Vanishing of the integral of the Hurwitz zeta function

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A proof is given that the improper Riemann integral of $\zeta(s,a)$ with respect to the real parameter a, taken over the interval $(0,1]$, vanishes for all complex s with $\Re(s) < 1$. The integral does not exist (as a finite real number) when $\Re(s) \geq 1$.

Key Words: Hurwitz zeta function, functional equation, improper Riemann integral.

MSC2000 11M35, 30E99.

1. INTRODUCTION

A number of authors have considered mean values of powers of the modulus of the Hurwitz zeta function $\zeta(s,a)$, see [3, 4, 5, 6, 7]. In this paper, the mean of the function itself is considered.

First a functional equation relating the Riemann zeta function to sums of the values of the Hurwitz zeta function at rational values of a is derived. This functional equation underlies the vanishing of the integral of the Hurwitz zeta function.

Consider the values of the function at negative integers:

$$\zeta(-n,a) = -\frac{B_{n+1}(a)}{n+1}, n \geq 0$$

where $B_n(a)$ is the n'th Bernoulli polynomial. The integral of the right hand side expression between 0 and 1 is zero for every n. This appears to be a side-effect of the properties of Bernoulli polynomials (namely for $n \geq 2$, $B_n(0) = B_n(1)$ and $B_n'(x) = nB_{n-1}(x)$), and nothing particularly intrinsic to the zeta function. However, as the theorem below will show, the integral vanishes at every value of the complex variable s to the left of the line $\Re(s) = 1$. The integral does not exist (as a finite real number), on or to the right of this line.

2. THE VANISHING THEOREM

The theorem is proved through developing a number of lemmas. The first is a fundamental, yet easy to derive, functional equation. See also, for example, [2].

LEMMA 2.1. *For all integers $k \geq 1$ and all $s \in \mathbb{C} - \{1\}$*

$$k^s \zeta(s) = \sum_{j=1}^{k} \zeta(s, \frac{j}{k}).$$

Proof. Consider the functional equation for the Hurwitz zeta function [1]:

$$\zeta(1-s, \frac{h}{k}) = \frac{2\Gamma(s)}{(2\pi k)^s} \sum_{j=1}^{k} \cos(\frac{\pi s}{2} - \frac{2\pi j h}{k}) \zeta(s, \frac{j}{k})$$

This formula holds for all s and all integers h, k with $1 \leq h \leq k$. Set $h = k$ and obtain

$$\zeta(1-s) = \zeta(1-s, 1) = \frac{2\Gamma(s)}{(2\pi k)^s} \cos(\frac{\pi s}{2}) \sum_{j=1}^{k} \zeta(s, \frac{j}{k})$$

Using the functional equation for the zeta function to write the left hand side in terms of $\zeta(s)$:

$$2(2\pi)^{-s}\Gamma(s)\cos(\frac{\pi s}{2})\zeta(s) = \frac{2\Gamma(s)}{(2\pi k)^s} \cos(\frac{\pi s}{2}) \sum_{j=1}^{k} \zeta(s, \frac{j}{k})$$

so the formula follows for all points except zeros of $\cos(\pi s/2)$ and poles of $\Gamma(s)$. But then it must hold at these points also since each side represents an analytic function, except for $s = 1$. ∎

COROLLARY 2.1. *If $\zeta(s_0) = 0$ then for all integers $k \geq 1$*

$$\sum_{1 \leq j \leq k, (j,k)=1} \zeta(s_0, \frac{j}{k}) = 0.$$

Proof. Let $\zeta(s_0) = 0$. If $k = 1$ then $\zeta(s_0, 1/1) = \zeta(s_0) = 0$ so assume it is true for all $m < k$. By the Lemma

$$\sum_{j=1}^{k} \zeta(s_0, \frac{j}{k}) = 0.$$

Divide the sum on the left up into groups of terms corresponding to indices (j, k) having the same gcd. By the inductive hypothesis, each of the groups with a common gcd greater than 1 will sum to zero. Omitting these terms we obtain the result of the corollary. ∎

Observation: It follows easily from the corollary that the sums of the values of the Hurwitz zeta function over the Farey fractions of a given order, other than zero, at a zero of zeta function, are all zero.

Lemma 2.2. *If* $\Re(s) < 1$ *then* $\lim_{n\to\infty} \sum_{j=1}^{n} \zeta(s, \frac{j}{n}) \frac{1}{n} = 0$.

Proof. By Lemma 2.1

$$n^{s-1}\zeta(s) = \sum_{j=1}^{n} \zeta(s, \frac{j}{n}) \frac{1}{n}.$$

Hence

$$n^{\sigma-1}|\zeta(s)| = |\sum_{j=1}^{n} \zeta(s, \frac{j}{n}) \frac{1}{n}|.$$

So if $\sigma < 1$, $\lim_{n\to\infty} n^{\sigma-1}|\zeta(s)| = 0$, and the lemma follows directly. ∎

Lemma 2.3. *Let* $f : (0, 1] \to \mathbb{R}$ *be a bounded* C^∞ *function. Extend* f *to a Riemann integrable function on* $[0, 1]$ *with* $f(0) = 0$. *If*

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(\frac{j}{n}) \frac{1}{n} = 0$$

then $\int_0^1 f = 0$, *because, in this case, the integral is the limit of the given Riemann sums.*

Lemma 2.4. *If* $\sigma = \Re(s) < 0$ *there exists a positive real number* $B = B(s)$ *such that for all* $a \in (0, 1]$, $|\zeta(s, a)| \le B(s)$.

Proof. Consider Hurwitz' formula for the zeta function in terms of the periodic zeta function [1], namely:

$$\zeta(1-s,a)=\frac{\Gamma(s)}{(2\pi)^s}\{e^{-\pi is/2}F(a,s)+e^{\pi is/2}F(-a,s)\}$$

where $0<a\le 1$, $1<\sigma$ and where

$$F(a,s)=\sum_{n=1}^{\infty}\frac{e^{2\pi ina}}{n^s}.$$

then

$$\zeta(s,a)=\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\{e^{-\pi i(1-s)/2}F(a,1-s)+e^{\pi i(1-s)/2}F(-a,1-s)\}$$

for $\sigma<0$. Hence

$$\begin{aligned}|\zeta(s,a)| &\le \frac{|\Gamma(1-s)|}{(2\pi)^{1-\sigma}}\{e^{-\pi t/2}|F(a,1-s)|+e^{\pi t/2}|F(-a,1-s)|\}\\ &\le \frac{|\Gamma(1-s)|}{(2\pi)^{1-\sigma}}\{e^{-\pi t/2}\sum_{n=1}^{\infty}\frac{1}{n^{1-\sigma}}+e^{\pi t/2}\sum_{n=1}^{\infty}\frac{1}{n^{1-\sigma}}\}\\ &= \frac{|\Gamma(1-s)|}{(2\pi)^{1-\sigma}}2\cosh(\frac{\pi t}{2})\zeta(1-\sigma)=B(s)\end{aligned}$$

▮

LEMMA 2.5. *If $0<\sigma<1$, there exists a positive real number $B=B(s)$ such that for all $a\in(0,1]$,*

$$|\zeta(s,a)|\le\frac{1}{a^\sigma}+B(s).$$

Proof. Consider the following expression for the zeta function [1], valid for $0<\sigma<1$ and all integers $N\ge 1$, namely

$$\zeta(s,a)=\sum_{n=0}^{N}\frac{1}{(n+a)^s}+\frac{(N+a)^{1-s}}{s-1}-s\int_N^\infty\frac{x-[x]}{(x+a)^{s+1}}dx.$$

Then

$$|\zeta(s,a)|\le\sum_{n=0}^{N}\frac{1}{(n+a)^\sigma}+\frac{(N+a)^{1-\sigma}}{|s-1|}+|s|\int_N^\infty\frac{1}{(x+a)^{1+\sigma}}dx.$$

Let $N = 1$ to derive the upper bound

$$\begin{aligned} |\zeta(s,a)| &\leq \frac{1}{a^\sigma} + \frac{1}{(1+a)^\sigma} + \frac{(1+a)^{1-\sigma}}{|s-1|} + \frac{|s|}{\sigma} \\ &= \frac{1}{a^\sigma} + B(s) \end{aligned}$$

where we may take

$$B(s) = 1 + \frac{2}{|s-1|} + \frac{|s|}{\sigma}.$$

▮

LEMMA 2.6. *Let $f : (0,1] \to \mathbb{R}$ be a C^∞ function. Let a positive real number M be such that, for some $\sigma \in (0,1)$*

$$|f(x)| \leq \frac{M}{x^\sigma}$$

for all x. Then f is Riemann integrable (proper if f is bounded). If $\lim_{n\to\infty} \sum_{j=1}^n f(\frac{j}{n})\frac{1}{n} = 0$, then $\int_{0+}^1 f = 0$.

Proof. Let σ_1 be such that $\sigma < \sigma_1 < 1$. Then

$$\frac{|f(x)|}{1/x^{\sigma_1}} \leq x^{\sigma_1 - \sigma} M$$

so

$$\lim_{x\to 0+} \frac{|f(x)|}{1/x^{\sigma_1}} = 0.$$

It follows that f is integrable on $[0,1]$.

Let $\int_{0+}^1 f = \alpha$ and suppose α is not zero. By replacing f with $-f$ if necessary we can assume $\alpha > 0$.

Since f is integrable there is an N_1 in $\mathbb{N}$ such that, for all $n \geq N_1$,

$$\int_{1/n}^1 f > \frac{\alpha}{2}$$

There exists an N_2 such that for all $l \geq N_2$

$$|\sum_{j=l}^{nl} f(\frac{j}{nl})\frac{1}{nl} - \int_{1/n}^1 f| < \frac{\alpha}{4}$$

so

$$-\frac{\alpha}{4} < \sum_{j=l}^{nl} f(\frac{j}{nl})\frac{1}{nl} - \int_{1/n}^{1} f$$

Therefore

$$\frac{\alpha}{2} < \int_{1/n}^{1} f < \frac{\alpha}{4} + \sum_{j=l}^{nl} f(\frac{j}{nl})\frac{1}{nl}$$

so

$$\frac{\alpha}{4} < \sum_{j=l}^{nl} f(\frac{j}{nl})\frac{1}{nl}.$$

By the given hypothesis

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(\frac{j}{n})\frac{1}{n} = 0$$

so there is an N_3 such that for all $l \geq N_3$

$$-\frac{\alpha}{8} < \sum_{j=1}^{ln} f(\frac{j}{ln})\frac{1}{ln} < \frac{\alpha}{8}$$

Therefore

$$-\frac{\alpha}{8} < \sum_{j=1}^{l-1} f(\frac{j}{ln})\frac{1}{ln} + \sum_{j=l}^{ln} f(\frac{j}{ln})\frac{1}{ln} < \frac{\alpha}{8}$$

and so

$$\frac{\alpha}{4} < \frac{\alpha}{8} - \sum_{j=1}^{l-1} f(\frac{j}{ln})\frac{1}{ln}$$

which implies

$$\begin{aligned}
\frac{\alpha}{8} &< \sum_{j=1}^{l-1} |f(\frac{j}{ln})|\frac{1}{ln} \\
&< M\sum_{j=1}^{l} (\frac{ln}{j})^{\sigma}\frac{1}{ln} \\
&= M\frac{l^{\sigma}n^{\sigma}}{ln}\sum_{j=1}^{l}(\frac{1}{j^{\sigma}}) \\
&< 2M\frac{l^{\sigma}n^{\sigma}l^{1-\sigma}}{ln}
\end{aligned}$$

which can be made arbitarily small for n sufficiently large. This contradiction shows we must have $\alpha = 0$, so completes the proof of the Lemma. ∎

Lemma 2.7. *If $\sigma = 0$ and $|t| \geq 1$ then*

$$|\zeta(it, a)| \leq B(t)$$

for some bound $B(t)$.

Proof. This follows directly from the inequality [1] valid for $-\delta \leq \sigma \leq \delta$ for $\delta < 1$ and $|t| \geq 1$

$$|\zeta(s, a) - a^{-s}| \leq A(\delta)|t|^{1+\delta}.$$

∎

Lemma 2.8. *If $\sigma = 0$ and $0 \leq t \leq 1$ then*

$$|\zeta(it, a)| \leq B(t).$$

Proof. If $t = 0$, $\zeta(0, a) = 1/2 - a$ so we may assume t is not zero.

To establish a bound we use two expressions for the Hurwitz zeta function derived with Euler summation and integration by parts [1]: For $\sigma > -1$ and $N \geq 0$

$$\begin{aligned}\zeta(s, a) &= \sum_{n=0}^{N} \frac{1}{(n+a)^s} + \frac{(N+a)^{1-s}}{s-1} \\ &- \frac{s}{2!}\{\zeta(s+1, a) - \sum_{n=0}^{N} \frac{1}{(n+a)^{s+1}}\} \\ &- \frac{s(s+1)}{2!} \sum_{n=N}^{\infty} \int_0^1 \frac{u^2}{(n+a+u)^{s+2}} du\end{aligned}$$

and if $\sigma > 0$

$$\begin{aligned}\zeta(s, a) &= \sum_{n=0}^{N} \frac{1}{(n+a)^s} + \frac{(N+a)^{1-s}}{s-1} \\ &- \int_N^{\infty} \frac{x - [x]}{(x+a)^{s+1}} dx.\end{aligned}$$

Substitute $\sigma = 0$ and $N = 0$ in the first formula to obtain the equation

$$\begin{aligned}\zeta(it,a) &= \frac{1}{a^{it}} + \frac{a^{1-it}}{it-1} \\ &- \frac{it}{2!}\{\zeta(it+1,a) - \frac{1}{a^{1+it}}\} \\ &- \frac{it(it+1)}{2!}\sum_{n=1}^{\infty}\int_0^1 \frac{u^2}{(n+a+u)^{it+2}}du\end{aligned}$$

so

$$\begin{aligned}|\zeta(it,a)| &\leq 1 + \frac{1}{|it-1|} + \frac{|t|}{2!}|\zeta(it+1,a) - \frac{1}{a^{1+it}}| \\ &+ \frac{|t|(|t|+1)}{2!}\sum_{n=1}^{\infty}\int_0^1 \frac{u^2}{(n+u)^2}du \\ &\leq 1 + \frac{1}{|it-1|} + \frac{|t|(|t|+1)}{2!}(\zeta(2)+1) + \frac{|t|}{2!}|C(t,a)|\end{aligned}$$

where

$$C(t,a) = \zeta(it+1,a) - \frac{1}{a^{1+it}}.$$

In the second formula let $N = 1$ and $s = 1 + it$ so $\sigma = 1 > 0$ giving

$$C(t,a) = \frac{1}{(1+a)^{1+it}} + \frac{(1+a)^{1-(1+it)}}{1-(1+it)} - (1+it)\int_1^{\infty} \frac{x-[x]}{(x+a)^{2+it}}dx$$

so

$$|C(t,a)| \leq 1 + \frac{1}{|t|} + \sqrt{1+t^2}.$$

∎

THEOREM 2.1. *For all $s \in \mathbb{C}$ with $\Re(s) < 1$ the (improper) Riemann integral of $\zeta(s,a)$ with respect to $a \in (0,1]$ exists and*

$$\int_{0^+}^{1} \zeta(s,a)da = 0.$$

Proof. The work has now been done. Simply apply the lemmas, valid in different subsets of $\sigma < 1$, to the real and imaginary parts of the integral of $\zeta(s,a)$:

If $\sigma < 0$ use Lemmas 2.2 and 2.4.

If $0 < \sigma < 1$ use 2.2, 2.5 and 2.6.
If $\sigma = 0$ and $|t| \geq 1$ use 2.2 and 2.7.
If $\sigma = 0$ and $0 \leq t \leq 1$ use 2.2 and 2.8. ▌

THEOREM 2.2. *For all $s \in \mathbb{C}$ with $\Re(s) \geq 1$ the (improper) Riemann integral of $\zeta(s,a)$ with respect to $a \in (0,1]$ does not exist.*

Proof. For every a, $\zeta(s,a)$ has a pole at $s = 1$, so the integral makes no sense at that value of s. The rest of the proof is straight forward, based on the non existence of the improper integral of a^{-s} on $(0,1]$ for $\sigma = \Re s \geq 1$ and $t = \Im s \neq 0$ decomposing this domain into subsets corresponding to $\sigma > 1$, $\sigma = 1$ and $|t| \geq 1$ and $\sigma = 1$ and $0 < t < 1$. ▌

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