Holomorphic flows on simply connected regions have no limit cycles

Version: 16th May 2002

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The dynamical system or flow $\dot{z} = f(z)$, where f is holomorphic on $\mathbb{C}$, is considered. The behaviour of the flow at critical points coincides with the behaviour of the linearization when the critical points are non-degenerate: there is no center-focus dichotomy. Periodic orbits about a center have the same period and form an open subset. The flow has no limit cycles in simply connected regions. The advance mapping is holomorphic where the flow is complete. The structure of the separatrices bounding the orbits surrounding a center is determined. Some examples are given including the following: if a quartic polynomial system has 4 distinct centers, then they are collinear.

Key Words: dynamical system, phase portrait, critical point, theoretical dynamics

MSC2000 30A99,30C10,30C15,30D30,32M25,37F10, 37F75

1. INTRODUCTION

There have been a number of studies of dynamical systems $\dot{z} = f(z)$ or $\dot{z} = f(z,t)$where f is holomorphic in z and $z \in \mathbb{C}^n$ or some subset. These are called holomorphic or conformal flows [1, 6, 7]. Some closely related work has been done on Newton flows, that is dynamical systems of the form

$$\dot{z} = -\frac{f(z)}{f'(z)}.$$

See [2, 3, 4, 5]. For example this type of flow is used by Benzinger in [2] to prove that holomorphic flows with rational function right hand sides do not have limit cycles.

In this article the primary interest is to explore holomorphic flows on $\mathbb{C}$ (i.e. $n = 1$) to better understand complex, even entire, functions. Applications to functions such as the Riemann Ξ function will be developed in a later article.

First it is shown that the flow characterizes the function. Section 2 is a compilation of local properties. These are straight forward and can be found in the literature, for example [8]. They are included here for ease of reference. Section 3 is the main part of this paper. There it is proved that, under suitable restrictions, the advance mapping is holomorphic. This is then used to prove that holomorphic flows have no limit cycle. In addition a description of the global neighbourhood of a center for every entire flow it derived.

Section 4 contains some examples, again indicating the restricted behaviour of holomorphic flows. We are especially interested in flows for entire functions which have only centers in the finite plane. It is shown that if the flow is polynomial of degree less than 5, with centers only, then the centers must be on a line.

THEOREM 1.1. *If the complex functions f and g are holomorphic on an open connected subset $\Omega \subset \mathbb{C}$ and not identically zero, and $\dot{z} = f(z)$ and $\dot{z} = g(z)$, have the same critical points and the same integral paths, then there exists an $\alpha \in \mathbb{R}$ with $\alpha \neq 0$ such that, for all $z \in \Omega$, $f(z) = \alpha g(z)$.*

Proof. Let Z be the (isolated) set of critical points. If z is not a critical point, then $f(z) = \alpha(z) \cdot g(z)$ where $\alpha(z) \in \mathbb{R}$ is non-zero. But then $\alpha(z) = f(z)/g(z)$, so $\alpha(z)$ is real and holomorphic and hence constant on $\Omega \setminus Z$, therefore on Ω. $\blacksquare$

It follows that the integral paths of $\dot{z} = f(z)$, for holomorphic f, determine f up to multiplication by a real constant.

EXAMPLE 1.1. Let

$$f(z) = (1 + \frac{z}{3i})(1 - \frac{z}{3i})^3.$$

The phase portrait of f it plotted below. It is a quartic polynomial with one simple zero, which is a centre, and one zero of order 3.

2. LOCAL PROPERTIES

THEOREM 2.1. *Let f be a meromorphic function on $\Omega \subset \mathbb{C}$ and let $a \in \Omega$ be such that $f(a) = 0$ and $f'(a) \neq 0$. Let $\dot{z} = f(z)$ be the corresponding flow. Then*

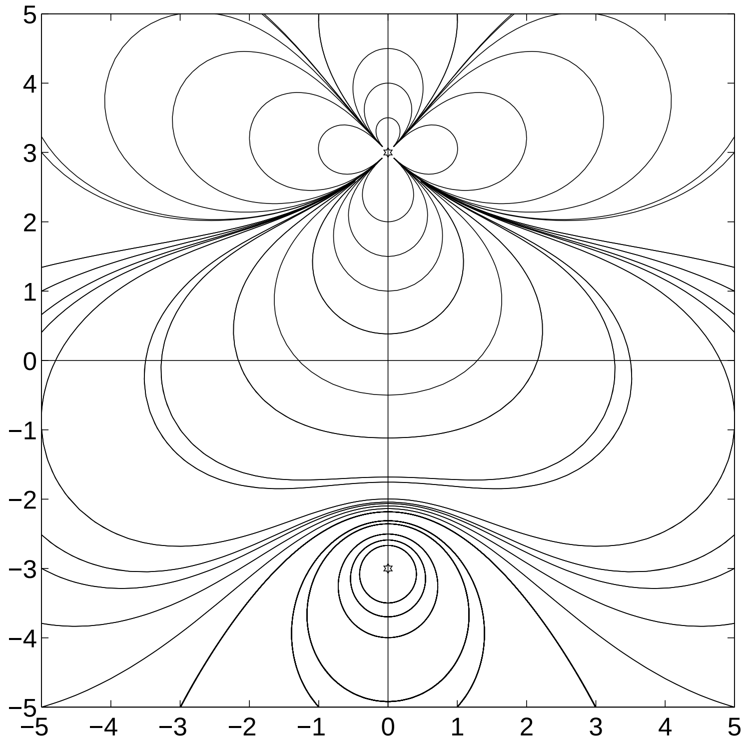

FIG. 1. Holomorphic flow for $f(z)$ a polynomial of degree 4.

(a) the critical point $z = a$ is non-degenerate,
(b) $\lambda_\pm = u_x \pm iv_x = \{f'(a), \overline{f'(a)}\}$,
(c) $\lambda_\pm$ is real if and only if $\lambda_1 = \lambda_2 = \Re f'(a)$,
(d) at the critical point the linearization of f has either a center, focus or node.

Proof.
(a) If $f(z) = u + iv$ then the characteristic polynomial

$$p(\lambda) = \lambda^2 - 2u_x\lambda + |f'(a)|^2,$$

so therefore $\lambda_\pm = u_x \pm iv_x$ and the critical point is non-degenerate since $f'(a) \neq 0$.
(b) Follows since if $\lambda_\pm$ is real then $v_x = 0$ and so $\lambda_+ = \lambda_- = \Re f'(a)$.
(c) Since $\lambda_+ = \lambda_-$ when the eigenvalues are real, saddles do not exist. ▮

THEOREM 2.2. *If $\dot{z} = f(z)$ has a simple pole at $z = a$ then, for some $\epsilon > 0$, the flow has the same orbits as a saddle on $B(a, \epsilon) \setminus \{a\}$*

Proof. The flow has integral curves which coincide with those of

$$\dot{z} = \frac{f(z)}{|f(z)|^2}$$

provided $f(z) \neq 0$. Assume, without loss of generality, the simple pole is at $z = 0$. Near $z = 0$,

$$f(z) = \frac{c_{-1}}{z} + c_0 + c_1 z + \cdots$$

where $c_{-1} \neq 0$. Therefore

$$\frac{f(z)}{|f(z)|^2} = \frac{\bar{z}}{c_{-1}} \frac{1}{\overline{1 + \frac{c_0 z}{c_{-1}} + \cdots}}$$

so the linearization near $z = 0$ is of the form

$$\dot{x} + i\dot{y} = (a + ib)(x - iy)$$

where a and b are real with $a^2 + b^2 \neq 0$. It follows that the characteristic polynomial is $\lambda^2 - (a^2 + b^2)$, so the singular point behaves as a saddle. ∎

THEOREM 2.3. *Let $\dot{z} = f(z)$ be a holomorphic flow on Ω with a center at z_o. Then all closed orbits in Ω with interior in Ω and z_o in the interior, have the same period, namely $2\pi i/f'(z_o)$.*

Proof. Let Γ be a closed orbit. Then, because there are no saddle points, Γ has one zero of f in its interior, say z_o, and that zero is simple. If T is the period:

$$T = \int_\Gamma \frac{dz}{f(z)} = \frac{2\pi i}{f'(z_o)}.$$

∎

THEOREM 2.4. *Let $\dot{z} = f(z)$ have a critical point at $z = z_o$. Then the order m of the zero at z_o is the index of the critical point at z_o.*

Proof. Let Γ be a simple closed curve with interior containing z_o and no other zero of f. Then if I is the index of z_o:

$$I = \frac{\Delta \arg f(z)}{2\pi} = \frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)} dz = m.$$

$\blacksquare$

Theorem 2.5. *If $\dot{z} = f(z)$ is a holomorphic flow and z_o a zero of f of order $m \geq 2$, $f(z) = (z - z_o)^m g(z)$, where $g(z_o) \neq 0$, then there are $2(m-1)$ sectors for the flow at z_o and all sectors are elliptic.*

Proof. Assume, without loss of generality, that $z_o = 0$ and that, in a neighbourhood of 0, f has the representation $f(z) = az^m(1 + z\phi(z))$ where ϕ is holomorphic on the neighbourhood.

Let $a = Re^{i\beta}$ and $z = re^{i\theta}$. Then since the index $m \geq 2$, the critical point 0 is not a focus or a center. Hence, ([9], Theorem 1.10.2), there are explicit directions θ at which the flow approaches or leaves 0. Each of these directions satisfies

$$\tan(\theta) = \frac{\dot{y}}{\dot{x}} = \frac{\sin(\beta + m\theta)}{\cos(\beta + m\theta)}.$$

Therefore

$$\theta = \frac{n\pi}{m-1} + \frac{\beta}{m-1}, n \in \mathbb{Z}$$

leading to $2(m-1)$ distinct directions.

But the index $I = 1 + \frac{e-h}{2}$, where e is the number of elliptic sectors and h the number of hyperbolic sectors, so $e - h = 2(m-1)$. Therefore $e = 2(m-1)$, since that is the total number of sectors, and therefore $h = 0$. $\blacksquare$

Theorem 2.6. *If $\dot{z} = f(z)$ is a holomorphic flow and z_o a pole of f of order $m \geq 2$, $f(z) = (z - z_o)^{-m} g(z)$, where $g(z_o) \neq 0$, then there are $2(m+1)$ sectors for the flow at z_o and all sectors are hyperbolic.*

Proof. The proof is similar to that of the previous theorem, replacing m by $-m$. $\blacksquare$

Theorem 2.7. *Let $\dot{z} = f(z) = u + iv = (\alpha + i\beta)z^m$ where $m \geq 2$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 \neq 0$ are such that $u(x,y) = u(-x,y)$ and $v(x,y) = -v(-x,y)$. Then if m is even $\beta = 0$ and if m is odd $\alpha = 0$.*

Proof. If m is even, $(x+iy)^m = A(x,y) + iB(x,y)$ where $A(x,y) = A(-x,y)$ and $B(x,y) = -B(-x,y)$. Hence

$$u + iv = f(z) = (\alpha + i\beta)(A + iB)$$

implies $u(x,y) = \alpha A(x,y) - \beta B(x,y)$ and so $\alpha A(x,y) - \beta B(x,y) = \alpha A(-x,y) - \beta B(-x,y)$ and therefore $2\beta B(x,y) = 0$, so $\beta = 0$.

The proof for m odd is similar. ▮

THEOREM 2.8. *Let f be homomorphic on $\mathbb{C}$, $\dot{z} = f(z)$, and $\overline{f(z)} = f(\overline{z}) = f(-\overline{z})$. Then the integral paths are symmetric with respect to reflection in the x and y axes. If any point on an axis is a center for the linearized system, then it is a center for the original system $\dot{z} = f(z)$.*

Proof. If $f(z) = u + iv$ then for all x, y (1) $u(-x,y) = u(x,y)$, (2) $v(-x,y) = -v(x,y)$, (3) $u(x,-y) = u(x,y)$, and (4) $v(x,-y) = -v(x,y)$.

The result then follows directly from ([9], Theorem 2.10.6). ▮

A key issue for any dynamical system is whether a distinction can be made between the behaviour of the linearization of a flow near a critical flow and that of the original flow. Recall the definitions: A point is called a stable node if each trajectory starting sufficiently close to the point approaches the point along a well defined tangent. It is called a stable focus if each trajectory starting sufficiently close spirals towards the point. For holomorphic flows we have the following local-global principle at each simple zero:

THEOREM 2.9. *Let $\dot{z} = f(z) = (z - z_o)g(z)$ where f is holomorphic on a neighbourhood of z_o and $g(z_o) = a = \alpha + i\beta \neq 0$ (so z_o is a simple zero of f). Then the flow has at z_o:*
(a) a focus if $\alpha \neq 0$ and $\beta \neq 0$,
(b) a node if $\beta = 0$,
(c) a center if $\alpha = 0$,
that is to say the critical point z_o has the same type as its linearization $f(z) = (z - z_o)g(z_o)$.

Proof. In cases (a) and (b), $\alpha \neq 0$, so the critical point is hyperbolic. Since the flow is holomorpic it is twice continously differentiable on a neigbourhood of the point so we can apply ([9], Theorem 2.10.4). Case (c) is the theorem of Benzinger [2]. ▮

THEOREM 2.10. *If $f(z) \neq 0$, the curvature of the orbit passing through z is given by*

$$\kappa = \frac{|v_x|}{|f(z)|}.$$

Proof. Let $\dot{z} = f(z) = u + iv$. If $\mathbf{r} = (x(t), y(t))$ is an integral path, the curvature

$$\kappa = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3}$$

where $\dot{x} = u$, $\dot{y} = v$, $\ddot{x} = uu_x + vu_y$, and $\ddot{y} = uv_x + vv_y$. The given expression for κ follows after substituting and simplifying. ∎

Corollary 2.1. *$\kappa = 0$ at $z = a$ if and only if $\Im f'(a) = 0$.*

Corollary 2.2. *If $f(a) = 0$ and $\Im f'(a) \neq 0$ then κ is unbounded in a neighbourhood of a. If $f(a) \neq 0$ then κ is bounded in a neighbourhood of a. Hence the curvature is unbounded in the neighbourhood of a center or focus. At a node it is zero. This follows directly from the previous corollary.*

Note: If f is holomorphic, every periodic orbit has at most a finite number of points of zero curvature. More generally, if $\gamma(z_o, t)$ describes an orbit then the map $t \to \kappa(\gamma(z_o, t))$ is analytic, so has at most a finite number of zeros on any bounded interval in its domain.

3. NO LIMIT CYCLES FOR HOLOMORPHIC FLOWS

The following theorem may be well known. It is used in the proof of Theorem 3.2 below

Theorem 3.1. *Let $\dot{z} = f(z)$ where f is holomorphic on the open region $\Omega \subset \mathbb{C}$ on which the vector field is complete. Then the solution $\gamma(z, t)$ satisfying $\gamma(z, 0) = z$ for all $z \in \Omega$ and*

$$\frac{d\gamma(z,t)}{dt} = f(\gamma(z,t)$$

for all $t \in \mathbb{R}$ is holomorphic on Ω in that, for fixed T the mapping $z \to \gamma(z, T)$ is holomorphic.

Proof. Fix $T \in \mathbb{R}$. Then

$$(1)\ \gamma(z,T) = z + \int_0^T f(\gamma(z,s))ds.$$

Let $f(z) = u(x, y) + iv(x, y)$ and $\gamma(z, t) = A(x, y, t) + iB(x, y, t)$. Then, A and B are real analytic in (x, y). Partially differentiate (1) with respect to x and y, use the Cauchy-Riemann equations, then equate the real and imaginary parts to obtain

$$\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y} = \int_0^T [u_x(\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y}) + u_y(\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x})]ds \tag{2}$$

$$\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x} = \int_0^T [-u_y(\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y}) + u_x(\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x})]ds \tag{3}$$

and so

$$\mathbf{C}(x,y,T) = \int_0^T \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix} \mathbf{C}(x,y,s)ds$$

where

$$\mathbf{C}(x,y,s) = \begin{pmatrix} \frac{\partial A}{\partial x} - \frac{\partial B}{\partial y} \\ \frac{\partial A}{\partial y} + \frac{\partial B}{\partial x} \end{pmatrix}$$

and where the elements of the 2×2 matrix appearing in the integral are evaluated at $(A(x,y,s), B(x,y,s))$.

Differentiate the integral equation with respect to T to obtain the system of two ODE's

$$\dot{\mathbf{C}}(x,y,t) = \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix} \mathbf{C}(x,y,t).$$

Since $\gamma(z,0) = z = A(x,y,0) + iB(x,y,0)$ it follows immediately that $\mathbf{C}(x,y,0) = 0$. By existence and uniqueness of solutions of the system, the map $z \to \gamma(z,T)$ satisfies the Cauchy-Riemann equations for each T and all z. Hence, for each T, it represents a holomorphic function. ▌

LEMMA 3.1. *Let Γ be a periodic solution to $\dot{z} = f(z)$, where f is holomorphic on the simply connected open subset Ω of $\mathbb{C}$. Then there is an open set G containing all points in the interior of Γ and on the graph of Γ on which the flow is complete.*

Proof. At each point z_o in the interior of Γ the orbit starting at that point is bounded, hence ([9],Corollary 2.4.2) $\gamma(z_o,t)$ exists for all $t \in \mathbb{R}$. The same is true for each point on Γ and therefore ([9], Theorem 2.4.4) on a neighbourhood of each point. Finally the union of these neighbourhoods and the interior of Γ is an open subset of $\mathbb{C}$ on which the flow is complete. ▌

THEOREM 3.2. *Let $\Omega \subset \mathbb{C}$ be a simply connected region and let f be holomorphic on Ω. Then the flow $\dot{z} = f(z)$ has no limit cycles in Ω.*

Proof. Assume there is at least one limit cycle. By the theorem of Poincaré, the flow has at most a finite number of limit cycles in any bounded subregion of Ω. Therefore there exists a limit cycle Γ with no other limit cycles in its interior. Inside this cycle there must be a single simple zero z_o, since there are no saddles.

(a) The point z_o cannot be a center, since in that case B would be filled entirely with periodic orbits, each of which has the same period by Theorem 2.3 below, leading to a return map for Γ which is the identity, contradicting Γ being a limit cycle. If it were not filled with periodic orbits than, by the Poincare-Bendixson Theorem, there would be a limit cycle interior to Γ, which is impossible.

(b) Let z_o be a focus or node (stable or unstable): Let $z \to \gamma(z,t)$ be the mapping describing the flow. Then, by Lemma 3.1, the mapping is holomorphic on Ω, the interior region of Γ, for each fixed $t \in \mathbb{R}$.

Let $0 < \delta < T$ where T is the period of the flow on Γ and let $g(z) = \gamma(z,-\delta)$ if z_o is stable and let $g(z) = \gamma(z,\delta)$ if z_o is unstable.

Then g is holomorphic on Ω. Let $U = \{z : |z| < 1\}$ be the open unit disk. Then by the Riemann Mapping Theorem, there is a conformal map $\theta : \Omega \to U$, which is injective and surjective.

Let $h(z) = \theta \circ g \circ \theta^{-1}(z)$. Then $h : U \to U$, $h(0) = 0$ and h is holomorphic on U. It follows from the Schwarz Lemma that h is either a rotation or satisfies $|h(z)| < |z|$ for all $z \in U$.

If h is a rotation, $h(z) = e^{i\alpha}z$ for some α with $0 \le \alpha \le 2\pi$. Let $z_1 = \frac{1}{2}$. Then there exists a subsequence of $\mathbb{N}$ and point $z_2 \in U$ such that $h^{n_j}(z_1) \to z_2$. If $z_1 = \theta(z_3)$ with $z_3 \in \Omega \setminus \{z_o\}$, then

$$g^{n_{j_k}}(z_3) \to \theta^{-1}(z_2) \in \Omega \setminus \{z_o\}.$$

But $g^{n_{j_k}}(z_3)$ converges to a point on Γ. This contradiction shows that z_o is not a focus if h is a rotation.

If $|h(z)| < |z|$ for all $z \in U$, $h^n(z_1) \to 0 \in U$. Hence $g^n(z_3) \to z_o$. But $g^{n_j}(z_3)$ converges to point on Γ. This contradiction completes the proof that z_o cannot be a focus or a node, so therefore the limit cycle Γ does not exist. ∎

THEOREM 3.3. *Let $\dot{z} = f(z)$ be an entire flow with center at x_o. Let P be the set consisting of x_o together with the union of all of the closed orbits of the flow which contain x_o in their interior. Then P is an open subset of $\mathbb{C}$ and ∂P consists of the at most countable union of a set of separatrices $\{\gamma(x_\lambda,t) : \lambda \in \Lambda, t \in D_\lambda\}$, D_λ being the maximum interval of existence of the flow through x_λ, where each $\gamma(x_\lambda,t)$ has an unbounded graph.*

Proof. 1. Let x_o be a center. Let

$$P = \{y \mid y \text{ is on a periodic orbit about } x_o\} \cup \{x_o\}.$$

Then, since $\dot{z} = f(z)$ has no limit cycle (Theorem 3.2), P is connected. If $y \in P$ and $y \neq x_o$, then by the continuous dependence of solutions on initial conditions, any trajectory starting sufficiently close to y will circle x_o, so must be a periodic orbit, since there are no limit cycles. Hence P is open.

2. $\partial P = B$ is closed in $\mathbb{C}$. Then if $B = \emptyset$ the proof is complete. Otherwise proceed as follows:

3. Let $D_\lambda = (\alpha, \beta)$ be the maximal interval of existence of $\gamma(x_\lambda, t)$. Consider $t \to \beta-$. The argument for $t \to \alpha+$ is similar. If the image of $[0, \beta)$ is bounded in $\mathbb{C}$ then necessarily $\beta = \infty$ and, since the flow has no limit cycle, $\omega(\gamma) = x_1$ which would be a critical point whith a hyperbolic sector, impossible for a holomorphic flow. Therefore

$$B_\lambda = \{\gamma(x_\lambda, t) \mid t \in D_\lambda\}$$

is unbounded.

4. Let

$$C_\lambda = \{t \in \mathbb{R} \mid \text{ for all } s \text{ with } 0 \leq s \leq t \text{ or } t \leq s \leq 0,\ \gamma(x_\lambda, s) \in B\}.$$

Then $C_\lambda = (r, s)$ where $\alpha \leq r \leq s \leq \beta$. Claim: $s = \beta$. (The proof that $r = \alpha$ is similar.) If $s < \beta$, continuous dependence in initial conditions applied to a neigbourhood of the point $x_1 = \gamma(x_\lambda, s)$ implies both that $x_1 \in B$ and that there are points on the orbit through x_λ with t in $(s - \epsilon, s + \epsilon)$ also in B, contradicting the definition of s. It follows that $B_\lambda \subset B$.

5. By 4. we can write

$$B = \sqcup_{\lambda \in \Lambda} \{\gamma(x_\lambda, t) \mid t \in D_\lambda\} = \sqcup_{\lambda \in \Lambda} B_\lambda$$

where the index set Λ is non-empty and the union disjoint.

6. Each B_λ is closed: If not there is an $x \in \omega(B_\lambda)$ or $x \in \alpha(B_\lambda)$. Since the flow has no limit cycle, x must be a critical point, so must be center, focus, node or point with only elliptic sectors. Since B is closed, $x \in B$, so it must have at least one hyperbolic sector, which is a contradiction.

7. $|\Lambda| \leq \aleph_o$: By 6. each B_λ divides $\mathbb{C}$ into three non-empty subsets: $\mathbb{C} = Q_\lambda \cup B_\lambda \cup P_\lambda$ where P_λ and Q_λ are open and $P \subset P_\lambda$. Then for $\alpha \neq \beta$, $Q_\alpha \cap Q_\beta = \emptyset$ so $|\Lambda| \leq \aleph_o$, since $\mathbb{C}$ is separable. ∎

An example with $B = \emptyset$ is $\dot{z} = iz$. The flow $\dot{z} = iz(z^n - 1)$ with center $x_o = 0$ has $|\Lambda| = n$. Constructing an example with $|\Lambda| = \aleph_o$ is an unsolved problem.

The above theorem and the theorem of Benzinger [2], showing that rational function flows on $\mathbb{C}$ do not have limit cycles, lead to the following natural conjecture.

Conjecture: Let $f : \Omega \to \mathbb{C}$ be meromorphic, where Ω is an open. Then the flow $\dot{z} = f(z)$ does not have a limit cycle.

4. EXAMPLES

Example 4.1. If $\{z_j : 1 \le j \le n\}$ are distinct points on a line L in $\mathbb{C}$ then there is a θ such that

$$f(z) = e^{i\theta} \prod_{j=1}^{n} (z - z_j)$$

has a center for $\dot{z} = f(z)$ at each z_j: If L cuts the real axis at $x = \eta$ and at an angle β set $w = e^{i\beta}(z - \eta)$ so each $w_j = e^{i\beta}(z_j - \eta)$ is real and thus $\dot{w} = i \prod_{j=1}^{n}(w - w_j)$ has a center at each w_j. Changing variables

$$\dot{z} = ie^{in\beta} \prod_{j=1}^{n} (z - z_j)$$

gives $\theta = \frac{\pi}{2} + n\beta$. If L is parallel to OX and cuts OY at γ, set $w = z - i\gamma$ and derive $\theta = \frac{\pi}{2}$.

Theorem 4.1. *Let $\dot{z} = f(z) = \alpha(z - z_1) \cdots (z - z_n)$ be a flow where $\alpha \in \mathbb{C} \setminus \{0\}$, the z_i are distinct, and each is a center (for the linearized flow). If $n \le 4$ then the z_i are collinear.*

Proof. If $n = 1$ or 2 there is nothing to prove. If $n = 3$ linearize about each of the points z_i to obtain the equations:

$$\begin{aligned}
\alpha(z_1 - z_2)(z_1 - z_3) &= i\alpha_1 \\
\alpha(z_2 - z_1)(z_2 - z_3) &= i\alpha_2 \\
\alpha(z_3 - z_1)(z_3 - z_2) &= i\alpha_3
\end{aligned}$$

where the α_i are non-zero real numbers. Dividing the first two of these equations leads to

$$\frac{z_1 - z_3}{z_2 - z_3} = \beta \in \mathbb{R} \setminus \{0\}$$

so $z_1 = z_3 + \beta(z_2 - z_3)$ and therefore $\{z_1, z_2, z_3\}$ are collinear.

If $n = 4$ proceed as above and derive the equations:

$$\begin{aligned}
&(1) \quad \alpha(z_1 - z_2)(z_1 - z_3)(z_1 - z_4) = i\alpha_1 \\
&(2) \quad \alpha(z_2 - z_1)(z_2 - z_3)(z_2 - z_4) = i\alpha_2 \\
&(3) \quad \alpha(z_3 - z_1)(z_3 - z_2)(z_3 - z_4) = i\alpha_3 \\
&(4) \quad \alpha(z_4 - z_1)(z_4 - z_2)(z_4 - z_3) = i\alpha_4
\end{aligned}$$

Divide (1) by (2), (3) by (4) and (2) by (3) to obtain:

$$\begin{aligned}
&(5) \quad \frac{(z_1 - z_3)(z_1 - z_4)}{(z_2 - z_3)(z_2 - z_4)} = -\frac{\alpha_1}{\alpha_2} \\
&(6) \quad \frac{(z_3 - z_1)(z_3 - z_2)}{(z_4 - z_1)(z_4 - z_2)} = -\frac{\alpha_3}{\alpha_4} \\
&(7) \quad \frac{(z_2 - z_1)(z_2 - z_4)}{(z_3 - z_1)(z_3 - z_4)} = -\frac{\alpha_2}{\alpha_3}
\end{aligned}$$

Multiplying (5) and (6):

$$\left(\frac{z_1 - z_3}{z_4 - z_2}\right)^2 = \frac{\alpha_1\alpha_3}{\alpha_2\alpha_4} \in \mathbb{R} \setminus \{0\}$$

Also, by symmetry,

$$\left(\frac{z_1 - z_2}{z_3 - z_4}\right)^2 =\in \mathbb{R} \setminus \{0\} \text{ and } \left(\frac{z_1 - z_4}{z_3 - z_2}\right)^2 =\in \mathbb{R} \setminus \{0\}.$$

If (Case I) $(z_1 - z_3)/(z_4 - z_2) \in \mathbb{R}$ then $z_1 - z_3 \parallel z_4 - z_2$. By (5), (6) and (7), $z_1 - z_4 \parallel z_2 - z_3$ and $z_2 - z_1 \parallel z_3 - z_4$. The only configuration of distinct points for which this is possible is when $\{z_1, z_2, z_3, z_4\}$ are collinear.

If (Case II) $(z_1 - z_3)/(z_4 - z_2) = i\beta_1$ for some non-zero real number β_1, then by symmetry, we may assume also that $(z_1 - z_2)/(z_3 - z_4) = i\beta_2$ and $(z_1 - z_4)/(z_3 - z_2) = i\beta_3$ for non-zero β_i, else the result would follow as in Case I. But then $z_1 - z_3 \perp z_4 - z_2$, $z_1 - z_2 \perp z_3 - z_4$ and $z_1 - z_4 \perp z_3 - z_2$, an impossible configuration for four distinct points in $\mathbb{R}^2$. ∎

Note that the same type of proof would enable the conditions to be relaxed so that ($n - 1$ of) the $f'(z_j)$ are parallel and lead to conclusions such as if $f(z)$ is a cubic with two nodes or two centers then the zeros are collinear. The same conclusion applies to a quartic with three nodes or three centers. The next example shows that the theorem cannot be extended to quintics.

EXAMPLE 4.2. Define a polynomial flow of degree $n = 5$ by

$$\dot{z} = f(z) = iz(z-1)(z+1)(z-i)(z+i).$$

Then the flow has centers at $\{0, \pm 1, \pm i\}$ and, more generally, if $\{r_1, \cdots, r_N\}$ are distinct strictly positive real numbers, then

$$\dot{z} = f(z) = iz \prod_{j=1}^{N} (z^4 - r_j^4)$$

has centers at $\{\pm r_j, \pm i r_j : 1 \leq j \leq N\} \cup \{0\}$.

Again, for all $n \in \mathbb{N}$ let

$$\dot{z} = f(z) = iz(z^n - 1)$$

has centers at 0 and each of the n'th roots of unity. To see this note that at each such root of unity, $f'(z_o) = i((n+1)z_o^n - 1) = in$.

EXAMPLE 4.3. Phase portrait for $\dot{z} = f(z) = iz(z-1)(z+1)(z-i)(z+i)$.

THEOREM 4.2. *Let $f(z)$ be a polynomial of degree $n \geq 2$ with simple zeros $\{z_1, \cdots, z_n\}$. Then*

$$\sum_{j=1}^{n} \frac{1}{f'(z_j)} = 0.$$

Proof. Integrate $1/f(z)$ over a circle of radius R sufficiently large to contain all of the zeros of f, and let $R \to \infty$. ▮

COROLLARY 4.1. *With respect to the flow $\dot{z} = f(z)$:*
(a) If $z_1, \cdots, z_{n-1}$ are nodes, so is z_n.
(b) If $z_1, \cdots, z_{n-1}$ are centers, so is z_n.
(c) If z_1 is a focus, then the remaining zeros cannot be all nodes or all centers.
(d) If there exists only centers and nodes then there is more than one center and more than one node.
(e) No cubic system has only centers and nodes.
(f) Each cubic system has at least one focus.

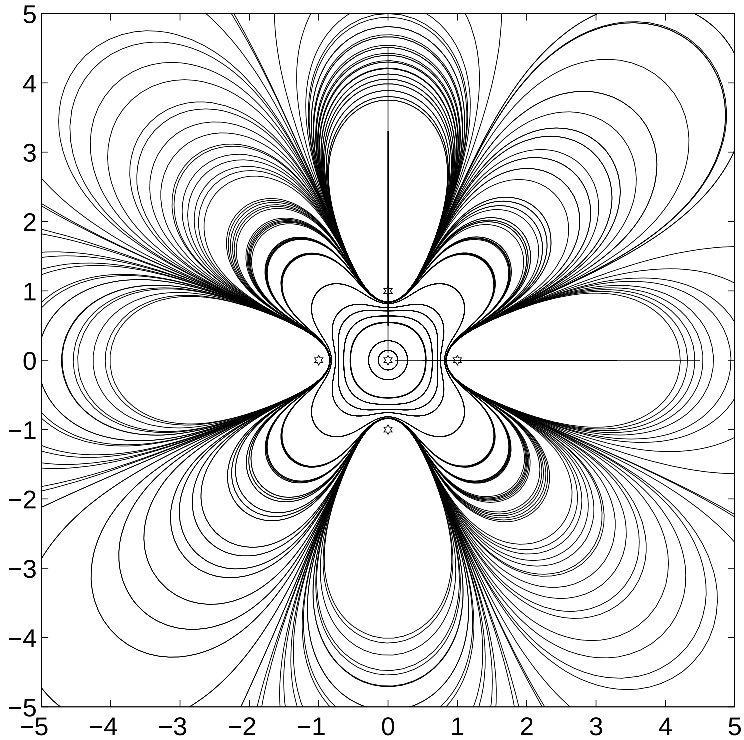

FIG. 2. Flow with five centers.

Conjecture: Let $\dot{z} = f(z)$ be a polynomial system with five simple zeros, each of which is a node. Assume that the system is normalized so that $f(0) = f(1) = 0$ and $f'(0) = -1$. Then

$$f(z) = z(z-1)(z+1)(z-i)(z+i) = z(z^4 - 1).$$

ACKNOWLEDGMENT

This work was done in part while the author was on study leave at Columbia University. The support of the Department of Mathematics at Columbia University and the valuable disussions held with Patrick Gallagher are warmly acknowledged. The figures were plotted first using a Java programme written by the author and Frances Kuo and then using a Matlab programme written by Ross Barnett. Their assistance is gratefully acknowledged. Finally the contributions of an anonymous referee are also gratefully acknowledged.

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