrast to the presupposition of discreteness (Stafylidou & Vosniadou, 2004). Note that these examples are all situated in the static existence of rational numbers. Operations involving them contravene another set of observed patterns that hold true for whole numbers such as 'multiplication makes bigger' and 'division makes smaller'. Recognizing that the mathematics itself produces many of the problems helps us avoid labeling students negatively, as though they alone are responsible for their lack of success.

Many students do not make associations between the concrete materials they use in class, their life experiences, and the symbols used to represent the fractions that are involved (Ball, 1992; Mack, 2001; Owens & Super, 1993; Saxe et al, 2005; Streefland, 1991). For example, these students are able to confidently describe dividing a pizza into quarters or eighths but are still regard  $1/8 > 1/4$  when presented with this situation symbolically. Tzur and Simon (2003) and Tzur (2007) describe the success seen by teachers observing students working with materials, and the subsequent failure seen in written tasks, as different stages of a learning continuum. Students may be able to cooperatively participate in activities but are unable to anticipate independently the use of knowledge required in a new situation.

There are also problems with the decoding of the symbols; for example, creating a new meaning for a '2' and a '3' when they are used to write '2/3', and recognizing that 1.23



needs to be interpreted with a new place-value concept than that used for 23 (Ball, 1992; Moss & Case, 1999).

The use of zero is especially problematic (Resnick et al, 1989, Sackur-Grisvard & Leonard, 1985; Schwartz et al, 2000). Often its presence is simply ignored, so 0.06 and 0.0006 are regarded as equivalent (Steinle & Stacey, 2001). Zero may also be regarded as a whole number and therefore bigger than a decimal, and so the inequality  $0 > 0.4$  is believed to be correct by some students (Stacey, Helme & Steinle, 2001b).

The lack of connection between context and symbolic form is likely to remain problematic for students unless teacher actions simultaneously address the referent systems students are employing as they grapple for meaning.

### **Pedagogical Responses**

In response to the phenomenon of whole-number thinking, Biddlecomb (2002) points out that ignoring or dismissing an underlying construct is neither viable nor desirable. If one holds the view that new knowledge *must* be constructed using existing schemes and knowledge, these must be employed in the construction of new knowledge. He makes a case for a pedagogical approach that helps students to reorganize their existing schema. The concept of reorganizing rather than replacing existing schema is also the focus of other researchers (e.g. Prediger, 2008; Steffe, 2001; Tzur, 1999). Activities that are initially accessible via the use of the existing construct but reach a point of discontinuity with that construct are seen as openings for students to construct new knowledge (Cobb & Yackel, 1996; Cobb et al, 2003; Tzur & Simon, 2003; Tzur, 1999). This can be viewed as operating within the ZPD to seek reconstructive generalization.

Tzur (1999) wrote that the essential role of the teacher is “generating tasks that are likely to promote transformation of these conceptions in the learners” (p. 391). Knowing that students draw upon their existing place-value construct for whole number need not be a barrier to the learning of decimal magnitude. If discussion begins around *why* a longer whole number is larger than a shorter one (e.g.  $4\ 000 > 800$ ), then students can make explicit self-reference to an important piece of their existing knowledge, viz. it is the ‘place-value’ that is the primary indicator of magnitude and not the ‘face-value’ of the digits involved. If this can be experientially tied to a physical model of tenths and

hundredths, then the fact that the same place-value rule applies it is more easily discerned, albeit with a slight difference in the appearance of the numbers.

As Irwin (2001) noted, students need to realise that decimals are always the result of *division* by ten in contrast to the place-value encountered with whole numbers where *collections* of ten are made. Situations that have students enact this divisive process may help them recognize the mathematical structure. We can find parallels with the initial teaching of place-value where researchers comment on the need for students to carry out grouping activities to internalize the collection of tens structure (Heirsfield & Cooper, 2004; Higgins, 2001; Young-Loveridge, 1998). Place-value understanding involves more than use of the canonical form. For example, the flexibility to see that 34 is 3 tens and 4 ones, and that 2 tens and 14 ones is required to use the standard subtraction algorithm. This re-presentation of a number into chosen powers of ten requires understanding of the multiplicative structure of our number system (Heirsfield & Cooper, 2004, Jones et al, 1996). Discussion around this grouping and ungrouping process can be extended into situations where decimal quantities are involved, especially if these quantities are represented in concrete form.

Teachers who are aware of which referent system the student is over-relying upon are able to draw attention to the other system to challenge faulty inferences. For example, the ‘longer is larger’ rule used by students to determine decimal magnitude draws from the whole-number place-value referent, especially when coupled with an interpretation that the decimal point separates two whole numbers. If a contrast is made with the fractional referent system where ‘longer is smaller’, then contradictory evidence is produced that the student may now try and resolve (Moskal & Magone, 2000).

Some traditional tools for learning decimal notation have unintentionally reinforced student misconceptions. This is particularly true with money; the place-value of the ‘2’ and ‘3’ in \$1.23 are perceived as tens and ones and not tenths and hundredths by students (Owens & Super, 1993; Stacey & Steinle, 1999). The selection of appropriate computational tasks such as requiring students to carry out ‘ragged decimal’ addition problems (decimal numbers of unequal length) may create a situation of cognitive conflict that students may recognize needs resolving (Wearne, 1990). Teachers must

avoid advocating the adoption of procedures such as ‘lining up the decimal points’. This masks the underlying conceptual issues for the short-term gain of task completion.

Language can help make the connections more explicit. While the reading of 0.52 as ‘zero point fifty-two’ can reinforce whole-number misconceptions, pronunciation as ‘zero point five two’ does little to promote understanding either. Verbalizing ‘no ones, five tenths and two hundredths’ – at least in the introductory phase – helps clarify the link between the two representations of the number and its two referent systems (O’Connor, 2001).

### **Operations involving Decimals: Multiplication**

With regard to fractions, children have to reorganize their basic numerical schemes by the equi-partitioning of the unit. These partitioned sections can then be iterated to produce multiples of the sub-unit (Bulgar, 2003; Steffe, 2001; Tzur, 1999). Thus multiplicative thinking is essential for conceptual understanding of fractions (Empson, Junk, Dominguez & Turner, 2005; Lamon, 07; Saxe et al, 2005; Thompson & Saldhana, 2003; Young-Loveridge, Taylor, Hawera & Sharma, 2007). This is true whether we are iterating sections of  $1/3$  or  $1/10$ , and so is connected with decimal understanding. Multiplicative thinking is required for initially understanding decimal numbers as static entities and not simply used when the student is asked to operate multiplicatively with them. Attention to the multiplicative equi-partitioning and iterative aspects of engaging with decimals is thus an important pedagogical response, and involves several layers of development of student understanding (Hackenberg, 2007).

In addition to the understanding of decimals as numbers, primary operational models influence how students extend multiplication with whole number to include multiplication by decimals. There are two epistemological obstacles that may arise. The first relates to the multiplication model employed. Some data supports the contention that the construction *decimal*  $\times$  *whole number* is more difficult for students than *whole number*  $\times$  *decimal* (Graeber & Tirosh, 1990). The initial multiplicative model of repeated addition involves replicative thinking, the collection of identical sets (Kieren, 1992). This is more easily extended to a problem such as  $6 \times 0.8$  than to  $0.8 \times 6$ . The

former example can be considered as gathering six groups of size 0.8. This is conceivable with a repeated addition model provided that the student has a model of what 0.8 represents.  $0.8 \times 6$  does not allow for a direct extension of the repeated addition model. Students who can access this model only, have to answer the internal question “How can you have 0.8 of a set?” For them the application of the commutative law is not obvious (Schliemann, 2002) and teacher insistence upon its use creates disjunctive, procedural thinking. The student must understand either *why* the commutative law is applicable (so that while  $0.8 \times 6$  is perceived as ‘unsolvable’ directly  $6 \times 0.8$  can be used), use the distributive law (the task may be re-written as  $1/10$  of  $8 \times 6$ ), or realise that a proportional adjustment can be made to a set of six (take 80% or  $4/5$  of a group of six). Using an array model of multiplication can allow students to independently verify that the commutative law is applicable for multiplication, having physical or diagrammatic models of tenths can help students appreciate the use of 48 tenths, while an area model of multiplication allows for proportional changes to be presented in diagrammatic form (Ministry of Education, 2007e).

The second obstacle involves the observation made by students (and often unwittingly reinforced by teachers) that multiplication makes things bigger. This concept will be coded as MMB following Harel and Sowder (2005). MMB is a well-documented student conception (Bonotto, 2005; Kieren, 1992). The data produced in repeated addition multiplicative situations supports this concept, whether whole number ( $4 \times 5$ ) or decimal ( $5 \times 0.8$ ) examples are used. The second example does not contradict MMB because the product of 4 is contrasted with the multiplicand 0.8, and not the multiplier of 5. Bana and Dolma (2004) report that almost two-thirds of Year 7 (Aust.) students believed that the product of 97 and 0.09 would be larger than 97, something they attribute to the influence of MMB thinking. If MMB is left unaddressed, its influence incapacitates learners from choosing correct procedures and/or accepting correct answers (Prediger, 2008).

There is evidence that students do not make reasoned estimates when operating with decimal numbers in multiplicative situations. This may be due to simple inability, but may also arise when students adopt disjunctive procedures and ignore their prior knowledge. One can envisage that at least some of the students who produced errors to

the power of ten in Burns (1990) when given the task  $3.04 \times 5.3$  would correctly adjust their response if challenged to consider the result of  $3 \times 5$ . As multiplicative procedures produce the same digits in the answer irrespective of the place-value of the digits concerned, there is no compelling reason to develop new rules for operations with decimals. If previous proficiency with basic operations can be combined with an ability to judge the reasonableness of the answer, then students should not find decimal multiplication difficult. While seemingly self-evident, Albert and McAdam (2007) showed that the invariance of digits produced in products cannot be assumed for adult learners, let alone for children. Knowing that  $6 \times 8 = 48$  does not automatically lead to the knowledge that  $0.6 \times 8$  must have the adjacent digits '4' and '8' in the answer.

A common response is for the teacher to provide another syntactic rule such as 'count the decimal places' which masks the underlying conceptual issue (Owens & Super, 1993; Resnick et al, 1989).

Work that involves different models of multiplication such as area models that allow multiplication can be conceived as 'stretching and shrinking' could help overcome the limitations of the intuitive additive model (Ministry of Education, 2007e). Deliberately linking estimation skills to operations should help students be aware of the need to establish the reasonableness of their calculations. Irwin and Britt's (2004) work showed that some students who had demonstrated understanding of compensation tasks with whole numbers could independently transfer their understanding to situations involving decimals. This implied that strategic facility, based upon an understanding of the underlying place-value construct of number, was of itself sufficient to empower these students to work with decimal numbers, even when these had not been explicitly studied.

### **Operations involving Decimals: Division**

The division of fractions is often considered the most mechanical and least understood topic in elementary school with very low success rates (Tirosh, 2000). One major reason for this is held to be that students have a narrow concept of division. The most common model used for division with early learners is that of sharing – the partitive model (Ministry of Education, 2007e). It answers the question; "How large will the pieces be (measure or group) when I share this unit into this many sets?" In classroom examples, an

object (e.g. a cake) or collection of objects (e.g. a bag of lollies) is equally divided into fragments or sub-collections. This has also been described as ‘natural division’ (Fischbein, 1985).

This model imposes three constraints (a) the divisor must be a whole number; (b) the divisor must be less than the dividend; (c) the quotient must be less than the dividend (Fischbein, 1985; Tirosh, 2000). If this model is held exclusively, it creates an effective barrier to solving many division problems as it implies that ‘division always makes a number smaller’ (Greer, 1987). (This concept will be labeled DMS). MMB and DMS are logical equivalents through the commutative law, but are not always linked experientially. People may reject MMB but still operate as though DMS were true, a logical paradox but an observed behaviour (Tirosh & Graeber 1989). Also, a person may state their belief that both MMB and DMS are false, and yet operate as though they were true. In studies of pre-service teachers, Tirosh found that the majority operated with these views across many problem-solving situations even when they could make explicit statements that MMB and DMS were not universally applicable (Tirosh & Graeber 1989, 1990; Tirosh, 2000; Tirosh et al, 2001).

Graeber and Tirosh (1990) respond to this by writing:

“An understanding of division by a decimal less than one almost demands the application of the measurement model” (p. 584).

Some writers distinguish between intuitive or natural concepts and acquired or algorithmic systems. MMB and DMS are both seen in the intuitive/natural domain and initial formal systems use whole numbers and so provide the learner with evidence that supports their natural belief (Tirosh, 2000). The measurement (quotitive) model answers the question; “How many of this size (measure or group) can be formed from this unit?” In this model getting quotients that are larger than dividends can be logically conceived. Experience with situations involving this model of division should allow students to accept data that is contradictory to the DMS misconception.

## **Research Methodology**

While concentrating on crucial moments, on unique turning points, we may lose from sight the normal developmental sequence of learning (Seeger, 2001, p. 292).

### **Zone of Study**

Lamon (2001) has explained the need for extensive study in the realm of how students learn rational numbers because students encounter many points of discontinuity with the whole-number system, both in the way symbols are interpreted and in the way operations are carried out. As a result of this discontinuity, the extension of student knowledge into rational numbers cannot occur by the expansive generalization of the whole-number schema they possess. Thus is provided an opportunity to view the construction of new knowledge by learners and to describe the influences upon this process. In the last section it was shown that knowledge has already been collected concerning the difficulties students have in learning decimals and these have been studied from epistemological and psychological perspectives. We also have a number of models that show how learning of new knowledge could occur and the mechanisms by which this may be facilitated. The depth within the field of learning decimal numbers has not been extensively researched yet remains highly problematic, which provides the opportunity to engage in meaningful research.

### **Research Directions**

Further research has been called for from within the research community. This recognizes the need to demonstrate and discuss the finer details of actual learning experiences. Goos (2004) calls for more attention to be given to the *detailed* practices of teaching and learning through which the reform approaches – often described in research reports only in general terms – are enacted in classroom communities. This is partly in response to data that has shown reform strategies to not produce universally higher student achievement (Tzur, 2007). According to Siegler (2007), typical studies in the past have exaggerated the ‘stages’ of growth and in doing so they miss the high variability of responses and the behavioural regressions that typify much learning.

Theories that have focused upon identifying the typical landmarks of children's developmental journeys are necessary but incomplete. There is a need to look at processes - particularly as they are involved in transitions - as this is where the learning actually takes place. Carpenter, Fennema and Franke (1996) believe that the research agenda must be based around *how* learning occurs rather than *what* learning occurs. While standard cross-sectional studies provide evidence of this learning having already taken place, they lack the temporal resolution that a microgenetic study provides to capture evolving – rather than evolved – competence (Siegler, 2007). He values 'microgenetic' methods as they span a period of changing competence and provide dense observations relative to change, allowing us to try to infer the processes involved with change. Variability not only exists between close peers, but is also observed within the individual within short time-frames, even during the performance of a single task. Given that many models of learning now include recursive elements (e.g. Lesh et al, 2003; Martin et al 2006; Pirie & Kieren, 1994; Siegler, 2007), it is important to collect evidence of these transitory states as they may prove to be critical in terms of our understanding of how learners negotiate their journeys to new knowledge.

Specific calls have been made with regard to the learning of decimal numbers. While decimal numbers are initially taught and tested as static units, relatively little has been written about how students might demonstrate understanding of decimals by using them dynamically (Irwin & Britt, 2004). Exceptions are Irwin (2001) and Irwin and Britt (2004). If we follow the call of Hiebert et al (1997), knowledge should be demonstrated by actions. Okazaki and Koyama (2005) report that while there have been several decades of research examining factors regarding students' failure with decimal operations, there has been a lack of clarity regarding the processes that might help design successful interventions. Studies that chart student progress *through* the journey from prior knowledge to the application of new knowledge have the potential to inform the educational community in a different way to studies that limit their scope to 'before and after' data sampling.



## **Type of Research**

The literature points to the need for ongoing research to capture and explain learning as it occurs regarding decimal numbers. Consideration of how to address this need will determine the type of research that is needed to produce the kind of evidence that is required. If the goal is simply to address the learning need of students – such as their lack of facility with decimal numbers - then an intervention study would suffice. Intervention studies are limited-term teaching experiments that target specific problems with the goal of producing more acceptable outcomes. In the domain of rational numbers, Lamon (2007) reports that where these studies have looked to teach to pre-determined mathematical ends, they have proven to be largely ineffective with disappointing results. Exceptions to these have taken account of student's informal knowledge and encouraged sense-making from the outset of instruction.

Collection of pre- and post-intervention data may inform the researcher about whether the intervention, as observed, has produced evidence of learning change. This evidential comparison serves to gauge the effectiveness of the intervention and thus its wider impact upon the educational community (Timperley, Wilson, Barrar & Fung, 2007). What this data does not capture are the transformations in thinking as they occur. A potential issue that arises from the examination of pre- and post-intervention data alone is that changes may be ascribed to what the researcher believed was the major pedagogical influence, whereas the students' perception was quite different. Thus, while a deliberate set of interventions and specific data collection points regarding outcomes may be incorporated as part of the research process, these actions alone will not provide the information needed to examine the process of learning.

The clinical interview is designed to allow the interviewees to 'tell their story' (Long & Ben-Hur, 1991). Its use can be seen as an extension of Piaget's research goal, to explicate the nature of thought (Ginsburg, 1997; Zazkis & Hazzan, 1998). It is termed 'clinical' in that the setting is outside of the usual learning environment – often an office. They are often called 'task-based interviews' as they typically employ activities for participants to carry out as a means for creating discussion concerning the thinking used to solve them. While a clinical interview typically has a pre-determined outline, the interviewer has the flexibility to vary both questions and tasks in order to probe for understanding or to seek

confirmation of other evidence (Burns, 2000; Cohen et al, 2000). The extra information gained from interviews is often different from that obtained from written tasks as the depth of understanding and rationale for the application of a procedure may be explored (Mitchell & Clarke, 2004). The NDP has a clinical interview, the *Numeracy Project Assessment* tool (NumPA) as its principle evidence-gathering device (Ministry of Education, 2007c).

However, Lamon (2007) advises that while Piagetian interviews reveal something about the changes underlying children's thought processes; they also fail to capture the dynamic processes underlying such change. So while this process may be also incorporated, it cannot in and of itself address the central issue being studied, which is of the change process as it occurs. This can only happen if a continuous data collection system is employed while the learning is occurring. Even when video- or audio-recording are used to collect continuous data during the teaching intervention, this does not guarantee that all learning will be captured. Trognon (1993) reminds us that not all cognitive action occurs simultaneously with the direct engagement in tasks and that not all of this cognition is made explicit.

In recognition of the complexity of learning, a study should have 'ecological validity' (Cobb, 2003). As described by Bronfenbrenner (1979) the 'ecological' part of the term means that the situation within which the study is conducted is critical to the interpretation of the results coming from that study. The 'validity' aspect is whether the environment is experienced by the participants in the manner that the researcher suggests. The investigation of student learning must try and capture, analyze, and reflect upon all of the contributing factors in the 'ecology'. Such a process will study the synergy of the interactive influences in the dynamic environment of the classroom and beyond (Lamon, 2007).

Studying the range of variables and their interactions stands in contrast to methodologies where all but one variable of learning is held constant. A criticism may be made that all that can be produced from such a study is anecdotal account. What qualifies this type of approach as research - and thus distinguishes it from simple teacher narrative - is firstly the systematic processes used to plan the intervention and gather data, and secondly, the

intention to produce analysis and reflective comment for the wider educational community (Burns, 2000; Cohen et al, 2000). The aim in describing the particular situation of learning is not to have this setting duplicated, as arguably this is impossible. Rather, it is to make explicit the conditions surrounding the application of the intervention so that the teaching process that was used is able to be modified for other settings.

If students are regarded as active participants in the learning process then they must be afforded a degree of agency. There must be a degree of flexibility within the framework of the research plan to allow for this. The planning involved for such research involves the creation of a hypothetical learning trajectory (HLT). This term was used by Simon (1995) to describe the teacher's prediction as to how the learning journey will look and therefore provide a rationale for choosing a specific instructional pathway. It is the use of the word 'hypothetical' that aligns this planning with the constructivist framework.

Simon (1995) describes his use of the this term by writing that

“It is hypothetical because the actual learning trajectory is not knowable in advance. It characterizes an expected tendency. Individual student's learning proceeds along idiosyncratic, although often similar, paths” (p. 135).

The hypothetical learning trajectory consists of two aspects; the learning journey which has as its focus the mathematical progressions, and the action plan for which the focus is the means of engaging students with these progressions. It is better to call this a 'plausible hypothetical learning trajectory' as there may be other, equally logical pathways that were not considered and/or taken.

Ideally then, the research approach should be able to respond to each of the challenges outlined above. These are summarized in the table below:

*Table 1*  
*Description of Valid Settings for Educational Research*

<b>Category</b>	<b>Description</b>
Location	Real-life settings where external stimuli and competing agendas abound and <i>not controlled settings</i>
Variables	Multiple dependent variables whose inter-relationships may also vary and <i>not a few clearly defined variables</i>
Focus	Characterizing the complexity of the learning situation and <i>not examining the effect of a measured adjustment to a single variable (or limited variables)</i>
Procedures	Flexible design that is altered in situ in response to the local environment and <i>not adherence to a carefully defined procedural structure</i>
Social Interaction	Complex; involves the inter-relationships of both researcher and learners and their interactions with each other. Both may have positive (e.g. scaffolding, emotional support, and sharing ideas) and negative (e.g. disempowering, distracting) influences on the learning and <i>not researchers viewing themselves as socially neutral with limited interactions beyond the subject matter reported.</i>
Findings	Examines multiple aspects of the design and reports upon a wide profile of results and <i>not testing a single hypothesis</i>
People	Participants who can shape the study due to its flexible design and are <i>not passive subjects who are relatively powerless</i>

This table was adapted from Barab & Squire (2004).

### **Design Experiments: Introduction**

All of the characteristics can be found in the type of research termed ‘Design Experiments’. The design experiment process draws from prior research but acts as “a test bed for innovation”. It engineers particular forms of learning, but studies the learning as it is taking place within that context. Its analysis is not as simple as judging the ‘right’ choices; rather, it recognizes that choices have been made, examines what arose from those choices, and speculates on alternatives (Cobb et al, 2003). It is a highly interventionist approach, innovating, and observing and responding to the results of the innovation. The concept of design experiments in educational research is often associated with Brown (1992), but the approach has been around for much longer in other disciplines. The use of the word ‘design’ comes from engineering situations. In that environment, testing a working model occurs to report on changes in outcome that follow changes in the design and not to simply determine if the design is good or bad. It is the exploratory phase before large scale-up is considered (Cobb et al, 2003). The engineering metaphor also directs us to think about modifications from models to new situations, to look to iterative adaptations in response to new data (Gorard, Roberts & Taylor, 2004).

Design experiment results are not final, but part of a process of continual change. Their role is in stark contrast to the epistemological research of past eras where a single best solution was sought. Sharp and Adams (2002) provide an example of this previous type of research – finding the ‘best’ algorithm for the division of fractions – and the problems raised by the imposition of a false dichotomy in having to choose between alternative approaches.

The main purpose of the design experiment is to develop and test theories that are grounded in specific settings. Broad patterns of emergent behaviour can shape regular instructional practice and detailed individual responses employed when the teacher has reason to do so (Moskal & Magone, 2000). This allows for the integrity of the original setting to be acknowledged, whilst providing direction that can be adapted into other situations.

As Lamon 2007 has pointed out:

“The design experiment allows the researcher to influence, to study, and to adjust the course of the longitudinal study based on the unstaged and serendipitous dynamics of a real classroom” (p. 633).

There are three levels involved in design experiment research: planning, intervention, and analysis.

### **Design Experiments: Planning**

Planning involves consideration of the potential learning that could take place – the hypothetical learning trajectory – and the pedagogical implications of this journey. Analytical and social scaffolding can be considered (Williams & Baxter, 1996). This will include an examination how prior student constructs will influence this learning. This has both a negative (likely inhibitors) but more importantly, a positive aspect. The positive aspect is that the teacher can plan to use existing mathematical constructs and prior experiences to establish linkages with prior knowledge or to engineer situations of cognitive conflict (Tzur, 2003). It will also take into account the types of group dynamics that will be encouraged, in order to create a risk-taking, goal-oriented, environment that encourages the verbalization of mathematical discussion around tasks (Yackel & Cobb, 1996).

Planning must also consider what will distinguish this from an anecdotal account and thus recognise and describe the data that will need to be collected. Initial data must

establish the students' prior understandings and serve as a benchmark with which to compare subsequent data. If we accept that students operate from pre-existing schema, then their answers to tasks are unlikely to be uninformed guesses or capricious in nature. Rather, they will stem from a prior understanding of the nature of the task, even though this understanding may not be valid in the wider community. Diagnostic tasks – whether written or oral – aim to give insight to these prior schemes rather than simply assessing the correctness of responses. The use of written tasks in a diagnostic (rather than summative) form regarding decimals has been practiced by earlier researchers (e.g. Hiebert & Wearne, 1985; Nesher & Peled, 1986; Resnick et al, 1989; Sackur-Grisvard & Leonard, 1985) and more recently exemplified in work produced from Melbourne University (Steinle & Stacey, 1998; Stacey & Steinle, 1999). In the latter's Decimal Comparison Test (DCT), student answers were classified according to the likely construct being employed to solve the tasks. Correlation of written results with oral interviews has confirmed the validity of the process.

Ways of assessing subsequent student learning need to be incorporated into the design matrix. In keeping with the constructivist view of knowledge being personal and demonstrable through usage, further data collection should capture evidence of the nature and extent of individual learning. This is likely to involve more than one type of evidence. Data gained from interviews, written tasks, and transcripts from video- or audio-taped discussions of teaching/learning times can be sifted and triangulated in order to search for commonalities and discrepancies, to generate links with existing research findings, and to provide the basis of new theories (Burns, 2000; Cohen et al, 2000; Wilson, 2001).

### **Design Experiments: Intervention and Analysis**

The intervention phase is when the planning of the teacher and the idiosyncratic nature of learners interweave. Models of learning predict that this will not be a simple, linear transition for students but a complex and recursive process of reorganization (Lambdin & Walcott, 2007; Pirie & Kieren, 1994). Scaffolding by the teacher will also be both continual and changing. Data collection needs to be an on-going process if these dual, reflexive actions of teachers and students are to be meaningfully captured. This will allow

for the examination of the effects and responses made by both parties to each other's actions. This analysis of cause and effect is at the heart of the design experiment approach.

Such analysis also looks for triangulation between data sets, as it is not accepted that a single tool provides sufficient evidence upon which to make claims. As examples, student written work may not always uncover misconceptions or students may voice confidence in their comprehension of a new concept but fail to complete a task that would demonstrate this understanding. Analysis of the pre- and post-intervention data provides information regarding the effectiveness of the innovation.

Results of this type of experiment do not claim to be universally applicable but analysis of the conditions present in their production help the educational community to assess their implications. The design experiment does not aim to produce grand theory but is not merely an individual account of learning either (Cobb et al, 2003).

In terms of wider application, the issues of 'scaling-up' must be addressed. Roschelle, Tatar, Shechtman and Knudsen (2008) point out that simply having a larger sample of students involved does not of itself more deeply inform the educational community. Instead a new set of questions may be investigated in order to "distinguish more from less robust innovations and examine sources of variation between implementations of an innovation in different settings" (p. 151). The first clause of that statement does not imply that reported results from an intervention study are invalid, but simply that they may or may not be consistently obtained in new settings. The second clause serves to remind us that imputed reasons for the success of an innovation cannot be verified until a deliberately engineered difference is made to the instructional design. This helps to separate essential requirements of the innovation from its attendant optional aspects.

Design experiments can take many forms. Indeed the very name indicates that particular choices are made to best serve the research questions and environment in which the study takes place.

The type of intense, small-scale study that captures the type of data yielded by design experiments has been termed 'microgenetic' by Siegler (2007). In an intensely-focused study, it is also possible to record the agency aspects of noted changes. Rather than ascribing all changes in data to the primary intervention, the researcher may note that

other influences have occurred. For instance, one student may make sense of a piece of mathematics from a piece of teacher-provided equipment. That is employed by this student as an artifact to convince a second student. These two then hold a discussion with others in their group which results in five students now able to act with this new understanding. A ‘before and after’ approach might wrongfully conclude that the introduction of the equipment was the essential feature for all five students, whereas more close observation of the learning process as it occurs can recognise the inter-relationship of a number of contributing factors. This may provide important statements about the variability of learning for individual students and help us understand how the learning environment as a whole operates.

## **Review of the New Zealand Curricula**

To be numerate is to have the ability and inclination to use mathematics effectively – at home, at work and in the community  
(Ministry of Education, 2001, p. 1).

Both phases of this research were carried out when the MiNZC document (Ministry of Education, 2002) was the official mathematics curriculum for New Zealand. On November 6<sup>th</sup> 2007, the revised curriculum document ‘*The New Zealand Curriculum*’ (NZC) was released (Ministry of Education, 2007a) and so both documents are included in this discussion.

The guidance to teachers regarding the place of decimals in the previous curriculum was slightly ambiguous and remains so with its revision. In the MiNZC document (Ministry of Education, 1992), ordering decimals was positioned as a Level 3 objective (p. 40)<sup>3</sup> Students were also expected to work at this level with problems involving decimals using each of the four basic arithmetic operations, though the suggested learning experiences (pp. 41-43) and common practice limited this to addition and subtraction. Level 4 (p. 44) explicitly listed the use of decimal numbers in problems involving multiplication and division. In NZC (Ministry of Education, 2007a), teachers are advised that at Level Three students are required to “use a range of additive and simple multiplicative strategies with

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<sup>3</sup> The MiNZC consistently uses numerical symbols, e.g. Level 3, while NZC uses the word form, e.g. Level Three. These conventions have been maintained to help distinguish which document is being referred to.



whole numbers, fractions, decimals, and percentages” but they are not required to understand the relative size and place-value structure of decimals (to three places) until Level Four! An unpublished (but widely disseminated) paper (Wright, 2007), advised the NZ educational community that “Full understanding of decimal place-value to three places has been delayed from the 1992 curriculum to allow foundational understanding of equivalent fractions to be developed”. This reasoning was not elaborated upon in the paper which was only intended as a short summary of the changes made to MiNZC. Consideration of what sense will be made by students of their Level Three experiences with decimals may lead to the conclusion that expansive and/or disjunctive generalizations will result. Also in Level Four is the direction that students should be using decimals in operations in conjunction with whole numbers. Meaningful interaction with decimals as static entities is postponed until Level Four, but at this level students are also expected to make sense of the problematic issue of multiplying by a decimal number. I interpret the phrase from NZC Level Five that students are required to ‘understand operations on fractions, decimals, percentages and integers’ to indicate that multiplicative processes involving finding a fraction of a fraction (for example) are included at this Level as is division by a decimal number. The NZC sets objectives involving decimals with slightly more emphasis on understanding and using strategies than its predecessor MiNZC.

Past and present maths curricula in New Zealand (Ministry of Education, 1992, 2007a) do not encourage an inter-weaving of the ‘strands’ of Number and Measurement beyond implying via Venn diagrams that there is a small degree of overlap.

While the NDP goal of Numeracy (Ministry of Education, 2001) implies a broad range of mathematical skills, the direction given to teachers in the NDP is that the mechanism of achieving gains in Numeracy is via the Number strand. There are references to students being able to use knowledge gained in the Number strand to help them in other strands; e.g. in *Book 9 Teaching Number through Measurement, Geometry, Algebra and Statistics* (Ministry of Education, 2005) it states that “number increasingly becomes a tool to be applied across the other strands” (p. 2).

Diagrammatically, this may be represented as below:

Number  Other Strands

There is but one explicit reference in that book to the idea that this process may operate in reverse, that working in the context of geometry or measurement may facilitate the learning of number, “There is also an extent to which spatial reasoning helps inform students’ number thinking” (Ministry of Education, 2005, p. 18). There is a great deal of implicit use of continuous models of number however. Previous editions of the NDP publication *Book 1 The Number Framework* made use of only one such model, the number line (Ministry of Education, 2006, p. 6) but the recently revised edition used four representations: line, strip, array and area (Ministry of Education, 2007b, p. 6). Each of these representations implicitly carries a continuous model of number.

While there is a strong emphasis on developing mental strategies in the NDP, it is also important to note that the teaching of algorithmic systems of solution is not regarded negatively per se. The NDP advises teachers that they should not introduce standard written algorithms to students *until* they are using part-whole mental strategies. Premature introduction is believed to hinder the development of number sense and students’ ability and desire to develop mental strategies. The standard algorithms are expected to be taught after students have demonstrated understanding of the basic mathematical structures of the contexts by using self-developed strategies (Ministry of Education, 2006, p. 14).

### **Rationale for this Research**

Theories and associated practices need to be rigorously evaluated in terms of their impact on students (Timperley et al, 2007, p. 199).

The purpose of this research was to examine how models of learning are worked out with students in mathematics. The mental activities of building upon prior knowledge and the resolution of cognitive conflict are at the heart of a constructivist perspective of learning and form the basis of this study. Research in the last ten years has made us more aware of the complexity of the learning process, in particular as non-linear, recursive models are

found to better fit observed student behaviour than more simplistic models. There are calls from within the mathematics educational community to further examine the fine detail of learning in order to better understand the processes by which learning occurs.

Studies that provide details of how students reconstruct their thinking are essential for the refinement of theories of learning. This refinement may result in more complete information being available for educators who may then alter their practice and so ultimately improve student achievement. This is especially pertinent in areas of mathematics that have proven problematic for students, as current practices are demonstrably not addressing key learning needs. This emphasis on *how* students are learning is most relevant where conceptual reorganization is identified as being the central issue, and not one of computational accuracy or factual recall.

In this research, the mathematical context within which these learning activities are explored is that of decimal numbers. An understanding of decimals is required for computational tasks within arithmetic and algebra, and is also necessary for work with measurement, geometry, and statistics (especially in probability). A growing awareness of the need for financial and statistical literacy also serves to underscore how important this area of mathematics is in terms of the numeracy level of the population as a whole. Poor attainment by students is well documented in the literature and is ascribed to a lack of conceptual understanding. This area of mathematics has been chosen because it is essential to student mathematical development and yet student success continues to be low.

The research community has responded to this need by work that has described the cognitive accommodations required of students and by reporting upon the partial constructions students have made as they have struggled to reorganize their thinking. However, while the conceptual issues surrounding the learning of decimals have been widely reported, there have been few studies that have documented both successful teaching innovations and described how these innovations have effected change in students' conceptual schema.

Consideration of recent models of learning and the acknowledged need to improve the teaching of decimals form the niche within which this research is situated and provide its main purpose.

This generated the research question:

**How can students' understandings of decimals be enhanced by a short-term teaching experiment?**

The next chapter provides a detailed description of the method used to answer this question.

## Chapter Three: Method

### Introduction

This chapter gives a factual account of the plans, procedures, and actions involved in the research. A key aspect of the method was that it was comprised of two phases. This allowed for wider exploration of the learning of decimal numbers than would have been possible had only one group of participants been involved. It also allowed for an iteration of process, so that the planning and delivery of the second intervention could occur after reflection upon the first phase of the research.

The structure of the chapter is as follows:

**Participants** The students involved in the research are introduced and described.

### Procedure

**Action Plan** This serves to outline the overall intent of each iteration of the research.

**Design and Implementation** This provides details and explanations of the planning decisions involved for each of the phases, and the settings, timings and summaries of what transpired in the learning sessions.

**Data Collection** A description of the types of data needed and the means of collecting this data are given.

**Materials** A description of the materials used is given.

### Participants

#### Introduction

At the beginning of each phase of the study, the students were believed to be at about the same stage of mathematical development. The classroom teachers' professional judgments were used to make these decisions. In Phase 1, this meant that members of the group were initially rated at Stage 5, and in Phase 2, at Stage 6 (Ministry of Education,

2007b). In brief, this implied that the Phase 1 students could add whole numbers by using at least one part-whole strategy, had started working with whole-number multiplication problems, and could order unit fractions. The Phase 2 group could add whole numbers using at least two strategies, knew most/all of their basic multiplication facts and could find a non-unit fraction of a set. The two groups of students were regarded as slightly above average in mathematics in their respective schools. Comparisons with national guidelines show them to be at - but not exceeding - expectations for their Year Levels (Ministry of Education, n.d.).

### **Participants: Phase 1 (2006)**

A teacher within a decile two Primary School with a roll of 70% Māori students agreed to let me approach her Year 5 - 6 students<sup>4</sup>. Students within this class who were at least Stage 5 on the Numeracy Project Framework (Ministry of Education, 2007b), were invited to participate. From those who returned parental permission slips<sup>5</sup>, six children met the minimum criteria but had not demonstrated proficiency with decimal numbers, and so they became the initial group. They were assigned pseudonyms that accurately reflected their gender and ethnic background.

*Table 2*

*Details of Participants in Phase 1*

<b>Pseudonym</b>	<b>Gender</b>	<b>Ethnicity</b>	<b>Age</b>	<b>Year</b>	<b>NDP Strategy Stage</b>
Tame	M	Māori	9	6	5
Ripeka	F	Māori	9	5	5
Mary	F	Pākehā	9	6	5
Grace	F	Pākehā	10	6	5
Aroha	F	Māori	10	6	5
Wini	F	Māori	10	6	5

<sup>4</sup> Decile rankings reflect the socio-economic status of the community and range from 1 (lowest) to 10 (highest).

<sup>5</sup> Examples of permission letters are provided in Appendix C

### **Participants: Phase 2 (2007)**

All of the Year 6 children in the first phase of the research had moved to Intermediate Schools. Mary, Grace and Wini attended the same school. (This Intermediate School was decile 3 with 75% of the students identified as Māori). These three students formed the core of the new group. As they were in two different classes, their teachers approached other children within those classes to make the numbers back up to six. Coincidentally, the new participants had also attended the same Primary School as the earlier students. Pseudonyms for the original students were retained, while new pseudonyms reflecting gender and ethnicity were created for the other three students.

*Table 3*

*Details of Participants in Phase 2*

<b>Pseudonym</b>	<b>Gender</b>	<b>Ethnicity</b>	<b>Age</b>	<b>Year</b>	<b>NDP Strategy Stage</b>
Mary*	F	Pākehā	11	7	6
Grace*	F	Pākehā	11	7	6
Wini*	F	Māori	11	7	6
Bridget	F	Pākehā	11	7	6
Kiri	F	Māori	11	7	6
Hoani	M	Māori	11	7	6

\*Original participant.

## **Procedure**

### **Action Plan**

Literature was reviewed that covered the broad themes of learning and specific studies of rational numbers in order to make initial preparation for the intervention phase. After giving explanations and gaining necessary permissions, a group of student participants was assembled. Diagnostic data was collected from the student group and this was used to modify the planned intervention. The intervention phase was carried out, involving both the delivery of activities and the collection of real-time data from the students. A further set of data was collected from the students that could be compared with the initial

diagnostic set. All of the collected data was analyzed. The scope of the literature being reviewed was extended.

This cycle was repeated in Phase 2.

### **Procedure: Design and Implementation**

Design: Phase 1

It was clear from the literature that the ability of students to order decimal numbers was problematic. A specific objective in MiNZC (Ministry of Education, 1992) was chosen,

“Within a range of meaningful contexts students will be able to order decimals with up to 3 decimal places” (p. 40).

That students would be required to *use* place-value knowledge of decimals is consistent with the view of knowledge described by Heinz et al (2000) and Hiebert et al (1996). Students would be asked to apply their knowledge to both contextual and non-contextual situations. A benchmark was needed that would provide a quantified measure of understanding. I had previously carried out extensive student trials ( $n > 2000$ ) of a diagnostic tool I had adapted from one produced by the University of Melbourne (Stacey & Steinle, 1999), the Decimal Comparison Test (DCT).<sup>6</sup> Its use would provide this benchmark, with additional information coming from a task used in the NDP diagnostic interview (Ministry of Education, 2007c, p. 29), and two tasks that ran parallel to those used in the literature describing common student misconceptions.<sup>7</sup>

The instructional design provided for as little knowledge to be transmitted as possible. Instead, it aimed to get students to utilize their prior knowledge by connecting this with new contexts, and to generate new concepts via their interaction with challenging tasks. This choice of pedagogical approach was made in response to the literature that described student learning using constructivist frameworks and to models of learning consistent with these frameworks. Specifically, the intention was to engage students in using concrete materials to provide contexts within which to discuss the understanding and procedures required to order decimal numbers successfully. It was hoped that the use of decimal place-value could be extended to involve situations requiring the addition and

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<sup>6</sup> Copies of both tools appear in Appendix A.

<sup>7</sup> These tasks are presented in the Results and Discussion chapter



subtraction of decimals and to the repeated addition model of multiplication. This potential learning trajectory is set out below.

#### Planned Learning Trajectory

Students will:

1. Understand fractions involving tenths.
2. Show they understand the iterative nature of fractions.
3. Recognise how tenths relate to the first decimal place and so start to interpret decimal symbols correctly.
4. Extend this understanding to include hundredths and so challenge the whole-number view of decimals.
5. Generalize this concept to include any number of decimal places.
6. Begin working with decimal numbers using the operations of addition, subtraction, and multiplication (the latter only as repeated addition).

It was anticipated that students at Stage 5 of the Numeracy Project would have working knowledge of these first two steps of this journey.

#### Implementation: Phase 1

This was held in March, 2006. Four, 45-minute periods of interaction were held in a medium-sized room away from the classroom. The final session was 30 minutes long.

No lectures or notes were given to students. Instead, lessons began with tasks involving the use of manipulatives, with questions and conversations arising from these tasks. Games and written recording were employed to reinforce new learning. Questions being posed by either the teacher or the students provided the forward impetus into new areas of learning. The actual implementation of the planned learning journey was continually adjusted so as to accommodate responses to the formative evidence being collected.

The instructional pathway was as follows:

Day 1 Iteration of unit fractions with denominators up to tenths using materials, and then as a mental process.

- Day 2 Introduction to the pipe numbers manipulative. Creating a situation where the paradox of whole-number thinking was exposed and resolved by the students.
- Day 3 Exploration of decimal place-value using measurement tasks.
- Day 4 Reinforcement of decimal numbers through the use of games. Extension of the concept of decimal place-value to addition.
- Day 5 Using diagrams to represent decimal numbers. Further reinforcement using games.

#### Design: Phase 2

Reading undertaken after the completion of Phase 1 convinced me that the problems students had with the use of decimals in additive contexts stemmed from place-value misconceptions, rather than being issues with the operation itself. For example, a student writing  $3.4 + 3.21 = 6.25$  is demonstrating understanding of the additive process, but misunderstanding the symbols used to describe the numbers. In response, I planned the next iteration to include work with additive contexts, but also to investigate how students would re-adjust their mental schema of multiplicative operations when faced with situations involving decimals. This re-conceptualizing was described as highly problematic in the literature.

I had become more convinced of the efficacy of using physical materials in measurement contexts as a result of the literature survey and my experiences in Phase 1. Examination of the use of these artifacts to promote learning in the context of multiplicative work with decimals became the investigative focus. The subtle shift in thinking involved is expressed in the following statements. In Phase 1, I wanted to see *if* the use of materials and realistic contexts could result in cognitive reconstruction, whereas in Phase 2, I wanted to investigate *how* the use of materials and contexts affected the reconstructive process.

It was intended that students would take part in a variety of experiences that would serve to expand their existing knowledge of decimal numbers. This was formalized as below.

## Planned Learning Journey

Students will:

1. Demonstrate that they can use their understanding of decimal place-value to order decimal numbers and then to add and subtract them.
2. Show that they have generalised the decimalization process through situations involving the ‘density of number’ property.
3. Be able to solve single-digit multiplication and division problems involving simple decimal numbers.
4. Begin to extend their multiplicative knowledge to more difficult problems involving decimals including double-digit multiplication and finding fractions of fractions.

## Implementation: Phase 2

This was held in August, 2007. Eight periods of interaction occurred. Various in-school factors meant that the learning sessions were held in three different rooms, at three different starting times and for periods of time ranging from 25 minutes to 1 ¼ hours. Flexibility was essential. For example, the water-pouring session had to be held on Day 3 as a suitable room was available then.

Again, no formal teaching took place, but a continued emphasis on engagement with practical tasks and discussion filled most of the learning sessions. A slightly higher proportion of time than in Phase 1 was spent on completing written tasks and in student recording of their practical results.

Another contrast with Phase 1 was that the effect of mood on student engagement was noticeably more pronounced, perhaps reflecting the developmental age of the students. The existence of off-task discussions regarding peers, music, television, and school would not surprise teachers of this age group. In particular, Wini was ‘growled at’ by the DP for uniform infringements on two occasions while walking to learning sessions and was disengaged for the beginning of each of those corresponding periods. At other times the group was totally focused and seemed very aware of the learning they were engaged in over and above the tasks they were performing.

The instructional pathway was as follows:

- Day 1 Collection of initial data, introduction to the pipe numbers, decimal addition using this model.
- Day 2 Decimal addition and subtraction using linear and area models.
- Day 3 Division using quantities of water and area as models.
- Day 4 (Short session). Review of earlier tasks. Density of numbers via Krispie biscuit tasks. Multiplication of a decimal number.
- Day 5 (Relatively short afternoon session). Density via number-line tasks and biscuits. Review of previous contexts.
- Day 6 Quotitive division using string lengths. Density using number-lines.
- Day 7 Further work with string contexts. Multiplication by a decimal number. Estimation of products using rounding.
- Day 8 Double-digit multiplication with whole numbers and then decimals using an area model.

### **Procedure: Data Collection**

Different types of data were gathered for this research. Diagnostic data was collected in order to shape the intervention, as this information would allow student prior knowledge to be built upon. Data collection concurrent with the teaching was needed to capture evidence of the evolving learning process. Data that could be used to provide a measure of change needed to be collected post-intervention.

#### Initial data

In Phase 1, diagnostic data was captured through the Decimal Comparison Test (DCT), and the addition tasks as mentioned earlier. A group interview was held prior to the intervention phase. The children had previously agreed to have individual discussions but become apprehensive when the audio recorder was produced. They agreed to a compromise whereby the group discussion was recorded.

In Phase 2, initial data was collected by the completion of written tasks. These tasks included a few diagnostic items and others whose purpose was to provide exposure to a range of multiplicative contexts involving decimals.

Data from the learning sessions

Audio recording, examples of student written work, personal notes, and photographic evidence were collected during both intervention phases to capture data of student experiences as they were learning.

The dialogue of the learning sessions was captured using audio equipment; a cassette recorder in 2006 and a digital recorder in 2007. Permission to collect and use this data was obtained from all participants prior to the start of the in-school component of the research. The portability of the equipment allowed me to record conversations with students who were undertaking measurement tasks in any part of the room. While all audio recordings were listened to, not all were transcribed. Occasional poor sound quality and off-task conversations were the main reasons for non-transcription. The transcripts were then analyzed and selected excerpts of conversation used in the Results and Discussion chapter.

[Video data was considered, but rejected as being impractical to collect. It may have offered some additional information (e.g. body language responses to statements). However, it would have required a research assistant to operate the camera as the students were moving around the room to complete tasks in most of the learning sessions. A fixed-position camera would not have supplied much additional data to the audio record].

Samples of student work were collected during both phases of research. During Phase 1, this was achieved by the collection of student work done on paper. In Phase 2, an exercise book was provided for each student in order that all drawings and calculations could be recorded and easily collected. At the end of each teaching session, I made brief field notes on what had occurred. These included informative details (such as the timing and rooms in Phase 2), and any important non-verbal data (such as notes on student confidence). This writing also served to promote reflection on the day's activities in order to prepare for the next session. This ability to reflect and respond to data mid-intervention is one of the strengths of the chosen methodological approach (Bonotto, 2005). Photographs were taken of both the artifacts being used and of written work produced from the use of artifacts to help present the story of the intervention phases.

### Measure of Change Data

At the end of Phase 1, evidence of the changes students had made in their thinking was obtained by the completion of written tasks and short, audio-taped interviews. Students were asked to complete a new DCT and to review their answers to the previous written tasks. They were also asked to complete two new tasks that required them to consider the ordering of decimal numbers in new contexts. Unlike the initial interview, each of the final interviews was on an individual basis. This no longer seemed to be an issue to the students. In these interviews, students were asked to compare their initial and final DCT papers (both of which were unmarked), and comment upon the reasons why changes had been made.

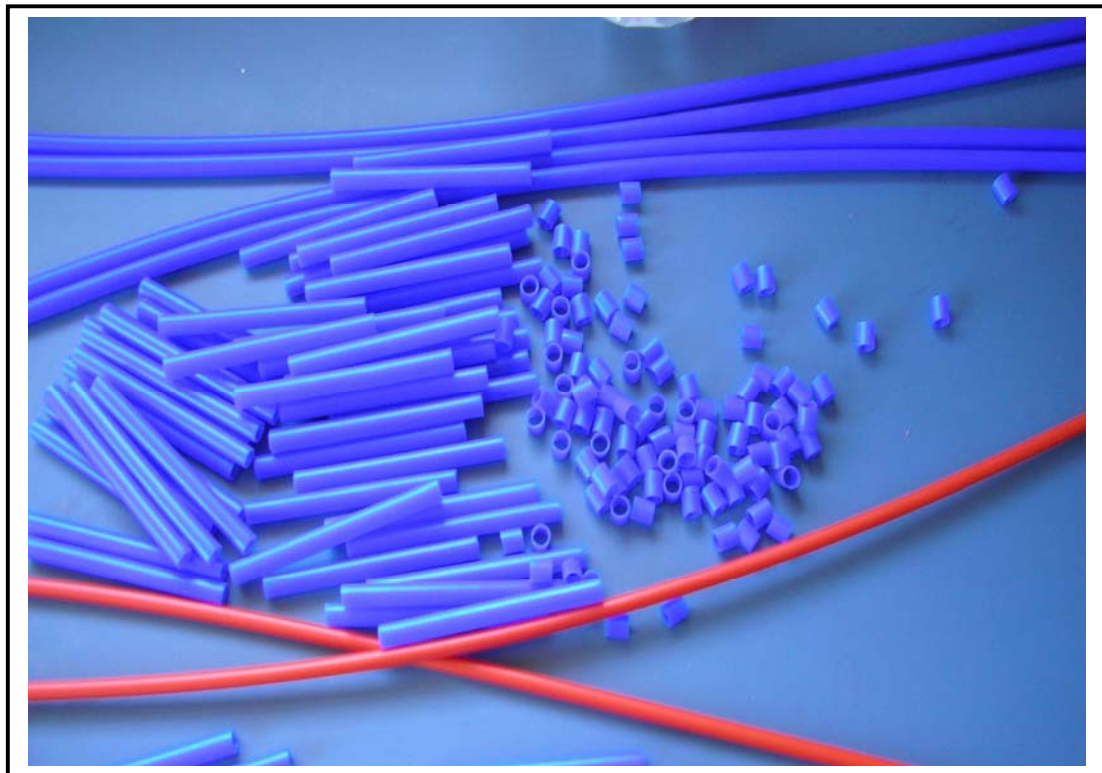
At the end of Phase 2, only written evidence could be collected post-intervention. [One student was in a negative frame of mind on the last day due to illness and voiced her decision not to take part in the planned interviews. This influenced the other students who decided that they too would complete the written tasks but not be interviewed. Work commitments did not allow me to arrange an alternative interview time].

### **Materials**

#### Pipe Numbers

I had previously worked with other students using equipment known as ‘pipe numbers’. This commercial product was adapted from my design of a set of materials that would serve to model decimal numbers. Pipe numbers are a linear model of the number system and are thus similar in structure to the number line. Conceptually they are identical to the Linear Arithmetic Blocks (LAB) described by Stacey and her colleagues (e.g. Helme & Stacey; 2000, Stacey et al, 2001). The use of a measurement-based system to represent decimals has logical support from the two referents of rational number understanding. The inverse relationship between the number of pieces needed to represent a given unit and the size of those pieces is a major aspect of fractional understanding. The groupings of ten extend the place-value system with which students are familiar. Furthermore, the drive to produce increasingly accurate descriptors of measurement through history can be re-duplicated in classroom settings to show the ‘numbers between the numbers’ concept – the density of number property (Lamon, 2001).

A photograph of the pipe numbers equipment appears below. The longest blue pieces are 1.25m long and are used as a representation of the number 1, considering measurement rather than count. The smaller pieces are tenths and hundredths respectively, cut to scale. All of this blue tubing is hollow. The red pipes are non-representational, but are used as assembling aids as the smaller pieces can fit over them. This allows for number representations to be portable. A more detailed explanation of the form and function of pipe numbers is provided in Appendix B.



*Figure 1 Photograph of Pipe Numbers Equipment*

## Phase 2

I wanted to create new artifacts. The pipe numbers could only be used in a repeated addition model of multiplication. The literature provided a small number of alternatives, and the models of buying by weight (Bonotto, 2005) and cutting ribbon lengths (Bulgar, 2003) were adapted for use in this research. Other studies had used volumes of water as static representations of rational numbers (e.g. Hunter & Anthony, 2003; Moss & Case, 1999). I envisaged students pouring water into containers as a means of enacting

quotitive division. Krispie biscuits provided a context for discussing thousandths. [This latter choice was serendipitous, I happened to notice the packet size and biscuit quantity while in the supermarket buying the containers for the pouring water activities].



*Figure 2      Photograph of Containers used for Quotitive Division Tasks*

The next chapter reports on the evidence that was obtained following the implementation of the Method and begins the discussion of this evidence.



## **Chapter Four: Results and Discussion**

### **Introduction: Structure of this Chapter**

Evidence obtained during the interventions is presented with accompanying discussion arising from its analysis. These results are presented in thematic progressions. These thematic layers are:

### **Decimals as Static Entities**

Most of the data for this section was gathered in Phase 1 of the research, with extra material generated in Phase 2.

### **Decimals in Additive Contexts**

This data was gathered in roughly equal amounts in the two phases of the research.

### **Decimals in Multiplicative Contexts**

The evidence for this sub-section was gathered in Phase 2 of the research.

Continuity between the three sections is provided by several features; one, the growing development of the same mathematical context (decimal numbers); two, half of the original student cohort participated in the second data collection phase; and three, the underlying research question that guided the research across all three thematic layers. Within each thematic section, the data is presented chronologically.

Numerical coding (e.g. 1.1.2) within each of these sections is used for ease of reference when these are referred to in discussion.

Student and teacher language is reported verbatim and was not tidied up to conventional forms. Background comments that did not relate to learning have been edited out. This has the disadvantage of not always showing the total learning environment, but does allow for a clearer progression through the learning conversations.

### **Introduction: Conventions**

I have taken the position that in general discussion, it is only important to label whether the speaker is the teacher (indicated by T:), a child (indicated by C:) or a chorus of simultaneous/overlapping responses (indicated by Chn:). Individual discussions have the child's pseudonym given.

Numbers are reported using digits unless the use of words is required to show changing use of language. The use of *italics* indicates that this word or phrase has been emphasized by the speaker. The use of three dots (...) indicates that a sentence is left hanging. A circumflex (^) has been used to show that the current speaker has been interrupted by the next speaker. A hash mark (#) has been used to indicate that a short pause has occurred within the conversation. This is to indicate that some reflection has occurred before the next line of conversation. The insertion of a blank line indicates that more than a minute has passed between lines of conversation.

The use of the Chn: and ^ symbols are intended to show that the reality of these learning conversations was messy and interactive rather than following conversational niceties.

The use of parentheses ( ), indicates actions that took place during the conversation or words that help explain the recorded dialogue while the use of square brackets [ ], indicates thoughts or reflections.

### **Decimals as Static Entities**

Students have to learn how the symbols for decimal fractions inform us of their magnitude. Understanding of decimals as numbers in their own right has been termed 'static' by Irwin and Britt (2004). Knowledge of the relative magnitude of decimal numbers allows students to order sets of decimal numbers reliably, a Level 3 MiNZC objective (Ministry of Education, 1992, p. 40). The revised curriculum, NZC, places this ability at Level 4 (Ministry of Education, 2007a).

## 1.1 Pre-Intervention Data Phase 1 (2006)

### 1.1.1 Decimal Comparison Test

The following table records the student results when they were asked to select the larger of pairs of decimal numbers from an adapted version of the Decimal Comparison Test (DCT).

Table 4  
*Students' Pattern of Responses to Ordering Decimals using DCT*

	<b>Initial Results 3/3/2006</b>	
<b>Name</b>	<b>Code</b>	<b>Exception Count (out of 30)</b>
Mary	No clear pattern	n/a
Grace	Longer larger	0
Wini	Shorter larger	3
Aroha	Shorter larger	4
Ripeka	Longer larger	0
Tame	Longer larger	0

#### Explanation of Codes

*No clear pattern* – sometimes longer decimals chosen, sometimes shorter ones, with no discernable logic being employed

*Longer larger* – increasing length of decimal believed to indicate relative size, 0.14 is considered larger than 0.9

*Shorter larger* – decreasing length of decimal believed to indicate relative size, 0.1 is considered larger than 0.99

*Exception Count* - the number of items that did not appear to comply with the overall system used by the student. E.g. a student who routinely selected the longer decimal as the bigger number may have a single contradictory answer, giving an exception count of 1.

The exception count served as an indication of the relative stability of the current constructs. Grace, Ripeka and Tame employed a totally consistent method to decide the ordering of decimals. Their decoding of the decimal symbol fits an existing schema, and they relied on this knowledge every time they solved one of the problems. Wini and Aroha operated with their systems for around 90% of the problems. Mary had no clear system. This may indicate that she had no way of decoding the decimal symbols but it could also be explained by her having two (or more) competing mental schema. Unable

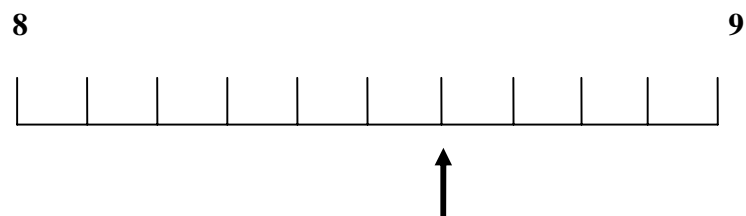
to choose between competing models, she may be working with both of them. This latter explanation would be consistent with the ‘within child’ variation noted by Siegler (2007).

The initial data established that none of the six students had achieved the Level 3 MiNZC objective of ordering decimals. It was inferred that this was because they had not understood how the place-value system extends to decimal numbers. The low exception counts show that at least five of the students came to the task of ordering decimal numbers with a strongly held prior understanding, even if that understanding was faulty. The students were not simply ‘blank slates’ awaiting the input of new data. This evidence is important to remember when we examine the role of cognitive conflict in their learning.

This data is consistent with similar studies in two aspects; both to the strong presence of whole-number thinking (e.g. Saxe et al, 2005; Steinbring, 1998; Streefland, 1991) and to the stability of the constructs used to make the decisions (e.g. Moss & Case, 1999; Nesher & Patel, 1986; Resnick et al, 1989; Sackur-Grisvard & Leonard, 1985; Steinle & Stacey, 1998, 2002). Within the data is evidence that the students did not understand the placeholder role of zero. For example, in selecting 0.061 as being larger than 0.53, the placeholder zero in the tenths column has been ignored. This accords with the research studies cited above and Steinle and Stacey (2001).

### 1.1.2 Other Initial Assessment Tasks

The students were also asked to identify the mark indicated by the arrow in the diagram below.



*Figure 3*      *Diagram of Number Line Task*

*Table 5*  
*Students' Responses to Number Line Task*

	<b>Mary</b>	<b>Grace</b>	<b>Wini</b>	<b>Ripeka</b>	<b>Aroha</b>	<b>Tame</b>
Correct answer 8.6 or $8\frac{6}{10}$	6	No answer	8.6	14	No answer	8.6

This task is parallel to one used in the NDP (Ministry of Education, 2007b, p. 29). The intention was to assess whether students could interpret the diagram in such a way so as to coordinate knowledge of whole numbers and fractional numbers to produce a single numerical answer. Ripeka and Mary had counted 'six'. Ripeka had added this to the 8 and seemingly ignored the information that the arrow was pointing to a number that exists between 8 and 9. Mary reported the count and ignored all other information. Ni's (2000) study showing the difficulties students have with interpreting the number line representation, suggests caution when considering the validity of this assessment item. It is possible that, if the students had not previously used a number line with fractional numbers, then problems with decoding the artifact may have masked the presence of the knowledge being sought. This might explain why Grace and Aroha did not record any answers. Later work with other representations of tenths was needed to provide evidence to support or refute the observation that only Wini and Tame could coordinate tenths and ones.

### 1.1.3 Initial Interview

A group interview was held. To promote discussion, I wrote pairs of decimal numbers that were similar to those used in the DCT and placed these in the centre of the group. Excerpts from the interview are given to demonstrate student thought.

a) Students were asked to choose between 9.3 and 9.21 as the larger number. Most of the students chose 9.21 ( $21 > 3$ ) including Mary.

T: How did you make your choice? (9.21)  
 Mary: I guessed because it is the highest  
 T: Why did you think it was the highest?  
 Mary: Because it is the higher number [We were going in circles].

Aroha appeared to get it right.

T: Why did you pick (that one)? (9.3)  
 Aroha: I just chose the lowest, and yeah, the lowest number there (Pointed to 9.3).  
 T: OK, so you were trying to pick the highest but you chose the lowest, why do you do that?  
 Aroha: I don't know. I just think it's the way it works

I consider it likely that Aroha had previously recognised that her application of a whole-number ordering system was incorrect. Her language showed that the longer decimal was actually believed to be the largest (“I just chose the lowest”). She also believed in the existence of some rule (“I just think that is the way it works”) which seems to imply that she reversed what she actually thought in order to get the right answer. If so, this is possible evidence of an external locus of control as she was not engaging with the problem conceptually or prepared to back her own judgment.

On the DCT evidence, she nearly always selected the shortest decimal as having the larger magnitude (e.g.  $0.5 > 0.75$ ,  $0.3 > 0.426$ ). Her written answers were consistent with students in earlier studies where further investigation revealed that they were focusing upon denominator size or thinking that decimals are in fact denominators (e.g. Hiebert & Wearne, 1985, Nesher & Peled, 1986; Resnick et al, 1989; Sackur-Grisvard & Leonard, 1985; Stacey & Steinle, 1999). A rational pedagogical response would have been to focus upon the difference between decimal places and denominators. If the DCT was the sole piece of evidence gathered, it would not have been discovered that Aroha was operating with a kind of reverse logic. The interview revealed that while Aroha *operated* with this ‘shorter is larger’ system, her *underlying construct* was the same as many of the other students, that ‘longer is larger’. This shows the importance of gathering more than one type of data in order to inform pedagogical decisions.

b) Students were asked to choose between 0.3 and 0.09 as the larger number. With 0.09 being the entire group’s choice I asked how that decision had been made.

Ripeka:	Because it’s higher
T:	OK, but how do you know it’s higher?
Ripeka:	Because it has a number nine
T:	Does that zero (pointing to the 0 in the tenths column of 0.09) mean anything?
Chn:	No

The students indicated that the zero in the tenths column could be ignored. This was in keeping with the written evidence in the DCT results. This conformed to the whole-number view of number where zeroes to the left of other digits only arise as part-solutions to problems and then are discarded. For example, when operating the vertical algorithm for  $83 - 78$ , a zero may be written in the tens column during the working out but then is not reported in the final answer.

e.g. 
$$\begin{array}{r} 83 \\ -78 \\ \hline 05 \end{array}$$
 The answer is 5

## 1.2 Teaching Interventions

### 1.2.1 Introduction of Pipe Number Decimal Equipment Session 1

The first part of the session was devoted to addressing the issue of iterating unit fractions and is reported in section 2.2. After working with area models of fractions including tenths, the decision was made to introduce the linear decimal model called ‘pipe numbers’. These were brought out and examined by the students. The students established that the pieces they were handling were tenths by assembling ten and comparing this with the ‘one’.

T: I want you to make  $7/10$  (and when completed) OK, do you know how to write that as a decimal?

Written answers include 7.10 and 10.7 No-one connected the use of symbols with the decimal numbers they had previously seen. This lack of understanding of decimal notation is consistent with that reported by other researchers (e.g. Moss & Case, 1999; Owens & Super, 1993; Saxe et al, 2005). Without a mutually comprehended language, any discussion concerning decimals would be meaningless to the students. Shared understanding is crucial (Godfrey & Thomas, 2003).

I decided to introduce the standard notation for decimals.

- T: OK, have a look at what I wrote (0.7).  
 C: Oh point seven  
 T: What does that first bit say?  
 Chn: Zero  
 T: Zero, zero point seven. That zero, can anyone think what that zero is telling us? (No one answered). Zero ones, ‘cause we haven’t got any of these big long ones, all we’ve got is these seven tenths, so this time I’m going to write a decimal number down and this time I want you to make it, so you just watch and I’ll write it here. (Wrote 0.4).  
 C: But how do you know that its tenths if it’s just like that? [The student felt safe in questioning the convention and was not passively accepting ‘expert’ knowledge without a justification].  
 T: Right, now that’s a good question. The way you know it’s a tenth is *that* zero tells you there’s no ones, no whole numbers, and then that point means that anything that comes after that point is going to be a tenth for the first one.

I realised that this explanation was neither convincing nor well worded and wrapped up the lesson. An important linkage was missed, the chance to reinforce that 0.7 and  $7/10$

were exactly the same number but with different representations. Referring students back to the language/place-value column link that they already knew (e.g. 4600 would be read as ‘four thousand six hundred’) would show them that the ‘thousand’ and ‘hundred’ were implied by the place-value column they occupied. This could then be linked to the new place-value column of tenths. I could also have used the term ‘alias’. (This word arose during discussion about confidentiality at the time of the initial interview. The students initially wanted to be called by an alias while they were being audio-taped). Transferring the use of the term to this setting may have helped them recognise that a number may exist under several names and symbols without actually changing its magnitude.

### 1.2.2 Challenging Whole-Number Thinking Session 2

The session began with the students working with tenths, manipulating the pipe numbers to model numbers in paired, written form such as  $7/10$ , 0.7 and  $3/10$ , 0.3. In doing so they were making the connection that decimal notation did not create a new number, but provided a new symbolic representation for a set of familiar fractions. That the students had a problem to solve, a physical means of accomplishing that task, and were working cooperatively ties in with recommended practice (e.g. Hiebert et al, 1997; Moyer, 2001). The students developed the knowledge of the symbolic form through the linkage of task, materials and discussion. This creation of meaning ‘in situ’ rather than ‘a priori’ is consistent with the statements of Cobb (1992) and Neuman (2001).

#### Hundredths

Following this work, I looked for a way to introduce hundredths and hoped that a measurement task would provide the impetus for this. The students were set the task of measuring the length and width of the room. This also served to demonstrate that decimal numbers were not always less than one. The children had used 3 ‘one’ lengths and 10 ‘tenth’ pieces.

- T: So you guys measured the width of the room and used 3 wholes and ten tenths. OK, so 3 wholes and ten tenths...
- C: Hey! That’s four wholes!
- T: Why?
- Chn: Cause ten tenths is just a whole.



The students had not automatically adjusted 3 and  $10/10$  into 4 but once given a prompt were able to do so.

- T: Now try the length of the room.  
C: We'll use some of these and some of these and some of these baby ones  
(The hundredths)  
C: We need a bit more, we'll have to use these ones (the hundredths).  
(The others ignored her).  
T: You've still got a bit on the end what are you going to do about that? (Trying to prompt).  
Chn: Six wholes and four tenths  
T: Six and four tenths as a decimal?  
Chn: Six point four

The extra distance (smaller than a tenth piece) was ignored, despite two students voicing a need to measure it and my prompt. I propose that the students were not going to allow the degree of difficulty in the task to rise beyond the level of tenths because they felt 'safe' with the current level of complexity. Dissonance was avoided by not acknowledging its existence.

- C: What are those little ones for?  
(One student persisted in drawing attention to the 'hundredth' pieces).  
T: We'll learn about those today  
C: What are those?  
C: We'll look at those?  
C: They're ones!

Interest was now being verbalized by four of the six students. It was time to promote a new learning site, capitalizing on the current student interest.

- T: Can they be ones (indicating the previously acknowledged representation of one) *and* these be ones (indicating the hundredths)?  
C: No, they're little tiny things  
C: They are twelfths!  
T: If they were twelfths, how many would we need to make a one?  
[How do we establish proof, by teacher decree or by student action?]  
C: They are twentieths!  
(They started organizing them by lining them up against 'tenths' pieces).  
C: Twentieths? (Same child as said this earlier).  
Chn: No! #  
Chn: Fiftieths? Hundredths? Hundredths! Hundredths!  
T: Why do you say that?  
Wini: Cause ten of those make one of those and ten times ten is a hundred. (Others nod).  
T: Hey! I didn't have to tell you that, you guys worked it out. That is good thinking.

A belief that the students could work it out had paid off. I could give legitimate feedback on the mathematical skills they employed to solve a problematic task rather than simply telling them the answer. I had drawn attention to their conflicting statements as Cobb

(1994) suggests, and let them seek to resolve the issue. This is also an example of Houssart's (2004) observation that equipment may facilitate students' appreciation of the magnitude of numbers in a more powerful manner than exposure to symbolic forms alone provides.

- C: But what are we going to write them down as though?  
[They seemed to have an ingrained acceptance that a symbolic form was required].  
T: Well, how would you write it as a fraction, do you think?  
[I wanted them to produce  $1/100$  and then compare this with  $1/10$ ].  
C: Ten over ten? [Presumably from a  $10 \times 10 = 100$  connection].  
T: How did you write one tenth? (Students wrote  $1/10$ ). Yes, that is right. You put a ten underneath for that one. What could you do for one hundredth? Have a go!  
C: As a fraction number?  
T: Yeah

Despite the instruction, all of the students proceeded to try and write  $1/100$  in decimal form. One child wrote it correctly, other presentations included .100, 1.00, 100.1, and 000.1 Proficiency with the linkage of tenths and the first decimal place had not been generalized. On reflection, this was to be expected. Consistent with variation theory (Runesson, 2005), unless students have been exposed to variation in a deliberate fashion, they are unlikely to detect the structural elements of the situations they have observed. Working with tenths had constituted a single data set with regard to the meaning of decimal symbolism, and generalization was not possible without variation (Mason, 2005).

#### Cognitive Conflict and Steps to Resolution

I had hoped that writing a number with hundredths might give a good lead-in to challenging the whole-number construct of decimals, but the students' difficulty with the symbols served as a barrier to progress. The next task was set to create a student-recognized cognitive conflict. The students would be confronted with a paradox by drawing together two acknowledged 'truths' that would contradict each other. Having established some links with prior knowledge and new learning, and having provided some positive experiences, I considered that the students were now cognitively and socially ready for the dissonance this task would produce. As Lambdin and Walcott (2007) have claimed, a supportive environment had to be created in order for situations of cognitive conflict to produce learning. It was not enough to simply challenge thinking

without attending to wider aspects of the learning situation, or students may have disengaged with the task once its difficulty was recognised.

T: Now I'm going to give you a tough one.  
Chn: Groans  
T: Yes, I'm hard and mean.  
Chn: Laughter

I did not want to pretend that the next task would not cause some anxiety. The fact that the students were laughing showed me that they had emotional capital to draw upon.

T: See if you can make this one, 0.12. (This was written on the board as well so that the symbol would be continually engaged with).  
C: That's twelve! [A typical whole-number response].  
T: OK, see what you will make. [Not giving validation or refutation, but simply asking that the task be carried out].  
C: Twelve! Got it! (Showing a model that used twelve tenths and then counting them out for me).  
T: So you've put twelve of those tenths on, OK, what does that symbol tell us (pointing to zero)?  
C: Zero  
T: How many ones is that? (Pointing to the written 0.12 on the board).  
C: Zero.  
T: But your one (meaning her model) is bigger than 1. (Pointing to her pipe representation).

The student was faced with two evident 'truths' that could not simultaneously co-exist. 0.12 must represent twelve tenths according to her current whole-number scheme, but the knowledge that ten tenths was equivalent to one whole was also known to be true, as it had been reinforced during the measurement task of the previous day. The use of the materials had produced a paradox. The whole-number thinking produced a result that conflicted with the fractional number thinking. These two referent systems had now been made explicit, an activity that Owens and Super (1993) believed would help students reorganize their thinking. The 'stress' created by this new awareness forced a re-examination of the prior constructs. The use of physical models embedded in a task was coupled with reminders of the students' prior statements to 'trouble the thinking' (Zazkis & Chernoff, 2008). That this was to prove effective is consistent with Moskal and Magone (2000). The emotional involvement with the conflict was predicted by research that had demonstrated the tenacity of intuitive models (e.g. Harel & Sowder, 2005; McNeil & Alibali, 2005).

Others produced the same model and went through the same conflict. Here another student was convinced that her model was correct, but was also convinced that ten tenths represented one.

C: But I've got ten tenths here.  
T: Yeah, and what do ten tenths make?  
C: One.  
T: So you've got a one there and two extra tenths, you've got 1.2, I want 0.12

The whole-number thinking being expressed by the students was predicted by the DCT results of Tame, Grace, Ripeka, and Mary, and by the interview comments of Aroha. Interestingly, the two students with the DCT result of 'shorter is larger' - Aroha and Wini - initially made the same arguments as the others but were the first to create correct models. This may be seen as supportive evidence for the earlier supposition (1.1.3a) that students using the 'shorter is larger' construct had started to recognise that the whole-number construct did not fit decimal situations and were looking for an alternative.

I was determined to allow the students to struggle with this cognitive disturbance and looked to shape their thinking with a range of question types (Fraivillig et al, 1999; Martino & Maher, 1999; Sahin & Kulm, 2008) rather than 'helping them out' with my mathematical knowledge. It would have been easy to let scaffolding questions slip into 'path-smoothing' actions such as supplying the correct model to copy. This would result in taking away from students their agency to learn, as Watson and De Greest (2005) have warned against. I decided to make explicit to the students the link between what they had physically modeled and its decimal symbol.

T: OK, you're using some of those little ones. You've got ten tenths and two of those little ones. (Wini was still making another model using twelve pieces). So you've made me a whole one and those little ones. You've made me 1.02.  
Wini: I don't get it!  
T: That's OK, I never said this one would be easy, it *is* hard.

Wini adjusted her model by replacing one of the one-hundredth pieces with a one-tenth piece. Her model still involved twelve pieces, but a new piece of feedback was possible.

T: That is 1.11 #  
Wini: Oh!

She then adjusted her current model to the correct one, gave me an expectant look, and then received a nod. She had used the information received from my responses - 1.02 and 1.11 - and had made the connection.

T: OK, one person's got it! When you have an idea, come up and show me.

While intended as encouragement – ‘if someone can get it then I can’ – this could also have been perceived as pressure – ‘I must be dumb if I can't get it’. I think that four students responded to this information as intended, but Grace may have felt under pressure to perform.

T: You've made 0.02. #

Aroha: Oh, I know! (She adjusted what she had to produce the correct model, probably because she had just made the connection between the smaller pieces and the hundredths place).

Mary showed me her model.

T: This one is 0.2. We need 0.12

(Mary adjusted her model by swapping two, tenth-pieces for two, hundredths-pieces).

T: So now you've made 0.02

Mary: Oh I know! I know how to do it now. (She made the correct model, possible also due to having just seen the link between hundredths pieces and the hundredths place).

Mary: That was easy!

The first three students took around 15 minutes to complete the task.

Those three were given other examples to create, and then told to write their own numbers and then to make models of them. They continued to do this while the others repeated earlier models or tried new ideas. Tame and Ripeka made correct models around five minutes later. Those five proceeded to their own models, checking them with each other and myself. I believed that it was important that they created their own problems to solve, as this would give them a sense of ownership of their new learning.

A total of 25 minutes had passed and Grace was now the only one who had not completed the initial task and was feeling stressed. I felt the tension of wanting to ease her stress but also my belief that if she could work through this task herself, it would be important to her long-term understanding and self-efficacy.

Grace: This is *hard!*

T: OK, what you have made is 0.44. We'll get everyone together soon and share ^

Grace: I can't do it, it's too hard (nearly crying) [To her credit, she had not tried to copy what the others had done, but wanted to work through the problem herself].

T: Remember when we had one of these, one tenth, we wrote like this, 0.1. You bring your page over; see where you did ones like that (reminding her both of her knowledge and her

previous success). Six tenths for that one eh? (Nod) So how did we write that as a decimal?

Grace: 0.6

T: That's right!

Grace: So why aren't we doing ten of those and two of those? (We've been at this point several times before).

T: Watch as I make up 0.6

Grace: (Tears in voice) But why aren't we doing ten of those and two of those?

T: Cause ten of those makes up a one. So I've put three of these on now and that makes this one 0.63. You write that down and then try making this one here. See where the two parts come from? Where does the 6 come from?

Grace: Those. (Pointed to the tenths)

T: And the three part?

Grace: Those. (Pointed to the hundredths)

T: Right.

Grace: But why aren't those 3's?

T: OK, those are not threes; remember how long a one is? They are three *hundredths*, they're not three as one, two, three, it is three of this *size*. [I tried to make the distinction between counting and quantity].

Grace: Oh. So when you do those, those are tenths and those are hundredths.

T: Yeah, that's right. So write 0.12. (She wrote it.)

Grace: So why don't you use ten of these? (Her tone was more positive now)

T: Because then you would make a whole one.

Grace: This is confusing (But the resignation has gone from her voice, she was re-engaged).

T: Yes it is at the start.

Grace: But why do we have a zero?

T: 'Cause that tells me that I have no ones in it. It is smaller than one, like a half is smaller than one. Tame, can you make 0.12 again?  
(He did and showed it to Grace.)

Grace: But why does it have a zero in it?

Tame: No full ones.

Grace: I get it now (Tentative tone)

T: Try 0.61

Grace: Is that it?

T: You've got six tenths and one hundredth, so that is 0.61

Grace: Why did I find it so hard? When you get it, it's easy but when you are learning it, it's hard

T: It's OK to struggle through with it

Grace then proceeded to make other models correctly along with the other students. Grace was reorganizing out loud through this process. The key aspect of her prior concept that was difficult to release (regarding all of the pieces as 'ones') probably related to her iterative skills with fractions still being very new and insecure. This is evidence of the powerful influence of 'how many?' over-ruling consideration of 'how much?' as described by Lamon (2001) and Sophian (2008).

It may have been better to have left her for a day to think about the problem, but this may also have meant that she would worry about it for a day longer. That she battled through this hurdle was important. She tackled all subsequent tasks involving the ordering of

decimals correctly and showed evidence of understanding at the final interview. Her statement about the learning that “when you get it, it’s easy” was important from the self-awareness aspect of learning.

The students made new numbers and built their own models. They were clearly aware of their achievement and pleased with themselves.

The use of counterexamples had proven to be powerful. By not diminishing the cognitive load, the students were forced to re-examine and restructure their thinking (Cobb, 1988). Using the label of Zazkis et al (2008), the students perceived this task as pivotal, one that challenged existing thinking to the point that a new understanding was possible. It needed time to ascertain that this moved to be a bridging task, allowing progress towards generalization. As Bronfenbrenner (1979) points out, learning must demonstrate its presence across a number of settings, such demonstration being termed ‘developmental validity’. This allows us to distinguish between a student only having a localized, situational proficiency or whether the underlying structures have been understood, allowing for a more general application.

### 1.2.3 Using Measurement to Create Place-Value Understanding Session 3

#### Review

A pipe number model was made up for 1.13

T: What is this one?  
Chn: 1.13, 1.03, 1.13, 1.31

Recursion in action! Fortunately research and experience had prepared me for this apparent lapse in understanding by several students (e.g. Pirie & Kieren, 1994; Martin et al, 2006). Rather than expect to have to re-teach, it was better to react to these answers as indicating that the students needed to reflect upon their newly acquired knowledge as Mack (2001) suggests.

T: OK, let’s have a look at it. Where is the ‘1’part?  
Chn: 1.13, 1.13

Ignoring my last question, they had returned to the original one and correctly answered it. The new knowledge was not lost; it existed simultaneously with the old knowledge and was retrieved independently. This was evidence of what Siegler (2000) called

‘overlapping waves’, where students simultaneously hold primitive and more advanced concepts and oscillate between their usage.

T: OK, ‘cause we have one of these one size, one of the tenth size and three of the (voice trails) can anyone remember the name of the other size?  
Chn: Hundredths  
T: Yeah, that’s right.

### Measurement

The students were set the task of using the pipe numbers to measure objects of their own choosing around the room and recording the lengths in decimal form to both paper and the whiteboard. The idea was to generate data that could be used to make sense of the process of ordering decimals by using actual objects as referents rather than exclusively relying on the number symbols. When first engaging with counting, students are expected to engage with both the symbols and concrete referents, with linkage between the two established through enacted tasks. We use whole numbers to enable us to count objects. It seemed logical to parallel the introduction of the new symbolic form (decimals) with a primary reason for their existence - we have decimals because they enable us to measure objects. I planned on the task lasting for around 10-15 minutes. I anticipated taking Grace (and possibly others) through the phase of linking hundredths as physical objects to the symbols again. As it turned out, all of the students were so engaged in the task and generating their own issues to discuss within the task, that I let it continue for about 30 minutes. As all students were actively engaged, I relinquished my plan of doing more direct teaching.

The difficult work of accepting the need for changing their decimal construct had changed into confident enthusiasm among the students. They had also found the task fun and enjoyed the freedom of making choices.

One advantage of this kind of task was that students were able to work at a self-determined pace. This allowed them the time to think about what they were writing and make comparisons as they noticed them, without having the pressure of having to complete an externally set number of tasks. As the teacher, I had time to allow students to come to me with questions or to pose my own questions directly into their immediate situations.



Measurement tasks are seen by Lamon (2001) as being an important mechanism of linking the realistic application of mathematics to the abstract symbolism. Hodge et al's (2007) criteria of effective instructional tasks – that they capture student attention and draw them towards important mathematics seem to be met in this session. For linear measurement tasks, standard metric rulers might have been expected to be used rather than the artificial pipe numbers equipment. My experience was that students interpreted linear metric measures in whole-number terms; for example, 1.43m being pronounced as 'one metre and forty-three centimetres' and thus avoiding the issues of decimal place-value. With no name for the sub-units other than their place-value terms, this avoidance can be circumvented by using the pipe numbers.

Issues that arose

Episodes of addressing conceptual issues arose in the context of the wider task of measurement. I believe that these small conversations - arising out of immediate student interest - were very powerful in helping continue the reorganization process.

a) Thousandths

The students had been measuring items and recording their results.

- C: So we use ones, tenths and hundredths...
- C: What about thousandths?
- C: Yeah we could have thousandths, what would they be like?
- C: Real big...No, real small, they would be like (shown with thumb and forefinger).
- T: Would thousandths be big or little? (They had no physical model of 1/1000; they could only make sense of it by extending their mental schema). (Pause).
- T: How could you make them?
- C: You'd have to cut this down to about there (pointing on hundredths pieces)
- C: You'd have to have heaps and heaps like a thousand of them
- T: You're right

I responded to the student-initiated discussion by eliciting rather than providing answers. The students had started to consider the decimalization process as potentially generalizable. They had started to discuss the equipment in terms of what it *could be*, rather than what it *was*. This may have been inspired by realizing that even using hundredths would not exactly capture the length of some objects being measured. It cannot be said that they were yet able to operate on the concept to the degree that Sfard (1991) described as 'reification' but progress was being made. It is these small glimpses of insight into how experience eventually leads to larger conceptual changes that microgenetic studies are especially suited to capture (Siegler, 2007). The reflective

language on thousandths (“Real big, no real small”) can be seen as an example of Tzur’s (2007) ‘reflection-upon-activity’ model where an unexpected result allows a re-shaping of thought. Whole-number thinking promoted the first thought – real big – but the newly acquired decimal knowledge competed with this first thought to provide self-correction – real small. Essential to this conversation were two elements; firstly, that it was student-initiated, and secondly, that there was time within the broad task instruction for this reflection to occur.

b) Zero as place holder

Tame had measured the sink bench and had used a ‘1’ and a ‘1/100’ piece, 1.01. He was looking at his measurement and his page with puzzlement.

- T: OK; this is a tricky one cause you’ve got a one and then you’ve got one of those hundredths, how many tenths do you need? (Tame shook his head) Have you got any tenths?  
 Tame: No.  
 T: No, so what do you reckon you’ll do when you write the number down?  
 Tame: One. (Pause).  
 T: Yeah one, and how will show that you’ve got no tenths?  
 Tame: You’ll go one point zero one.  
 T: Excellent. Excellent. That’s probably the first time that you’ve done that and you saw what to do, worked it out yourself, well done.

The students wrote their measurements onto sheets of A3 paper. Here is one pair’s record.

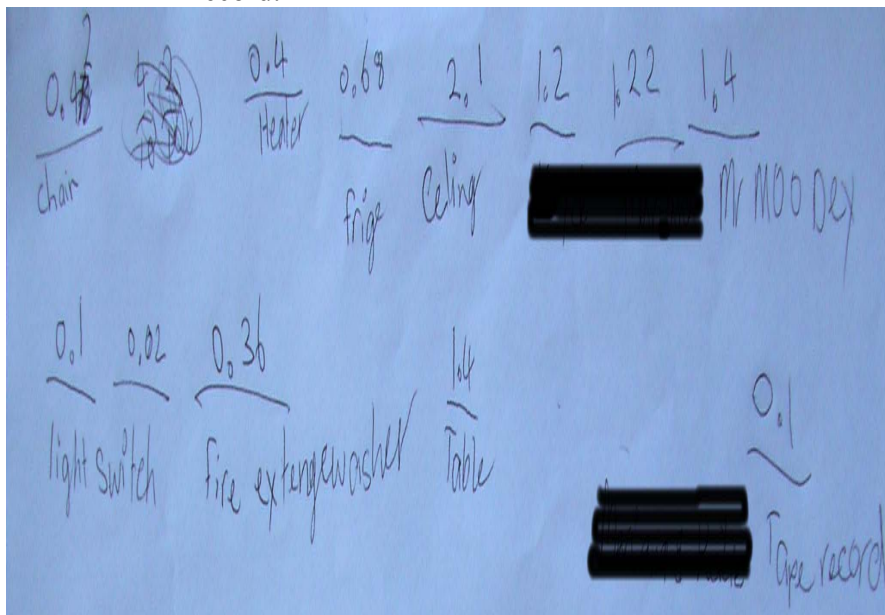


Figure 4 Photograph of Student Recording following Measurement Task

From this photograph, we can identify a number of learning experiences. In the top left corner can be seen the fractions  $\frac{4}{10}$  and  $\frac{2}{100}$  crossed out and re-written as 0.42 chair. The students had initially written the actual descriptions of the pipe pieces used (four, tenth pieces and two, hundredth pieces) but had then re-written these fractional forms into the decimal equivalent without prompting. At the bottom left, we see how a place-holder zero has been used to indicate that the switch is much smaller than the light. The relative sizes of the heater (0.4) and the fire extinguisher (0.36) can be referred to as a means of reinforcing the distinction between whole-number and place-value methods of determining size. The heights of students (names blanked out) can be compared with each other and with me. Each pair of students recorded their own results. In doing so, rich data sets were collected and were available for further discussion. Some of this discussion is reported below.

T: Tame, can you tell me what the sink bench was?  
Tame: 1.01  
T: Can someone tell me what Tame did to measure the bench?  
(Asking for the measurement to be imaged, not performed).  
Aroha: One whole and one hundredth.  
T: OK, so what does that zero tell us?  
Aroha: That there's no tenths.  
T: So look at what these guys did, they did the actual switch part of a light switch and they got 0.02. So what does *that* zero mean Tame? (Indicating the left-most zero).  
Tame: That there was no wholes.  
T: And what does *that* zero mean Ripeka? (Indicating the right-most zero).  
Ripeka: No tenths.

T: You've written 1.00. OK, so was it *exactly* one of those sticks?  
C: Yep.  
T: That is very accurate measuring. [I actually meant to say 'accurate recording'. The child wanted to show that no pieces other than the ones pipe was used and found a way to record this].

The students had discovered contexts where the use of zero as a placeholder had meaning. In the initial interview (1.1.3c), the students had voiced a belief that the use of a zero in the tenths column carried no meaning; which is why they thought 0.09 was larger than 0.3. Now they were voicing the fact that zeroes in different place-value columns convey different pieces of information. As predicted by research (e.g. Goldin & Shteingold, 2001; Jones et al, 1996; Lachance & Confrey, 2001), concrete referents were

instrumental in producing a deeper awareness of how digits are employed in decimal symbols, a development of semiotic understanding.

c) Density property

Students were measuring each other's heights.

T: So how big is Grace?

Mary tried 1.2, then 1.3 as Grace's height was somewhere in-between. Mary had a puzzled look, and then went to get hundredths pieces to complete the task. She recognised a need for hundredths to exist in order that the task could be completed.

d) Absolute difference between decimal numbers

T: So Mary is 1.18, Wini is 1.23; how much bigger is she?  
(Answers were shown by a spread of hands, the hoped-for quantifying did not occur)

.  
Later on they measured me.  
Aroha: Hey you're 1.4. That means you are 2 tenths bigger than her.  
T: Why do you say that?  
Aroha: She's 1.22 and you're 1.4 so that's 2 tenths.  
T: Great!

This student had done some further processing in the interval between my asking the question and this situation. While the answer was not *exactly* correct, it was evidence that she was starting to think of difference numerically and not simple spatially. With regard to place-value, the crucial advance was that the difference was not being expressed as a whole number, nor was it being calculated in whole-number operations (the difference between 4 and 22). This small - and almost certainly transitory - phase was part of Aroha's journey of learning to deal with decimal subtraction in a more formal way. It was important not to correct her calculation but to acknowledge the new extension of logic that had occurred.

During discussions, the students had been comparing their numbers with the actual measurements of the objects. I now wanted to introduce some numbers that were independent of the pipe number representation to see whether the ordering of decimal numbers was starting to be generalised. The 'somethings' I referred to were not actual objects; the students were aware that we were now playing a 'let's pretend' scenario.

T: I measured a 'something' and it was 1.25; was it bigger than Mr. Moody?  
 Chn: One 'yes' drowned in a chorus of 'no's'.  
 T: OK, so Mr. Moody was 1.4 and the something was 1.25, which is the biggest?  
 C: The something.  
 Chn: Mr. Moody.  
 T: How can we tell, Aroha?  
 Aroha: 'Cause that has 4 tenths and the something has two tenths

T: This 'something number two' was 0.3. Is there anything that we measured up there (on the whiteboard) that was smaller than 0.3? What do you think Grace?  
 Grace: The switch on the light switch.  
 T: How big was that?  
 Grace: 0.02

I realised that this example - though correct - did not pose a challenge to whole-number thinking as the relative size of the digits alone would settle the issue.

T: Here is 'something number three', 0.27. Which is the biggest, 'something number two' or 'something number three'?

Chn: 'Something number three'. [Whole-number thinking,  $27 > 3$ ]

T: OK.

Wini: No, 'something number two'. [This better choice of numbers has provoked debate].  
 (After a few seconds this was agreed to by the others).

T: Well, you guys are changing your minds; did you all just copy Wini? What do you think Ripeka? [I wanted to challenge them to defend their reconsideration and so voice to themselves the reason for the change].

Ripeka: Two's bigger  
 T: Why?  
 Ripeka: (Pointed to the tenths) Cause that one has two, and that one has three.

Social interaction had worked to create a central pool of knowledge that individuals could respond to. One voice in the 'measure Mr. Moody' example re-thought her position after hearing the majority. The majority re-thought their position after hearing one voice in the latter example. This could be seen as support of the ideas of Nyikos and Hashimoto (1997) and Martin et al (2006) that the existence of group discussion may create synergy where the pooling of knowledge promotes new learning.

The fact that the students kept moving between schema was predicted by the recursive nature of learning as described in the literature (e.g. Pirie & Kieren, 1994; Pirie & Martin, 2000; Siegler, 2007). Being aware that this was normal allowed me to focus upon what was acceptable as proof, rather than simply giving positive and negative feedback to responses.

### 1.3 Post-Intervention Data

#### 1.3.1 Proficiency with ordering pairs of decimals.

The following table compares student results from their initial and final DCT.

Explanation of new codes

‘Money’ level – correct ordering to hundredths place, but inconsistent results with pairs such as 1.23 and 1.234 where the numbers are identical to two decimal places.

Task Expert – all items correctly ordered

*Table 6*

*Comparison of Students’ Initial and Final Responses to DCT*

	<b>Initial Result 3/3/2006</b>		<b>Final Result 17/3/2006</b>	
<b>Name</b>	<b>Code</b>	<b>Exception Count</b>	<b>Code</b>	<b>Exception Count</b>
Mary	No pattern	n/a	Money level	3
Grace	Longer is larger	0	Task Expert	1
Wini	Shorter is larger	3	Task Expert	0
Aroha	Shorter is larger	4	Task Expert	0
Ripeka	Longer is larger	0	Task Expert	2
Tame	Longer is larger	0	Task Expert	1

The final data indicated that all students had improved in their ability to order decimal numbers. It was inferred that this was due to an enhanced understanding of place-value. The low exception counts showed that the students were working from stable schema. This does not imply that the previous schema had been totally eliminated from their thinking (Tirosh et al, 1998), but currently the new conception was dominant (Siegler, 2000).

#### 1.3.2 Context-based questions

All students were asked to complete two tasks involving decimals with contexts different from any we had discussed in the learning sessions (See Figure 5 below). Each child was asked for their answers in an individual interview.

1. Here are two tins of paint for sale. One holds 4.5 litres and the other 4.25 litres. Which one holds the most paint? How do you know this?



4.5 L

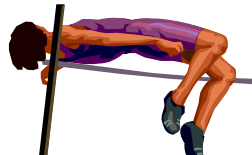


4.25L

How much extra is in the larger tin?

2. Here are the results of the high jump. Can you put them in order from 1<sup>st</sup> place to 4<sup>th</sup> place?

Jenni	1.6m
Hoani	1.55m
Lee	1.09m
Sasha	1.37m



*Figure 5 Contextual Tasks used as part of the Post-Intervention Interview*

All students correctly identified the 4.5L tin as holding more paint and all correctly ordered the high jump results. Only Wini attempted the question regarding the difference between the cans. Each student was asked to explain their answers, to see whether the new procedure they had employed was founded upon a change in thinking about decimal numbers. While each student was interviewed individually, I have collected their responses into groups for ease of reporting.

Teacher: Why do you think that tin was bigger?

Wini, Aroha, and Grace

Wini: That's just hundredths (pointing at the '5' in 4.25) that's not that big, and five (the '5' in 4.5) is bigger than two.

Grace: Because that's two tenths and that's five tenths

Aroha: Two tenths and little bit of hundredths and that one has five tenths

Their explanations were varied, which showed that they were using their own language rather than trying to remember a teacher-provided script. This is suggestive of intellectual autonomy as Yackel and Cobb (1996) presented as being an important signal that conceptual learning has taken place. These three explanations all showed evidence that the students were able to explain a reason as to why 4.5L represented the larger quantity. They incorporated their newly acquired place-value vocabulary as though the use of these

terms settled the issue. They had subconsciously entered a new domain in the wider mathematical community where shared knowledge of these terms was implicit. Their answers contrasted with procedurally-based answers such as “lining up the numbers underneath each other”, “adding zeroes to make them match”, and similar statements heard in classes of similarly-aged students. They also contrasted with the responses made during the first interview (1.1.3).

These responses provide triangulation of evidence that these three students had mastered the MiNZC objective concerning the ordering of decimal numbers because they had developed a more refined place-value construct rather than having acquired a change in procedure. It is worth noting that Grace, who struggled so much with the challenge to her prior construct, emerged as one of the sub-group with the greatest evidence of conceptual change.

These students were able to transfer their decimal ordering process to new contexts, indicating a measure of robustness in their system (up to two decimal places). This transfer can be described as empirical or factual generalization following Harel and Tall (1991), and Radford (2003), respectively. They had shown an ability to anticipate, using Tzur and Simon’s, (2003) use of this term in that they transferred the use of their decimal ordering schema into two unfamiliar settings. This skill meets the demand of the MiNZC objective, but there was insufficient evidence that the degree of generalization that has occurred will allow students to operate in other ways with this knowledge. I did not have the confidence that their knowledge of the decimal system was sufficiently developed so as to warrant the label of reconstructive (Harel & Tall, 1991; Zazkis et al, 2008) as it had yet to be tested in a longer time frame. Also, as knowledge needed to be seen in what it could do (Hiebert et al, 1997), further work was needed to build upon this important first step. Wini was able to say that the difference between the two paint tins was ‘0.25’. That she was able to do so was a measure of how strong her new construct was.

The other students also correctly answered the two contextual tasks but were less clear in their explanations.

Tame

Tame: I sorted it out, that one has five tens and that one has two.



It was unclear whether Tame was simply getting used to the tens/tenths vocabulary, was not hearing the critical suffixes, or was still regarding digits after the decimal point as being mirrored by the place-value before it. That his DCT gave him Task Expert status and had a low exception count gave some weight to the first option, as did his use of ‘hundreds’ in a subsequent task.

#### Mary and Ripeka

Mary: ‘Cause that one (pointing to the 4.25L tin, but not exactly on the 0.2 part of the number) hasn’t got to five yet

Ripeka: (pointed to 4.5L) Can I change my answer? (Received nod) Yeah, that one. (Pointed to 4.25L)

T: Why did you change your mind? (Said neutrally)

Ripeka: I dunno. (Paused) No, I’ll change back.

T: Why change back?

Ripeka: That one has two five and that one is five

These explanations did not use any place-value language.

#### 1.3.3 Interview Results

Each student was directed to three pairs of examples from their two DCT scripts and asked to explain why they had made changes, or why they had kept the same answer. Their scripts were unmarked so as not to provide external validation of either response. The student’s new answer is underlined. I have again grouped the responses, though the interviews were held individually.

Sample comments are given and the actual decimal pair shown in brackets.

Wini: (0.55 and 0.555) Five thousandths more

Grace: (0.75 and 0.8) It’s larger, it has an extra tenth; when I first started I thought that (pointed to 0.75) was the highest because of seventy-five

Aroha: (4.45 and 4.4502) It’s got a little bit, thousandths bigger [I considered that her saying ‘thousandths’ rather than the more correct ‘ten-thousandths’ to be inconsequential at this stage of her learning].

These three students also used place-value language in their other two explanations.

Tame: (4.63 and 4.8) That was eight *tens* and that one is only six *tens* and three *hundreds*

It appeared that Tame had not adopted the ‘ths’ suffix, but was not thinking in tens and ones (as in the whole-number construct) either. His reference to ‘*hundreds*’ was evidence that he was not considering the point six three as ‘sixty-three’ but was able to consider the digits with reference to their correct place-value. Two possible explanations are that

he was not hearing any difference between tens and tenths (a slight hearing impairment or he subconsciously suppressed the unfamiliar sound); or that he heard, but did not sufficiently value the suffix enough to be careful with his use of it.

Mary: (3.073 and 3.72) 'Cause of the zero  
T: What does it tell you?  
Mary: No *tens*  
Mary: (17.35 and 17.353) I don't know  
Mary: (0.3 and 0.4) There's one more *ones*  
Mary: (17.35 and 17.353) No verbal response

Mary displayed inconsistency of language use, describing tenths as both 'tens' and 'ones' and she offered no reason for why she believed 17.35 was larger than 17.353.

Ripeka: (0.216 and 0.37) Cause on that one (earlier DCT) I didn't really know decimals, now I know what to do and what's right.  
T: How do you tell?  
Ripeka: That one had uh three that is bigger than two, (pause) that one had seven that is bigger than one, no thousands

Letting Ripeka talk on (rather than stopping her with a 'well done' after the first clause), allowed her to reveal that for her, the issue was not settled by simply examining the tenths digits.

Ripeka: (4.08 and 4.7) I dunno  
Ripeka: (5.62 and 5.736) I dunno

Analysis of the new DCT results labeled Ripeka as Task Expert, and Mary as showing mastery to two – but not three – decimal places. When all of the data was examined, there were counter-indications. Neither could identify 8.6 on the number line task. Neither used any place-value language in response to the 'bigger paint tin' question. Neither could give consistent explanations when asked to explain why they had altered previous answers in the DCT.

In Mary's responses, the first decimal place was described using two different language codes; 'tens' in the first response and 'ones' in the third response. This, along with the lack of generalization in DCT box D, led me to think that Mary did not have a conceptual structure of decimals that was independent of the decimal equipment. Her involvement with the equipment had provided her a reason to change her procedure for ordering decimal numbers but this had yet to become conceptual re-adjustment.

Ripeka's written and verbal evidence were in conflict with her DCT result. When explaining why she now believed that 0.37 was larger than 0.216; she appeared not to realise that the comparison of tenths was all that was needed to order those numbers. She did not elaborate on her other choices. It was not possible to determine if she was unwilling or unable to verbalise her decisions.

It seemed appropriate to describe these two students as still in a transitional phase of learning how place-value operates with decimal numbers. They showed evidence of being able to undertake the procedures for some tasks but their rationale for doing so was still being formed. This was only uncovered by personal interviewing, demonstrating the need to collect more than one type of student data before conclusions about learning are made (Long & Ben-Hur, 1991; Mitchell & Clarke, 2004).

When the data for the other four students was triangulated, it allowed complementary results to be examined. Their final DCT responses were matched by an ability to order decimal numbers in context and explained with reference to the structure of the place-value system being employed.

#### **1.4 Between Data Collection Times**

Just over six months after the final interviews from Phase 1, I was able to meet with Wini, Aroha, Mary and Tame again. (Ripeka had changed schools and Grace was ill that day). The students had been working in class to learn strategies to carry out the four basic operations with whole numbers, but had not had further instruction on decimal numbers.

After a few minutes re-acquaintance with the pipe number equipment (with no teaching from myself involved), the students demonstrated what they done in the learning sessions to a group of teachers. (The teachers were assembled for a workshop on the teaching of decimal numbers that I was running). Despite not having formally re-visited decimal numbers since the learning sessions with me, all of the students were able to communicate how decimals were ordered, how zero acts as a placeholder, and how simple decimals are added. This 'residue' was powerful evidence of the efficacy of the approaches used in those sessions.

As detailed in the Method chapter, Mary, Grace and Wini were able to take part in this second phase of data collection, but the other students were not attending the same

school. Three new students, Bridget, Kiri, and Hoani made the group up to six again. The students had done ‘three or four lessons’ on adding decimals according to their teachers. No equipment had been used, and decimals had only been added by applying the standard algorithm using vertical columns.

The intention in Phase 2 was for students to use decimals as operators in multiplicative situations and develop sufficient understanding of place-value to make sensible estimates in order to judge the accuracy of their calculations. It was hoped that the action/reflection-upon-action cycle involved (Tzur, 2007) would promote a reorganization of student thought concerning decimals and multiplication.

### 1.5 Re-engagement with Decimal Numbers Session 1

A check was made that the group understood how to order decimal numbers by asking the questions, “Which is the larger of these two numbers, 2.6 or 2.27?” and “How do you know which is the larger?”

All of the students answered the first part of this question correctly (indicating correct procedure), but their explanations varied (showing conceptual ambiguity).

*Table 7  
Students’ Responses to the Ordering of Decimal Numbers Task*

<b>Task</b>	<b>Mary</b>	<b>Grace</b>	<b>Wini</b>	<b>Bridget</b>	<b>Kiri</b>	<b>Hoani</b>
Which is the larger of these two numbers? 2.6    2.27	2.6	2.6	2.6	2.6	2.6	2.6
How do you know which is the larger?	0.6 bigger than 0.2	Not sure	.6 = -6 0.27 = -27	2.27 is larger (contradicting herself)	Not sure	2.6 is like 260, 2.27 is like 227

The first line of the table indicated that none of the students were operating with a whole-number procedure for choosing the larger of the decimals. The second question probed the reasons for the correct procedure being employed. Asking students to explain their thinking revealed that the conceptual base of this procedure was not consistent between the students and the correct explanation was given by only one student. Mary appeared to have the weakest understanding during Phase 1; but was the only one of the original trio who provided a correct explanation. Grace was now unsure. Wini’s answers were

difficult to interpret. It appeared as though some newly acquired knowledge concerning negative numbers had been intermingled with her previous decimal knowledge. The other three students gave procedurally-correct answers, but their verbal responses indicated that the conceptual reason for these answers was not known or not accessed for this task.

A 15-minute period of interaction with the pipe numbers was held to help students frame their language around tenths and hundredths. It only took a few minutes for the original trio to re-connect with both the equipment and the language associated with it and the new students quickly shared in this knowledge.

(The majority of time in Sessions 1-3 was devoted to working with decimal operations and is reported separately).

## 1.6 Density of Numbers

### 1.6.1 Context for Thousandths Session 4

The students were shown a packet of Mini-Krispie biscuits. There were 50 biscuits in the packet and the packet weighed 200g. The group agreed that this weight could be re-written as 0.2kg. The students were told that they had to solve my task to get a biscuit.

- T: Your job is to work out how much one biscuit is.  
 Chn: 0.04 (called out, others agreed)  
 Hoani: 0.25 grams (then changed his mind to join the others). Look,  $200 \div 50$  is 4 so 0.04. Each biscuit is 4 grams.  
 T: OK, so in kilograms?  
 C: 0.04  
 T: So how much was the packet?  
 [I wanted to model a checking mechanism; their answers needed to be referred back to the concrete situation from which they had arisen to assess their reasonableness].  
 Chn: 0.2. 0.4  
 T: (I picked up on the 0.4 response). So what is 0.4kg in grams?  
 Chn: 400  
 T: So one biscuit is twice as heavy as the packet?  
 (Other children started chorusing 0.04)  
 C: The answer's 0.04.  
 T: So what would 40 grams be?  
 Chn: 0.04 (chorused and supported). [The conflict was unrecognized].  
 C: It can't be 0.4kg so it must be 0.04kg [Apparently believing there were only two options].  
 C: 0.4 is 400 grams  
 T: Ok, what you are saying is right, (started to make table on the board) 0.4kg is 400 grams, so how big will that one (writing 0.04kg) be?



*Figure 6 Photograph of Krispie Packet*

Table on Board

0.4kg must be 400g
0.04kg must be ?g
? must be 4g

- Chn: 4 grams  
 Hoani: No, 40 (Had the conflict has been noticed or simply the pattern?).  
 Chn: (two) Oh, yeah  
 C: Yeah 'cause 0.04kg must be 40g.  
 Chn :(two) 0.004!  
 (I wrote this up for the others to contemplate).  
 Hoani: Can I have a biscuit now? (The expression on his face shows that he knows he has it right). #  
 Grace: Can you have three decimal places?  
 Chn: I dunno. I dunno either. I dunno but apparently. Yeah, you must.  
 T: OK, this is a new thing because we normally have only done things with two decimal places. (The students were re-reading the table and nodding).  
 Do you think you could get 4 decimal places?  
 Kiri: Yeah, you could get a million!  
 Mary: Can we have a biscuit now?  
 T: No you can't have one but I'll let you have two. OK, when you eat your biscuits, you need to write this down; as this is new thinking isn't it?

Biscuits in hand.

- Kiri: Does the weight include the packet?  
 T: No, they have to sell things by what is inside, otherwise, someone might make a really heavy packet and put like three biscuits inside.  
 Chn: Laughter.  
 T: Legally, they have to say the weight of the inside; that was a good point.  
 (Students finish eating their biscuits).  
 Later on.  
 T: So look back to here (The table on the board).  
 Bridget: One (biscuit) is 0.004. (Said with confidence).  
 T: Yeah, some people think that you only have two<sup>^</sup> (decimal places)  
 Kiri: But you can have to infinity!

Again, discussion centred on a manipulative had lead to the extension of the students' decimal schema. That this occurred is consistent with earlier work (e.g. Goldin & Janvier, 1998; Tzur & Simon, 2003). This discussion differed significantly from those centred around the pipe number equipment in that the biscuits could not directly model the numbers they were representing. Kiri's last comment suggested that she was beginning to entertain the idea of decimalization being completely generalizable.

That the task was experientially real was evidenced by Kiri's question about the weight of the packet. This activity was seen to link to the real world, an important pedagogical factor according to research (e.g. Hiebert et al, 1996; Hiebert et al, 1997; Meyer, 2001) and in particular for decimal understanding (e.g. Irwin, 2001; Moss & Case, 1999).

### 1.6.2 Dense Number Property Session 5

(Wini absent)

#### a) Number Lines

The students were asked to create their own number lines and place sets of numbers on them. This was in line with recommendations by Diezmann and Lowrie (2006) and Saxe et al (2007), that the use of number lines is conducive to the learning of the density of number property. This property involves appreciating that the process of decimalization can be continued to any required degree. When understood, this serves as a powerful internal counter to residual whole-number thinking, and also helps students make sensible rounding decisions in order to make estimates.

Set One was 0.5, 1.09, 1.53, 0.8 and 1.25.

Set Two was 13.95, 13.06, 14.75, 14.222 and 13.41783249.

Choosing where to place numbers required the students to engage deeply with the relative sizes of the numbers. They had to relate the markings they had on their scale to the numbers they represented and then make adjustments according to whether the given decimal was larger or smaller. Of the two sets, only the markings of 0.5 and 0.8 could be *known* to be accurate by the students. All of the others required a degree of estimation.

Hoani and Grace had them all well placed; the others had a mixture of correct and incorrect placements. Hoani's number lines are presented below.

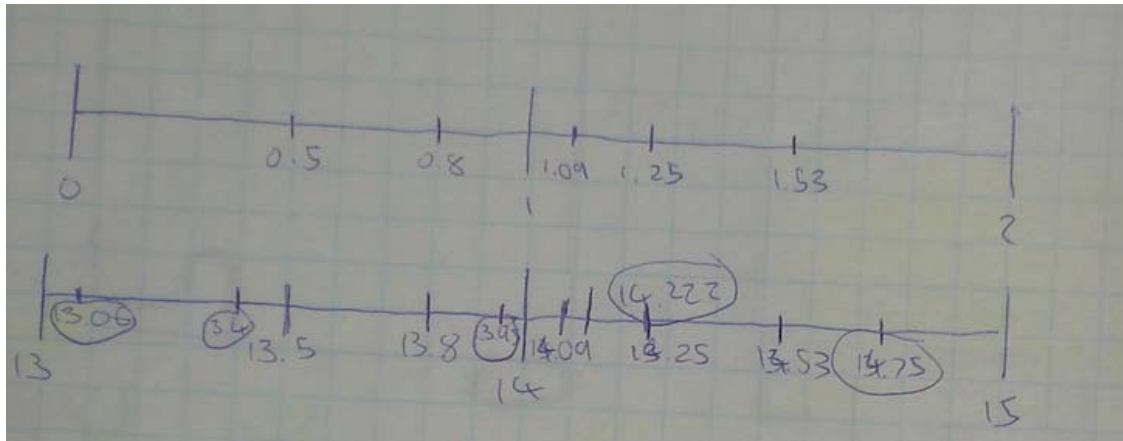


Figure 7 Photograph of Hoani's Number Lines

In the photograph above, we can see that Hoani had correctly placed 1.09 and 13.06, demonstrating that he understood the use of the placeholder zero in the tenths column. He had also shown a link between fractional and decimal understandings in the lower line in that he marked a valid position for 14.25 (which was not part of the data set) and then

used this to place 14.222. He had also recognised that 13.41783249 is possible to be represented (the length of this number would cause major difficulties to a whole-number thinker) as it can be considered as being approximately 13.4. The positioning of the numbers was discussed once all of the students had completed the task.

b) Iteration with thousandths

It appeared in discussions that Grace still had difficulty accepting that the third decimal place had any representational meaning. She could accept it to the extent that she could order decimals with three places or more and could position long decimals on a number line, but had expressed doubts that this had any actual reality. This highlights again the need for teachers to allow students to talk about the mathematics they are engaging with. It would have been easy to look at Grace's written work and believe that she had no ongoing conceptual issues.

I sought to address this by creating a situation where digits in the thousandths would have to be engaged with and related to a physical meaning. This was not solely for Grace's benefit, but I hoped that she would produce more self-created data to reflect upon.

T: Right, we'll get the biscuits out and put one in front of you but you can't eat it.  
Chn: Groans. Temptation, temptation.  
T: Well you can only eat it after we do these. How much was one biscuit?  
Hoani: 0.04  
Chn: 0.004. (Hoani quickly agreed with them, first earlier, last now!).  
T: Right, so I have three biscuits in front of me (Bit of by-play about my having three to their one).  
What will it^  
Kiri: Zero point zero twelve or oh point oh one two

The students all wrote the correct answer (0.012); none 'lost' the sense of place-value and put 0.0012, or put 0.12. They were able to aggregate using a decimal multiplicand. This was important new learning, but does not yet address the 'multiplication makes bigger' issue, as replicative thinking will still allow this result (Kieren, 1992).

The students were then given 5, 25 and 26 biscuit weights to work out. There was a dual purpose in this task; firstly, to apply multiplications of the form *Whole number*  $\times$  *decimal*  $<1$ , and secondly, to provide contextual practice of considering the thousandths place. All students recorded these as 0.020, 0.100, and 0.104, respectively. Grace appeared troubled, and she pulled a face at the announcement of the last answer and then sought



confirmation from me that this was in fact correct by looking up at me. Even though she had written 0.104, she still had a problem accepting its existence.

I asked them to sketch a number line and mark where 0.104 would be placed. While five students had no difficulty with this, Mary positioned 0.104 at 0.14. This was evidence that 'zero as a placeholder' was a remaining issue for her. Within all of the issues surrounding the learning of decimal numbers, problems with the role of zero are regarded as especially difficult to overcome (Steinle and Stacey, 2001).

The students were then given the following task: We have one biscuit, (weight of) 0.004, so what about 7, 15, 150 biscuits?

All completed these correctly. I then wanted to test the understanding of the group by deliberately operating with a common whole-number procedure used by many students.

T: OK, so 0.004 for one, ten biscuits must be 0.0040 by putting a zero on the end. #  
Chn: No, no.  
T: Doesn't it work?  
C: No, that's just the same.  
T: OK, so writing a zero on the end doesn't work for decimals?  
Chn: No.  
Grace: Can we write that down? (Despite her earlier misgivings about the reality of the thousandths place, Grace was determined to capture any piece of information that would help her make sense of what was happening).

My enactment of the 'add a zero' rule for multiplying by ten is advocated by researchers as a means of forcing students to defend their new understandings or betray the weakness of them (e.g. Moss & Case 1999; Siegler, 2000).

### c) Less-scaffolded Responses

T: Next mission, put down any numbers you can think of between these two numbers; 2.1 and 2.2. [To see what students would do in the absence of the scaffolding of number lines and biscuits].  
Hoani: How many?  
T: Just some you can think of.  
Hoani: But there is a zillion! [It looked like he had already understood the density of numbers principle].

In their written recording, all of the students limited themselves to the hundredths place. When challenged to next write numbers between 2.11 and 2.12; Hoani and Kiri quickly wrote nine correct examples each. Kiri now used thousandths while Hoani included

numbers such as 2.144444444444 and 2.16954, further evidence that he had generalised the density principle.

Mary and Grace had not written any.

T: Do you think that there are any numbers between 2.11 and 2.12?  
Grace: No [The third decimal place is still a conceptual hurdle for her despite the biscuit examples].  
Mary: Yes  
T: Well Mary, you try and convince her.  
Hoani: You can have 2.11111 (Mary nodded).  
T: Is that bigger than 2.11?  
Mary: Yes  
T: Is it smaller than 2.12?  
Mary: Yes (but voices in background call both yes and no and she changed her mind several times).  
Bridget: What about 2.115?  
T: Yeah that would work.  
Grace: But... (Shaking her head)  
T: Remember yesterday, you had a biscuit, that was 0.004. #  
Grace: But those other places aren't really real  
T: But you could hold the biscuit... (She remained unconvinced).

An attempt had been made to use collective student voice to expose these two students to the other's truth and so challenge their cognitive status quo (Cobb, 1994). Grace seemed to have an obstacle that could be termed epistemological (Harel & Sowder, 2005; Henningsen & Stein, 1997) in that she would not accept evidence even when she was involved in its production.

### 1.6.3 Further exploration of density

#### Session 6

A challenge was given for the students (in two groups of three) to mark given decimals onto a number line. A number line marked in tenths was provided.

Group 1 0.62, 0.095, 0.389 0.095 was incorrectly positioned, the other two were correct

Group 2 0.812649, 0.702, 0.035, all correctly positioned

Group 1 then adjusted their incorrect answer.

#### Session 7

The students were asked to draw a number line and mark the positions of these decimals: 0.21, 0.05, and 0.129

Kiri, Wini, Grace, Hoani all correct.

Bridget had 0.05 marked as 0.5. [Zero as placeholder issue again].

Mary had 0.129 marked close to 0.2 [I wondered if she read this as 0.19].

### 1.7 Final Results

Task: Which is the larger of these two numbers, 2.6 or 2.27, and how do you know?

Table 8

*Comparison of Students' Initial and Final Responses to Ordering Decimals Task*

	<b>Mary*</b>	<b>Grace*</b>	<b>Wini*</b>	<b>Bridget</b>	<b>Kiri</b>	<b>Hoani</b>
Pre-Intervention Response	2.6 0.6 is bigger than 0.2	2.6 Not sure	2.6 .6 = -6 0.27 = -27	2.6 2.27 is larger (?)	2.6 Not sure	2.6 2.6 is like 260, 2.27 is like 227
Post-Intervention Response	<i>Agreed with earlier answer</i>	<i>2.6 is bigger because it has 6 tenths which is more than 2 tenths</i>	<i>Because 6 tenths is bigger than 2 tenths</i>	<i>2.6 No reason given</i>	<i>2.6 is 2.60 and 2.27 is 2.27</i>	<i>2.6 No reason given</i>

\* Original Participant

Grace and Wini had now re-connected with the reasoning used when they completed the post-intervention interview in 2006. Bridget, Kiri and Hoani's thinking is still unclear.

Understanding the process of creating decimal numbers beyond tenths and hundredths was examined by asking the group to provide examples of numbers that were between fixed values. The first example allowed students to transfer knowledge from activities we had worked upon (pipe numbers, water measuring, area models, and biscuits). The second example went beyond any situation that had been discussed or modeled and thus served to check whether the students could reason that a number could exist with more than three decimal places.

Table 9  
*Students' Responses to Density of Numbers Task*

<b>Task</b>	<b>Mary</b>	<b>Grace</b>	<b>Wini</b>	<b>Bridget</b>	<b>Kiri</b>	<b>Hoani</b>
Show numbers between: 1.6 and 1.7	Correct Examples given	Correct Examples given	Correct Examples given	Correct Examples given	Correct Examples given	Correct Examples given
7.578 and 7.579	Incorrect example	Correct Examples	Incorrect examples	Correct Examples	Correct Examples	Correct Examples

Students were asked to show more than one response as the mid-point may be provided by students who cannot provide other answers. All students were able to present numbers such as 1.64 as being in-between 1.6 and 1.7, something that was not possible under the whole-number construct. Mary and Wini did not provide evidence that they had recognised that the decimalizing process could be continued indefinitely, in contrast to Grace, Bridget, Kiri, and Hoani. That there was a difference in difficulty for students between the case where hundredths could be used (1.6 and 1.7) and where a generalized application was required (7.578 and 7.579) is consistent with other findings (e.g. Greer, 1987, Vamvakoussi & Vosniadou, 2004).

## **Decimals in Additive Contexts**

### **Introduction**

Place-value knowledge of decimals is essential if students are going to make sense of using them in additive operations. Initially, many students process decimal numbers by applying a set of rules developed from working with whole numbers and ignore decimal place-value (Saxe et al, 2005; Stacey & Steinle, 1999; Steinle & Stacey, 1998). This results in digits to the right of the decimal point being combined as though a second whole-number system exists, so that statements such as  $0.4 + 0.12 = 0.16$  are made (Irwin & Britt, 2004; Saxe et al; 2005). However, having correct decimal place-value knowledge does not automatically translate into fluency with addition.

As noted by research (e.g. Empson, et al, 2005; Lamon, 07; Saxe et al, 2005; Thompson & Saldhana, 2003; Young-Loveridge et al, 2007), understanding of fractional numbers requires the ability to iterate with units other than one. This ability is multiplicative in

nature. With this perspective, the addition of decimals is regarded as conceptually more difficult for students than decimal ordering tasks, though not as difficult as using decimals dynamically as operators in multiplicative tasks. This is reflected in the NDP Framework (Ministry of Education, 2007b, pp. 14-16) where identification of decimal place-value is rated as Stage 6, additive problems with decimals at Stage 7 and multiplicative problems at Stage 8.

The data regarding student experiences with decimals in additive situations has been selected from both intervention phases and is presented in this section.

## 2.1 Pre-Intervention Data: Phase 1 (2006)

### 2.1.1 Written Data

As part of the pre-intervention data collection, students were asked to carry out two tasks involving fractional addition. Their responses are reported below.

*Table 10*  
*Students' Responses to Fractional Addition Tasks*

	$3/10 + 4/10$	$1.3 + 1.13$
Mary	27	Left blank
Grace	7	2.16
Wini	Left blank	2.16
Aroha	Left blank	2.16
Ripeka	Left blank	Left blank
Tame	Left blank	2.16

None of the students were able to complete either example correctly. The low decimal result was anticipated but not the fractional result. This meant that the initial planned teaching sequence had to be modified. The inherent flexibility of the design experiment approach allowed for contingencies such as this (Cobb et al, 2003).

It appeared that none of the students could picture (literally or conceptually) a joining of two sets where the numbers involved were tenths. The decimal answer of 2.16 was indicative of using a construct that perceived a second whole-number system recommencing after the decimal point, and so  $3 + 13 = 16$  was calculated (Steinbring, 1998). Students were relying upon existing procedures that had previously proven

effective with familiar tasks, rather than recognizing the need for change (Charalambous & Pitta-Pantazi, 2007).

### 2.1.2 Interview Results

In the initial interview, I created another problem involving the addition of tenths, and asked the students to solve, and then to describe their solution methods.

The task was  $2/10 + 2/10$

Tame:  $4/20$ ,  $2 + 2 = 4$  and  $10 + 10 = 20$  [Failure to recognise that the two digits of  $2/10$  constitute one number]

Mary: Same

T: Aroha, you wrote 20 then a dot and then 4, (20.4) why did you write that?

Aroha: I don't know, I'm just guessing

T: You've got the numbers from somewhere, where did you get that 20 from? [Seeking to elicit a response, rather than accepting the 'guess' comment].

Aroha:  $10 + 10 = 20$  and then  $2 + 2 = 4$

T: OK, so why did you put the dot there?

Aroha: I don't know

T: Ripeka, you wrote four then a dot then twenty (4.20); why did you do that?

Ripeka: (no answer, just a shrug).

I propose that the DCT had sensitized the students to decimal numbers. Aroha and Ripeka believed that they had received a cue to express their answer in decimal form. Neither could give any rationale for their choice of using a dot in their answers. (I do not believe that this 'dot' was for them a decimal point as commonly understood).

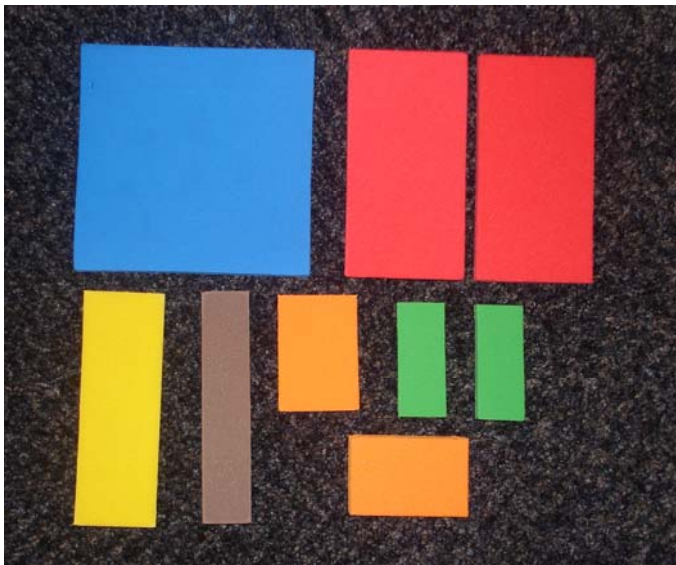
What was revealed as a result of probing questions in the interview - and would not have been evident if written evidence alone was used - was that the thinking of all of the students was similar. They all processed the numerators and denominators separately to produce sub-results of 4 and 20, but they presented those digits in markedly different fashions. The detailed data collection of the microgenetic study (Siegler, 2007) enabled this to be noticed, whereas other types of study are unlikely to have uncovered it.

The students' comments revealed that they did not appreciate that the use of a symbol (e.g.  $2/10$ ) denoted a single number. They were interpreting each digit as an independent unit. As Baroody et al (2004) have noted, it was unlikely that the students would make further decimal progress until the iterative process of combining fractions of like denominators was understood. This was also supported by research into how students develop understanding of the equi-partitioning process with its attendant multiplicative thinking (e.g. Empson et al, 2005; Saxe et al, 2005; Thompson & Saldhana, 2003; Tzur,

1999; Young-Loveridge et al, 2007). In response, the first learning session centred on iteration.

## 2.2 Iteration of Unit Fractions Session 1

The lesson used hard foam rectangular-shaped pieces of different colours with the largest piece being assigned a value of one.



The largest piece was designated as representing '1'. The other pieces in this figure are then  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{5}$ ,  $\frac{1}{6}$  and  $\frac{1}{10}$ .

Figure 8 Photograph of Foam Fractional Materials

The following exchanges indicated that while the students were able to recognise the equivalence of two halves and one whole, they were not able to confidently iterate with units of size one-half.

- T: So if we've got two of them, the two halves are the same as the one whole? (Nods received). OK, so if we have three of them, then that's...
- C: Two [meaning that the two pieces constituting a one, not literally the number 2] and a half
- T: When we get to four halves...
- C: Two wholes
- T: Yeah, four halves alias two wholes, what's after four halves?
- C: Ummm four and a half? [Again, an indication that the halves that would now be joined to form two ones were different from the remaining piece]

I did not think that these students were meaning  $2\frac{1}{2}$  and  $4\frac{1}{2}$  as normally understood. They recognised that the two and four halves respectively should be grouped together and the remaining half was to be treated differently. In trying to describe this difference both

exchanges show that the students were confused between the number in the count (two and four) and the size of piece being counted. This corresponds to Ball's observation (1993) concerning students confusing 'how many?' with 'how much?' and the comments of Sophian (2008) that over-emphasis on enumeration can effectively exclude consideration of quantity.

A decision was made to use the manipulative more explicitly. The children were seated in a circle and there was a central location that all had easy access to. I distributed one red '1/2' piece to each child to place into the middle on their turn. I placed the first piece into this zone and announced its value, 'one half'. The students were then invited to place their pieces next to mine in turn and to say what quantity was now represented.

C: (Placed their half next to mine) One.  
T: So how many halves?  
C: Two.  
C: (Placed their half) Three.  
C: (Place their half) Four.  
T: So what would come next? (Without materials being used).  
C: Ummm Five halves?  
T: Yeah, then?  
C: Six halves  
T: Right; and that is the same as...  
C: The same as three.  
T: Then my piece makes seven halves and then...  
C: Eight. # Alias four?

Some evidence of iteration was now apparent, but whether it was from continuing the counting pattern or from a developing concept of number using fractions was unclear.

Students then worked with the other-sized pieces; quarters, fifths, sixths, eighths, and tenths. Tasks were given that required the students to make sense of statements that involved the collection of unit fractions. For example, 'show me four sixths', 'show me six quarters'. The aim was for them to create empirical evidence that the process of iteration was transferable across any (and therefore every) denominator.

The transcript notes pick up approximately ten minutes later.

T: OK, we now have ten tenths or a whole.  
C: How do you write it as a whole?  
T: I just do a 1 for a whole. What if I put down two more...?  
C: But you can't do that! [Possibly a rejection of the concept of an improper fraction].  
C: Yes you can, 'cause you've got a whole and two-tenths



The statement that rejected the twelve-tenths being created can be described as evidence of a fractional referent scheme that understands the bipartite symbol as being ‘out-of’. Using this vocabulary makes it impossible to have twelve out of ten and so the number twelve-tenths was rejected. This response was predicted and explained by work such as Lamon (2002) and Tzur (1999). Four students now seemed confident that you could amalgamate unit fractions while two were still unsure. I decided to introduce the pipe numbers. I hoped that it would be an easier model to work with than the area model as the physical variation would only exist in one dimension. While the area model had epistemic fidelity, it seemed to lack transparency insofar as these two students were concerned (Meira, 1998; Stacey et al, 2001).

## **2:3 Decimal Addition**

### 2.3.1 Adding Tenths Session 2

The pipe numbers were brought out again, and the students made up two examples with tenth language being used. (Neither  $4/10$  or  $3/10$  were used as examples). I then posed a new question.

- T: What is  $4/10 + 3/10$ ? (Some students reached for equipment, others immediately said “ $7/10$ ”, I paused, and all students ended up agreeing on  $7/10$ ).
- T: And add another  $4/10$
- Chn:  $11/10$  (All students wrote  $\frac{11}{10}$  or  $1\frac{1}{10}$  with no apparent problems).
- T: And how do we write that (showing a model of twelve-tenths) as a decimal? Write down how big that number is. (The students all wrote 1.2).

In the 24 hours following session one, the students had made adjustments to their schema independently of further input from me. They had further processed the data they received in that session, perhaps unconsciously. This served as evidence of the theory that students are *active* learners and do not always require an adult, or even a social context, in order to continue learning. As Trognon (1993) asserts, any data-collection system in research will always fail to capture *all* of the data, as cognitive action is not always occurring simultaneously with direct task engagement.

### 2.3.2 A Game as a Context for Addition

The students had engaged with hundredths in sessions two and three and were starting to reconstruct their place-value schema.

#### a) Session 4

- C: Can we play Nasty?<sup>8</sup>  
T: No 'cause we're doing decimals. [I thought that they wanted to just play a familiar whole-number game for fun].  
C: No, *with* decimals  
Chn: Yeah!

An opportunity was presented. The playing of the game provided a meaningful reason to learn how to add decimals. Addition could further reinforce their new consideration of decimal place-value as successful outcomes depend upon an understanding of the nature of the addends. I modeled an addition example with the pipe numbers [Did I not trust the students to work it out themselves?].

- T: I'll put an adding question,  $0.35 + 0.2$  (modeled at either end of the pipe decimal equipment).  
C: 0.55  
C: I agree  
C: 0.55  
C: Same  
T: Hey that's good! Do you know when I came in two weeks ago? There was one like this and no-one got it right then, you guys have really learned a lot.  
Wini: What if you had like  $0.3$  plus  $0.35$  plus  $0.2$  plus...  
T: OK let's try  $0.25 + 0.2 + 0.2$  (written on the board and not modeled with equipment)  
Wini: 0.65  
C: No, yes, 0.65  
T: What do you think Tame? (He had not joined in with the 0.55 response earlier).  
Tame: 0.65  
T: Hey that's good.

The same students who could not add  $2/10 + 2/10$  using pen and paper at the time of the initial interview were now solving the addition of three simple decimals mentally. The expected whole-number thinking response to the task of 0.29 was not heard.

Two teams were organised for the Nasty game and the random digits were produced and positioned.

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<sup>8</sup> Details of this, and other games that were used, are given in Appendix D

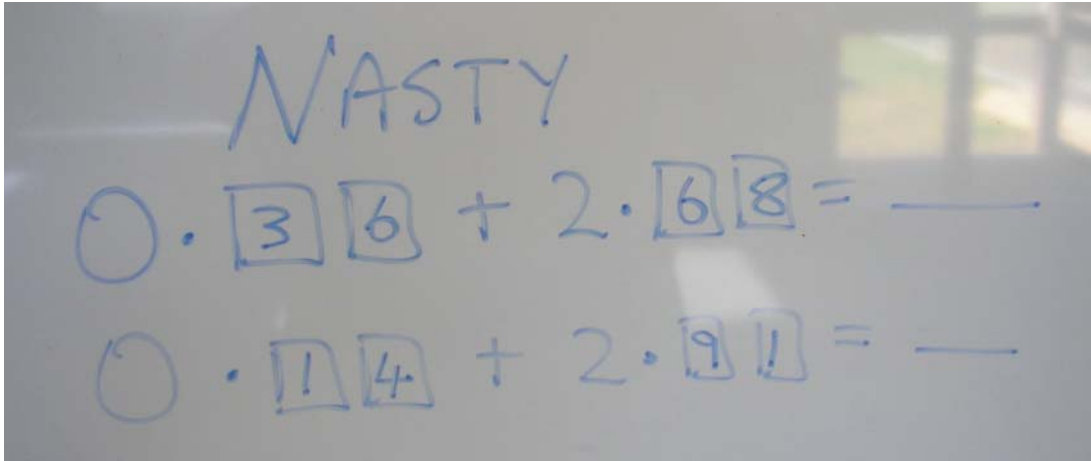


Figure 9 Photograph of a Nasty Game

Note that both equations required renaming, hundredths and tenths in the top example and tenths in the bottom example.

- T: So try and add that up. (0.14 + 2.91, the students were mentally processing the problem, not writing it)  
 C: 3.5 no 3.05  
 The other team had 0.36 + 2.68  
 C: 3.4 # No, no it isn't. 3.04  
 First Group: We won!

The representation of decimals provided by the pipes was not used to carry out the addition tasks. This showed that the students were starting to think of decimal numbers as entities in their own right that could be operated upon. This is evidence of these students moving from Imaging to Properties as described in the NDP teaching model (Ministry of Education, 2007d, p. 5) and evidence of moving from participatory to anticipatory knowledge as Tzur and Simon (2003) have described. This coordination of place-value columns provides evidence of understanding that a sum such as 0.66 + 2.25 does not as no renaming is required in the latter.

Both students who carried out the Nasty additions initially gave answers where the hundredths digit was mis-placed (3.5 changed to 3.05 and 3.4 changed to 3.04). Their unprompted self-corrections showed that they could use zero as a place-holder once they realised that the last digit had to be in the hundredths column. This reflection and adjustment behaviour was seen as a strong indicator of an internal locus of control being developed. Making informed choices between conflicting alternatives is evidence that the

students were re-organising their thinking according to Kazemi and Stipek (2001). These actions stand in contrast to the self-doubting statements heard earlier such as Aroha’s justification of her ‘shorter is larger’ procedure and the previous lack of understanding of the role of zero ((see section 1.1.3).

The examples of addition described above provided evidence that the work with the pipe numbers was not only instrumental in creating cognitive conflict but also in its resolution for these students. In Zazkis and Chernoff’s terms (2008), some students had appropriated meaning in such a way that the task of working with the pipe numbers could be termed not only pivotal, but also constituted a bridging example.

#### b) Nasty again

##### Fifth learning session

T: OK so you have 8.7 plus 2.3, how will you add that?

C: Well you get 10 and umm...

[The student had transferred a commonly used whole-number addition strategy to a decimal situation. In this method, the larger units were combined first, then the smaller units combined, (e.g.  $37 + 44$  becomes  $70 + 11$ . I supported the student in continuing this transfer by recapitulating what they had done and suggesting the next step].

T: OK, 10 from 8 and 2, what about the other bit?

C: That’s 1, so its 11

## 2.4 Post-Intervention Data: Phase 1

Table 11

*Comparison of Students’ Initial and Final Responses to Fractional Addition Tasks*

	<b>3/10 + 4/10</b>		<b>1.3 + 1.13</b>	
<b>Name</b>	<b>Initial</b>	<b>Final</b>	<b>Initial</b>	<b>Final</b>
Mary	27	7/10	Left blank	1.33
Grace	7	7/10	2.16	2.43
Wini	Left blank	7/10	2.16	2.43
Aroha	Left blank	7/10	2.16	2.43
Ripeka	Left blank	Left blank	Left blank	2.16
Tame	Left blank	7/10	2.16	2.16

Ripeka was still unable to complete the  $3/10 + 4/10$  task. Ripeka, Tame, and Mary were unable to transfer the knowledge of tenths they had demonstrated with the pipe number materials to a situation where no scaffolding was provided. Grace, Wini, and Aroha had taken the experience of a few decimal additions and had been able to calculate without physical or written materials. In Tzur’s (2007) analysis, the students could all *participate*

in decimal addition but only three could independently *anticipate* its application when no social or physical scaffolding was present.

### 2.5 Initial Data: Phase 2 (2007)

I presented the students with the following situation: Shane had to add 3.4 and 4.15. He got the answer of 7.19. Is that what you get? Can you explain what you have done to get the answer? I was providing a misconception to probe the depth of understanding as Moss and Case (1999) recommend.

*Table 12*  
*Students' Responses to Misconception Question*

	<b>Mary*</b>	<b>Grace*</b>	<b>Wini*</b>	<b>Bridget</b>	<b>Kiri</b>	<b>Hoani</b>
Is $3.4 + 4.15$ equal to 7.19?	Yes	Yes	Yes	Yes	Yes	Yes
Explanation	None given	$3 + 4 = 7$ $15 + 4 = 19$	$3.4 + 4.15 = 7.19$	$3 + 4 = 7$ $15 + 4 = 19$	None given	$0.15 + 0.4 = 0.19$

\* Original Participant

The entire group was prepared to accept a whole-number interpretation of the digits appearing after the decimal point ( $4 + 15$ ). Grace and Wini had shown evidence 17 months earlier that they could operate with decimal place-value. Their responses to this task indicated that the new construct co-existed with the previous whole-number interpretation. The exposure to working with decimals in Phase 1 had proven insufficient to displace the earlier thinking. This finding was consistent with the overlapping waves model of Siegler (2000, 2007). It was important to ascertain whether the knowledge of place-value needed to be re-established, or simply re-activated. I decided to address this by using the pipe number equipment. Half of the group was familiar with this from Phase 1 of the study in 2006.

## 2.6 Decimals in Additive Contexts

### Session 1: Place-Value

The students involved in Phase 1 introduced the new students to the decimal pipes equipment and they all made up models of a few decimal numbers. I now wanted to directly address the whole-number construct that had been employed in the initial data collection.

- T: We had a problem like (wrote  $3.4 + 4.15$  on board) and you guys went  $3 + 4$  is 7 which is right<sup>^</sup>
- Hoani: But we did  $15 + 4$  eh? (He had already recognised the issue).
- T: Try to put the decimal part together
- Hoani: (Without touching the materials) 7.19; no, point five five! [He had produced the expected answers from the whole-number and place-value constructs respectively. It seemed that both schemes co-existed in his thinking. '7' was implied in the latter answer as he later made clear].
- Chn: 7.55
- Hoani: It is 7.55 (Emphatically spoken).
- Bridget: Point one nine (Whole-number thinking).
- T: Ok, make up the decimal part of it. (Children made up a model of the addition in pairs).
- Bridget: Ooohh! OK!
- T: So what is it?
- Hoani: Seven point five point five, no, wait, seven point five five. Ooh wee! Skills!
- Mary: You're brainy at maths
- Hoani: First time using these things.

His reference to the pipe numbers ('these things') made it unclear if Hoani was implying that his answer of 7.55 was true mathematically, or true with *these* materials - but perhaps not generally - and so not a universal truth. Working with decimal additions in other contexts was needed to check this.

- Kiri: Oh no! This means that I got my homework wrong! (From her teacher).
- Bridget: Same
- Kiri: But Miss ticked it right! (Giggles from both girls).

It was clear that these students had connected this learning experience with some earlier work. They had not only remembered it, but were clearly able to recall the process used to generate their answers and knew that this process was incorrect. (*Note, this does not necessarily mean that the teacher was mathematically incorrect, only that the work was incorrectly marked or at least remembered as such*).

This demonstrated the strength of evidence attributed by these students to self-generated results from using physical materials. Their enactment of the addition using a concrete

model was sufficiently powerful to overturn the external locus of mathematical authority, the classroom teacher. This could be interpreted as evidence of the operation of the tiered models of mathematical engagement, where student use of concrete materials may produce powerful mental images (e.g. Biddlecomb, 2002; Herscovics, 1989; Sfard, 1991). Without the students' belief in the power of their own evidence, one could equally imagine the conversational opener running on these lines:

*“No, these current answers must be wrong because I did similar ones for homework a different way and they were marked correct by the teacher”.*

## Session 2

### a) Renaming with Decimal Numbers

The purpose was to show how subtraction with renaming could be thought about using equivalent decimal fractions. The pipe numbers were being used.

- T: Now I got you to think about that one, (1.2), and you said that could be twelve tenths or it could be one and two tenths Now have a look at this one here, (3.2) you said thirty-two tenths and this one (2.5)
- Chn: Twenty-five tenths
- T: So if you did 32 take-away 25 you'd get ...
- C: 7
- T: So if you had 3.2 take-away 2.5 you'd get ...
- C: 7, 7/10! (Connection made).
- T: So we could write that, how big are they?
- Chn: Tenths, point seven (Again, the connection has been made).
- T: So we could write that as seven tenths, point seven?
- Chn: Yeah
- T: Ok, so if we had 0.41, and we were taking away 0.27, how many tenths there?
- C: 4
- T: But if we were making it up *just* with hundredths, how many of those hundredths would I need?
- Chn: 4, 40. Yeah, 40
- T: And for the whole number?
- C: 41
- Chn: Yeah
- T: Yes 'cause the smart way to *make* it up would be to use four tenths and one hundredth eh?
- Chn: Yeah
- T: But sometimes you can *think* of making it up with hundredths; and the same for this one here (2.5) you could make that up with two wholes and five tenths but you could make it up with twenty-five tenths. Sometimes with subtraction that can be useful. Now I'm going to give you this one (0.41 – 0.27), how many hundredths are there?
- C: 41 (Equivalent fraction noted).
- C: 1 (Visual signal of one, 1/100 piece noted).
- T: How many could you use for the whole thing?
- C: 41
- T: Here?

C: 27  
 T: So you can choose to ^  
 Kiri 14  
 T: Use that many hundredths minus that many hundredths  
 (The children were modeling these steps on the pipe numbers as the conversation took place).

Re-presenting decimals into alternative forms both uses and builds upon knowledge about the multiplicative structure of the number system (Heirsfield & Cooper, 2004; Jones et al, 1996).

b) Area Representation

The group was then introduced to the idea of modeling decimals by using a ten-by-ten grid in their books to represent 1. I wanted them to explore alternative representations so that learning would be independent of any single model. While the mathematics may be represented very clearly with one particular material, it was necessary to bear in mind the cautions concerning over-reliance upon a single physical representation, and the dangers of assuming that generalization could occur without variance being noted (e.g. Kamii et al, 2001; Lamon, 2001). Keeping the tasks very similar but varying the representation could be interpreted as controlling the variation so as to highlight the underlying structural invariance as per Runesson (2005).

The first example showed how the addition of 0.2 and 0.25 could be represented. Students were then set other examples including this one,  $0.36 + 0.17$ .

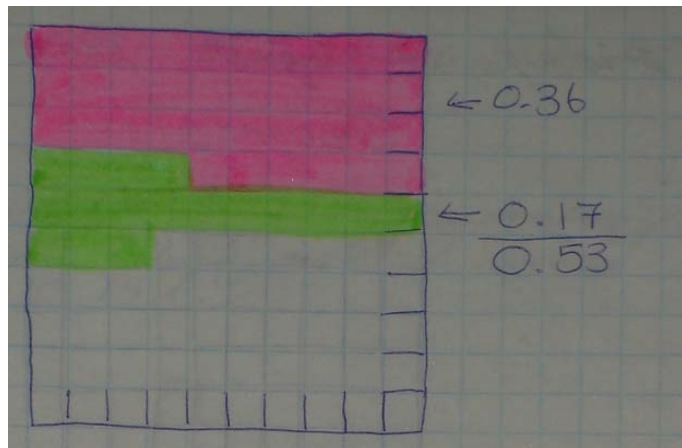


Figure 10 Photograph of Student use of the Area Representation of Number

We then examined how  $0.6 - 0.17$  could be shown by crossing out hundredth boxes from a depiction of 0.6. Subsequent problems ( $0.8 - 0.27$ ,  $0.41 - 0.05$ ,  $1.03 - 0.56$ ) were set



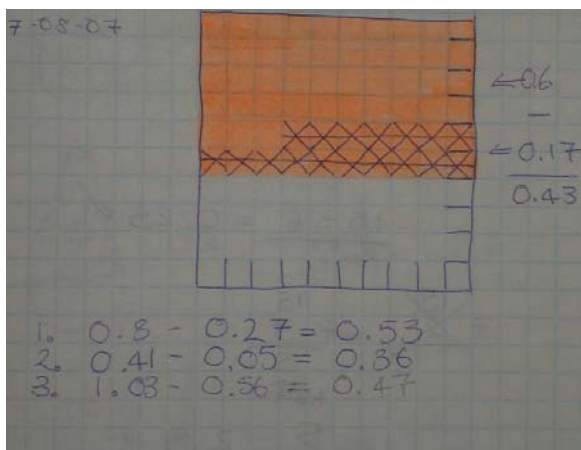
with the students having free choice about how they solved them. Two of the problems had unequal numbers of digits, and two involved having zero as an intermediate placeholder. This was because these features have been shown to increase the difficulty of the problems (Wearne & Kelly, 2003).

Grace, Wini and Hoani worked directly with the numbers and had all the answers correct. It appeared that these students no longer required a physical representation, but were able to operate with their symbolic forms. It is not possible to know whether these students had mental images of the materials (logico-physical abstraction as in Herscovics, 1989; or figurative units as in Biddlecomb, 2002) or whether they were starting to operate solely with the abstract form.

Kiri and Bridget worked in part with the pipe numbers and had all answers correct. These students needed to enact the situations to confirm their thinking.

Mary drew grids out, but failed to turn the number of boxes (e.g. 43) back into the decimal it was representing (e.g. 0.43). While the process of using the squares was used, Mary had not connected its use with the mathematics it was seeking to display. This observation helps confirm the statements of Schliemann (2002) and Schoenfeld (1989) concerning the use of artifacts and their interpretation. They note that the connections between concrete referents and the mathematics they are seeking to represent are not automatic and cannot be assumed to exist.

An example of student work appears below:



*Figure 11 Photograph of Student work with Subtraction*

## Session 4: Contextualized Situations

The group was asked to begin the session by answering this question: You had 3.5L and poured in another 1.25L, how much is this?

All of the students except Hoani wrote the correct answer. For some reason (that he did not know either) he divided the two amounts, writing  $3.5 \div 1.25 = 2 \frac{4}{5}$ . He had correctly carried out a much harder task than what was intended but had misinterpreted the question. The group was also asked to work out what would remain if 0.75L was poured from a 2.25L bottle. Kiri, Bridget and Grace wrote 1.5L, Hoani correctly divided 2.25 by 0.75, but in doing so did not recognise that the situation required subtraction. The other two students did not complete the task.

Hoani and Kiri spontaneously carried out a correct decimal subtraction later that session as when the group was asked if six 0.4L bottles would fill a 2.25L bottle, they pointed out that there would be a 0.15L overflow.

### **2.7 Summary of Additive Work**

The intention of the work with decimal numbers in additive situations was to reinforce the meaning of place-value and to introduce the area model of number as a representation of decimals. It was not considered a high priority to collect further results on situations involving addition in Phase 2, as the multiplicative work was more demanding and so would reveal more about student learning.

## **Decimals in Multiplicative Contexts**

### **Introduction**

No data on multiplicative work with decimals was collected in Phase 1.

In Phase 2, the research was intended to expose the students to a number of different situations that required multiplicative thinking and to observe how they responded to the conceptual challenges, rather than concentrating on any one particular skill until it was 'learned'. Practitioners reading these results might otherwise question why such a broad scope of skills was attempted in a short time frame.

As described earlier, research had noted a number of critical changes in thinking that are required if students are to make sense of processes where decimals are used as operators (e.g. Graeber & Tirosh, 1990; Harel & Sowder, 2005; Saxe et al, 2005).

In response to this literature, key ideas being explored in these sessions were:

- Recognition of situations that required multiplicative approaches.
- Extending multiplicative schema to include situations where products are smaller than multiplicands.
- Extending multiplicative schema to include situations where quotients are larger than dividends
- Using quotitive division to show that divisions with divisors less than one are meaningful.
- The use of benchmarks to calculate and/or estimate answers.
- Coordination of the decimal and the whole-number parts of numbers in multiplicative contexts

An ability to multiply and divide with decimal numbers is a MiNZC objective at Level 4 (Ministry of Education, 1992, p. 44). In the revised curriculum it is less clear, the application of simple multiplicative strategies involving decimals is listed at Level Three, with additive operations involving decimals are listed at Level Four and understanding operations with decimals is given as Stage Five (Ministry of Education, 2007a).

### **3.1 Initial Data**

#### 3.1.1 Base Constructs

‘Multiplication makes bigger’ and ‘division makes smaller’ are well-documented beliefs (e.g. Bonotto, 2005; Kieren, 1992; Tirosh & Graeber, 1989) and were evident in many of the student responses to initial questioning. Examples other than whole-number problems did not occur to them.

Table 13  
Students' Responses concerning Decimal Operations

Question	Mary	Grace	Wini	Bridget	Kiri	Hoani
Someone has said that 'multiplication always makes things bigger'. Do you agree with this?	Yes	Yes	No	Yes	Yes	Yes
Can you write an example to show what you think?	No example	Other than $0 \times$ and $1 \times$	$3 \times 1 = 3$	$37 + 46 = 83$ (?)	Except 0 and 1	No example
Someone else has said that 'division always makes things smaller'. Do you agree with this?	Yes	Yes	No	Yes	Yes	Yes
Can you write an example to show what you think?	No example	Except for $\div 1$ and $\div 0$	$2 \div 2 = 1$	$72 \div 3 = 24$	Except $\div 1$	No example

None of the students considered multiplicative situations involving fractions. Apart from the exceptions of one and zero, the statements of 'multiplication makes bigger' and 'division makes smaller' were accepted as true.

### 3.1.2 Multiplicative Proficiency

Other tasks were set to examine how the students would solve a series of problems that involved multiplication with decimals of types *whole number*  $\times$  *decimal*, *decimal*  $\times$  *whole number*, and *decimal*  $\times$  *decimal*. These are regarded in the literature as being in order of increasing difficulty for students.

#### Task One

I poured four 1.25L bottles of orange juice into a bowl. Can you write the maths sentence to work out how much juice is in the bowl?

Table 14  
Students' Responses to Whole-number  $\times$  Decimal Task

	Mary	Grace	Wini	Bridget	Kiri	Hoani
$4 \times 1.25$	Left blank	$4L + 4 \times 25 = 100$ Not sure	$1 \times 4 = 4$ $0.25 \times 4 = 100$ 5L	4.1L	5L 4 + 1	$1.25 \times 4 = 5$

Three of the students were able to successfully coordinate the whole number and decimal parts of the quantity. Kiri and Hoani appeared to have used the fact that 0.25 is another representation of  $\frac{1}{4}$  as they processed the problem mentally. Wini had used a whole-number fact ( $25 \times 4 = 100$ ) and then been able to correctly apply this result into its decimal context.

Task Two

Peanuts cost \$7.00 for one kilogram. I get 0.9kg in a bag. Can you write what the maths sentence to work out how much I will pay?

Table 15

*Students' Responses to Decimal  $\times$  Whole-number Task*

	Mary	Grace	Wini	Bridget	Kiri	Hoani
$0.9 \times \$7$	\$6	\$6.50	\$6.30	Don't know	$\$7.00 \div 100 =$ \$6.50	\$6.50

While only Wini had the correct answer, four other students recognised that the required amount must be less than \$7. What was unclear was whether these four students regarded the situation as multiplicative. I suspected that they recognised that 0.9 kg was slightly less than 1 kg and made a small, uncalculated adjustment to the initial amount of \$7. Either they did not recognize the situation as being multiplicative, or, if they did, they avoided this knowledge by choosing to estimate and not calculate. Only Wini seemed to realise that she already possessed the key piece of multiplicative knowledge ( $9 \times 7$ ) that would provide her with the digits of the answer.

In recognizing that the answer must be less than \$7, the students exhibited number sense that may later allow them to accept multiplicative statements where the product is less than the multiplicand.

The selection of Tasks One and Two was based upon similar research tasks where realistic problems were set for students to help them make connections with the underlying mathematics. In particular Irwin (2001) included discussion of soft drink bottle size and Bonotto (2005) had found that the examination of till receipt artifacts assisted students in overcoming the MMB misconception.

A non-contextualized task was also presented.

Task Three Calculate  $4.2 \times 2.6$

Table 16  
Students' Responses to Decimal  $\times$  Decimal Task

	Mary	Grace	Wini	Bridget	Kiri	Hoani
$4.2 \times 2.6$ (10.92)	6.8	9.2	9.2	6.8	9	Left blank

Mary and Bridget added the amounts. They knew what the  $\times$  sign meant with whole numbers, but had reinterpreted this meaning for decimals. This served to show that meaning is not inherent in the symbol but in the mind of the user of that symbol (Godfrey & Thomas, 2003). It is not that the  $\times$  symbol was learned 'once for all time' but new extensions of meaning may arise as new situations are encountered (Cobb, 1992).

Grace and Wini both calculated in similar fashion, combining two, rather than four, partial products. Whereas Mary and Bridget's answers result from a misinterpreted symbol, these students have misapplied a process.

Wini's working (copied from her workbook)

$$\begin{array}{r} 4.2 \\ \times 2.6 \\ \hline 9.2 \end{array}$$

Wini reproduced the layout of the standard vertical-form for multiplication but did not appreciate how the multiplicative process it represented could operate for these numbers. Use of standard algorithms without understanding has been cautioned against (e.g. Ebby, 2005; Hiebert, 1999; Hiebert et al, 1997). As Stage 6 students, the group will have worked with the standard vertical addition algorithm. In first learning this, two partial sums are created and then combined to produce a final answer.

E.g. 
$$\begin{array}{r} 46 \\ + 37 \\ \hline 13 \\ \hline 70 \\ \hline 83 \end{array}$$

The units and tens are treated separately in order to create two partial sums. It is possible that this process had been transferred to the new multiplicative situation. Doing this

would not have been regarded as an error by the students but a logical extension of prior experience; that is, an expansive generalization. This implied that a situation needed to be engineered whereby the students would recognise the need for a method that produced four partial-products.

Kiri doubled 4.2 to get 8.4 and then added 0.6. Doubling the 4.2 may be seen as partitioning the 2.6 and first multiplying by 2. It seemed that she did not know how to incorporate a multiplication involving 0.6 into her schema and so resorted to an operation she could make sense of – addition – in order to satisfy herself that she had used both parts of her partitioned multiplier.

It was not expected that the students would be able to successfully complete Task Three, given that their teachers had described them as Early Multiplicative using the NDP Framework (Ministry of Education, 2007b). This task was set two stages higher. It was important to see what knowledge the students attempted to bring to the problem though, and to see if they were socially prepared to attempt a task that they recognised as more difficult than what they were used to. Hoani seemed to have recognised the high novelty of the problem and chosen not to further engage with it despite his being regarded by the others in the group as more mathematically able than themselves.

### **3.2 Measurement Contexts**

Session 3: Environmental notes

This session was held in a technology classroom. This was ideal for pouring water but, due to staff illness, some biotechnology experiments had been left unattended so the room had a bad smell. It was the school's Wild Hair Day, a fund-raiser for charity with students and staff decorating their hair or wearing wigs or funny hats. Excitement levels were high, as was the degree of distraction when decorative pupils walked past the room.

#### **3.2.1 Area Representation of Number for Division**

The group had each drawn a pictorial representation of 1.2 using 120 grid squares in their exercise books. The instruction was given to use their diagram to work out  $1.2 \div 4$ . That it took 6 minutes to resolve indicates how much the students were struggling with the

concept. I then asked them to show what four lots of 0.35 would equal. It took 3 minutes before a correct answer was found.

There were several issues to consider when accounting for the difficulties the students had with the initial task of  $1.2 \div 4$ . It was believed that the students were employing a partitive model as they were looking to divide 1.2 into 4 equal groups rather than the much more difficult interpretation of wondering how many portions of 4 there were in 1.2 as in a quotitive model. The number choice and creation of the diagram before the division task was set in a way that the simple number choice would be sufficient scaffolding, as it was expected that a link with  $12 \div 4$  would be made with 12 tenths. This support proved to be barely sufficient as the link with the intended whole-number parallel was not obvious to the students. They seemed to struggle with the concept of applying division when a decimal number was involved. The area model did not seem to help. This model is the most common representation of fractions – which is why I thought it would have provided a strong physical representation to work from - though studies have warned against expecting transfer from area models to generalized number Lamon (2001, 2002). A third example of  $1.5 \div 6$  was then set.

Grace: I don't know what to do  
T: Can you draw a picture like the other one?  
Grace: Drawing doesn't help me (plaintive tone).  
T: Well, what *will* you do?

I did not require her to draw a diagram, but did expect her to draw upon her own resources. After waiting for a couple of minutes, she carried out a calculation. She appeared to have used the known fact that  $2 \times 6 = 12$ , and then recognised that the remainder of 3 is half of 6. Her initial answer was recorded in her book as '2.5'. While incorrect, it seemed that she was trying to link 1.5 to 15 and this process of looking for linkage can lead to effective strategies.

Bridget and Kiri both employed a common division strategy used in the NDP, that of carrying out two 'known' divisions to find an unknown (Ministry of Education, 2007e).

Bridget: Well  $15 \div 3 = 5$ , halve it so 2.5

In her book, the working appeared as below:  
 $15 \div 3 = 5$       $\frac{1}{2}5 = 2.5$  (interpreted as 'half of 5 is 2.5')  
 $1.5 \div 6 = 0.25$



All the students initially calculated  $15 \div 6$  and then adjusted their answers from '2.5' to '0.25' as a second step. They applied a strategy of using their knowledge of whole-number operations and then adjusting their initial answer to make it reasonable. This approach stood in contrast to Task Two (0.9kg of peanuts at \$7/kg) where only Wini thought to use  $9 \times 7$  as part of the solution process.

One contributing factor for this change may have been the use of the grid squares. That none used a diagram to calculate  $1.5 \div 6$  led me to conclude at the time that this method was not considered useful by the students. Later reflection modified this view. As noted above, the calculation of  $1.2 \div 4$  was not easy for the students even though they had already drawn up a 12 by 10 grid to model the 1.2. The link between  $1.2 \div 4$  and  $12 \div 4$  was not immediately recognised. The grid arrangement was seen by the students as 120 small squares. Its division by 4 was unproblematic. Re-assigning a decimal meaning to the newly created zones of 30 squares took time. The students had to adjust 1.2 (in the problem) to 12 rows of 10 (in the drawing) to 120 small squares (the total area) to 30 squares (each sub-zone) to 3 rows of 10 (tenths) and then to the final answer of 0.3. While the method of using the squares was quickly discarded by most of the group, the place-value adjustments inherent in its use may have provided meaningful help by linking whole number and decimal operations. The example may have been 'bridging' for the students in that it promoted cognitive re-consideration, even though its efficacy was not obvious at the time.

There were (at least) two plausible hypothetical learning trajectories with regard to decimal multiplication, to further explore the decimalization process by looking at the effect of taking a tenth of a tenth using 'decimats'<sup>9</sup> (as recommended in Ministry of Education, 2007f, p. 37), or to utilize the students' approach of employing whole-number knowledge and seeking to use estimation to judge the reasonableness of their answers. The approaches are complementary and ideally would both be used, but time constraints meant a choice had to be made. My choice was shaped by the students' overwhelming preference to process by using whole numbers initially and then to adjust their answers.

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<sup>9</sup> Decimats are a progressively more complex set of cards where area is used to model number. Initial cards are marked in tenths, with subsequent cards marked in hundredths and thousandths. Decimats are a NDP resource available from [http://www.nzmaths.co.nz/numeracy/2007matmas/Bk7/MM%207\\_3.pdf](http://www.nzmaths.co.nz/numeracy/2007matmas/Bk7/MM%207_3.pdf)

### 3.2.2 Water Pouring Division

I decided to introduce the idea of pouring water between containers as a primary model for quotitive division. I had planned that the pouring out tasks would involve seeing how many of a particular size could be filled from a larger container, thus setting the dimensions of variability (Marton & Trigwell, 2000; Mason, 2005). By not introducing partitive problems such as ‘share this bottle out into five containers and see how much you get in each, I hoped that the students would start to acquire a new division vocabulary. This could enable them to re-frame a division involving a decimal divisor into a ‘how many of *these* are in *that*?’ type of question. This has been regarded as essential for developing understanding of this type of problem (e.g. Graeber & Tirosh, 1990; Tirosh, 2000). I had not previously read of this activity being employed, though others had used water with other interactions with decimals (Hunter & Anthony, 2003; Moss & Case 1999). I believed that this activity would link the experiential knowledge students had of pouring out situations (e.g. sharing drinks) with a decimal context with which they were familiar, the size of liquid-filled containers (e.g. milk, soft drink). The use of these artifacts was thus hoped to be transparent (Meira, 1998; Stacey et al, 2001). It was anticipated that the enactive process of division (Mason, 2005; Meyer, 2001) would allow the students to reflect upon the results they were producing and thus upon the mathematics (Tzur, 2007). In particular, the ‘division makes smaller’ (DMS) concept would be challenged by student-produced evidence. This had the potential to lead to cognitive reconstruction as suggested by research (e.g. Moskal & Magone, 2000; Tirosh & Graeber, 1990; Zazkis & Chernoff, 2008).

The other pedagogical reason was simply that water pouring would be fun! While Moyer (2001) found that materials could be viewed as interesting for students but not carriers of meaning, attention to the mathematics and the enjoyment of students seemed possible here (Ball, 1993). Engagement is a major aspect of the design of effective instructional tasks (Hodge et al, 2007).

I produced a wide range of containers that had all previously contained liquids. We talked about the sizes of the various containers by having the students reading the measurements

from the printed labels. Some students were unfamiliar with capital L being used as the symbol for litre; but simple metric conversions (e.g. 600mL = 0.6L) did not seem to be a problem.

We started working out  $0.6 - 0.4$  by using a 600mL and a 400mL bottle. Looking at the remainder, the students quickly identified this as both 200mL and 0.2L.

T: If we start with 0.6 and we share it into 0.4 bottles, how many will we get?  
(This was carried out by the students using water and a funnel over a sink).  
T: You've filled one up right? (Nods). So our answer has 1 in it (Yeah) but we've still got some left.  
Chn: 1.2, 1.5, 1.05  
T: OK, so let's pour that out (the full 0.4L bottle) and see. (Pouring done).  
C: Hey, that looks like half of it!  
T: So what have we got?  
Chn: 1.2, Eh? 0.15, One, one slash 2 ( $1 \frac{1}{2}$ ) I reckon it is 1.2  
T: We have one and a half...  
Chn: 1.2, 1.5,  $1 \frac{1}{2}$  (these latter answers received nods)  
T: So what is that as a decimal?  
Chn: 1.5

With hindsight, a different bottle choice for the first example may have been preferable. As the physical operation could have been seen as repeated subtraction, the cognitive proximity of the subtractive remainder of 0.2 to the multiplicative relationship of  $\frac{1}{2}$  (both involving the digit 2) possibly clouded the thinking.  $0.75\text{L}$  into  $0.25\text{L}$  may have been better. It needs to be considered too that the students were very excited at the water pouring as well as with it being Wild Hair Day.

We looked at the written representation,  $0.6 \div 0.4 = 1 \frac{1}{2}$  or 1.5

T: So the 0.6 and the 0.4 refer to the size of the bottles and the 1.5 to how many bottles.  
Hoani: Eh? (Incredulous tone)  
T: Well we had one full bottle (yeah) and then half a bottle.  
Hoani: Yeah. # Oh yeah.

Hoani recognised the (apparent) incongruity of the written statement. I consider that his thoughtful reviewing of this answer allowed him to be the only student to successfully predict the outcome for  $2.25 \div 9$  later.

The group was now selecting items to work with.

T: Now you guys started with a 0.4L bottle and you filled two of those cups yeah?  
Chn: Yeah

T: The 200mL cups, point two. So 0.4 divided by 0.2 gave us 2. How many 0.2s are there in 0.4?  
Chn: 2

I modeled the language of quotitive division so that the exercise was not simply seen as a practical activity of water pouring. The practical experiences would have little value unless the mathematics was made explicit.

T: So this one's 1.5, and we take that 750mL, decimal? (Pointing to the respective bottles the students have chosen).  
Chn: 0.75  
T: So how many of these will fit into there?  
Chn: Two. Zero point two.  
T: How can we know which one is right?

Two girls commandeered the bottles and filled the 0.75L from the 1.5L (repeated subtraction) then said 'two'.  
Some voices agreed, but others said '½' (Looking at what was left in the 1.5L bottle).

T: So let's look at the question again, if we read it as 'how many 0.75s are there in 1.5?'  
C: Two (But the '2' and 'point 2' answers continue to be voiced in the background).

Later on.

T: We are now pouring out a (real) 2.25L L & P into these 0.25L cups  
Hoani: It will be 9 full cups.  
(I started pouring).  
T: I've filled 3 cups, how much of a litre have I poured?  
Chn: 1/3, 1/3, 3/10 (I think having actual soft drink is hindering logic!)  
T: How much is one cup?  
C: 250mL  
T: Decimal?  
Chn: 0.25  
T: So how much for three cups?  
C: 0.5, 0.75 (Has skip counted in units of 0.25)

Later

T: How much have I poured out?  
C: Seven cups so ..... 1.75

Later, while sipping L & P.

T: So we shared out the 2.25L bottle out into those cups, 0.25L and we got 9.  
Now many people think that when you divide you must end up with smaller numbers^ (I was invoking imaginary 'others' while in essence referring to their own previous constructs).  
Kiri: But not with decimals  
T: Yeah, not with decimals, 'cause we had  $2.25 \div 0.25$  and got 9. Because this number was less than one, we ended up with a bigger number here. Did we end up with more drink?  
Chn: No (chorus)  
T: So we didn't get more drink, we just got more^  
Kiri: Portions  
T: Yeah, we got more portions.

We had enacted a situation whereby the quotient exceeding the dividend has made practical sense. We had looked at how division re-distributes an original quantity into

different sized quantities to create a different number of portions (to use the student's word). This quotitive scheme of division allowed students to accept a solution where the answer was 'bigger' in a way that the partitive division scheme does not. By interlinking the students' prior experiences with both decimals and bottle sizes, the data that was created forced a re-examination of the previous conception and so initiated the process of reconstruction.

The students then worked in two groups of three to make up their own pouring problems, enact them, and then record them into their books. This was to parallel the method used with the students in Section 1.5. Giving them a choice of materials and hands-on practical tasks coupled with a requirement for written recording was designed to promote sense-making experiences. This is what the NDP describes as moving from Materials to Imaging. The students had to be reminded to record their findings. The mathematization of the task was not the goal of the students but of the teacher. While the goals were not identical, there was sufficient overlap for both parties to view the activity as successful. It was clear from overheard conversations and the written work that the students had established that what they were doing was considered by them to be division and not simply repeated subtraction.



*Figure 12 Photograph of Students Pouring Water for Quotitive Division*

The students' books had the following written work:

*Table 17*  
*Quotitive Division work taken from Student Workbooks*

Group	Context	Equation
1	Soft drink bottle to water bottle	$1.5 \div 0.6 = 2.5$
1	Large to medium soft drink bottle	$2.25 \div 1.25 = 1.85$
1	Water bottle to plastic wine cup	$0.65 \div 0.175 = 3.5$
2	Detergent bottle to small soft drink bottle	$2 \div 0.33 = 6$
2	Detergent bottle to water bottle	$2 \div 0.75 = 2 \frac{2}{3}$
2	Water bottle to plastic cup	$0.4 \div 0.2 = 2.25$

While some 'equations' were not strictly accurate, there was clear evidence that the students were able to record division problems that would conflict with the statement that 'division makes things smaller'.

### 3.2.3 Further Multiplicative Tasks

#### Session 4: Environmental Notes

In the smelly Technology Room again. The students arrived 45 minutes late due to an assembly so only a short session was held.

The group was asked to work out a number of written tasks. Kiri and Bridget worked together and had identical working out, the others worked independently.

*Table 18*  
*Students' Responses to Multiplicative Tasks*

	Mary	Grace	Wini	Bridget	Kiri	Hoani
I had six, 0.4L; will this fill a 2.25L bottle?	No answer	$0.4 \times 6 = 2.4$ It would overflow	$0.4 \times 6 = 2.4$	Overflow by 0.15L (repeated addition used)	Overflow by 0.15L (repeated addition used)	0.15 left over
I had eight, 2.25L; will this fill a 20L bucket?	No answer	$8 \times 2 = 18$ (!) $8 \times 0.2 = 1.8$ $8 \times 5 = 0.40$ 20.2L	$2 \times 8 = 16$ $0.2 \times 8 = 1.6$ $0.05 \times 8 = 0.4$ 18L	No answer	No answer	$8 \times 2.25 = 20$
I had 3L, shared into 0.6L bottles, how many would it fill?	No answer	$3L \div 0.6L = 5$	5	$3L \div 0.6L = 5$	$3L \div 0.6L = 5$	$3L \div 0.6L = 5$

After the written work was attempted, these tasks were then acted out by the students.

The group had no problems with  $4 \times 0.6$  (except for Mary), but only Wini correctly calculated  $8 \times 2.25$ . I think that the change from one to three digits in the multiplicand caused the problem. This is unsurprising when considering the students' Numeracy Project stage as the entire group was listed as being at Stage 6 according to the Framework document (Ministry of Education, 2007b). At this Stage, students are able to process multiplicatively (as opposed to additively) for problems involving double digit by single digit numbers *when* given a prompt.<sup>10</sup>

With the division example, only Mary did not produce the correct result. One might have been tempted to assume that the skill of single-digit decimal division had now been internalized by most of the group; however, a task in the next session challenged that conclusion.

#### Session 5: Environmental notes

This was held in the afternoon after a weekend's break in an unused classroom. Wini was away and Kiri and Bridget were unfocussed.

A written task was presented,  $3 \div 0.6 =$

No-one had the correct answer, the most common was 0.5

It seemed that without the context of having visible bottles and the unit of measure given as one litre, the group reverted back to the pre-existing belief that division makes smaller. A single set of experiences of enacting quotitive division had not been sufficient to displace this earlier belief, a not unforeseen circumstance (e.g. Fischbein, 1985; Harel & Sowder, 2005; McNeil & Alibali, 2005; Siegler, 2007).

I produced the relevant bottles and mentioned their sizes. Hoani immediately wanted to change his answer. The stimuli of seeing the bottles and hearing the unit of measure was sufficient for him to re-connect with the mathematics of the tasks in the previous session. The others (bar Mary) started saying the correct answer once the problem was re-worded to "how many 0.6 bottles can I fill from the 3L bottle?" They had not been able to re-phrase the question independently, and this had prevented them from carrying out the

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<sup>10</sup> E.g. If  $3 \times 20 = 60$ , what will  $3 \times 18$  equal? (Ministry of Education, 2007c, p. 41)

division. The mechanics of the division were not the issue as these correct answers came without any written work. The new concept had not been lost, but the old concept overrode the new concept until prompts were provided as scaffolding.

Mary was still confused, and the task had to be physically acted out for her to look happy with the answer of 5. Unlike the others who only needed verbal evidence to re-select the new way of thinking, Mary needed more concrete evidence. These can be interpreted with reference to the Teaching Model of the NDP with its broad labels of Materials, Imaging and Properties (Ministry of Education, 2007d). Mary was operating at the Materials level; she required manipulatives in order to carry out calculations. While the others had not yet moved to the Properties level – where abstraction of the concept would have enabled them to solve the problem without any contextual clues – there was evidence that they were operating at the Imaging (sense-making) level. Here students are described as needing concrete experiences as referents before being able to perform mental operations. This dependency initially requires close modeling of the initial situations and high scaffolding - as Grace, Wini, Bridget, and Kiri needed - and progresses to where brief reminders are all that is required – as Hoani needed.

### 3.2.4 Using Length as a Measurement Context Session 6

One type of experience with a concrete operation might prove to be insufficient as it may have led to the resulting knowledge being too strongly associated with only that one situation. String and scissors were chosen as a new context within which division could be explored as it allowed both partitive and quotitive problems to be created.

Students worked in pairs and measured out 3m of string per pair.

T: If we cut this into 5 equal pieces, how long is a piece?  
C: Six? ( $30 \div 5 = 6?$ )  
Chn: 0.6! (Immediately agreed to by the original speaker).

The sharing (partitive) problem did not seem to be problematic in either concept or calculation.

T: OK, cut 0.6 off. How much is left?  
Chn: 2.4



This answer came without formal calculation; that the subtraction involved the coordination of whole numbers and tenths did not seem to pose problems. There was some discussion about whether the amount should be recorded as 2.4 or 2.40; before group consensus was achieved that it didn't matter.

Moving to some measuring (quotitive) situations for division seemed to be the next logical step. The students had their 2.4m string and access to a 1m ruler. They worked by agreeing on an answer with their partner and then checking its veracity by 'stepping it out' on the ruler for the first 2 – 3 problems. After this, they simply agreed on answers with their partner. This can be seen as movement into the second tier of the 3-tier learning models (e.g. Bruner, in Mason, 2005; Herscovics, 1989; Sfard, 1991) or as moving from 'Materials' to 'Imaging' using the NDP language (Ministry of Education, 2007d, p. 5).

A scenario was written on the board. Students understood that with regard to their 2.4m length of string, they had to say how many pieces would result if they were to cut it into each of the following lengths; 0.8m, 0.6m, 0.2m, 0.15m, 0.08m, and 0.03m. The first question was recorded on the board to provide scaffolding into the quotitive approach; "How many 0.8's are in 2.4m?"

I wanted the students to become accustomed to seeing quotients that were larger than the dividend, and so had them record each result in their books. Exposure to the situation was intended to be practically meaningful and so provide another referent (along with the water pouring situation) to division problems of this type. Again, having the results recorded by each student provided them with a personally collected body of evidence where quotients were larger than dividends. It was hoped that this would allow them to overcome any cognitive conflict they would experience when processing non-contextualized problems later.

Once the task was completed, the results were discussed by the group and me. I wanted the students to read out their answers to further familiarize them with these 'quotient larger than dividend' problems and to de-sensitize them to their answers appearing larger than 2.4. The written and spoken answers provided a double data source for them to access. This self-referencing of data is believed have a powerful influence (Lamon,

2007). It was clear that they could rationalize out the distinction between the number of pieces versus the size of those pieces in this marking session as they assigned the metric unit to the divisor and never to the quotient. (By way of contrast; if a student had answered ‘four *metres*’ to the question “how many 0.6m lengths are there in 2.4m?” this would serve as a counter-indication).

One pair had the 0.15 answer incorrect, all other answers from the students were correct. It is possible that there was a computational, rather than conceptual reason behind this error. 0.15 was the only problem to have two, non-zero digits. At their reported NDP stage, teachers had previously only recorded success with single digit division problems for these students.

The students said their first answers and I re-modeled the equations on the board to provide another visual reminder to show that they had carried out ‘quotient larger than dividend’ division problems. I was also interested in the strategies used for the latter exercises.

T: 0.15, how did you do it?  
Hoani: I just turned the 0.15 into 0.3 and that’s 8 and then (doubled) the answer  
T: 0.08?  
Grace: I just used that one (0.8) and the (new) answer must be ten times bigger

These two students have shown evidence of deriving new division facts by drawing upon existing knowledge of both division and decimals and successfully combining these. This is evidence of Stage 8 behaviour in the NDP (Ministry of Education, 2007b, p. 17).

### Session 7

a) The students were given a series of tasks that required them to switch between partitive and quotitive division models.

Task: Imagine that you have a 3.6m length of string.

1. How long are the pieces if you cut it into 4? Into 12?
2. Imagine you are cutting it again, but this time into lengths that are 0.2m long. How many pieces do you get? What about for 0.05m pieces?

The difference between this and the last session’s work was that the literal presence of string was not longer there as a scaffold. Hoani and Kiri were able to choose and apply the correct division construct to calculate the answers. Grace carried out the partitive but

not the quotitive division tasks. Bridget appeared to copy from Kiri, while Wini and Mary were disengaged from the problems. Hoani and Kiri did not need the visual prompts of materials, let alone being tied to using them.

### 3.2.5 Food Purchases

Multiplication with one decimal can be of the form *whole number*  $\times$  *decimal* or (as here) *decimal*  $\times$  *whole number*.

This latter type of problem is recognised in the research as being more difficult (e.g. Graeber & Tirosh, 1990; Schliemann, 2002). Whole number (as multiplier)  $\times$  decimal (as multiplicand) type problems allow the solver to conceive of entire sets (even though these sets contain decimals) being grouped together; an expansion of the repeated addition model. Where the multiplier is a decimal, the accumulation of sets does not make sense easily. Instead of building upon repeated addition, a means of scaling the multiplicand is required.

The students' first problem was:

Oranges cost \$6 per kilogram. How much would you pay for 0.7kg?

Hoani: That is 70% of \$6  
T: Yes, can you use that? (No response)

Bridget: How much does one orange cost?  
T: You don't know that, you only know the weight.  
Wini:  $6 \times 7$  is 42 [Tellingly, she had made the  $9 \times 7$  connection in the peanuts problem in Section 3.1 as well].  
T: Yes, can you use that?  
Wini: \$4.20  
Hoani: Agree  
Bridget: But how much does *one* orange cost?  
T: You don't know that

They were set the problem of finding out how much 0.3kg would cost.

All completed the task correctly, but I was unconvinced that Mary and Bridget had worked independently. Bridget's comments led me to think that she was searching for a means to resolve the problem into a repeated addition base.

0.3kg at \$6/kg

T: How did you do it?  
Mary:  $3 \times 6 = 18$   
T: And then...  
Mary: Make it into \$1.80

She appeared to have made sense of it by using whole-number facts and then adjusting the answer. Was this a new procedure or the start of a new concept? As Hiebert et al (1997) have described, teachers can often infer that students have grasped the mathematics of a situation because of their correct use of a procedure but remain unaware that a conceptual reconstruction has not occurred. In Tzur and Simon's (1993) model, these students are at the participatory but not anticipatory stage.

Written task: Tamarillos cost \$8 per kg. How much for 0.6kg?

Bridget: I don't know what to do.  
 T: Ok, if you bought 3kg, what would you do?  
 Bridget:  $3 \times 8$   
 T: 5kg?  
 Bridget:  $8 \times 5$  [Commutative law applied, but this loses the practical sense of what happens in this scenario].  
 T: 15 kg?  
 Bridget:  $8 \times 15$   
 T: 0.6kg  
 Bridget:  $8 \div 0.6$   
 T: Why divide it? # What did you do here? (Pointing to the 3kg example).  
 Bridget: Multiplied; but this is less than one.  
 T: But you multiplied all of the others  
 Bridget: Yeah, but this is less than one.

The size of the numbers involved seemed to be a clue to Bridget that she should change operations. The cognitive demand of the scaling effect of a decimal as multiplier was being avoided by searching for a different procedure. As this was not proving to be any easier ( $8 \div 0.6$ ), she was stalled. There was a self-imposed learning barrier in her mind that would not allow the pattern of multiplication to continue. This fits with McNeil and Alibali's (2005) model of change resistance. With hindsight, it may have been better to have produced a model of the \$8 and enacted buying amounts of tamarillos to reinforce the fact that in this situation the \$8 is the multiplicand. While I allowed Bridget to apply the commutative law in the hope that she would find  $8 \times 0.6$  easier to work with, I think that she lost sight of the fact that the original problem required her to think in groups of  $\frac{1}{2}$  a kg priced as  $0.5 \times 8$  may have been a suitable intermediate stage to resolve in order to accomplish the 0.6 task or possibly 0.1 kg.

### 3.3 Further Multiplication

#### 3.3.1 Estimation: Session 5

The group was told that they were to each estimate the answer to the problems and then they would take turns at using a calculator to get the exact answer.

The first task was  $38 \times 0.948$

- T:  $38 \times 0.948$  I'm not going to ask you to do all the multiplying, just think about what the answer would be, roughly. (Pause while students think).
- T: Ok, Grace thinks around 14, Hoani thinks about 37
- Kiri: What's half of 38?
- Bridget: 19
- Kiri: Yeah 19 so around 19
- Mary: Yeah  
(I was puzzled at the answers of 14 and 19).
- T: OK, so what we have to think about at the start is this, 0.948. So what number is that really close to?
- Kiri: 1, one whole
- T: Yeah so if we had 38 times one whole we'd get...
- Chn: 38
- C: It's half, we said it ^
- Hoani: So I put 37^
- Kiri: But its about half, 19, where's the calculator?
- Hoani: It's 0.052 away from it
- Bridget: Even if you calculate you can't prove me wrong!
- Hoani: So what's the exact answer?
- T: I don't know. I need a calculator. # What we are looking to do is to use numbers we *do* know to work out these other ones.  
Mary, you do this for us, work it out. #
- Mary: 36.024
- Grace: Oh, I was wrong
- Hoani: I got 37
- T: Yes, a good estimate.

Even when attention was drawn to the agreed fact that 0.948 was close to 1, the students did not adjust their answers. This surprised me. It was only as I was listening to the transcript tape that I have been able to come up with a hypothesis on why the majority of the group seemed to focus on  $\frac{1}{2}$ . The example in my planning book was 0.998, but I wrote it on the board as 0.948 and read it out as this. Hoani verbalized the observation that I think the others made, that it was 0.052 away from 1. I believe it is possible that the others interpreted this as 0.52 away from 1, i.e. close to  $\frac{1}{2}$  and hence the confusion. My original example may well have provided a different result. Not realizing this, I

persevered with the first example before abandoning it. As Irwin (2001) points out, task selection is crucial!

- T: OK so let's write down what we used for our estimate, 'cause we used 1 didn't we? (Waiting for a chorus of recognition about the previous, incorrect answers, it didn't come).
- T: Could the answer have been more than 38?
- Kiri: No, because that number is less than 1. Having that less than one makes that less than 38 (But no modification of her original answer was forthcoming).
- T: OK, try this one,  $38 \times 0.51876$
- Bridget: Nineteen (immediately)  
(Hoani and Grace have also immediately written correct answers down in their books).
- T: OK, so 0.51867 is about...
- Chn: Half
- T: So it should be around^
- Chn:  $19^{18-20}$  about 21
- Bridget: (mock serious) Now if my calculations are correct...
- T: Now a lot of you did that,  $38 \times \frac{1}{2}$
- Hoani: 19
- T: Will it be bigger or smaller than 19?
- Hoani: Bigger, no smaller
- Chn: Bigger!
- Hoani: Oh what, yeah, bigger?
- T: Bigger than 19?
- Hoani: Smaller than 38 'cause that's not even 1
- T: Bigger than 19?
- Hoani: Oh yeah, bigger than 19
- Kiri: By one or two. 20, 21
- T: Do you *know* the answer?
- Kiri: No, only round about. (I wanted her to be reminded that the process is not exact, but still meaningful).  
(Grace had the calculator and started the calculation).
- Bridget: OK Grace, tell me what I want to hear!
- Grace: 19.7288 (The group was pleased).

### 3.3.2 Double digit Multiplication: Session 8

#### Environmental Note

Mary was ill and was coughing deeply throughout the session.

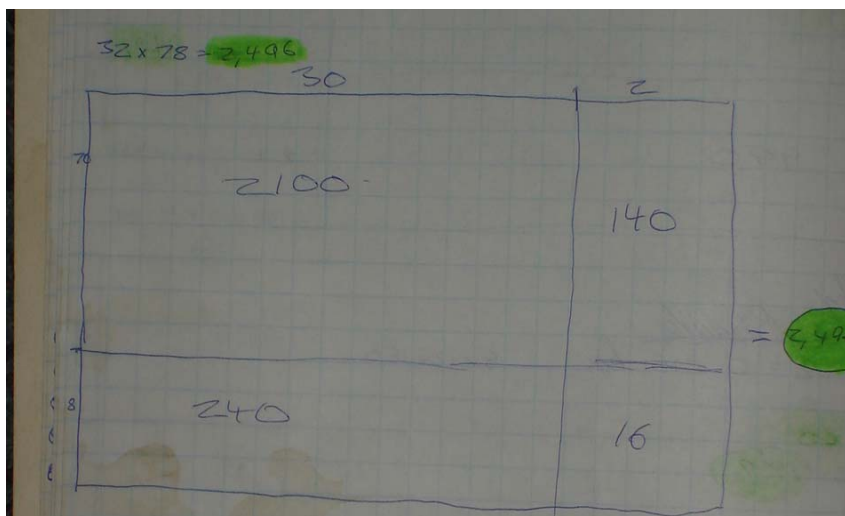
The group was asked if they could do double-digit multiplication but they responded by saying that it was too hard – conforming to the NDP Stage 6 behaviour as described earlier.

I asked the group to calculate  $15 \times 18$ . Kiri immediately responded “140” and then elaborated, “ $10 \times 10$  is 100 and  $5 \times 8$  is 40”. Kiri did this even though she could easily calculate  $10 \times 18$  and know that this was 180, more than her stated answer of 140. A faulty procedure was chosen ahead of any application of number sense. This seemed to

be the same approach used previously by Wini and Grace (3.1.2, Table 13). I decided to work with double-digit whole-number multiplications before I introduced their decimal parallels. I chose to work with an area model as this demonstrates why there are four sub-products (Young-Loveridge, 2005). This would challenge the faulty, two sub-product system that had been observed to date. The area model draws upon existing knowledge of place-value partitioning and the procedure for calculating rectangular area, both of which were existing knowledge for these students. In this way, a situation of cognitive dissonance could be created without the simultaneous introduction of a range of new procedures.

The students confirmed that they knew how to find the area of a rectangle. They were then shown asked how they would split a double-digit number when used as the side of the rectangle and, as expected, they proposed standard place-value partitioning. By drawing this partitioning onto their rectangles, it allowed the students to recognise the four sub-areas. The students readily accepted the fact that the process produced four - and not two – sub-products. The working shown in their books always had four sub-areas calculated. The area model was so clear that it may not have provided enough reflective contrast with the mis-applied algorithm. This earlier method was not referred to by the students during these calculations, yet it re-appeared in Task Three of Bridget’s work at the end of the study. If more time had been available, more deliberate linking of the area model and the correct application of the standard algorithm may have been worthwhile.

An example of an early calculation appears below:



$32 \times 78$   
 The four sub-areas have been calculated separately and the total area written to the right and highlighted. Note the independence of scale, the side labeled '30' is much longer than the '70' side.

Figure 13 Photograph of Student Work showing Area Model in Multiplication

It was interesting to note how the students used the diagrammatic approach. The rectangles were drawn and partitioned without reference to the relative sizes of the numbers being multiplied. They have adopted the diagram as it assists them to organise their work and are not reliant upon it for any factual evidence (Diezmann, 2000, 2005). Grace typified the dominant approach (four students) that I have labeled the ‘flag’ style. All ten of her recordings were drawn in this way with the aspect ratio averaging 2.7. Hoani was alone in drawing near squares that I have labeled the ‘box’ style. Here the aspect ratio averaged 1.1. Hoani’s diagram for  $73 \times 27$  (73 vertical) had an aspect ratio of 1.5, when a scale drawing would have 0.37).

That the styles of diagrams were largely independent of the numbers that they were representing suggested that they were being used as tools to aid the process (Imaging) rather than being attempts to concretely model the situation (Materials). They did not require a direct numerical equivalence in order to be functional. Occasionally, weak times-table knowledge caused errors to be made. I did not allow the group access to calculators as I thought that these might become the tool of choice. This would lower the cognitive demand of the task into becoming a simple rote procedure.

Bridget:           What’s  $7 \times 7$  Hoani? 42? 48?  
 Hoani:             49  
 Bridget:           35, 42, 49 Yeah. (Skip-counting from a known fact).

Another calculation issue was the size of the products.

Here Wini’s first attempt at  $42 \times 56$  has been copied from her workbook.

	50	6
40	200	240
2	100	12

T:                    So you’ve got  $40 \times 50 = 200$  (Neutral tone, trying to see if self-correction will occur).  
 Wini:                Yeah, ‘cause  $4 \times 5$  is 20  
 T:                    But next door you’ve got  $40 \times 6 = 240$   
 Wini:                Is that wrong?  
 T:                    No, that looks OK because  $4 \times 6$  is 24, but you’ve got ten times as much.  
 (No response).



T: So this area is bigger than this one? (Pointed to the 240, then the 200 zones).  
 Wini: Oh, so this one should have another zero? (Wrote it in).  
 T: Well, yes, but think about why. What is  $4 \times 50$ ?  
 Wini: 200 # Oh yeah! I get it.

Her subsequent examples were correctly calculated.

The group - except the ill Mary - showed great enthusiasm at being able to solve these 'hard' problems.

Wini: Can we do another one? Mister, can we do another one?  
 Mary: I hate maths so mmmm! I just keep coughing this morning.  
 (Wini completed another problem).  
 Wini: This is fun!  
 Mary: You reckon? (Incredulous).  
 Wini: Yeah, this is fun. (To me) We want another one!  
 Mary: NO!  
 T: It's alright; I know you are not feeling well.  
 Mary: Oooh! (Slumped over her desk).

This was the third time that the students had been really excited about doing mathematical activity; working with the pipe numbers (and thus learning decimal place-value), pouring the water between containers (and thus investigating decimal division) and this activity that involved drawing diagrams (and thus learning how to do double digit multiplication).

Enthusiasm could be ascribed in the first two activities to being 'fun'. There was a high novelty factor (new equipment in the first pipe tasks, with lots of choice of familiar object and an unexpected chance to pour water around in the second) but this could not explain the reaction in the third. One area of commonality between these three situations was the high degree of initial difficulty from the students' perspective. Gaining partial and then complete success was highly engaging and seen as enjoyable. As Williams (2003) asserts, anticipated success is a powerful motivator.

I decided that the group (except Mary) was ready to combine the earlier work of relating decimals to close whole numbers (benchmarking on the number line) with this new multiplicative skill.

T: Ok, if  $18 \times 15$  is 270, what about  $18 \times 14.93$ ?  
 Chn: 270, bit less than 270  
 (It was checked on calculators, 268.74).  
 T: This time we'll try and estimate first.  $5.9 \times ^$   
 Chn: Around 6

T:  $4.93^{\wedge}$   
 Chn: Around 5  
 T: So around...  
 Chn: 30  
 (Checked on the calculator as 29.087)  
 T:  $3.2 \times 7.8$   
 Kiri: Round about 8 and 3 so 24

They were then set  $32 \times 78$  to work out using an area model. When five of them had finished, I read out the answer from one of their books.

T: Ok so 2 496. So what is  $3.2 \times 7.8$ ?  
 Hoani/Wini: 24.96 (Immediate response).  
 T: Can you see what they have done? (To the others).  
 Kiri: They used the other answer^  
 Hoani: And then you just put in the dot!  
 T: Yeah, you use your estimate to make it a decimal.  
 T: Try  $42 \times 23$  #  
 Kiri: 966 (She had sketched a flag style diagram and had the product in around 30 seconds).  
 T: OK, so work out  $4.2 \times 2.3$   
 Chn: 9.66 (Immediately called out by several students).  
 T: If you start with one of these (decimal version) you could^  
 Kiri: Just use the top way (Meaning that she could easily adapt the area model with whole numbers to the new situation).  
 T: Yes, many high school kids can't do that.  
 T: So have a go at this one,  $3.7 \times 6.2$  (They have not been given the scaffold of working out  $37 \times 62$  first).

Five members of the group got the right answer. Mary had set out  $67 \times 32$ , a transpositional error, possibly due to her poor health.

### 3.3.3 Final Tasks

I produced a packet of Tim Tam chocolate biscuits and a bottle of Coke to wind up this final teaching session.

T: OK we have 180grams and 12 biscuits  
 (The students showed pleasure at the food and mock dismay as they knew they would have to earn their treat by doing some form of calculation).  
 T: How much is one biscuit in kilograms?

Hoani and Wini worked together. They first reasoned that they could use the packet weight of 180g and divide by 6 and then by 2 to effect a total division by 12. They then were able to re-interpret the quotient of 15 in order to make their final answer appear reasonable. They did not do this by counting decimal places but by applying number sense.

Their actual written work is reproduced:  $180 \div 6$  is 30 so  $180 \div 12$  is 15 so 0.015

Kiri and Bridget worked together using a different approach. They converted the 180g into kilograms and then applied a succession of simple divisions until they knew that they had performed an equivalent operation to dividing by 12.

Their actual written work looked like this: 0.180kg so 6 is 0.090, 3 is 0.045, 0.015kg

Grace worked by herself. Her working was more difficult to interpret. She seemed to have used the idea of solving  $\square \times 12 = 180$ , finding 15 as this answer, and then expressing this as 0.015kg.

Her actual recording was  $5 \times 12 = 180$  (15  $\times$  12 meant?)  $180 \div 12 = 0.015\text{kg}$

Mary's work was also unclear. She seems to have used  $180 \div 12$ , and then expressed the answer as 0.15

It needs to be remembered that none of these students had been taught a long-division algorithm and so dividing by 12 was problematic. They needed to create methods of solution as they could not simply draw upon knowledge of any of the common standard systems that adults use.

- T: OK, we have this division to do now, we have 1.5 (a bottle of Coke) and  
C: One, two, seven people  
(I was happy to be excluded to make the division easier as  $1.5 \div 6$  would produce a tidier quotient but a cup had been produced for me and the students clearly wanted me to be included).
- T: OK, so  $1.5 \div 7$ , what will that be, approximately?  
Hoani: 250mL, 2.5  
T: Is it 2.5?  
Hoani: Yeah, 2.5, its 2.5  
T: But this (bottle) is only 1.5.  
Hoani: Oh, 0.25  
Grace: I reckon about 0.2

The others agreed it was somewhere around these two answers and as exact calculation was going to be tedious, I suggested that we get on with the pouring. We then drank the Coke and ate the Tim Tams.

### 3.4 Final Results

Table 19

Comparison of Students' Initial and Final Responses to Whole-number  $\times$  Decimal Task

Task One: You pour four 1.25L bottles of orange juice into a bowl. How much is this?

		Mary	Grace	Wini	Bridget	Kiri	Hoani
$4 \times 1.25L$	Initial	Left blank	$4L + 4 \times 25 = 100$ Not sure	$1 \times 4 = 4$ $0.25 \times 4 = 100$ 5L	4.1L	5L $4 + 1$	$1.25 \times 4 = 5$
	Final	5.8	$4 \times 1.25 = 5$	$1 \times 4 = 4$ $0.25 \times 4 = 1$ 5L	5L	$1.25 \times 4 = 5L$	Same answer

Mary was unable to correctly carry of the task without equipment or other scaffolding. The others could coordinate their multiplications by either combining partial products or by direct calculation.

Table 20

Comparison of Student Initial and Final Responses to Decimal  $\times$  Whole-number Task

Task Two: Peanuts cost \$7 per kg. How much would you pay for 0.9kg?

		Mary	Grace	Wini	Bridget	Kiri	Hoani
$0.9 \times \$7$	Initial	\$6	\$6.50	\$6.30	Don't know	\$6.50	\$6.50
	Final	\$6	$9 \times 7 = 63$ so \$6.30	$9 \times 7 = 63$ so \$6.30.	Left blank	\$6.50 I think	\$6.30

Hoani and Grace have joined Wini in recognizing the connection between this problem and the times-table fact  $9 \times 7$ .

Table 21

Comparison of Students' Initial and Final Responses to Decimal  $\times$  Decimal Task

	Timing	Mary	Grace	Wini	Bridget	Kiri	Hoani
$4.2 \times 2.6$	Initial	6.8	9.2 $4 \times 2 = 8$ $2 \times 6 = 12$	9.2	6.8	9	Left blank
Answer is 10.92	Final	6.8	8.8	8.4 Area model	9.2	10.92 Area model	10.92 Area model

Kiri and Hoani were able to employ their recently-learned knowledge of the area model. Wini was also able to use this model; she had the four correct partial products written in decimal form but then only used two of them for her total. Consideration of her *working* shows an advance in her thinking that her answer alone does not reveal.

Wini's working

	<b>2</b>	<b>0.6</b>
<b>4</b>	<b>8</b>	<b>2.4</b>
<b>0.2</b>	<b>0.4</b>	<b>0.12</b>

Wini had processed the multiplicative steps using the correct decimal place-value for each partial product. The group had seen examples of *whole number*  $\times$  *tenth*, (e.g. Task One as above) and *tenth*  $\times$  *whole number* (e.g. Task Two as above), but had not been explicitly taught how the multiplication of *tenth*  $\times$  *tenth* produces hundredths. Wini has independently realised the place-value of the fourth partial product.

With no scaffolding provided, the other three did not use the area model or any other system successfully.

Having examined the data in each sub-section, the next chapter discusses the general themes that emerged and what implications arise from the study as a whole.

## Chapter Five: General Discussion

### Introduction

Here the general themes from the Results and Discussion chapter are examined. This involves further analysis of evidence with regard to the literature, the recording of implications and the presentation of suggestions for further inquiry and debate. This discussion has been arranged as follows:

Student Learning

Content

Process

Individuality

Implications and Suggestions for Teaching

The Research Process

### Student Learning

#### Content

Review of the Results and Discussion chapter shows that a broad range of mathematical content was covered in the intervention phases.

Summary of mathematical content covered:

- Iteration of unit fractions
- Connection between decimal symbols and their fractional equivalents
- Understanding of decimal symbols to enable the correct ordering of numbers
- Application of decimal place-value to simple additive problems
- Understanding of the density principle of number
- Extension of multiplicative schema to include situations involving:
  - a) *Whole number*  $\times$  *decimal*
  - b) *Decimal*  $\times$  *whole number*
  - c) *Division by a decimal*
  - d) *Decimal*  $\times$  *decimal (to 2 d.p.)*

The iteration of unit fractions was not explored in detail but only as it related to tenths. In that context only, evidence showed that all of the participants in Phase 1 (except Ripeka) learned how to combine tenths during the learning sessions (2.4, Table 11). The connection between decimal form and magnitude (as understood by reference to their specific fractional equivalents) was considered to be pre-requisite to the ordering decimal tasks and so was not reported separately. All of the students made substantial progress in terms of their ability to order decimal numbers (1.3.1, Table 6). Initially, none of the students had successful procedures for the ordering task but by the end of Phase 1 most of the group ordered decimals correctly. The exception was Mary, but even she was correct provided that the right answer could be determined solely by the examination of the first two decimal places. Comparative NEMP results showed that only 50% of Year 8 students could identify tenths and hundredths (Flockton et al, 2006). Interviews with students in the present study showed that conceptual generalization lagged behind procedural success for two of them, Mary and Ripeka. The other four provided evidence that they had adjusted their mental scheme of decimal numbers by combining their knowledge of fractions with an extension to their knowledge of place-value, with Grace, Aroha, and Wini using correct mathematical language as the basis of the justification of their answers (1.3.2, 1.3.3). I do not believe that it was coincidental that it was these three students who were also able to add simple decimals together (2.4) but that this ability was a product of the deep conceptual change they had undergone.

In Phase 2, the students who had not participated in Phase 1 had procedures that enabled them to correctly order decimals (1.5). However, they were not able to describe the conceptual reasons behind their procedures. Exposure to equipment allowed them to model the numbers and acquire a vocabulary which was then used in subsequent discussions. The students were then able to transfer their place-value understanding to situations involving the addition and subtraction of decimals (2.6).

Four students could envisage numbers in the ten-thousandths column and beyond (1.7, Table 9). This was taken as evidence that the process of extending the number system to produce decimals as required had been internalized. Extension by students of the place-value construct in this manner was interpreted as an example of reification - the term used by Sfard (1991). They could operate with the concept of decimalization as though it

were an object in its own right. Their degree of intellectual autonomy (Yackel & Cobb, 1996) meant that they could apply mathematics as required to answer the task independently. These results are encouraging given the degree of student difficulty with the density property that has been reported in earlier studies (e.g. Greer, 1987; Vamvakoussi & Vosniadou, 2004). The students also demonstrated their awareness of the relative magnitude of numbers through their ability to round decimals when required to estimate answers (3.4.1, 3.4.2). Studies have previously shown that general awareness of decimals does not always translate into an ability to use rounding in this way (e.g. Bana & Dolma, 2004; Yang, 2005).

The extension of multiplicative schema was a ‘work in progress’. As Lesh et al (2003) have noted, major conceptual change is an evolving process that takes considerable time to establish. Problems of the type *whole number*  $\times$  *decimal* seemed easier than the other three types (*decimal*  $\times$  *whole number*, *division by a decimal* and *decimal*  $\times$  *decimal*). This observation conforms to other studies (e.g. Graeber & Tirosh, 1990; Schliemann, 2002) as this first type of problem does not conflict with the ‘multiplication makes bigger’ (MMB) or ‘division makes smaller’ (DMS) constructs. It is also less demanding operationally than the other types. All but Mary could solve this first kind of task (3.8). The fact that most of the group could solve these problems compares favourably with other findings (e.g. Bana & Dolma, 2004; Irwin & Britt, 2004). Evidence is provided of some student success in each of the other types, *decimal*  $\times$  *whole number* (3.4, Table 20), *division by a decimal* (3.3.3), and *decimal*  $\times$  *decimal* (3.4, Table 21). It should be remembered that these students had only worked with these mathematical situations for just over an hour on each topic. The fact that they could demonstrate any competence at all provides a marked contrast with the findings of Burns (1990), where most 17-year-olds struggled with these types of problems.

I anticipated that the production of data that was contrary to the MMB and DMS concepts would create obvious signs of cognitive conflict as these epistemological obstacles are well-documented in the literature (e.g. Bana & Dolma, 2004; Harel & Sowder, 2005; Prediger, 2008; Tirosh, 2000). The context of buying small quantities of fruit priced per kilogram was designed to link life experience and estimation skills in order to confront the MMB concept. It was not initially obvious to the students (except for Wini), that the



non-zero digits appearing in products produced by multiplication with decimals were the same as those produced by whole-number operations (e.g.  $9 \times 7 = 63$  was not applied to solve  $0.9 \times 7$  automatically). This is in keeping with the findings of Albert and McAdam (2007). All of the students used sensible estimation, and while five of the group began to link their estimation skills with their whole-number multiplication knowledge, Bridget continued to struggle (3.3.2). It was unclear whether MMB, the decoding of the task, or some combination of the two was the obstacle. Hoani, Wini, and Grace were able to complete the parallel task at the end of the intervention independently and without scaffolding (3.8, Table 20).

The water-pouring and string-cutting tasks (3.2.2, 3.2.4) were expected to result in vigorous debate about how quotients could be larger than dividends, as these results violate the DMS rule. The anticipated conflict did not eventuate and the results were accepted by the students as being valid. Kiri verbalized the view that the group could 'see' that DMS was not true with decimals as they had just enacted a number of these divisions. I initially considered that they may have been ignoring the conflicting evidence, or had categorized the data as being true for the first context (water) but not being generally applicable. However the introduction of a second context of string length (3.3.1) also failed to produce debate. An important piece of evidence regarding how DMS might be addressed was produced in this study in that the results of quotitive division by a decimal were accepted as making sense. When a non-contextualized problem was presented in the form of an equation, ( $3 \div 5 = ?$  3.2.4), the students reverted to DMS thinking. This recursion to intuitive schema is predicted by research (e.g. Fischbein, 1985; McNeil & Alibali, 2005). It may be that more exposure to situations of this type and a more gradual removal of the contextual scaffolding may have resulted in a more robust new construct being formed.

Double-digit multiplication was itself new learning for all of the students (3.7.2) even before decimal double-digit numbers were introduced. The students were able to use their place-value knowledge to round decimal numbers in order to estimate the magnitude of products (e.g.  $5.9 \times 4.93$  was estimated to be around 30). The brief exposure (one learning session) to double-digit multiplication using the area model was sufficiently powerful enough for two students (Hoani and Kiri) to be able to apply this to situations

involving one-place decimals independently (3.8, Table 18) and for another (Wini) to demonstrate the key understanding of the process.

Evidence of new learning was also gathered that was not directly related to the context of decimals. For example, while all of the participants were identified as Stage 6 on the NDP Framework at the start of Phase 2, there were instances of them operating with a range of mental strategies to complete multiplication and division problems (3.2.1, 3.3.1, 3.4.2) – activities designated as Stage 7. It was unclear whether the students' reported NDP stage was incorrect or whether the demands of the tasks required students to self-invent new procedures. My suggestion is that the latter is the better explanation as throughout the research there were other instances of students making sense of new material in response to situational challenges. The increased cognitive demands created a need to invent successful procedures.

### **Student Learning: Process**

The Piagetian view that learning can occur through both accommodation of new material into existing constructions and through the reconstruction of schema in response to situations of cognitive conflict (Zazkis et al, 2008) shaped the design and implementation of the teaching interventions. The planned pedagogical strategy was two-fold; firstly, to encourage students to retrieve and apply prior knowledge and attempt to create their own meanings; and secondly, to not transmit knowledge unless deemed necessary (for example, when students revealed that they did not possess certain required facts such as writing  $7/10$  as  $0.7$ ).

More important than the creation of cognitive conflict were steps towards its resolution. This was deemed important not only from the mathematical perspective but was also an ethical consideration. I considered that if I had deliberately engineered a stressful situation, then some accountability rested with me to resolve that stress. The literature describes the durability of intuitive schema (e.g. Tirosh, 2000; McNeil & Alibali, 2005), and describe the important role of the teacher to help the learner to reconstruct thinking when contradictory stimuli are encountered, instead of ignoring or dismissing such evidence.

The work of Zazkis and her colleagues (Zazkis & Chernoff, 2008; Zazkis et al, 2008) was especially useful when considering how students interacted with tasks. They provided the terms ‘pivotal’ and ‘bridging, that distinguish between tasks in terms of their efficacy from the learner’s perspective. This re-framing of language making the learner’s perspective paramount has a parallel in the description applied to materials as having ‘transparency’ (Meira, 1998; Stacey et al, 2001). Transparency refers to the ease at which the mathematical content that sits behind the artifact is interpreted by the learner. I consider that the activities of measuring with the pipe numbers, dividing using water containers and cutting string, and the area model of multiplication were all transparent processes and many of them were pivotal in terms of student learning.

I found the terms ‘participatory’ and ‘anticipatory’ (Tzur & Simon, 2003; Tzur, 2007) useful in thinking about the development of learning in the participants. These terms allow for observed success in scaffolded situations that did not automatically translate into independent application of knowledge. Participatory learning can still be valued, even though it is incomplete. The learning sessions allowed for reconstructive generalization to occur. Evidence for this can be seen in the fact that at least some students were able to demonstrate this for each of the conceptual issues raised. New concepts had been abstracted from interactions with tasks. Abstraction seemed to occur concurrently rather than subsequently to interaction with tasks, though initial abstractions were not necessarily robust. This seems to fit models that allow for iterative views of learning such as those of Pirie and colleagues (e.g. Martin et al, 2006; Pirie & Kieren 1992, Pirie & Kieren, 1994; Pirie & Martin, 2000) and Tzur (2007), and those that acknowledge the co-existence of alternative schema in students’ minds (e.g. Harel & Sowder, 2005; Lesh et al, 2003; McNeil & Alibali, 2005; Siegler, 2000; 2007). As Mason et al (2007) assert, learning is not a simple, linear process.

The study was also able to capture moments of non-learning, where students could not make sense of the evidence before them, or were unable to recognise the incongruity of their statements. Examples include the conversation with Bridget regarding the purchase of tamarillos (3.3.2) and where a small mistake by me precipitated confusion among the group (3.4.1). This evidence is quite different from the collection of pre-intervention diagnostic data. Diagnostic data is important to gather as it helps show the initial

constructs of students and how this can be used to shape interventions. Evidence of cognitive struggling *while* the intervention was occurring, reflected much behaviour that teachers observe. This was only made possible by the use of the microgenetic approach. I faced challenges that are common to practitioners such as students who ‘just don’t get it’. The transcripts showed that I was able to draw upon pedagogical content knowledge to address these situations successfully on some occasions but at other times was not able to find an appropriate response.

### **Individuality**

As expected in a constructivist view of learning, the learning journeys of the students had both similarities and marked differences. As Mason et al (2007) have suggested, learning does not occur uniformly across a group. This was despite the students having had relatively similar prior decimal knowledge and identical exposure to new material during the intervention periods. Their personal experiences and reflections were unique, and so one of the common themes was that of variability. Each of the students had times when they were among the first to grasp a new idea but at other times of being among the last. For example, Grace was the very last to engage with a place-value construct of decimals (1.2.2) and was the last to accept meaning for the third decimal place (1.6.2). She was also among the first to no longer require physical materials to add and subtract decimal numbers (2.4), was able to apply the density of number property (1.7, Table 9) and showed some proficiency with multiplicative situations (3.8). In contrast, Mary was among the first to recognise how the size of decimal numbers related to their symbolic form but was least able to transfer her knowledge to additive and multiplicative situations. Hoani (3.2.2) was the only student to register surprise at the result of the first situation involving quotitive division but then applied the understanding almost immediately to solve a new problem. This variability demonstrates the need for teachers to be continually monitoring the comments made by students in order to assess their immediate learning needs. Proficiency in one area cannot be assumed to be transferred by a student to another context, and a student who takes longer to think through a concept may well be the one with the deepest understanding.

## **Implications and Suggestions**

Trying to separate out items of influence when examining episodes of learning proved to be very difficult, consistent with the views of researchers who regard the learning environment is not only highly complex, but essentially irreducible (e.g. Barab and Squire, 2004). Despite this caution, it is still possible to suggest approaches that could be transferable into other situations without implying that attendant factors are uninvolved.

### **Implications and Suggestions: Measurement**

One potentially generalizable approach was that of having students involved in performing measurement tasks in order to learn about decimal numbers. These include linear measurement tasks in order to understand place-value, and the distribution of water and string to enact division. Student's involvement appeared to be successful in promoting the development of new thinking; in particular regarding the role of place-holder zeroes in the former and the extension of multiplicative schema via quotitive division in the latter. Other applications that involved a continuous view of number included the use of number lines, area models, weight, and length. The use of measurement situations was chosen to scaffold students into re-consideration of their concept of number as these are contexts in which decimal numbers are naturally found. In this study, students' prior exposure to place-value symbols and terminology did not translate to demonstrated facility with ordering decimals (1.1.1). This is consistent with many other studies (e.g. Moss & Case, 1999, Steinle & Stacey, 1998, 2002; Young-Loveridge, 2007). Instead, it was the experiences involving the manipulation of physical representations of decimal numbers in order to complete practical tasks that resulted in the adoption of the place-value terminology and the correct use of decimal notation (1.3). Students did not understand decimal symbols until they had worked with analogous quantities. This is consistent with other findings (e.g. Helme & Stacey, 2000; Hunter & Anthony, 2003). This is also consistent with Sfard's (1991) assertion that what mathematics *does* (its operational conception) is for many people the better starting point than what it *is* (its structural elements). Streefland (1991) noted that the failure of students to learn rational numbers was in large part due to the lack of connections made with other knowledge. I believe that it is worth considering the promotion of numeracy

via measurement for rational numbers in order to address Streefland's observation and in agreement with Sophian (2008).

The use of measurement as a mechanism for engaging students appeared to be very effective with regard to both affect and mathematical content. In my view these tasks were instrumental in both challenging existing student thought and scaffolding them through change, both pivotal and bridging (Zazkis & Chernoff, 2008). I believe that this was partly due to connections being formed between the symbols, the physical representations, and the activities, simultaneously. This type of engagement both required, and facilitated links to be formed between the three factors. These links co-evolved, it was not that one form was the necessary precursor to the others, though typically the materials promoted the first discussions.

In my view, the other main promoter of learning was that the tasks were actually experienced by the students. I believe that having students *discuss* realistic contexts is more engaging than presenting de-contextualized mathematics, consistent with the findings of Irwin (2001). I also believe that having students *enact* realistic contexts is even more powerful, as other recent research findings have shown (e.g. Bonotto, 2005; Bulgar, 2003; Brousseau et al, 2004; Hunter & Anthony 2003).

I will outline a tentative rationale for this second belief. If we consider the psychological state of the learner, situations of cognitive conflict lead to a student having doubts about what is true and how truth can be verified. This creates stress. (This stress need not be considered in purely negative terms as students may relish the challenge of new learning). Carrying out practical tasks may reduce stress in two ways. Firstly, there is enjoyment in being active, in having stimuli through many of the senses, and in the social aspects of the task. Secondly, there is the time to reflect (perhaps at a sub-conscious level) that is not available when new information is simply presented by an authority, or where the teacher probes for understanding that the student does not yet possess. The re-equilibration process described by Zazkis and Campbell (1996) needs both time and mental space within which to occur.

Alongside these elements is the body of self-produced data. When a person is experiencing internal discord, it may be hard to know one's own mind, as this oscillates between old and new constructs. Awareness of this internal conflict raises another issue,

how does one resolve this tension when the main resource for resolving conflict (one's mind) is itself part of the problem? If an external authority (e.g. the teacher) pronounces which thinking is correct, the students may not try to re-organize their own thoughts but simply suppress the conflict. This response may also reinforce a view of mathematics that assumes that verification of truth can only occur through an external agent. By gathering one's own evidence, the agency of deciding what is true rests more with the learner. The externalized actions of hands and eyes are to some degree independent of the competing constructs in the mind. In this study, students completed tasks (e.g. measuring items around the room as in 1.2.3 or pouring water as in 3.2.2), and in doing so created independent sources of data to address the internal cognitive conflict. Students may accept this evidence as crucial in determining which of their competing constructs has the most validity in the situation. This may be an example of Tzur's (2007) 'reflection on activity-effect relationship'. Thoughts are tested by actions whose results create new thinking. The authority of verification depends to a greater extent upon the learner's own reflections. The teacher is still part of the verification process as the student is likely to want to test their 'truth' against an accepted expert. The framing of the question may be quite different however, from "Is this correct?" to "This is correct, isn't it?"

### **Implications and Suggestions: Numeracy Development Project (NDP)**

If the view that contexts involving measurement can inform number knowledge is accepted, then there are implications in terms of the NDP. These could involve making explicit the use of the continuous representations of number that are currently implicit in many NDP tools (e.g. number lines), by the demonstration of activities whereby the learning of numeracy may be enhanced by work in other mathematical strands, and by the encouragement of teachers to integrate work across strands to achieve the overall goal of numeracy. Investigation into other sources of numeracy could alter the current uni-directional model that has Number being the foundation of numeracy in these other strands.

The literature (e.g. Graeber & Tirosh, 1990) regards the quotitive model as essential for understanding division by rational numbers, and evidence shows that many NZ primary teachers lack understanding of how this model applies in rational number contexts (Ward

& Thomas, 2007). The quotitive model of division is mentioned with regard to whole numbers in the recently revised NDP publication *Book 6 Teaching Multiplication and Division* (Ministry of Education, 2007e, pp. 38 & 54-55). Its application to situations involving rational numbers as divisors is one of the additions made to the revised edition of *Book 7 Teaching Fractions, Decimals, and Percentages* (Ministry of Education, 2008, pp. 68-69). Rational number applications in the NDP seem to be limited to working with linear and area models (e.g. paper strips and the decimat diagrams). Given the reported difficulties that *teachers* have with quotitive division with rational numbers, it may also be useful to promote activities that have people enacting experientially concrete situations before more abstract tasks (such as decimats) are introduced. Examples could include the water-pouring and string-cutting tasks carried out in this study.

The issue of practicality might be raised following these suggestions. While accepting that the learning sessions in this study had a low student/teacher ratio, this situation is not too different from recommended NDP practice where classes are typically arranged into three groups of 8 – 10 students each. The total teaching time was also brief (Phase 1 had around 200 minutes of contact time while Phase 2 had approximately 340 minutes of contact time). Teachers would be able to extend the intervention time allocated to these topics which I was unable to do. An important aspect of the intervention was that students were given sufficient time to carry out the measurement tasks. Strict adherence to rotational time-frames (such as 15-20 minutes per activity before change is required) may be inappropriate for these kinds of learning experiences. Time has to be given to build appropriate data sets and discuss the issues that arise from them without interruptions.

### **Implications and Suggestions: Challenge**

Another potential area of transferability is that of the degree of challenge presented to students. Working in the ZPD entails positioning tasks that are beyond students' current ability but are accessible via appropriate scaffolding (Baroody et al, 2004; Shayer, 2003). In this research, it was planned that students would work at two stages above their reported NDP attainment. It should be remembered that the participants were at - but not exceeding - national expectations in mathematics, they were average students by these measures (Ministry of Education, n.d.). In Phase 1, students at Stage 5 were asked to



engage with tasks whose outcomes are described as Stage 7, i.e. the ordering of decimal numbers and simple additions of these numbers. In Phase 2, students at Stage 6 were asked to work with situations involving the multiplication and division of decimals, activities regarded as Stage 8 (Ministry of Education, 2007b, pp. 17 & 21). These students became most engaged when they started to achieve success in mathematics that they regarded as difficult (i.e. their first engagement with decimal place-value, division by decimals and completing double-digit multiplication), all of which were beyond what teachers would expect to teach to those students at their current NDP stages. The findings of this study raise questions about whether the close aligning of students in the NDP to 'stages' is always pedagogically advantageous. This does not mean that the stages themselves are necessarily faulty or that they cannot provide useful diagnostic information, but teachers need to recognise that these are indicators of past performance and they should not feel constrained by the Number Framework to teach only to an immediate horizon. One is reminded of the earlier quote from Harel and Sowder (2005) to the effect that limiting students' exposure to new mathematical concepts also limits their potential to learn. Studies such as Irwin and Britt (2004) showed that some students started to make independent connections to new material involving decimals. Those students did not all have to proceed through the same pathway to knowledge. The students in this study were also able to transcend their stage levels in an environment that both encouraged and scaffolded them to do so.

The revision in NZC of when students are expected to engage with decimal place-value (after equivalent fractions) was presumably based upon the epistemological consideration that the general re-naming of fractions as decimals requires a sound knowledge of fractional equivalence. However, this study has shown that the generalization of equivalence of fractions was not pre-requisite knowledge for these students to understand how to order decimal numbers. This finding is consistent with the Piagetian reasoning levels presented by Adhami (2002) where the place-value of decimals is listed at a lower level than the generalization of equivalent fractions. It is not necessary to think in terms other than tenths, hundredths and thousandths in order to know the relative magnitude of simple decimal numbers. Review of the student NDP data presented by Young-Loveridge (2007, pp. 158-159) shows that if knowledge of equivalent fractions is regarded as pre-

requisite to decimal place-value understanding, then relatively few students would currently have this learning opportunity. Five-year averaging of Years 6, 7, and 8 end-of-year data for knowledge of equivalence of fractions produces figures of 13%, 20% and 31% respectively. If iteration of unit fractions is taken as the pre-requisite skill, those figures increase to 34%, 43% and 55%.<sup>11</sup> These ‘average’ students at early Year 6 in this study could engage with the ordering of decimals with a relatively small amount of instructional time. Encouraging early meaningful interactions with decimals was advocated by Irwin and Britt (2004). Instead, such engagement is likely to be further delayed with this latest curriculum revision, which is of major concern to me.

### **Implications and Suggestions: Novel Task Responses**

Synthesis of ideas from the literature and reflection upon the results led to the creation of a framework to consider student and teacher responses to the introduction of a novel task and the potential outcomes of those responses.

The diagram overleaf (Figure 14) presents this framework, and is entitled *A Framework of Responses and Outcomes following Novel Task Introduction*.

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<sup>11</sup> Equivalence of fractions data relates to the NumPA knowledge task (Ministry of Education, 2007c, p. 35), where recognition that  $\frac{2}{3}$  and  $\frac{6}{9}$  are equivalent is deemed to show evidence that the student is at Stage 7. The ‘iteration of unit fraction’ task relates to the same reference, and involves the student being able to link the fraction  $\frac{8}{6}$  to either  $1\frac{2}{6}$  or  $1\frac{1}{3}$ . (One might argue that the former task does not test for generalization of equivalence to any greater degree than the latter task).

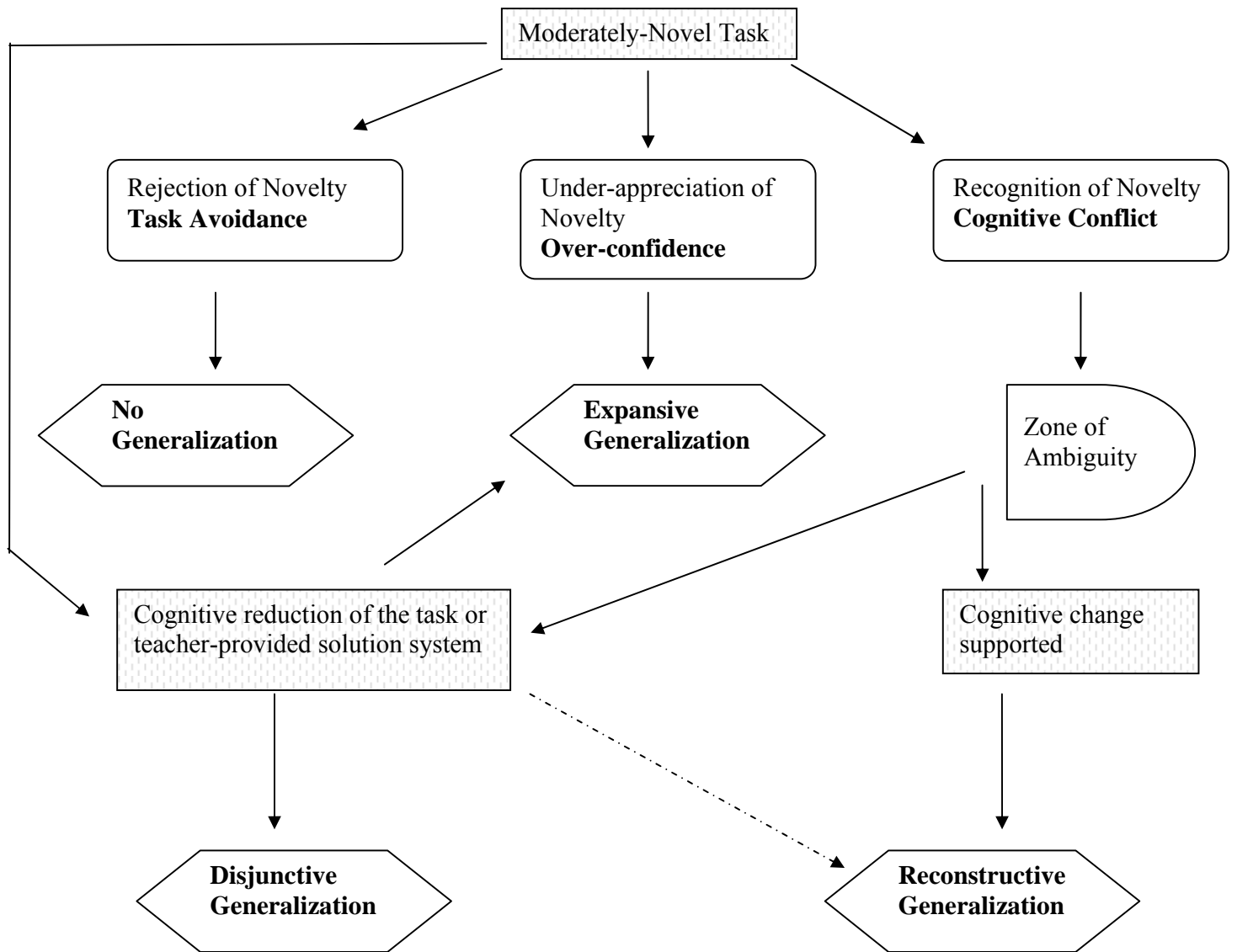
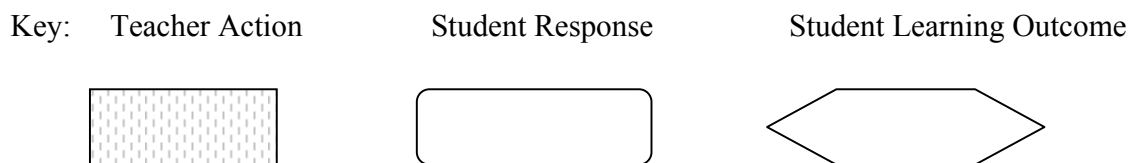


Figure 14 A Framework of Responses and Outcomes following Novel Task Introduction.



### Explanation of the Framework

A first, essential step in the framework is that a moderately-novel task must be introduced. If the teacher does not expose students to cognitively challenging problems the potential for reconstructive generalization is absent. The introduction of problems having excessive novelty does not permit student engagement.

The left-most arrow indicates a situation whereby the teacher may introduce a challenging problem, but immediately presents a solution system and insists upon its procedural adoption, irrespective of current student understanding or their reaction to the introduced novelty. (The teacher presumably hopes that practice of the procedure may lead to subsequent student understanding). Alternatively, while the initial task may have been challenging, the teacher may diminish this challenge by breaking the task into a set of highly-scaffolded small steps that do not require understanding, but simply imitation. The cognitive load is reduced to a minimum. Likely outcomes of these teacher actions are that students have immediate success in terms of task completion, but as no internalization of the concept has occurred, there is little likelihood that future applications and/or extensions of the concept will be understood. The likely student learning outcomes are either that no generalization will occur, or students will perceive the outcome to be a new set of procedures to remember, which is disjunctive generalization.

Student responses are described according to their reaction to the novelty of the new situation. Novelty may be noticed and feared. Instead of the task providing a stimulus to new learning, the reaction may be distress. Students who respond in this way may show task avoidance behaviours. Their reluctance to engage with the task may be expressed in actions such as ‘forgetting’ their books and equipment and in long delays in getting started with work on the task. The students may not share the teacher’s agenda of learning the new concept as they do not expect to complete the task successfully and/or independently. This belief is often self-fulfilling. Contributing factors to this response include the absence of a safe environment within which risks may be taken and low student self-efficacy. If students have experienced humiliation (whakamaa) by either the

teacher or their peers when incorrect answers have been given previously, there may be considerable reluctance to engage in future risk-taking with regard to other learning situations. The student learning outcome is likely to be that no generalization takes place.

If the novelty is unnoticed, students may believe that they already have sufficient knowledge with which to solve the task. They have over-confidence in their current understanding. These students will engage with the task independently by including the new task into a previous schema, thereby ignoring its novelty. An example of this is where students order decimal numbers using whole-number thinking, i.e. on the basis of the number of non-zero digits and without reference to place-value. The likely student learning outcome following unrecognized novelty is expansive generalization.

If the students perceive the novelty, but regard this stress as a stimulus to learning rather than as a threat, they may enter into a state of cognitive conflict. The students are unsure which pieces of prior knowledge are applicable and which pieces of new evidence are important, hence the label 'zone of ambiguity'. To use a common phrase, they are in 'two minds', perhaps more accurately we can describe them as simultaneously managing competing constructs.

What happens next depends to a large extent on how supportive the learning environment is regarding cognitive change. The students may be helped via the actions of the teacher and their peers to reflect upon and adapt their current thinking. Conditions that assist this include the transparency of the artifacts used in the task, classroom attitudes to risk-taking, the kinds of justification and explanation that are valued, and the time allowed for engagement with concepts via discussion. Within this environment, abstraction is possible (though there are no guarantees of this). The student learning outcome is much more likely to be reconstructive generalization in these circumstances than in any of the other pathways.

From the zone of ambiguity another path is possible. The teacher may recognize the students' task engagement and also their uncertainty. In a desire to help address this uncertainty, the teacher may take inappropriate actions. They may prematurely direct the

students to the adult solution system for the task or reduce task complexity for the students rather than allowing them to do this by their own actions. The teacher may also reduce the potential for abstraction to occur by not providing sufficient time for students to reflect upon their current activity/result data. The requirement to complete a set number of exercises can be a contributing factor in this. Failure to engage students in mathematical discussion may leave them unable to articulate and examine their competing constructs and so also unable to resolve the cognitive conflict. Reversion to intuitive schema may result. The likely student learning outcomes are that there will be a recursion to expansive generalization or that the teacher-provided assistance will not be understood and so result in disjunctive generalization.

Finally, the dotted arrow acknowledges the agency of the students to achieve reconstructive generalization independently of the teacher. This may be possible through individual reflection, group collaboration, or by the assistance of an agent not initially involved with the task such as a parent or more knowledgeable peer. This latter possibility is acknowledged by Martin, Towers and Pirie (2006).

## **Research Process**

### **Data**

The design experiment approach yielded rich, varied data. While most learning takes place ‘off-camera’ in the student’s mind, the capture of actions and conversations as the tasks were engaged with afforded insight into the process of accommodation and re-organization of concepts that another type of study could not provide. The combination of audio data and student written recording served to capture progress being made towards new ideas, rather than simply reporting on the products once learning had already taken place. These dual data sources provided additional insight into the progress towards generalization as some of this evidence proved to be contradictory. At times, written work demonstrated a procedural ability that a subsequent interview revealed was not yet embedded as conceptual understanding, for example, Ripeka (1.3.2). It was also found that Aroha (1.3.2) was thinking using one system but operating with another. Wini’s double-digit multiplication was presented as an incorrect answer (3.8), but her working

showed that she had made the important step of being able to quantify each of the partial-products correctly.

The audio data allowed for an added dimension to be presented in terms of the ecology of the learning environment. Aspects of the personalities of the students were recorded and their emotions could be shown as well as their mathematical reasoning. The realities of discussions with interruptions, repetitions, circular arguments, and flashes of insight that resulted in immediate group acceptance are all part of the learning environment as teachers experience it. Educators may believe that this added dimension helps authenticate the findings of the research as the participants are recognizable individuals with behaviours similar to those they encounter in classes.

The recorded instances of recursion and doubt served as examples of what constituted part of the actual learning process. These would not be recognised in pre- and post-intervention data collection methods. They are examples of what Seeger (2001) described as the ‘normal developmental sequence of learning’ and what Siegler (2007) believed was missing from studies that emphasized ‘stages’ of growth without attending to the non-linear pathways of learning he observed occurring in practice.

Gains in place-value understanding (ordering decimals and the density principle) were more obvious than gains in multiplicative applications of decimals. Part of the reason for this may have been the number and types of mathematical connections that the students were required to make in the limited time available. When ordering decimal numbers, students had to coordinate knowledge of fractions, place-value, and the symbolic form of decimals, but there was a single focus for the learning. No calculational proficiency was required nor were any operational strategies involved. With the multiplication and division of decimals, the inherent complexity of those operations and the conceptual reorganizations that were simultaneously required resulted in substantially increased cognitive demand compared with the place-value component of the intervention. I believe that this is a better explanation of the multiplicative gains being less obvious than an alternative suggestion that the pedagogical approach is less suited to operational contexts. Possible pedagogical responses during the intervention to address the issue of increased cognitive demand include extending the length of the intervention (though this was impossible in the context of full-time employment), limiting the study to either

multiplication or division, and limiting the range of problems that students were exposed to during the sessions. Limiting the scope of the study in Phase 2 may have meant that some areas were taken to completion and richer insights into those particular areas may have resulted. This would have come at a cost, however. Convention would have dictated that multiplication would precede division, and so it might not have been possible to gather information regarding the successful use of quotitive division. A standard approach might not have discovered that at least two (and almost three) students could learn two-digit decimal multiplication very quickly. I believe that the decisions that were made regarding the scope of inquiry were ambitious. This desire to push the boundaries of what was possible and to expose students to many complex ideas made it possible for some students to go further than what would have resulted from a more conservative approach.

### **Research Process: Responses**

The methodology also allowed for teaching responses to perceived student needs that other research types do not permit. This may resonate with the practice of classroom teachers, as they too have to respond to actual student needs and cannot simply adhere to an external agenda. An example of this was in Phase 1 when the initial interview revealed that the students did not have the knowledge of how to iterate unit fractions that I expected them to possess. I was able to adjust the learning sessions to accommodate this. In Phase 2, the proficiency and interest the students showed in using approximations helped me to decide which of two alternatives to employ when exploring multiplication, (the tenth-of-a-tenth approach using decimats, or estimating from a whole-number calculation base).

Having two phases allowed for iteration of process. The examination of data from Phase 1 had an influence on the design of Phase 2. This can be seen in the creation of tasks that involved student measurement, as these had been conducive to much of the learning in that earlier phase. Ideally, the same set of students would have been participants in both phases. Having an overlap of half of the group provided some continuity and some additional verification of the first results. The students had had very little teaching on decimals between the two phases of the research, yet even Mary – who showed the



weakest understanding in Phase 1 – was operating at Stage 7 of the NDP in decimal ordering (1.5, Table 7).

The interventions used in the study were positioned in a wide body of literature that described the kinds of teacher actions that were known to help students construct knowledge. There is value in presenting a detailed account of the particular interventions of this study as the domain of decimal numbers is known to be problematic. There are relatively few studies detailing student learning in this mathematical area. The results suggest that there are sufficient grounds to warrant further investigation and ‘scaling-up’ of this study for further research. The microgenetic approach provides the richness of data that makes it possible for this study to contribute to the educational community to both practitioners and researchers. The former group may find both immediate application to their own classrooms when teaching decimals and may also find a stimulus to pedagogical inquiry through reference to the framework presented in Figure 14. The detailed description of student learning provided by the record of actual conversations and attendant actions may be of interest to researchers investigating the development of mathematical thinking by students.

One drawback of the microgenetic approach is that the data analysis is extremely labour-intensive. There is also the danger of a lack of focus, in that because it captures such a wide range of data, many avenues of possible exploration arise. Adherence to clearly-defined objectives overcomes this potential problem.

### **Research Process: Self-involvement**

Being both the researcher and the teacher of the students allowed for reflection upon the actions I made and also what permitted those choices to exist in my mind. The key element was the amount of pedagogical content knowledge (PCK) relating to decimals that I was able to draw upon. My thought processes were shaped by my immersion in the literature surrounding student learning, both in general mathematical terms and specifically relating to decimals. This reflection is consistent with claims that sustainable improvement in the achievement of students depends to a large extent upon improving the understanding of those who teach them (e.g. Ball, 2000; Hill et al, 2004; Hill & Lubienski, 2007). Improving the PCK of teachers allows for the possibility that they will

be able to be more contextually-responsive to the needs of their students. This is the thinking behind the NDP being framed as a professional development programme rather than a set of useful classroom resources. As stated in Curriculum Update 45, “Teachers' understanding of subject matter and of pedagogy are critical factors in mathematics teaching” (Ministry of Education, 2001). This approach is similar to that used in the Cognitively Guided Instruction Project (CGI) where increasing teacher understanding was seen as necessary to produce sustainable improvement in student achievement (Carpenter et al, 1996). This present study may help improve the PCK of teachers and thus facilitate their own interactions with students and encourage the creation of their own innovations. It may also serve to reinforce the view that there are potentially large gains to be made in student achievement resulting from the increased professional development of teachers.

The issue of objectivity is important to consider. This is addressed in the research design and reporting style that allowed for the presentation of ‘warts n all’ insights into actual learning sessions. These included occasions where I was unable to find immediate, clear explanations (e.g. at the end of 1.2.1), or where students faced blockages that my responses at the time could not address (e.g. Bridget in 3.3.2). Far from diminishing the value of the research, I contend that reporting such incidents adds to the overall perception of veracity by the reader. It also serves to underscore the point that student learning is highly complex, and is to a high degree unpredictable. Teachers may be reassured that they do not have to expertly address every learning issue in real time in order for an overall progression of student learning to occur.

Finally, conclusions are presented in the form of the main findings of the research and the key implications for the educational community.

## **Chapter Six: Conclusion**

The research resulted in three main findings, with important implications for teachers and suggested directions for future research.

## **Findings**

1. The learning sessions were designed around pedagogical actions where the learning of decimals relied upon measurement contexts. This provided students with experiences where meaningful enactments of tasks, combined with the cognitive challenge of new, student-generated data, promoted the re-shaping of their thinking. Students were afforded a degree of agency in their operation within these tasks, which allowed them to raise questions and reflect on the results of their actions at their own pace. This approach is consistent with recommendations made by earlier research and also helps address long-standing pedagogical issues regarding the teaching of decimal numbers. The research literature has established that student achievement in many applications of rational number is still of major concern. An innovative approach that can demonstrate mechanisms whereby students address many of the problematic issues surrounding decimals is therefore worthy of careful consideration.

2. Having high expectations of the students coupled with the teaching mechanisms described above enabled significant learning to occur in relatively short time-frames. Generating learning involved the engineering of situations that were engaging both in terms of interest for students and with regard to important mathematical ideas. Scaffolding was initially required in order for students to complete tasks and its removal was important to gauge independent student knowledge. Students appeared to enjoy both the experiences of being challenged as well as their resulting successes. In the learning sessions, the practice of making explicit links between prior knowledge/experience and new learning was considered to be more important in facilitating student learning than close attention to student levels of learning. The careful consideration of pedagogical options was to result in activities that overcame many of the students' cognitive conflicts and compensated for their being a Numeracy Development Project (NDP) stage below where this content would normally be taught. The students invented new strategies and

thought their way through challenging concepts in response to the high demand of the tasks before them in the context of a supportive environment.

3. Close observation of student learning was achieved by the use of a microgenetic study. This methodology provided instances of the complexity of student learning as described by recent theoretical models. Evidence was obtained of temporary and simultaneously competing constructs, recursion, and movement towards abstraction. Data from oral, enacted, and written sources would at times complement and at times contradict each other. Student learning in this research was idiosyncratic and non-linear, and generalization appeared to be an evolving process. The methodology allowed for this complexity to be captured so that it might be examined and reflected upon. It also accommodated changes made to planned teaching sequences in order to be contextually responsive to the students' immediate questions and concerns. Pre- and post-intervention measures of success served to set the in-situ learning data into its educational context.

### **Implications**

Two of the innovations used in the research have not been reported in the literature previously. The purposeful use of the pipe numbers equipment in measurement tasks goes beyond the reports of the use of this, and similar tools. In previous accounts, the purpose was to provide static representations of decimal numbers. Here, the actions of measuring actual objects provided the opportunity for a range of decimal issues to be discussed as they were perceived to be important by the *students* as well as providing student-generated sets of data that could be used to reinforce the consideration of place-value. Prior to this study, the action of engaging students through the pouring of water in order to enact quotitive division involving decimal numbers has not been documented. These activities resulted in high levels of engagement by the students and had both epistemic fidelity and transparency. They appeared to promote learning, and thus can be described as counterexamples to expansive generalizations that are both pivotal and bridging. These innovations could be part of a scaled-up study to determine their robustness, as they may prove to have similar efficacy in other settings.

Of wider interest is the overall approach of using measurement contexts to engage students both in completing tasks and considering the mathematical ideas implicit in those tasks. When the use of manipulative materials has been criticized, it is often because the materials have been seen by teachers and/or students as ends in themselves. Tasks involving measurement situations provide motivation for students in that there is a self-determined agenda being enacted, i.e. the students want to find something out (a measure) and in doing so encounter new learning issues. Measurement also provides meaningful interaction with the continuous model of number in which decimals (and other fractions) are naturally found. This is an alternative approach to the NDP ‘number-first’ recommendation and to the NZC division of mathematics into three ‘strands’ with little overlap between these strands.

There is an implicit challenge provided to the current NDP practice of teaching students according to their ‘stage’. This study provides a record of how the participants in both phases of this research not only showed success at higher levels, but also relished the opportunity to try and make sense of ‘hard’ tasks. These results support my conviction that delaying exposure to problematical ideas in mathematics is not always in the best interests of students. Instead, more attention could be given to the professional development of teachers to give them the confidence and ability to address these mathematical issues with their students. Making teachers more aware of examples of successful interventions and the models of learning that explain these successes may improve teacher pedagogical content knowledge and ultimately raise student achievement. While teachers may be generally aware of the need for having high expectations of their students, they may not be acquainted with many instances of lesson sequences that model this in mathematics. This research serves as an example of how the principle of having high expectations may be enacted in work with students.

While the intent of the research was to investigate the learning process, the study also provides a record of the teaching process of providing students with challenging tasks and appropriate concrete materials and how this approach generated learning. Many teachers have not had the experience of observing a sequence of lessons presented with this pedagogical approach. While the NDP provides opportunity for facilitator modeling,

this is nearly always on a single-lesson basis. Many published research articles document the results of such pedagogical practice, and journals aimed directly at teachers tend to describe single innovative lessons. Having these sequences of lessons to examine and critique provides an informative source for teachers who want to learn more about the learning process, particularly as it applies to this problematic area.

The revised curriculum (NZC, 2007, p. 35) requires teachers to investigate their own practice, and ways changes to their practice have an impact on student learning. The *Framework of Responses and Outcomes following Novel Task Introduction* provides a tool that may be useful for teachers to reflect upon as they consider this process in regard to their own practice.

## **Directions**

The routes to numeracy (other than through number) require further study of the literature and further investigation with students. This is not a case of rejecting the NDP's goal for numeracy, but a suggestion that the mechanism of achieving this (number-first) may be too simplistic. In particular, the interaction between continuous models of number and measurement needs further exploration.

Each facet of the multiplicative work warrants its own study. This research was unable to investigate specific contexts deeply, but provided examples of potential lines of inquiry to address subtle inherent epistemological distinctions such as the difference for students in addressing situations of *whole number*  $\times$  *decimal* versus *decimal*  $\times$  *whole number*.

This study serves as an example of the microgenetic approach. The type of data produced offers both insights into learning that other methodologies cannot provide and a degree of connectedness with teacher experience that may prove to be engaging - and therefore informative - for this community. The evidence of in-situ learning produced in this study may be combined with the results from other microgenetic studies from different contexts and different aspects of mathematics, allowing meta-analysis to take place. This may result in general themes emerging that could shape future learning models and future teacher practices.

Further studies exploring how models of teaching that feature high expectations coupled with a practical problem-solving approach, might impact upon student achievement have

the potential to provide information upon which recommendations regarding best practice can be formed.

*Te otinga*

## **References**

- Adhami, M. (2002). Cognitive acceleration in mathematics education in Years 5/6: Problems and challenges. Retrieved March 28, 2008, from [www.sunderlandschools.org/southmoorcs/Images/SMP/SMP.pdf](http://www.sunderlandschools.org/southmoorcs/Images/SMP/SMP.pdf)
- Albert, L. (2000). Outside-in - inside-out: Seventh-grade students' mathematical thought processes. *Educational Studies in Mathematics*, 41(2), 109-141.
- Albert, L., & McAdam, J. (2007). Making sense of decimal fraction algorithms using base-10 blocks. In G. Martin, M. Strutchens & P. Elliot (Eds.), *The learning of mathematics sixty-ninth yearbook* (pp. 303-315). Reston, VA: NCTM.
- Anthony, G. (1996). Active learning in a constructivist framework. *Educational Studies in Mathematics*, 00, 1-21.
- Ball, D. L. (1992). Halves, pieces and twos: Constructing and using representational contexts in teaching fractions. In T. Carpenter, E. Fennema & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 158-195). Hillsdale, N.J.: Lawrence Erlbaum Associates.
- Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal*, 93(4), 373-397.
- Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education*, 51(3), 241-247.
- Ball, D., Hill, H., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator* (pp. 14-22, 43-46). Retrieved August 1, 2008, from [http://www.aft.org/pubs-reports/american\\_educator/issues/fall2005/BallF05.pdf](http://www.aft.org/pubs-reports/american_educator/issues/fall2005/BallF05.pdf)
- Bana, J., & Dolma, P. (2004). The relationship between the estimation and computation abilities of Year 7 students. In I. Putt, R. Faragher & M. McLean (Eds.), *Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 63-70). Townsville: MERGA.
- Barab, S., & Squire, K. (2004). Design-based research: Putting a stake in the ground. *The Journal of the Learning Sciences*, 13(1), 1-14.
- Barnes, M. (2003). Patterns of participation in small-group collaborative work. In L. Bragg, C. Campbell, G. Herbert & J. Mousley (Eds.), *Proceedings of the 26th Conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 120-127). Geelong, Vic.: MERGA.
- Baroody, A. J., Cibulskis, M., Lai, M.-I., & Li, X. (2004). Comments on the use of learning trajectories in curriculum development and research. *Mathematical Thinking & Learning*, 6(2), 227-260.
- Biddlecomb, B. D. (2002). Numerical knowledge as enabling and constraining fraction knowledge: An example of the reorganization hypothesis. *The Journal of Mathematical Behavior*, 21(2), 167-190.
- Bills, C. (2003). Metaphor in young children's mental calculation. Retrieved September 11, 2007, from [http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG1/TG1\\_bills\\_cerme3.pdf](http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG1/TG1_bills_cerme3.pdf)
- Bliss, J., Askew, M., & Macrae, S. (1996). Effective teaching and learning: Scaffolding revisited. *Oxford Review of Education*, 22(1).
- Boaler, J. (1993). The role of contexts in the mathematics classroom: Do they make mathematics more 'real'? *For the Learning of Mathematics*, 13(2), 12-17.



- Bonotto, C. (2005). How informal out-of-school mathematics can help students make sense of formal in-school mathematics: The case of multiplying by decimal numbers. *Mathematical Thinking and Learning*, 7(4), 313-344.
- Boufi, A., & Skaftourou, F. (2002). Supporting students' reasoning with decimal numbers: A study of classroom's mathematical development. In A. Cockburn & E. Nardi (Eds.), *Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 153-160). Norwich: PME.
- Bronfenbrenner, U. (1979). *The ecology of human development*. Massachusetts: Harvard University Press.
- Brown, T. (2008). Lacan, subjectivity and the task of mathematics education research. *Educational Studies in Mathematics*, 68, 227-245.
- Brousseau, G., Brousseau, N., & Warfield, V. (2004). Rationals and decimals as required in the school curriculum: Part 1: Rationals as measurement. *The Journal of Mathematical Behavior*, 23(1), 1.
- Brousseau, G., Brousseau, N., & Warfield, V. (2007). Rationals and decimals as required in the school curriculum part 2: From rationals to decimals. *Journal of Mathematical Behavior*, 26(4), 281-300.
- Brown, A. (1992). Design experiments: Theoretical and methodological challenges in creating complex interventions in classroom settings. *The Journal of Learning Sciences*, 2(2), 141-178.
- Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, 41, 32-41.
- Bruner, J. (1997). Celebrating divergence: Piaget and Vygotsky. *Human Development*, 40(2), 63.
- Bulgar, S. (2003). Children's sense-making of division of fractions. *The Journal of Mathematical Behavior*, 22(3), 319-334.
- Burns, M. (1990). The math solution: Teaching groups of four. In N. Davidson (Ed.), *Cooperative learning in mathematics: A handbook for teachers*. Menlo Park: Addison-Wesley.
- Burns, R. (2000). *Introduction to research methods* (4th Ed.). Frenchs Forest NSW Australia: Pearson Educational Australia.
- Carpenter, T., Fennema, E., & Franke, M. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal*, 97(1), 3-20.
- Carraher, D., & Schliemann, A. (2002). The transfer dilemma. *The Journal of the Learning Sciences*, 11(1), 1-24.
- Charalambous, C., & Pitta-Pantazi, D. (2007). Drawing on a theoretical model to study students' understandings of fractions. *Educational Studies in Mathematics*, 64(3), 293-316.
- Clarke, D. (1995). The learners' perspective study. Retrieved April 30, 2008, from <http://extranet.edfac.unimelb.edu.au/DSME/lps/assets/lps.pdf>.
- Clarke, D., Roche, A., & Mitchell, A. (2007). Year Six fraction understanding: A part of the whole story. In J. Watson & K. Beswick (Eds.), *30th Annual Conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 207-216). Hobart: MERGA.

- Cobb, P. (1988). The tension between theories of learning and instruction in mathematics education. *Educational Psychologist*, 23(2), 87-103.
- Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher*, 23(7), 13-20.
- Cobb, P. (2006). Supporting a discourse about incommensurable theoretical perspectives in mathematics education. *Philosophy of Mathematics Education* Retrieved Oct 2 2007, from <http://www.people.ex.ac.uk.ezproxy.waikato.ac.nz:2048/PErnest/pome19/>
- Cobb, P., Confrey, J., Di Sessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, 32(1), 9-13.
- Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, 31(3/4), 175-190.
- Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of the mind in mathematics education. *Journal for research in mathematics education*, 23(1), 2-33.
- Cohen, L., Manion, L., & Morrison, K. (2000). *Research methods in education 5th edition*. London: RoutledgeFarmer.
- Confrey, J., & Kazak, S. (2006). A thirty-year reflection on constructivism in mathematics education in PME. In A. Gutiérrez & B. Paolo (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 305-345). Rotterdam/Taipei: Sense Publishers.
- Cramer, K., & Henry, A. (2002). Using manipulative models to build number sense for addition and fractions. In B. H. Litwiller & G. W. Bright (Eds.), *Making sense of fractions, ratios and proportions 2002 yearbook* (pp. 41-48). Reston, VA: National Council of Teachers of Mathematics.
- Cramer, K., Post, T. P., & del Mas, R. C. (2002). Initial fraction learning by fourth- and fifth- grade students: A comparison of the effects of using commercial curricula with the effects of the rational number project curriculum. *Journal for Research in Mathematics Education*, 33(2), 111-145.
- Dahl, B. (2003). A synthesis of different psychological learning theories? Piaget and Vygotsky. *Philosophy of Mathematics Education Newsletter* (17).
- Diezmann, C. (2000). The difficulties students experience in generating diagrams for novel problems. In T. Nakahara & M. Koyama (Eds.), *25th Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 214-248). Hiroshima.
- Diezmann, C. (2005). Primary students' knowledge of the properties of specially-oriented diagrams. In H. Chick & J. Vincent (Eds.), *30th Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 281-288). Melbourne.
- Diezmann, C., & Lowrie, T. (2006). Primary students' knowledge and errors on number lines. Retrieved Feb 15, 2008, from [www.eprints.qut.edu.au](http://www.eprints.qut.edu.au)
- Druyan, S. (2001). A comparison of four types of cognitive conflict and their effect on cognitive development. *International Journal of Behavioural Development*, 25(3), 226-236.

- Ebby, C. B. (2005). The powers and pitfalls of algorithmic knowledge: A case study. *The Journal of Mathematical Behavior*, 24(1), 73.
- Ellerton, N., & Clements, K. (1994). Fractions: A weeping sore in mathematics education. *SET: Research Information for Teachers*, 2, 93-96.
- Empson, S., Junk, D., Dominguez, H., & Turner, E. (2005). Fractions as the coordination of multiplicatively related quantities: A cross-sectional study of children's thinking. *Educational Studies in Mathematics*, 63(1), 1-28.
- Ernest, P. (1996). The nature of mathematics and teaching. *Philosophy of Mathematics Education Newsletter*, 9, 40-45.
- Falle, J. (2003). Mathematics as conversation: A model for a mathematics retrieval programme conducted with small groups. In L. Bragg, C. Campbell, G. Herbert & J. Mousley (Eds.), *Proceedings of the 26th Conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 324-331). Geelong, Vic.: MERGA.
- Fernandez, E. (2007). Learning from the 'unknown' in mathematics teacher education: One teacher educator's reflections. In G. Martin, M. Strutchens & P. Elliot (Eds.), *The learning of mathematics sixty-ninth yearbook* (pp. 322-343). Reston, VA: NCTM.
- Fischbein, E. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education*, 16(1), 3-17.
- Flockton, L., Crooks, T., Smith, J., & Smith, L. (2006). NEMP Report 37: *Mathematics assessment results 2005*: Ministry of Education, New Zealand.
- Flores, A. (2001). Profound understanding of division of fractions. In A. A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics* (Vol. 2001 Yearbook, pp. 237-246). Reston, VA: National Council of Teachers of Mathematics.
- Forno, D. M. (1929). Notes on the origin and use of decimals. *Mathematics News Letter* Retrieved April 17, 2007, from <http://links.jstor.org/sici?sici=1539-557X%28192904%293%3A8%3C5%3ANOTOAU%3E3.0.CO%3B2-H>
- Fraivillig, J. L., Murphy, L. A., & Fuson, K. C. (1999). Advancing children's mathematical thinking in everyday mathematics classrooms. *Journal for Research in Mathematics Education*, 30(2), 148-170.
- Franke, M., Carpenter, T., Levi, L., & Fennema, E. (2001). Capturing teachers' generative change: A follow-up study of professional development in mathematics. *American Educational Research Journal*, 38(3), 653-689.
- Galligan, L. (2008). Using Valsiner. In M. Goos, R. Brown & K. Makar (Eds.), *Proceedings of the 31st Annual Conference of the Mathematics Research Group of Australasia (MERGA)* (Vol. 1, pp. 211-218). Brisbane: MERGA.
- Gilbert, J. (2003). Catching the knowledge wave? The knowledge society: What does it mean for education? *SET: Research Information for Teachers* (3), 31-31.
- Ginsburg, H. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge, MA: Cambridge University Press.
- Godfrey, D., & Thomas, M. (2003). Student perspectives on equation: Constructing the mathematical object. In L. Bragg, C. Campbell, G. Herbert & J. Mousley (Eds.),

- Proceedings of the 26th Conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 396-403). Geelong, Vic.: MERGA.
- Goldin, G., & Shteingold, N. (2001). Systems of representations and the development of mathematical concepts. In A. A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics* (Vol. 2001 Yearbook, pp. 1-23). Reston, VA: National Council of Teachers of Mathematics.
- Goldin, G. A., & Janvier, C. (1998). Representations and the psychology of mathematics education. *The Journal of Mathematical Behavior*, 17(1), 1-4.
- Goos, M. (2004). Learning mathematics in a classroom community of inquiry. *Journal for Research in Mathematics Education*, 35(4), 258-291.
- Goos, M. (2008). Towards a sociocultural framework for understanding the work of mathematics teacher-educator-researchers. In M. Goos, R. Brown & K. Makar (Eds.), *Proceedings of the 31st Annual Conference of the Mathematics Research Group of Australasia (MERGA)* (Vol. 1, pp. 235-242). Brisbane: MERGA.
- Gorard, S., Roberts, K., & Taylor, C. (2004). What kind of creature is a design experiment? *British Education Research Journal*, 30(4), 577-590.
- Graeber, A., & Tirosh, D. (1990). Insights fourth and fifth graders bring to multiplication and division with decimals. *Educational Studies in Mathematics*, 21(6), 565-588.
- Gravemeijer, K. (2001). Fostering a dialectic relation between theory and practice. In J. Anghileri (Ed.), *Principles and practices in arithmetic teaching* (pp. 147-162). Buckingham: Open University Press.
- Greer, B. (1987). Nonconservation of multiplication and division involving decimals. *Journal for Research in Mathematics Education*, 18(1), 37-45.
- Greer, B. (2004). The growth of mathematics through conceptual restructuring. *Learning and Instruction*, 14(4), 541-548.
- Hackenberg, A. (2007). Units coordination and the construction of improper fractions: A revision of the splitting hypothesis. *Journal of Mathematical Behavior*, 26(1), 27-47.
- Hall, R. (2001). Cultural artifacts, self regulation, and learning: Commentary on Neuman's Can the Baron von Munchausen phenomenon be solved? *Mind, Culture and Activity*, 8(1), 98-108.
- Harel, G., & Sowder, L. (2005). Advanced mathematical-thinking at any age: Its nature and its development. *Mathematical Thinking and Learning*, 7(1), 27-50.
- Harel, G., & Tall, D. (1991). The general, the abstract, and the generic in advanced mathematics. *For the Learning of Mathematics*, 11(1), 38-42.
- Heinz, K., Kinzel, M., Simon, M., & Tzur, R. (2000). Moving students through steps of mathematical knowing: An account of the practice of an elementary mathematics teacher in transition. *The Journal of Mathematical Behavior*, 19(1), 83-107.
- Heirdsfield, A., & Cooper, T. (2004). Factors affecting the process of proficient mental addition and subtraction: Case studies of flexible and inflexible computers. *The Journal of Mathematical Behaviour*, 23(4), 443-463.
- Helme, S., Clarke, D., & Kessel, C. (1996). Moments in the process of coming to know. In P. Clarkson (Ed.), *Mathematics Research Group of Australasia (MERGA) 19th Annual Conference* (pp. 269-276). Melbourne: MERGA.

- Helme, S., & Stacey, K. (2000). Can minimal support for teachers make a difference to students' understanding of decimals? *Mathematics Teacher Education and Development*, 2, 105-120.
- Henningsen, M., & Stein, M. (1997). Mathematical tasks and student cognition: Classroom factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, 28(5), 524-549.
- Herscovics, N. (1989). The description and analysis of mathematical processes. In C. A. Maher, G. Goldin & R. B. Davis (Eds.), *Proceedings of the Eleventh Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 1-28). New Brunswick, New Jersey: PME.
- Hiebert, J. (1999). Relationships between research and the NCTM standards. *Journal for Research in Mathematics Education*, 30(1), 3-19.
- Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., et al. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational Researcher*, 25(4), 12-21
- Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Wearne, D., Murray, H., et al. (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth: Heinemann.
- Hiebert, J., & Wearne, D. (1985). A model of students' decimal computation procedures. *Cognition and Instruction*, 2(3/4), 175-205.
- Higgins, J. (2001). *Developing numeracy: Understanding place value*. Wellington, NZ: Ministry of Education.
- Higgins, J. (2005). Equipment-in-use in the Numeracy Development Project: Its importance to the introduction of mathematical ideas. In J. Higgins, K. Irwin, G. Thomas, T. Trinick & J. Young-Loveridge (Eds.), *Findings from the New Zealand Numeracy Development Project 2004* (pp. 89-96). Wellington, NZ: Ministry of Education.
- Hill, H., Ball, D., & Schilling, S. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal*, 105(1), 11-30.
- Hill, H., & Lubienski, S. (2007). Teachers' mathematics knowledge for teaching and school context: A study of California teachers. *Educational Policy*, 21(5), 747-768.
- Hodge, L., Visnovska, J., Zhao, Q., & Cobb, P. (2007). What does it mean for an instructional task to be effective? In J. Watson & K. Beswick (Eds.), *30th Annual Conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 392-401). Hobart: MERGA.
- Houssart, J. (2004). *Low attainers in primary mathematics: The whisperers and the maths fairy*. London: RoutledgeFalmer.
- Hughes, P. (2002). A model for teaching numeracy strategies. In B. Barton, K. Irwin, M. Pfannkuch & M. Thomas (Eds.), *Proceedings of the 25th Annual Conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 350-357). Auckland: MERGA.
- Hunter, R., & Anthony, G. (2003). Percentages: A foundation for supporting students' understanding of decimals. In L. Bragg, C. Campbell, G. Herbert & J. Mousley

- (Eds.), *Proceedings of the 26th Conference of the Mathematics Education Research Group of Australasia* (Vol. 2, pp. 452-459). Geelong, Vic.: MERGA.
- Hunting, R., & Sharpley, C. (1988). Fraction knowledge of pre-school children. *Journal for Research in Mathematics Education*, 19(2), 175-180.
- Irwin, K. (2001). Using everyday knowledge of decimals to enhance understanding. *Journal for Research in Mathematics Education*, 32(4), 399-420.
- Irwin, K., & Britt, M. (2004). Operating with decimal fractions as a part-whole concept. In I. Putt, R. Faragher & M. McLean (Eds.), *Proceedings of the 27th annual conference of the Mathematic Education Research Group of Australasia* (Vol. 1, pp. 312-319). Townsville: MERGA.
- Jaworski, B. (2004). *Grappling with complexity: Co-learning in inquiry communities in mathematics teaching development*. Paper presented at the Proceedings for the 28th conference of the International Group for the Psychology of Mathematics Education, Bergen, Norway.
- Jones, G., Thornton, C., Putt, I., Hill, K., Mogill, A. T., Rich, B., et al. (1996). Multidigit number sense: A framework for instruction and assessment. *Journal for Research in Mathematics Education*, 27(3), 310-336.
- Kamii, C., & Dominick, A. (1997). To teach or not to teach algorithms. *The Journal of Mathematical Behavior*, 16(1), 51-61.
- Kamii, C., Kirkland, L., & Lewis, B. A. (2001). Representation and abstraction in young children's numerical reasoning. In A. A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics* (Vol. 2001 Yearbook, pp. 24-34). Reston, VA: National Council of Teachers of Mathematics.
- Kazemi, E., & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *The Elementary School Journal*, 102(1), 59-81.
- Kemmis, S. (2008). Praxis and practice architectures in mathematics education. In M. Goos, R. Brown & K. Makar (Eds.), *Proceedings of the 31st Annual Conference of the Mathematics Research Group of Australasia (MERGA)* (Vol. 1, pp. 17-30). Brisbane: MERGA.
- Kieren, T. (1980). The rational number construct - its elements and mechanisms. In T. E. Kieran (Ed.), *Recent research on number learning* (pp. 125-150). Columbus Ohio: Ohio State University.
- Kieren, T. (1992). Rational and fractional numbers: From quotient fields to recursive understanding. In T. Carpenter, E. Fennema & T. Romberg (Eds.), *Rational numbers an integration of research* (pp. 49-84). Hillsdale NJ: Lawrence Erlbaum Associates.
- Kirschner, D. (2002). Untangling teachers' diverse aspirations for student learning: A cross-disciplinary strategy for relating psychological theory to pedagogical practice. *Journal for Research in Mathematics Education*, 33(1), 46-58.
- Kirschner, P., Sweller, J., & Clark, R. (2006). Why minimal guidance during instruction does not work: An analysis of the failure of constructivist, discovery, problem-based, experiential, and inquiry-based teaching. *Educational Psychologist*, 41(2), 75-86.

- Lachance, A., & Confrey, J. (2001). Helping students build a path of understanding from ratio and proportion to decimal notation. *The Journal of Mathematical Behavior*, 20(4), 503-526.
- Lambdin, D., & Walcott, C. (2007). Changes through the years: Connections between psychological learning theories and the school mathematics curriculum. In G. Martin, M. Strutchens & P. Elliot (Eds.), *The learning of mathematics sixty-ninth yearbook* (pp. 3-25). Reston, VA: NCTM.
- Lamon, S. (2001). Presenting and representing from fractions to rational numbers. In A. A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics* (Vol. 2001 Yearbook, pp. 146-165). Reston, VA: National Council of Teachers of Mathematics.
- Lamon, S. (2002). Part-whole comparisons with unitizing. In B. H. Litwiller & G. W. Bright (Eds.), *Making sense of fractions, ratios and proportions 2002 yearbook* (pp. 79-86). Reston, VA: National Council of Teachers of Mathematics.
- Lamon, S. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning*. (Vol. 2, pp. 629-668). Reston, VA: NCTM.
- Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, 27(1), 29-63.
- Leikin, R., & Levav-Waynberg, A. (2007). Exploring mathematics teacher knowledge to explain the gap between theory-based recommendations and school practice in the use of connecting tasks. *Educational Studies in Mathematics*, 66(3), 349-371.
- Leikin, R., & Zaslavsky, O. (1997). Facilitating student interactions in mathematics in a cooperative learning setting. *Journal for Research in Mathematics Education*, 28(3), 331-354.
- Lesh, R., Doerr, H., Carmona, G., & Hjalmarson, M. (2003). Beyond constructivism. *Mathematical Thinking and Learning*, 5(2), 211-233.
- Long, M. J., & Ben-Hur, M. (1991). Informed learning through the clinical interview. *Arithmetic Teacher*, February 1991, 44-46.
- Lukhele, R., Murray, H., & Olivier, A. (1999). Learners' understanding of the addition of fractions. Retrieved 4 July, 2005, from [www.wescape.school.za/malati/Files/Fractions992.pdf](http://www.wescape.school.za/malati/Files/Fractions992.pdf)
- Mack, N. K. (2001). Building on informal knowledge through instruction in a complex content domain: Partitioning, units, and understanding multiplication of fractions. *Journal for Research in Mathematics Education*, 32(3), 267-296.
- Martin, L. (2008). Folding back and the dynamical growth of mathematical understanding: Elaborating the Pirie-Kieren theory. *Journal of Mathematical Behavior*, 27(1), 64-85.
- Martin, L., Towers, J., & Pirie, S. (2006). Collective mathematical understanding as improvisation. *Mathematical Thinking and Learning*, 8(2), 149-183.
- Martinie, S., & Bay-Williams, J. (2003). Investigating students' conceptual understanding of decimal fractions using multiple representations. *Mathematics Teaching in the Middle Schools*, 8(5), 244-247.

- Martino, A. M., & Maher, C. A. (1999). Teacher questioning to promote justification and generalization in mathematics: What research practice has taught us. *The Journal of Mathematical Behavior*, 18(1), 53-78.
- Marton, F., & Trigwell, K. (2000). Variatio est mater studiorum. *Higher Education Research and Development*, 19(4), 381-395.
- Mason, J. (2001). Tunja sequences as examples of employing students' powers to generalize. *the Mathematics Teacher*, 94(3), 164-169.
- Mason, J. (2005). Frameworks for learning, teaching and research: Theory and practice. *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* Retrieved March 31, 2008, from [http://convention2.allacademic.com/index.php?cmd=pmena\\_guest](http://convention2.allacademic.com/index.php?cmd=pmena_guest)
- Mason, J., Drury, H., & Bills, L. (2007). Studies in the zone of proximal awareness. In J. Watson & K. Beswick (Eds.), *30th Annual Conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 42-58). Hobart: MERGA.
- McClain, K. (2002). Teacher's and students' understanding: The role of tools and inscriptions in supporting effective communication. *The Journal of the Learning Sciences*, 11(2&3), 217-249.
- McNeil, N., & Alibali, M. (2005). Why won't you change your mind? Knowledge of operational patterns hinders learning and performance on equations. *Child Development*, 76(4), 883-899.
- Meira, L. (1998). Making sense of instructional devices: The emergence of transparency in mathematical activity. *Journal for Research in Mathematics Education*, 29(2), 121-142.
- Merriam, S. (2001). *Qualitative research and case study applications in education*. San Francisco: Jossey-Bass.
- Mewborn, D., & Cross, D. (2007). Mathematics teachers' beliefs about mathematics and links to students' learning. In G. Martin, M. Strutchens & P. Elliot (Eds.), *The learning of mathematics sixty-ninth yearbook* (pp. 259-269). Reston, VA: NCTM.
- Meyer, M. R. (2001). Representation in Realistic Mathematics Education. In A. A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics* (Vol. 2001 Yearbook, pp. 238-250). Reston, VA: National Council of Teachers of Mathematics.
- Ministry of Education. n.d. Retrieved July 30, 2007, from [http://www.nzmaths.co.nz/numeracy/lead\\_teacher/PLC/Expectations/Expectations.doc](http://www.nzmaths.co.nz/numeracy/lead_teacher/PLC/Expectations/Expectations.doc)
- Ministry of Education. (1992). *Mathematics in the New Zealand curriculum*. Wellington, NZ: Learning Media.
- Ministry of Education. (2001). *Curriculum update No. 45: The Numeracy Story*. Retrieved July 29, 2008, from [http://www.tki.org.nz/r/governance/curric\\_updates/curr\\_update45\\_e.php](http://www.tki.org.nz/r/governance/curric_updates/curr_update45_e.php)
- Ministry of Education. (2005). *Numeracy Professional Development Projects (2005) (draft) Book 9: Teaching number through measurement, geometry, algebra and statistics*. Wellington, New Zealand: Author.
- Ministry of Education. (2006). *Numeracy Professional Development Projects Book 1: The number framework*. Wellington, NZ: Author.



- Ministry of Education. (2007a). *The New Zealand curriculum*. Wellington, NZ: Learning Media Ltd.
- Ministry of Education. (2007b). *Numeracy Professional Development Projects book 1: The number framework*. Wellington, NZ: Author.
- Ministry of Education. (2007c). *Numeracy Professional Development Projects book 2: The diagnostic interview*. Wellington, NZ: Author.
- Ministry of Education. (2007d). *Numeracy Professional Development Projects book 3 getting started*. Wellington, NZ: Author.
- Ministry of Education. (2007e). *Numeracy Professional Development Projects book 6 teaching multiplication and division revised edition 2007: Draft*. Wellington, NZ: Author.
- Ministry of Education. (2007f). *Numeracy Professional Development Projects book 7: Teaching fractions, decimals and percentages*. Wellington, NZ: Author.
- Ministry of Education. (2008). *Book 7 teaching fractions, decimals, and percentages Revised edition 2008: Draft*. Wellington, NZ: Author.
- Mitchell, A., & Clarke, D. (2004). When is three-quarters not three-quarters? Listening for conceptual understanding in children's explanations in a fraction interview. In I. Putt, R. Faragher & M. McLean (Eds.), *Proceedings of the 27th annual conference of the Mathematic Education Research Group of Australasia* (Vol. 1, pp. 367-373). Townsville: MERGA.
- Moody, B. (2007). *Connecting the points. Unpublished Dissertation*. University of Waikato, Hamilton, NZ.
- Moskal, B., & Magone, M. (2000). Making sense of what students know: Examining the referents, relationships and modes students displayed in response to a decimal task. *Educational Studies in Mathematics*, 43(3), 313-335.
- Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education*, 30(2), 122-137.
- Moyer, P. S. (2001). Are we having fun yet? How teachers use manipulatives to teach mathematics. *Educational Studies in Mathematics*, 47(2), 175-197.
- Myhill, D., & Dunkin, F. (2002). What's a good question? *Literacy Today* (33), 8-9.
- Nesher, P., & Peled, I. (1986). Shifts in reasoning. *Educational Studies in Mathematics*, 17(1), 67-79.
- Neuman, Y. (2001). Can the Baron von Munchausen phenomenon be solved? An activity-oriented solution to the learning paradox. *Mind, Culture and Activity*, 8(1), 78-89.
- Ni, Y. (2000). How valid is it to use number lines to measure children's conceptual knowledge about rational numbers? *Educational Psychology*, 20(2), 139-153.
- Nyikos, M., & Hashimoto, R. (1997). Constructivist theory applied to collaborative learning in teacher education: In search of ZPD. *The Modern Language Journal*, 81(4), 506-517.
- O'Connor, M. (2001). Can any fraction be turned into a decimal? A case study of a mathematical group discussion. *Educational Studies in Mathematics*, 46(1), 143-185.

- Okazaki, M., & Koyama, M. (2005). Characteristics of 5th graders' logical development through learning division with decimals. *Educational Studies in Mathematics*, 60(2), 217-251.
- Op 'T Eynde, P., De Corte, E., & Verschaffel, L. (2006). Accepting emotional complexity: A socio-constructivist perspective on the role of emotions in the mathematics classroom. *Educational Studies in Mathematics*, 63(2), 193-207.
- Owens, D., & Super, D. (1993). Teaching and learning decimal fractions. In D. Owens (Ed.), *Research ideas for the classroom: Middle grades mathematics* (pp. 137-158). New York: Macmillan.
- Oxford, R. (1997). Constructivism: Shape-shifting, substance, and teacher education applications. *Peabody Journal of Education*, 72(1), 35-66.
- Pesek, D., & Kirschner, D. (2000). Interference of instrumental instruction in subsequent relational learning. *Journal for Research in Mathematics Education*, 31(5), 524-540.
- Pirie, S., & Kieren, T. (1992). Creating constructivist environments and constructing creative mathematics. *Educational Studies in Mathematics*, 23(5), 505-528.
- Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics*, 26, 165-190.
- Pirie, S., & Martin, L. (2000). The role of collecting in the growth of mathematical understanding. *Mathematics Education Research Journal*, 12(2), 127-146.
- Prediger, S. (2008). The relevance of didactic categories for analyzing obstacles in conceptual change: Revisiting the case of multiplication of fractions. *Learning and Instruction*, 18(1), 3-17.
- Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning*, 5(1), 37-70.
- Resnick, L., Nesher, P., Leonard, F., Magone, M., Omanson, S., & Peled, I. (1989). Conceptual bases of arithmetic errors: The case of decimal fractions. *Journal for Research in Mathematics Education*, 20(1), 8-27.
- Roschelle, J., Tatar, D., Shechtman, N., & Knudsen, J. (2008). The role of scaling up research in designing for and evaluating robustness. *Educational Studies in Mathematics*, 68, 149-170.
- Ron, G., Herschkowitz, R., & Dreyfus, T. (2008). The identification of partially correct constructs. In M. Goos, R. Brown & K. Makar (Eds.), *Proceedings of the 31st Annual Conference of the Mathematics Research Group of Australasia (MERGA)* (Vol. 2, pp. 421-428). Brisbane: MERGA.
- Roth, W.M., & Hwang, S. W. (2006). Does mathematical learning occur in going from concrete to abstract or in going from abstract to concrete? *Journal of Mathematical Behavior*, 25(4), 334-344.
- Runesson, U. (1999). Teaching as constituting a space in variation. Retrieved April 1, 2008, from <http://www.ped.gu.se/biorn/phgraph/civil/graphica/ur.pdf>
- Runesson, U. (2005). Beyond discourse and interaction. Variation: A critical aspect for teaching and learning mathematics. *Cambridge Journal of Education*, 35(1), 69-87.

- Sackur-Grisvard, C., & Leonard, F. (1985). Intermediate cognitive organizations in the process of learning a mathematical concept: The order of positive decimal numbers. *Cognition and Instruction*, 2(2), 157-174.
- Sahin, A., & Kulm, G. (2008). Sixth Grade mathematics teachers' intentions and use of probing, guiding, and factual questions. *Journal of Mathematics Teacher Education*, 11(3), 221-241.
- Saxe, G., Shaughnessy, M., Shannon, A., Langer-Osuna, J., Chinn, R., & Gearhart, M. (2007). Learning about fractions as points on a number line. In G. Martin, M. Strutchens & P. Elliot (Eds.), *The learning of mathematics sixty-ninth yearbook* (pp. 221-237). Reston, VA: NCTM.
- Saxe, G. B., Edd, V., Taylor, C. M., & Gearhart, M. (2005). Representing fractions with standard notation: A developmental analysis. *Journal for Research in Mathematics Education*, 36(2), 137 - 157.
- Schliemann, A. (2002). Representational tools and mathematical understanding. *The Journal of the Learning Sciences*, 11(2&3), 301-317.
- Schoenfeld, A. H. (1989). *Exploring the process problem access space: Notes on the description and analysis of mathematical processes*. Paper presented at the Proceedings of the eleventh annual meeting, North American Chapter of the International Group for the Psychology of Mathematics Education, September 20 -23 1989, New Brunswick, New Jersey.
- Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, meta-cognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.
- Schwarz, B., Neuman, Y., & Biezuner, S. (2000). Two wrongs may make a right...If they argue together! *Cognition and Instruction*, 18(4), 461-494.
- Seeger, F. (1998). Representations in the mathematics classroom: Reflections and constructions. In F. Seeger, J. Voigt & U. Wascgescio (Eds.), *The culture of mathematics classrooms* (pp. 308-343). Cambridge: Cambridge University Press.
- Seeger, F. (2001). Research on discourse in the mathematics classroom: A commentary. *Educational Studies in Mathematics*, 46(1), 287-298.
- Sfard, A. (1991). On dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22(1), 1-36.
- Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher*, 27(2), 4-13.
- Sharp, J., & Adams, B. (2002). Children's constructions of knowledge for fraction division after solving realistic problems. *The Journal of Educational Research*, 95(6), 333-347.
- Shayer, M. (2003). Not just Piaget; not just Vygotsky, and certainly not Vygotsky as alternative to Piaget. *Learning and Instruction*, 13(5), 465-485.
- Shayer, M., & Adhmi, M. (2007). Fostering cognitive development through the context of mathematics: Results of the came project. *Educational Studies in Mathematics*, 64(3), 265-291.
- Shepard, L. (2000). The role of assessment in a learning culture. *Educational Researcher*, 29(7), 4-14.

- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Siegler, R. (2000). The rebirth of children's learning. *Child Development*, 71(1), 26-35.
- Siegler, R. (2007). Cognitive variability. *Developmental Science*, 10(1), 104-109.
- Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26(2), 114-145.
- Simon, M. (2006). Key developmental understandings in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning*, 8(4), 359-371.
- Simon, M. A., Tzur, R., Heinz, K., & Kinzel, M. (2004). Explicating a mechanism for conceptual learning: Elaborating the construct of reflective abstraction. *Journal for Research in Mathematics Education*, 35(5), 305-329.
- Skemp, R. (1976). Relational understanding and instrumental understanding. Retrieved February 5, 2008, from <http://www.tallfamily.co.uk/david/skemp/pdfs/instrumental-relational.pdf>.
- Sophian, C. (2008). Rethinking the starting point for mathematics learning. In O. Saracho & B. Spodek (Eds.), *Contemporary perspectives on mathematics in early childhood education* (pp. 21-44). Charlotte, NC: Information Age Publishing.
- Stacey, K., Helme, S., Archer, S., & Condon, C. (2001a). The effect of epistemic fidelity and accessibility on teaching with physical materials: A comparison of two models for teaching decimal numeration. *Educational Studies in Mathematics*, 47(2), 199-221.
- Stacey, K., Helme, S., & Steinle, V. (2001b). Confusions between decimals, fractions and negative numbers: A consequence of the mirror as a conceptual metaphor in three different ways. Retrieved April 30, 2008, from <http://extranet.edfac.unimelb.edu.au/DSME/decimals/SLIMversion/backinfo/refs/PME2001.pdf>
- Stacey, K., Helme, S., Steinle, V., Batur, A., Irwin, K., & Bana, J. (2001b). Pre-service teachers' knowledge of difficulties in decimal numeration. *Journal of Mathematics Teacher Education*, 4, 205-225.
- Stacey, K., & Steinle, V. (1999). Decimal comparison test. Retrieved February 8, 2006, from <http://extranet.edfac.unimelb.edu.au/DSME/decimals/SLIMversion/tests/comptest.shtml>
- Stafylidou, S., & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction*, 14(4), 503-518.
- Steencken, E. P., & Maher, C. A. (2003). Tracing fourth graders' learning of fractions: Early episodes from a year-long teaching experiment. *The Journal of Mathematical Behavior*, 22(2), 113-132.
- Steffe, L. P. (2001). A new hypothesis concerning children's fractional knowledge. *The Journal of Mathematical Behavior*, 20(3), 267-307.
- Steffe, L. P., Cobb, P., & von Glasersfeld, E. (1988). *Construction of arithmetic meanings and strategies*. New York: Springer-Verlag.
- Steffe, L. P., & Kieren, T. (1994). Radical constructivism and mathematics education. *Journal for Research in Mathematics Education*, 25(6), 711-733.

- Steinbring, H. (1998). Mathematical understanding in classroom interaction: The interrelation of social and epistemological constraints. In F. Seeger, J. Voigt & U. Wascgescio (Eds.), *The culture of mathematics classrooms* (pp. 344-376). Cambridge: Cambridge University Press.
- Steinle, V., & Stacey, K. (1998). The incidence of misconceptions of decimal notation amongst students in grades 5 to 10. In C. Kanes, M. Goos & E. Warren (Eds.), *Proceedings of the 21st annual conference of the Mathematics Education Research Group of Australasia* (Vol. 2, pp. 548-555). Gold Coast, Qld.: MERGA
- Steinle, V., & Stacey, K. (2001). Visible and invisible zeros: Sources of confusion in decimal notation. In J. Bobis, B. Perry & M. Mitchelmore (Eds.), *Proceedings of the 24th annual conference of the Mathematics Research Group of Australasia* (Vol. 2, pp. 434-441). Sydney: MERGA.
- Steinle, V., & Stacey, K. (2002). Further evidence of conceptual difficulties with decimal notation. In B. Barton, K. Irwin, M. Pfannkuch & M. O. J. Thomas (Eds.), *Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia* (Vol. 2, pp. 633-640). Auckland: MERGA.
- Streefland, L. (1991). *Fractions in realistic mathematics education, a paradigm of developmental research*. Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Thompson, P., & Saldhana, L. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. Martin & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 95-113). Reston, VA: NCTM.
- Timperley, H., Wilson, A., Barrar, H., & Fung, I. (2007). *Teacher Professional learning and Development: Best evidence synthesis iteration [BES]*. Wellington, NZ: Ministry of Education.
- Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education*, 31(1), 5-25.
- Tirosh, D., Fischbein, E., Graeber, A., & Wilson, J. (2001). Prospective elementary teachers' conceptions of rational numbers. from [www.jwilson.coe.uga.edu/](http://www.jwilson.coe.uga.edu/)
- Tirosh, D., & Graeber, A. (1989). Pre-service elementary teachers' explicit beliefs about multiplication and division. *Educational Studies in Mathematics*, 20(1), 79-96.
- Tirosh, D., & Graeber, A. (1990). Evoking cognitive conflict to explore pre-service teachers' thinking about division. *Journal for Research in Mathematics Education*, 21(2), 98-108.
- Tirosh, D., Stavy, R., & Cohen, S. (1998). Cognitive conflict and intuitive rules. *International Journal of Science Educations*, 20(10), 1257-1269.
- Trognon, A. (1993). How does the process of interaction work when two interlocutors try to resolve a logical problem? *Cognition and Instruction*, 11(3/4), 325-345.
- Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. *Journal for Research in Mathematics Education*, 30(4), 390-416.
- Tzur, R. (2007). Fine grain understanding of students' mathematical understanding: Participatory and anticipatory stages in learning a new mathematical conception. *Educational Studies in Mathematics*, 66(3), 273-291.

- Tzur, R., & Simon, M. A. (2003). *Distinguishing two stages of mathematics conceptual learning*. Paper presented at the International Conference of Science and Mathematics Learning, Taiwan.
- Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction, 14*(4), 453-467.
- Van de Walle, J. (2007). *Elementary and middle school mathematics: Teaching developmentally (6th ed.)*. Boston, MA: Pearson.
- Ward, J., & Thomas, G. (2007). What do teachers know about fractions? In *Findings from the New Zealand Numeracy Development Projects 2006* (pp. 128-138). Wellington: Learning Media.
- Wascgescio, U. (1998). The missing link: Social and cultural aspects in social constructivist theories. In F. Seeger, J. Voigt & U. Wascgescio (Eds.), *The culture of the mathematics classroom* (pp. 221-244). Cambridge: Cambridge University Press.
- Watson, A. (2003). Opportunities to learn mathematics. In L. Bragg, C. Campbell, G. Herbert & J. Mousley (Eds.), *Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 29-40). Geelong, Vic.: MERGA.
- Watson, A., & De Geest, E. (2005). Principled teaching for deep progress: Improving mathematical learning beyond methods and materials. *Educational Studies in Mathematics, 58*(2), 209-234.
- Watson, J., & Kelly, B. (2003). Predicting dice outcomes: The dilemma of expectation versus variation. In L. Bragg, C. Campbell, G. Herbert & J. Mousley (Eds.), *Proceedings of the 26th Conference of the Mathematics Education Research Group of Australasia* (Vol. 2, pp. 728-735). Geelong, Vic.: MERGA.
- Wearne, D. (1990). Acquiring meaning for decimal fraction symbols: A one year follow-up. *Educational Studies in Mathematics, 21*(6), 545-564.
- Wilkins, J. (2008). The relationship among elementary teachers' content knowledge, attitudes, beliefs, and practices. *Journal of Mathematics Teacher Education, 11*(2), 139-164.
- Williams, G. (2003). Associations between student pursuit of novel mathematical ideas and resilience. In L. Bragg, C. Campbell, G. Herbert & J. Mousley (Eds.), *Proceedings of the 26th Conference of the Mathematics Education Research Group of Australasia* (Vol. 2, pp. 752-759). Geelong, Vic.: MERGA
- Williams, S., & Baxter, J. (1996). Dilemmas of discourse-oriented teaching in one middle school mathematics classroom. *The Elementary School Journal, 97*(1), 21-38.
- Wilson, J. (2001). Methodological difficulties of assessing meta-cognition: A new approach. Retrieved July 4, 2005, from [www.aare.edu.au/01pap/wil01001](http://www.aare.edu.au/01pap/wil01001)
- Winn, J. (1994). Promises and challenges of scaffolded instruction. *Learning Disability Quarterly, 17*(1), 89-104.
- Wood, D., Bruner, J., & Ross, G. (1976). The role of tutoring in problem solving. *Journal of Child Psychology and Psychiatry, 17*, 89-100.
- Wright, V. (2007). In a nutshell mathematics and statistics curriculum *Unpublished*. Hamilton, NZ

- Yackel, E., & Cobb, P. (1996). Socio-mathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 27(4), 458-477.
- Yang, D.-C. (2005). Number sense strategies used by 6th-grade students in Taiwan. *Educational Studies*, 31(3), 317-333.
- Young-Loveridge, J. (1998). How does place value understanding develop? In A. Else (Ed.), *Exploring issues in mathematics education* (pp. 49-62). Wellington, NZ: Ministry of Education (NZ).
- Young-Loveridge, J. (2005). Fostering multiplicative thinking using array-based materials. *Australian Mathematics Teacher* (May 2005).
- Young-Loveridge, J. (2007). Patterns of performance and progress on the Numeracy Development Projects: Findings from 2006 for Years 5-9 students. In *Findings from the New Zealand Numeracy Development Projects 2006* (pp. 16-32, 154-177). Wellington, NZ: Learning Media.
- Young-Loveridge, J., Taylor, M., Hawera, N., Sharma, S. (2007). Year 7-8 students' solution strategies for a task involving addition of unlike fractions. In *Findings from the New Zealand Numeracy Development Projects 2006* (pp. 67-86). Wellington, NZ: Learning Media.
- Zaslavsky, O. (2005). Seizing the opportunity to create uncertainty in learning mathematics. *Educational Studies in Mathematics*, 60(3), 297-321.
- Zazkis, R., & Campbell, S. (1996). Divisibility and multiplicative structure of natural numbers: Pre-service teachers' understanding. *Journal for Research in Mathematics Education*, 27(5), 540-563.
- Zazkis, R., & Chernoff, E. (2008). What makes a counterexample exemplary? Retrieved March 28, 2008, from [www.springerlink.com/index/lh626777872168g7.pdf](http://www.springerlink.com/index/lh626777872168g7.pdf)
- Zazkis, R., & Hazzan, O. (1998). Interviewing in mathematics education research: Choosing the questions. *The Journal of Mathematical Behavior*, 17(4), 429-439.
- Zazkis, R., & Liljedahl, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics*, 49(3), 379-402.
- Zazkis, R., Liljedahl, P., & Chernoff, E. (2008). The role of examples in forming and refuting generalizations. *ZDM Mathematics Education*, 40(1), 131-141.

# *Appendices*

## **Contents**

- A Decimal Comparison Tests
- B Pipe Numbers
- C Letters of Information
- D Games

## **Appendix A: Decimal Comparison Tests (DCT)**

### **University of Melbourne DCT**

This tool was developed as part of a project whereby teachers could receive diagnostic assessment of student behaviours with ordering decimal numbers online. This is the print version of that tool. There are parallels between the types of pairs chosen here and the earlier work undertaken in France, Israel, the UK and the USA mentioned in the literature review.

### **My Version**

One of the authors of the University of Melbourne, Prof. K. Stacey kindly allowed me to adapt the DCT for ease of teacher use and to directly explore another behaviour that I had observed.

The main modifications I made were to standardize the 30 items into six boxes. A cursory examination of student answers to boxes A and B was all that was required to form the classification of most students. Whole-number thinkers (longer is larger) have Box A wrong but Box B right. 'Shorter is larger' thinkers have Box A right and B wrong. Experts have both sets of items correct.

Box C identifies issues with zero as a place-holder (often accompanies whole-number thinking).

Box D identifies those who have rejected the longer is larger view but do not accept the validity of decimal places beyond two. Typically they cannot decide between the pairings.

Box E identifies those 'shorter is larger' thinkers who confuse decimal places with denominators, 0.3 being perceived as  $\frac{1}{3}$  for example.

Box F identifies a small group of students who think of decimals as a type of negative number. The decimal component of the number reduces its value from the whole number stem. (I have observed teachers presenting this as mathematical truth).

Copies of both tools are reproduced overleaf.



Name: \_\_\_\_\_

Class: \_\_\_\_\_

Date: \_\_\_\_\_

For each pair of decimal numbers, circle the one which is the LARGER.

0.4	0.457	1.06	1.053	4.4502	4.45
0.86	1.3	4.08	4.7	17.353	17.35
0.3	1.3	3.72	3.073	8.245	8.24563
1.85	1.84	2.621	2.0687986	3.2618	3.26
3.71	3.76	8.052573	8.514	3.741	3.746
4.8	4.63	5.62	5.736	0.35	0.42
0.5	0.36	0.5	0.75	2.186	2.954
0.75	0.8	0.426	0.3	0.872	0.813
0.37	0.216	2.516	2.8325	0.038	0.04
3.92	3.4813	7.942	7.63	0.006	0.53

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Name \_\_\_\_\_ Class \_\_\_\_\_ Year \_\_\_\_ Date \_\_\_\_\_ School \_\_\_\_\_

**For each pair of decimal numbers, circle the one that is LARGER.**

<b>4.8</b>	<b>4.63</b>	
<b>0.5</b>	<b>0.36</b>	<b>A</b>
<b>0.75</b>	<b>0.8</b>	
<b>0.37</b>	<b>0.216</b>	
<b>3.92</b>	<b>3.4813</b>	

<b>4.08</b>	<b>4.7</b>	
<b>3.72</b>	<b>3.073</b>	<b>C</b>
<b>2.621</b>	<b>2.06879</b>	
<b>8.052573</b>	<b>8.514</b>	
<b>0.53</b>	<b>0.061</b>	

<b>0.3</b>	<b>0.4</b>	
<b>1.85</b>	<b>1.84</b>	<b>E</b>
<b>3.71</b>	<b>3.76</b>	
<b>0.2</b>	<b>0.5</b>	
<b>22.1</b>	<b>22.6</b>	

<b>5.62</b>	<b>5.736</b>	
<b>0.5</b>	<b>0.75</b>	
<b>0.426</b>	<b>0.3</b>	<b>B</b>
<b>2.516</b>	<b>2.8325</b>	
<b>7.942</b>	<b>7.63</b>	

<b>17.353</b>	<b>17.35</b>	
<b>4.45</b>	<b>4.4502</b>	
<b>8.24</b>	<b>8.24563</b>	
<b>0.55</b>	<b>0.555</b>	<b>D</b>
<b>3.741</b>	<b>3.74</b>	

<b>0.03</b>	<b>0.00</b>	
<b>0</b>	<b>0.6</b>	
<b>0.1</b>	<b>0.0</b>	<b>F</b>
<b>2</b>	<b>2.3</b>	
<b>10.7</b>	<b>10</b>	

## Appendix B: Pipe Numbers

Pipe numbers are a linear model of the number system and thus similar in structure to the number line.

The intention behind having all pieces the same colour and abutting on the same inner support was to reinforce the point that this is a continuous representation of number as length. The reason for this is because students often interpret the use decimal points to denote the separation of units (e.g. dollars and cents, metres and centimeters). They do not recognise that they are subdivisions of the same unit.

I had originally designed the pipe numbers to be made of black irrigation tubing and wooden dowelling rods and so the pipes were clearly of different material to the rods which were simply an assembling aid. The 'one' had to be big enough to manufacture meaningfully-sized hundredths. I had made the prototype when I read a description of the Melbourne University LAB model, parallel evolution! They had extensively trialed their model before mine was created. I adopted their practice of having washers available to model thousandths, but discarded this after school trials showed that it was unnecessary. At the time, I was attempting to engage teachers in recognizing the potential of using diagnostic information to plan lesson sequences rather than relying upon the existence of a scheme of work to become the de facto curriculum. The DCT and pipe numbers became tools in this process.

The primary purpose of the design was to allow the direct comparison of decimal representations, chiefly to promote dissonance between naïve student beliefs about decimals (the whole number construct) and the new physical model they have made sense of. It is also easy to show how addition works, both in the senses of addressing the whole number misconception ( $0.2 + 0.11 = 0.13$  sic) and demonstrating that the commutative law still applies and to show that systems of 'trading' tens are still valid. The concept of rounding was also easy to demonstrate.

Simple multiplication problems using repeated addition can also be modeled; three lots of 0.4 demonstratively are equivalent to 1.2 for example.

The version of the pipe numbers used in this study was commercially manufactured. I have no commercial interest in this product.

In their version, pipe numbers are made from hollow, blue plastic tubing of identical diameter and with scaled lengths of 1, 1/10 and 1/100. Red inner tubing is provided to aid the assembling of number models and to achieve portability of the representations.

## **Appendix C: Informative Letters**

### **Contents**

Student Consent Form	(used in both Phases)
Parental Consent Form	(from Phase 2)
Letter to Principal	(from Phase 1)
Letter to Teacher	(from Phase 2)

### **Student consent form - Student Copy**

I have had the research work explained to me. I agree to take part in the interviews and the class work and I agree that these will be tape recorded for sound. I know that it is ok for me to change my mind later and decide not to go ahead.

I understand that no-one will be able to find out my real name when Mr Moody writes up his report or in any other articles from his research.

I have been able to ask any questions about this that I wanted to.

Name ..... Date .....

*Letterhead with contact details*

Date

To the Parents/caregivers of .....

Kia ora,

My name is Bruce Moody. I work for the University of Waikato as a Numeracy Adviser and in this role I have worked in the contributing schools to *School Name* Intermediate and in *School Name* itself.

I want to carry out some research on how children learn about decimal numbers; a topic your child will be doing this year. I would like to find out what he/she already understands about decimals, and then follow this through with some teaching sessions to see how this understanding changes and develops. This will form the basis of a thesis I will be writing to gain a Masters qualification in education.

The classroom work has three parts. The first involves an initial interview. I will then work with a small group of children for around six teaching sessions. I will be taking those sessions for the teacher during their normal maths time. Following this I will re-interview the children. I will then write a report based upon what happens which will be compared with what other people have written and so form a thesis. Information in the report may be used to write articles for publication and/or presentations at conferences.

Your child's name will not be used in the report, in any other publications or used verbally at any conference. In the writing process I will assign pseudonyms to each child so that their identity is hidden behind this false name. Everything that is said during the sessions and interviews will remain confidential. When completed, a copy of the report will be available at the school.

In order to accurately capture what the children will be saying, I will audio-tape the interviews and teaching sessions as well as take notes. These tapes and notes will not be available to anyone else but kept securely by myself.

Can you please talk with your child about what is involved with the research and see if they are happy to take part? If you also agree with their involvement, then please complete the consent form below and return it to school by Friday of next week. It is important that the form is returned as I cannot work with any children without consent from home.

I will also be explaining the process to the children and asking them to complete a personal consent form if you give permission. They will be free to withdraw their participation at any time from any question, task or even the entire project.

If you have any questions or require further information before making a decision, please contact me through the email address or phone number listed above.

If at any time during the research you have concerns, please feel free to contact School Name Intermediate School on (phone number) or my supervisor, Dr Jenny Young-Loveridge (*phone number*).

Yours sincerely,  
Bruce Moody

Numeracy Adviser

.....  
.....

Parent/Caregiver Consent Return Slip

I agree to (child's name) ..... taking part in the research work as described in the letter I have received. I understand that audio-tapes of interviews and teaching sessions will be made with my child's permission. I realise that all information will be kept private and that my child's name will not appear on any documents published from this project. I understand that my child can skip any questions or tasks at any time if they choose to and/or withdraw from the project at any time.

Signed .....

Name

Date .....

Date

Dear (*Principal*)

As part of the work towards a post-graduate qualification in education, I want to undertake a small research project in your school in 2006.

The focus of the project is to document the changes children make in their thinking about decimal numbers as they start to engage in classroom activities that target this area of Numeracy.

I am looking to involve children whom their teacher will have already identified as being ready for formal teaching in decimals. This is likely to be about 6 children. I would then interview each child, the times being set in coordination with their classroom teacher so as to minimize disruption to the normal class programme. I would then work with the children for each maths time for one week; I have the period of March 13<sup>th</sup> – 17<sup>th</sup> in mind. At a suitable time later in Term 1, I would re-interview each child. The interviews are likely to take 20 – 30 minutes each.

Participation in this research is entirely voluntary. The children may choose not to answer a question, or to stop the interview at any time. The interviews and teaching sessions will be audio-taped with the children's consent. The children's names will not be used in the final research report and everything they tell me will remain confidential. I will be the only person to have access to the audio-tapes.

The school and class will also not be identified in any report that is written as a result of the research.

I will provide you with a copy of the research report upon completion.

I would value your help in agreeing to host me for the research activity, suggest a teacher who may be interested in having their children involved, and providing me with a room in which to interview and work with the children.

I will send letters of information and consent forms for the parents or caregivers of the children to be interviewed. A separate letter of information for teachers will also be sent. If you have any questions or require further information, please feel free to call me or email me using the information given above.

Yours sincerely,

Bruce Moody  
Numeracy Adviser

*Contact information provided*

Date

Dear *Colleague*,

As part of the work towards a Masters degree in education, I want to undertake a small research project in your school.

The focus of the project is to document the changes children make in their thinking about decimal numbers as they start to engage in classroom activities that target this area of Numeracy. Specifically I will be looking at how students adjust their concepts of multiplication and division. I will then write a thesis based upon what happens and comparing this to the current research literature. Information from the thesis may be also be used to write articles for publication and/or presentation at conferences.

I am planning to work with a group of children in one class according to the following protocols:

1. The children will be those that you were already intending to teach decimals to according to your Numeracy plans. This will probably mean that they are at least Stage 6 on the Numeracy Project Framework.

2. The children agree to be part of the research and will also have caregiver approval.

3. I would interview the children individually prior to the teaching sessions at some mutually convenient time where disruption to your teaching programme will be minimal.

4. I would come in and work with this group during your usual maths time, I am hoping that the dates of August 16<sup>th</sup> – 23<sup>rd</sup> will prove suitable. I would take the children for a teaching session of around 40 mins each time.

5. On Aug 24<sup>th</sup>, I would re-interview the children.

6. The interviews and teaching sessions would be audio-taped (with children's permission) but no child's real name will be used in the research and no identification will be made of either the class or the school.

7. I will attach a proposal of the Action Plan I intend to carry out, bearing in mind that I will adjust it as the children's needs are continually re-assessed during the research.

8. At any time in the process you may contact me using the phone or email contacts as given above if you require more information.

9. If there any problems, my supervisor - Dr Jenny Young-Loveridge - can be contacted at this number (*phone number*).

Yours sincerely,

Bruce Moody

*Contact details provided*

Numeracy Adviser



## Appendix D: Game Descriptions

Four games were used during the intervention section of Phase One.

### Nasty

The purposes of this game are to engage children in considering place value magnitude e.g. “should I put my eight in the ones or tens column?” and to consider strategies of addition. An equation ‘blank’ for each team is slowly filled in as randomly generated digits are produced and then assigned by students to specific places.

e.g.  $\square\square + \square\square = \underline{\quad}$  becomes  $6\square + \square3 = \underline{\quad}$  becomes  $67 + 23 = 90$  as the game progresses and concludes. The ‘nasty’ aspect comes from the rule that you can choose to place any of ‘your’ digits in the opposition’s equation at the time the digit is first produced. Thus ones are frequently placed in the opposition’s tens column to prevent them placing a higher digit in that position. A decimal version of Nasty was suggested by the students and samples of play are described in chapter 5.

### Buzz

A game of repetitive counting that has ‘target numbers’ that invoke a response of “buzz” rather than saying the actual number. E.g. counting up in ones, but with reference to groupings of three produces “one, two, buzz, four, five, buzz etc”. My variation is that the target numbers should be emphasized, not replaced by the ‘buzz’. E.g. “One, two, THREE, four, five, SIX, etc”. On day four, Buzz was used to practice counts in units of 0.1 with the main purpose of getting students to replace ‘point ten’ with ‘one whole’ in their counts.

### Alias

A variation of Buzz. Students struggle with the idea that a number can have more than one name.

The game of alias gets students to practice saying these alternative labels. E.g. one quarter, two quarters alias one half, three quarters, four quarters alias one whole.

The game was used in two versions; one to practice iteration of a unit fraction (day one) and then to gauge whether students would remember that the tenth after 9/10 and 0.9 is the number one (and not 0.10, 0.11 etc) on day four in Phase One.

### Bust

Randomly generated digits are first assigned a place value and then added to make a cumulative total. Going over the pre-set limit makes you ‘bust’ and lose, getting closer to the total than your opposition makes you win. In this decimal variant students first chose which ten-sided dice they would roll (tenths, hundredths or thousandths) and then added the resulting number to their cumulative total. Playing decimal Bust formed part of the last session’s work in Phase One.