ASYMPTOTIC ORDER OF THE SQUARE-FREE PART OF N!

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Abstract

The asymptotic order of the logarithm of the square-free part of $n!$ is shown to be $(\log 2)n$ with error $O(\sqrt{n})$.

1. Introduction

If the standard prime factorization of $n!$ is considered over a range of values of n then a number of patterns are apparent:

$$\begin{aligned} 10! &= 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \\ 20! &= 2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \\ 30! &= 2^{26} \cdot 3^{14} \cdot 5^7 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \\ 40! &= 2^{38} \cdot 3^{18} \cdot 5^9 \cdot 7^5 \cdot 11^3 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37. \end{aligned}$$

All the primes up to n appear. If p and q are primes appearing in the factorization with $p < q$ and α, β are the highest powers of p and q dividing $n!$ respectively, then $\alpha \geq \beta$, i.e. the smaller the prime, the larger the power. Even though sometimes a given power does not appear (the power 3 is missing from 20! even though the powers 2 and 4 appear), the power 1 always appears.

The square-free part of $n!$ is the number a, with no square factors, which appears in the factorization

$$n! = ab^2.$$

It is easy to see that a is exactly the product of each of the primes which appear to an odd power in the standard factorization, and in particular is divisible by the primes appearing to power 1 in that factorization.

Two natural questions arise: what is the size of the square-free part a of $n!$ and what proportion of a is the product of the primes which occur to power 1 ? In this note it will be shown that, asymptotically, the square-free part of $n!$ has order 2^n and that the proportion of primes to power 1 is about 72%.

2. Integer Square Roots

For each whole number n let the integer lower square root be defined by

$$r_-(n) = \prod_{p^\alpha || n} p^{\lfloor \frac{\alpha}{2} \rfloor}$$

and the integer upper square root by

$$r_+(n) = \prod_{p^\alpha || n} p^{\lceil \frac{\alpha}{2} \rceil}.$$

If $n = ab^2$ and $cn = d^2$ with a and c square-free, then

$$b = r_-(n), d = r_+(n), a = c = \frac{r_+(n)}{r_-(n)}.$$

This pair of functions $r_\pm$ is quite useful. They are multiplicative, can be generalized to integer k'th roots and are related to the integer conductor or square-free core. For examples and applications see [3, 4].

3. Computing the square-free part of n!

To obtain some idea of the behavior of the square-free part of $n!$, for large n, it pays to do some computations. However, for numbers of quite small size, say $n = 400$, $n!$ is a number with over 800 digits, so finding the square-free part should not be attempted directly. The following strategy was adopted:

For each $n \geq 1$, let θ_n be the square-free part of $n+1$, i.e.,

$$\theta_n = r_+(n+1)/r_-(n+1).$$

Because $a_{n+1}b_{n+1}^2 = (n+1)n! = (n+1)a_n b_n^2$ and $n+1 = \theta_n c^2$ for some integer c, we have $\theta_n a_n b_n^2 = a_{n+1} b_{n+1}^2$.

If a prime $p \mid (\theta_n, a_n)$, then p occurs as a factor in both θ_n and a_n, so must occur to an odd power in both $n!$ and $n+1$, and therefore to an even power in $(n+1)!$. Hence it does not occur in a_{n+1}. If a prime occurs in just one of θ_n and a_n, then it must occur in a_{n+1}. This leads directly to the formula:

$$a_{n+1} = \frac{a_n \theta_n}{(a_n, \theta_n)^2}. \tag{1}$$

Note that this formula can be used to evaluate the sequence (a_n) recursively, so the values of $\log a_n$ can be plotted, revealing a nice approximately linear dependence on n. See Figure 1.

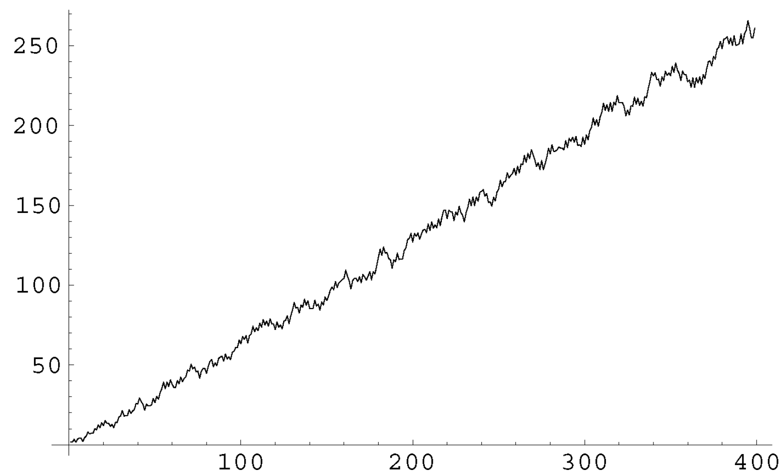

Figure 1. The sequence $\log a_n$ as a function of n.

4. Asymptotic orders

The result of these computations of the square-free part of $n!$ leads to two natural tasks: determining the slope of a line approximating the graph of $\log a_n$, and finding an upper bound for the error in this approximation. The completion of both tasks is summarized in the next theorem.

Theorem 1: For each $n \in \mathbb{N}$ let $n! = a_n b_n^2$ where $a_1 = b_1 = 1$ and where for all $n \geq 1$, a_n is square-free.

Then
$$\log a_n = n \log 2 + O(\sqrt{n}),$$
and
$$\log b_n = \frac{1}{2} n \log n - \frac{1 + \log 2}{2} n + O(\sqrt{n}).$$

Proof: Consider the central binomial coefficient $\binom{2n}{n} = t_n s_n^2$ where t_n is square-free. Then
$$b_{2n}^2 a_{2n} = (2n)! = (n!)^2 s_n^2 t_n$$
so $t_n = a_{2n}$ for all $n \in N$. By the main result in [7], there is a real strictly positive constant c such that for all $\epsilon > 0$ and all n sufficiently large
$$(c - \epsilon)\sqrt{n} < 2 \log s_n < (c + \epsilon)\sqrt{n}.$$
Therefore $\log s_n = O(\sqrt{n})$.

Stirling's approximation for $n!$ [8] is $n! \approx \sqrt{2\pi n}(n/e)^n$. It leads to the formula:

$$\log n! = n\log n - n + O(\log n).$$

Consequently:

$$\begin{aligned}
\log a_{2n} &= \log\binom{2n}{n} - 2\log s_n \\
&= 2n\log 2n - 2n - 2n\log n + 2n + O(\sqrt{n}) \\
&= 2n\log 2 + O(\sqrt{n}).
\end{aligned}$$

By equation (1)

$$\begin{aligned}
\log a_{2n+1} &= \log a_{2n} + \log\theta_{2n} - 2\log(a_{2n},\theta_{2n}) \\
&= \log a_{2n} + O(\log n) \text{ since } \theta = O(n) \\
&= (2n+1)\log 2 + O(\sqrt{n})
\end{aligned}$$

and therefore

$$\log a_n = n\log 2 + O(\sqrt{n}).$$

But, by Stirling's approximation again and this estimate for $\log a_n$:

$$\begin{aligned}
2\log b_n &= n\log n - n - n\log 2 + O(\sqrt{n}) \\
&= n\log n - (1+\log 2)n + O(\sqrt{n})
\end{aligned}$$

and therefore $\log b_n = \frac{1}{2}n\log n - \frac{1+\log 2}{2}n + O(\sqrt{n})$. This completes the proof of the theorem.

It follows also that the square-free part of $\binom{2n}{n}$, namely t_n, satisfies $\log t_n = 2n\log 2 + O(\sqrt{n})$, giving the asymptotic order. This relates to the solved conjecture of Erdős [5] that the binomial coefficient $\binom{2n}{n}$ is not square-free for $n > 4$. It relates also to the parity of the exponents of the prime factors of $n!$, [2].

5. Primes dividing n!

Lemma 1: Let $k \geq 1$ and let p be a prime integer. If $n \geq k(k+1)$ then $p^k \| n!$ if and only if $\frac{n}{k+1} < p \leq \frac{n}{k}$.

Proof: If $\frac{n}{k+1} < p \leq \frac{n}{k}$ then $k \leq \frac{n}{p} < k+1$, so therefore

$$k = \lfloor \frac{n}{p} \rfloor.$$

Since $k(k+1) \leq n$ we have $k \leq \frac{n}{k+1} < p$, so that

$$\lfloor \frac{n}{p^2} \rfloor < \frac{k+1}{p} \leq 1.$$

It follows that $\lfloor \frac{n}{p^2} \rfloor = 0$, by Legendre's formula

$$\alpha_p = \sum_{j=1}^{\infty} \lfloor \frac{n}{p^j} \rfloor = \lfloor \frac{n}{p} \rfloor = k.$$

Conversely, if $p^k \| n!$ then $k = \lfloor \frac{n}{p} \rfloor + \cdots$. Thus $\lfloor \frac{n}{p} \rfloor \le k$, which implies $\frac{n}{k+1} < p$, so $k < p$. In addition $k < \frac{n}{k+1}$, therefore $\frac{n}{p^2} \le \frac{k}{p} < 1$ so $\lfloor \frac{n}{p^2} \rfloor = 0$ and $k = \lfloor \frac{n}{p} \rfloor$, which shows $p \le \frac{n}{k}$. This completes the proof of the lemma.

For $x > 0$ let

$$\theta(x) = \sum_{2 \le p \le x} \log p,$$

Chebyshev's function [1], where the sum is over all primes less than or equal to x. If $x \ge 563$ then $\theta(x)$ is close to x in that [6]

$$x\big(1 - \frac{1}{2\log x}\big) < \theta(x) < x\big(1 + \frac{1}{2\log x}\big).$$

If follows that if $n \ge n_k$

$$|\theta\big(\frac{n}{k}\big) - \theta\big(\frac{n}{k+1}\big) - \frac{n}{k(k+1)}| \le \frac{n}{k \log \frac{n}{k}}.$$

By Lemma 1, the logarithm of the product of primes which appear in $n!$ to the k'th power is

$$\begin{aligned} \log \prod_{\frac{n}{k+1} < p \le \frac{n}{k}} p &= \sum_{\frac{n}{k+1} < p \le \frac{n}{k}} \log p \\ &= \theta\big(\frac{n}{k}\big) - \theta\big(\frac{n}{k+1}\big) \\ &= \frac{n}{k(k+1)} + O_k\big(\frac{n}{\log n}\big), \end{aligned}$$

so the asymptotic order of the product is $\frac{n}{k(k+1)}$ as $n \to \infty$.

Therefore, by Theorem 1, the asymptotic proportion of the square-free part of $n!$ due to primes appearing to powers $1, 3, \ldots, 2k-1$ is

$$\frac{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2k-1} - \frac{1}{2k}}{\log 2}.$$

For example, primes to power one contribute $\frac{1/2}{\log 2}$ or about 72%, and those to power one or three to $\frac{7/12}{\log 2}$, or about 84% of the square-free part.

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