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DETERMINATION OF THE RANK OF AN INTEGRATION LATTICE *

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Abstract.

The continuing and widespread use of lattice rules for high-dimensional numerical quadrature is driving the development of a rich and detailed theory. Part of this theory is devoted to computer searches for rules, appropriate to particular situations. In some applications, one is interested in obtaining the (lattice) rank of a lattice rule $Q(\Lambda)$ directly from the elements of a generator matrix B (possibly in upper triangular lattice form) of the corresponding dual lattice $\Lambda^{\perp}$. We treat this problem in detail, demonstrating the connections between this (lattice) rank and the conventional matrix rank deficiency of modulo p versions of B.

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1 Introduction

We start with background material on lattice rules. An s-dimensional *lattice* is an infinite set of points in $\mathbb{R}^s$ that is closed under addition and subtraction and has no limit points. The *unit lattice* Λ_0 comprises all points $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_s)$, all of whose components λ_i are integers. An *integration lattice* Λ is a lattice that contains Λ_0 as a sublattice, that is, $\Lambda \supseteq \Lambda_0$. An s-dimensional *lattice rule* $Q(\Lambda)$ is a quadrature rule for $[0,1)^s$ that assigns equal weight to each point of the specified integration lattice Λ that lies in $[0,1)^s$, that is,

$$Q(\Lambda)f = \frac{1}{N(\Lambda)} \sum_{\mathbf{p}\in[0,1)^s\cap\Lambda} f(\mathbf{p}),$$

where $N(\Lambda)$ is the number of abscissas.

The *reciprocal* (or *dual*) *lattice* $\Lambda^{\perp}$ corresponding to Λ is conventionally defined by

$$\mathbf{x} \in \Lambda^{\perp} \Leftrightarrow \mathbf{x}\cdot\mathbf{p} = \text{ integer } \forall \mathbf{p} \in \Lambda.$$

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$\Lambda^\perp$ plays a major role in the theory of lattice rules. The discretization error of $Q(\Lambda)$ can be expressed in a Fourier series whose nonzero elements are related to $\Lambda^\perp$ [Ly89], [SJ94]. Developing that result leads to a criterion for the trigonometric degree of $Q(\Lambda)$ in terms of the properties of $\Lambda^\perp$ [CL01].

An $s\times s$ matrix A is termed a *generator matrix* of Λ when Λ comprises precisely all points of the form

$$\mathbf{p} = \sum_{i=1}^{s} \lambda_i \mathbf{a}_i = \boldsymbol{\lambda} A, \quad \boldsymbol{\lambda} \in \Lambda_0,$$

where $\mathbf{a}_i$ is the ith row of A. When Λ is generated by A, $\Lambda^\perp$ may be generated by $B = (A^T)^{-1}$. When Λ is an integration lattice, $\Lambda^\perp$ is an *integer lattice*; that is, all the components of $\mathbf{x} \in \Lambda^\perp$ are integers, and all the elements of B are integers. Moreover, the abscissa count of the rule is

$$N = |\det A|^{-1} = |\det B|.$$

This is also termed the order of $\Lambda^\perp$.

A unimodular matrix U is an integer matrix for which $|\det(U)| = 1$. It is known [Sc86] that any two generator matrices, B and B', that generate the same lattice $\Lambda^\perp$ are related by $B = UB'$, where U is a unimodular matrix. Moreover, B can always be chosen to be in the form given by the next definition.

DEFINITION 1.1. *An $s \times s$ integer matrix B is in* upper triangular lattice form (utlf) *when B is upper triangular and*

$$0 \le b_{ij} < b_{jj} \text{ for } 1 \le i < j \le s.$$

There is a one-to-one correspondence between a lattice rule and the generator matrix B in utlf. This circumstance has been used extensively to classify and enumerate lattice rules.

In whatever manner the rule is specified, for use in cubature it has to be expressed in a form involving abscissas in a reasonably accessible form. Such a form is a t-cycle $D - Z$ form, namely,

$$\begin{aligned} Qf &= Q[t, D, Z, s]f \\ &:= \frac{1}{d_1 d_2 \cdots d_t} \sum_{j_1=1}^{d_1} \sum_{j_2=1}^{d_2} \cdots \sum_{j_t=1}^{d_t} f\left(\left\{\sum_{i=1}^{t} j_i \frac{\mathbf{z}_i}{d_i}\right\}\right); \end{aligned} \tag{1.1}$$

here d_i is a positive integer, an element of a $t\times t$ diagonal matrix D; $\mathbf{z}_i$ is a row of a $t \times s$ integer matrix Z; and $\{\mathbf{x}\} \in [0, 1)^s$ denotes the vector whose components are the fractional parts of the components of $\mathbf{x}$. A rule Q has many different t-cycle $D - Z$ representations employing different t, D, and Z.

The *lattice rank*, r, of Q may be defined as the smallest value of t for which such a representation exists. A *canonical form* is a form $Q[r, D, Z, s]$ in which the elements of D, now denoted by n_i in this context, satisfy

$$n_{i+1} \mid n_i; \; n_r > 1. \tag{1.2}$$

There are many canonical forms of the same rule, all having the same D matrix. This particular set of diagonal elements n_i (which satisfy (1.2)) are unique to the rule Q and are known as the *invariants* of Q. It is occasionally convenient to define *trivial* invariants $n_i = 1$, $i = r+1, \ldots, s$. The rank and invariants are properties of the underlying group-theoretic structure of the abscissa set under addition modulo 1. The rank and invariants are discussed in detail in references [SL89] and [SJ94]. The somewhat involved process of reducing a particular rule from a specified $D - Z$ form to its canonical form is described in some detail in [LK95]

When Q has lattice rank 1, representation (1.1) reduces to the familiar (number-theoretic) rule

$$Qf = Q[1, N, \mathbf{z}, s]f = \frac{1}{N}\sum_{j=1}^{N} f\left(\left\{j\frac{\mathbf{z}}{N}\right\}\right).$$

Much of the research into lattice rules has involved programmed computer searches ([Za72], [LS91]) for rules having particular properties such as a high Zaremba index ([Ma72], [KZ74], [BP85], [La96]) or a high trigonometric degree [CL01]. Many searches for lattice rules are limited to rank-1 rules and occasionally to rank-1 simple rules, in which $\mathbf{z}$ can be chosen to have at least one unit component. This is partly because for a given N, a large proportion of the lattice rules of order N are rank 1 and a search over rank-1 rules can be reduced to an s-parameter search; to take advantage of this, rules of rank 1 have to be recognized and treated differently from those of higher rank.

The purpose of this paper is to develop methods of rank recognition. The specific problem is to determine the lattice rank of $Q(\Lambda)$ in terms of the elements of B, a generator matrix of $\Lambda^{\perp}$. We refer to this *lattice* rank as either $r(\Lambda)$ or (where no confusion is likely to arise) as $r(B)$.

In a previous companion paper [LJ03] we treated the problem of determining how many lattice rules of given abscissa count N and rank r exist. To that end, we expressed an integration lattice, uniquely, as the lattice sum of its Sylow components

$$\Lambda = \Lambda^{(p_1)} + \Lambda^{(p_2)} + \cdots + \Lambda^{(p_q)}.$$

Here $\Lambda^{(p_j)}$ is a prime power lattice of order $p_j^{\alpha_j}$ for some positive integer α_j. We showed that the lattice rank $r(\Lambda)$ is

$$r(\Lambda) = \max_{1 \le j \le q} r(\Lambda^{(p_j)}).$$

These terms are defined in [LJ03] but are not needed in this paper. Unfortunately, as in many other aspects of lattice rules, the theory that led to elegant overall results was not helpful in dealing with individual cases.

As mentioned above, the situation envisioned in this paper is one in which the generator matrix B of $\Lambda^{\perp}$ is available, and only the lattice rank $r(B)$ of $Q(\Lambda)$ is required. One approach might be to construct the Smith Normal Form of B (see (2.3) below) and the lattice rank is simply the number of nontrivial invariants appearing there. However, in a situation where the values of the invariants are not required, this underlying approach may be abbreviated out of all recognition so as to avoid any calculation of invariants.

The principal result of this paper is Theorem 2.1 (in Section 2), in which a prescription for the rank of $Q(\Lambda)$ is given in terms of a generator matrix B of $\Lambda^{\perp}$. This prescription is straightforward to apply, involving the calculation of the *rank deficiency* of various matrices B_p each of which is related to B in a simple manner.

The remaining sections of this paper apply this theorem to provide methods for recognizing this rank deficiency from the structure of the matrix B in utlf (Definition 1.1 above). In Section 3 we show that the prime factor decompositions of the diagonal elements of B play a key role, and in some cases the rank deficiency can be recognized immediately from these. The rest of the paper is devoted to cases where this is not possible.

In Section 4 we assemble some straightforward results relating the rank deficiency of an upper triangular matrix to the rank deficiencies of some of its submatrices. These results are exploited in Section 5 where a technique involving reducing the matrix B to an *essential part* $\widetilde{B}$ and then constructing smaller *rank recognition matrices* $\widetilde{B}'_p$ is described. In Section 6 we provide an example that illustrates the key points in the theory.

2 A Basic Theorem for the Lattice Rank $r(B)$

In this section we establish Theorem 2.1 in which the (lattice) rank $r(B)$ of B is related to standard (matrix) ranks $\rho(B)$ and $\rho_p(B)$ defined below.

In any specification in which the abscissa count N is available, it is well known that the rank may be bounded in terms of the integers α_j appearing in the prime factor decomposition

$$N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_q^{\alpha_q}. \tag{2.1}$$

This bound is

$$r(\Lambda) \leq \max_{1 \leq j \leq q} \alpha_j \tag{2.2}$$

and can readily be established by considering the possible forms of invariants, which satisfy (1.2) and satisfy $N = n_1 n_2 \cdots n_r$.

Note that when $N = |\det B| > 1$ has no square factor, we have immediately $r(\Lambda) = 1$. In this paper we are interested in the problem of determining the lattice rank of Q in terms of the matrix B for more general N.

It has been known since the middle of the nineteenth century that corresponding to every nonsingular integer matrix B, there exist unimodular integer matrices U and V and integers r, n_i, $i = 1, 2, \ldots, r$, such that B can be diagonalized in the following particular way:

$$S := \mathrm{snf}(B) = UBV = \mathrm{diag}\{n_1, n_2, \ldots, n_r, 1, \ldots, 1\}, \tag{2.3}$$

where $n_{i+1} \mid n_i$ and $n_r > 1$. This matrix S, which for given B is unique, is known as the Smith normal form of B and may be denoted by $\mathrm{snf}(B)$. An elementary discussion of the Smith normal form emphasizing its relevance to lattice rules is given in [LK95].

The lattices generated by S and by B are closely related. Premultiplication by a unimodular matrix has the effect of carrying out elementary row operations which do not affect the lattice. Postmultiplication carries out elementary column operations, thus applying an affine transformation in the coordinate space. Geometrical properties such as rank and invariants are unaffected. It is readily verified that the rule whose dual lattice is generated by S in (2.3) is a Cartesian product of n_i-panel one-dimensional rectangle rules of orders $n_1, n_2, \ldots, n_r$, respectively; its rank r and invariants n_i are as stated.

Definition 2.1. *We shall denote by* $\rho(M)$ *and* $\bar{\rho}(M) = s - \rho(M)$ *the* conventional *rank and rank deficiency of an* $s \times s$ *integer matrix* M.

We denote by $\rho_p(M)$ *and* $\bar{\rho}_p(M)$ *the* conventional *rank and rank deficiency of* M *when the elements of* M *satisfy modulo* p *arithmetic.*

On the other hand, $r(M)$ *denotes the* lattice *rank of a rule* $Q(\Lambda)$*; specifically the rule whose dual lattice* $\Lambda^{\perp}$ *is generated by* M.

Note that $\rho_p(M)$ is the rank of M calculated by using elements of $M_p = M \bmod p$ and by using modulo p arithmetic for calculations involving matrix elements. Thus $\rho_p(M) = \rho_p(M_p)$.

Let S be the Smith normal form of B as given in (2.3), and let p be any prime number. Then

$$S \bmod p = \mathrm{diag}\{n_1 \bmod p, n_2 \bmod p, \ldots, n_r \bmod p, 1, 1, \ldots, 1\}.$$

The rank deficiency of the matrix $S \bmod p$, denoted by $\bar{\rho}_p(S)$, is the number of zero diagonal elements. This number cannot exceed r. Thus

$$r \geq \bar{\rho}_p(S) \text{ for all prime } p. \tag{2.4}$$

However, let p be a prime factor of n_r. Since n_r is itself a factor of all the n_i, so is p and $n_i \bmod p = 0$ for all $i \leq r$. Thus

$$r = \bar{\rho}_p(S) \text{ when } p \text{ is a prime factor of } n_r. \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$r = \max_{p \text{ prime}} \bar{\rho}_p(S). \tag{2.6}$$

However, if N does not have factor p, that is, $p \notin \{p_1, \ldots, p_q\}$, then $n_i \bmod p \neq 0$. In this case $\bar{\rho}_p(S) = 0$, and the corresponding term in (2.6) may be omitted giving

$$r = \max_{p \in \{p_1, p_2, \ldots, p_q\}} \bar{\rho}_p(S). \tag{2.7}$$

We have already noted that S is obtained from B by elementary row and column operations only. These affect neither the rank nor the modulo rank of the matrices. Thus, for all prime p,

$$\bar{\rho}_p(S) = \bar{\rho}_p(B).$$

This together with (2.6) and (2.7) leads to the following theorem.

THEOREM 2.1. *Let $Q(\Lambda)$ be an s-dimensional lattice rule and let B be any generator matrix of $\Lambda^{\perp}$. Then r, the lattice rank of $Q(\Lambda)$, is given by*

$$r(B) = \max_{p \in \mathcal{P}} \bar{\rho}_p(B), \tag{2.8}$$

where the set $\mathcal{P}$ includes (but need not be limited to) all primes occurring in the prime factor decomposition of $N = |\det B|$, and $\bar{\rho}_p(B)$ is the modulo p rank deficiency (defined in Definition 2.1 above) of the matrix B.

This theorem is quite general in that it applies to any generator matrix B. It is basic in that it indicates a direct method for calculating $r(B)$ in terms of the elements of B. The rest of this paper deals with cases where B is in upper triangular lattice form. We exploit this theorem to provide, in some cases, rapid ways to determine the rank. In these cases $N = |\det B|$ is readily available, as is its prime factor decomposition.

3 $r(B)$ for B in Upper Triangular Lattice Form

In this section we treat some of the more straightforward consequences of Theorem 2.1. Some arise simply from the prime factor decomposition of N in (2.1) above. Marginally more sophisticated results stem from the prime factor decomposition of each of the diagonal elements b_{ii} of B in utlf. To introduce an idea of the sort of results we obtain, we start with a simple example.

EXAMPLE 3.1. *Determine the ranks of all lattices in which B in utlf has leading diagonal elements* $8, 1, 12, 1$. In the terminology of [LSK91], these B matrices belong to the *upper class* $[8, 1, 12, 1]$. The lattices of this upper class form a two-parameter system

$$B = \begin{bmatrix} 8 & 0 & b_{13} & 0 \\ 0 & 1 & b_{23} & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.1}$$

with $b_{13}, b_{23} \in [0, 11]$. There are 144 members of this upper class. Since $N = |\det B| = 96 = 2^5 \times 3$, inequality (2.2) is not useful. To apply Theorem 2.1 above, we note

(3.2)

$$B \bmod 2 = \begin{bmatrix} 0 & 0 & b_{13} \bmod 2 & 0 \\ 0 & 1 & b_{23} \bmod 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B \bmod 3 = \begin{bmatrix} 2 & 0 & b_{13} \bmod 3 & 0 \\ 0 & 1 & b_{23} \bmod 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The rank deficiency of B mod 3 is clearly $\bar{\rho}_3 = 1$ independently of the values b_{13} and b_{23}. However, the rank deficiency of B mod 2 depends on the value of b_{13}. When b_{13} is odd, $\bar{\rho}_2 = 1$, and when b_{13} is even, $\bar{\rho}_2 = 2$. In view of (2.8), the lattice rank is $\max(\bar{\rho}_2, \bar{\rho}_3)$. Thus $r(B) = 1$ when b_{13} is odd, and $r(B) = 2$ when b_{13} is even.

The circumstances that the value of any element b_{ij} is immaterial to the rank when $b_{ii} = 1$ is general. Here, precisely half the matrices of this upper class have lattice rank 1 and half have lattice rank 2. In this example, the immediate recognition of the rank of the 4×4 matrices $B \bmod 2$ and $B \bmod 3$ is trivial. In general, however, it may be much more difficult.

Let us look at a modification of this example in which the upper class $[8, 1, 12, 1]$ is changed to $[2^{\beta_1}, 1, 2^{\beta_2}3^{\beta_3}, 1]$ for some *positive* integers $\beta_1, \beta_2, \beta_3$. We find that apart from replacing 8 and 12 in (3.1), almost all the rest of the description is identical, including (3.2), and the conclusion is just the same; that is, the rank is 1 or 2 depending on whether b_{13} is odd or even. This motivates the next result.

LEMMA 3.1. *Suppose $\bar{B}$ is a generator matrix that is upper triangular. Let $\bar{b}_{mm}$ have a prime factor p and let B be identical to $\bar{B}$ except that $b_{mm} = \bar{b}_{mm}p$. Then the lattice ranks of $\bar{B}$ and B coincide.*

PROOF. The lattice ranks of B and $\bar{B}$ are given, by (2.8), to be

$$r(B) = \max_{p_j \in \mathcal{P}} \bar{\rho}_{p_j}(B) \quad \text{and} \quad r(\bar{B}) = \max_{p_j \in \mathcal{P}} \bar{\rho}_{p_j}(\bar{B}),$$

respectively. $B \bmod p_j$ differs from $\bar{B} \bmod p_j$ only in that the element $b_{mm} \bmod p_j$ may differ from $\bar{b}_{mm} \bmod p_j$.

If p_j is any prime factor of b_{mm}, then both elements $\bar{b}_{mm} \bmod p_j$ and $b_{mm} \bmod p_j$ are zero. For these primes, $B \bmod p_j$ and $\bar{B} \bmod p_j$ are identical, and so $\bar{\rho}_{p_j}(B) = \bar{\rho}_{p_j}(\bar{B})$ when p_j is a factor of b_{mm}.

When p_j is not a factor of b_{mm}, then both elements $b_{mm} \bmod p_j$ and $\bar{b}_{mm} \bmod p_j$ are nonzero. The rank deficiency of any upper triangular matrix is not changed by replacing one nonzero diagonal element by a different nonzero element. Thus $\bar{\rho}_{p_j}(B) = \bar{\rho}_{p_j}(\bar{B})$ when p_j is not a factor of b_{mm}.

Thus all these individual rank deficiencies are identical for all primes; so, in view of (2.8), $\bar{B}$ and B have the same rank. □

The result of Lemma 3.1 may be iterated to allow any individual nonunit diagonal element with prime decomposition $p_1^{\alpha_{1i}} p_2^{\alpha_{2i}} \cdots p_q^{\alpha_{qi}}$ to be changed to $p_1^{\gamma_{1i}} p_2^{\gamma_{2i}} \cdots p_q^{\gamma_{qi}}$ so long as all indices α_{ji} and γ_{ji} are positive, without altering the lattice rank. The underlying theorem is as follows.

THEOREM 3.2. *Let B be an $s \times s$ matrix in utlf and let*

$$N = |\det B| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_q^{\alpha_q}$$

be the prime factor decomposition of its order. Let its diagonal elements be

$$b_{ii} = p_1^{\alpha_{1i}} p_2^{\alpha_{2i}} \cdots p_q^{\alpha_{qi}}, \quad i = 1, 2, \ldots, s.$$

Let $\bar{B}$ coincide with B except that its diagonal elements are different and given by

$$\bar{b}_{ii} = p_1^{\bar{\alpha}_{1i}} p_2^{\bar{\alpha}_{2i}} \cdots p_q^{\bar{\alpha}_{qi}}, \quad i = 1, 2, \ldots, s,$$

where

$$\bar{\alpha}_{ji} = \begin{cases} 0 & \text{when } \alpha_{ji} = 0, \\ 1 & \text{when } \alpha_{ji} \geq 1. \end{cases}$$

Then the lattice ranks of B and $\bar{B}$ coincide; that is, $r(B) = r(\bar{B})$.

As an example, suppose $q = 3$, $p_1 = 2$, $p_2 = 3$, $p_3 = 7$, and let

$$B = \begin{bmatrix} 3^4 & b_{12} & b_{13} \\ 0 & 3^2 \cdot 7 & b_{23} \\ 0 & 0 & 2^4 \end{bmatrix}.$$

Then in the notation of the theorem we have

$$\alpha_{11} = 0,\ \alpha_{21} = 4,\ \alpha_{31} = 0,\ \alpha_{12} = 0,\ \alpha_{22} = 2,\ \alpha_{32} = 1,\ \alpha_{13} = 4,\ \alpha_{23} = 0,\ \alpha_{33} = 0.$$

The corresponding values for the $\bar{\alpha}_{ji}$, $i, j = 1, 2, 3$, are given by

$$\bar{\alpha}_{11} = 0,\ \bar{\alpha}_{21} = 1,\ \bar{\alpha}_{31} = 0,\ \bar{\alpha}_{12} = 0,\ \bar{\alpha}_{22} = 1,\ \bar{\alpha}_{32} = 1,\ \bar{\alpha}_{13} = 1,\ \bar{\alpha}_{23} = 0,\ \bar{\alpha}_{33} = 0.$$

The theorem then shows that the matrix

$$\bar{B} = \begin{bmatrix} 2^0 \cdot 3^1 \cdot 7^0 & b_{12} & b_{13} \\ 0 & 2^0 \cdot 3^1 \cdot 7^1 & b_{23} \\ 0 & 0 & 2^1 \cdot 3^0 \cdot 7^0 \end{bmatrix} = \begin{bmatrix} 3 & b_{12} & b_{13} \\ 0 & 3 \cdot 7 & b_{23} \\ 0 & 0 & 2 \end{bmatrix}$$

has the same lattice rank as B. (In this case, the rank is 2 or 1 depending on whether b_{12} is a multiple of 3 or not.)

We remark that in the theorem, although B is in utlf, $\bar{B}$ is necessarily upper triangular but need not be in utlf. This theorem leads directly to a significant strengthening of the result stated in (2.1) and (2.2). Using that result applied to $\bar{B}$, we obtain the following.

COROLLARY 3.3. *The lattice rank $r(B)$ satisfies the inequality*

$$r(B) \leq \max_{1 \leq j \leq q} \sigma_{p_j},$$

where $\sigma_{p_j} = \sum_{i=1}^{s} \bar{\alpha}_{ji}$ is the number of diagonal elements of B having p_j as a factor.

In the previous example, $\sigma_2 = 1$, $\sigma_3 = 2$, and $\sigma_7 = 1$. Corollary 3.3 then shows that $r(B) \leq 2$.

4 Some Rank Reduction Lemmas

Before proceeding with the theory, we assemble some straightforward results relating the rank deficiency of an upper triangular (integer) matrix with the rank deficiencies of some of its submatrices.

The first lemma is straightforward, but basic.

LEMMA 4.1. *Let M be an $s \times s$ upper triangular matrix. Suppose m_{ii} is a diagonal element of M and all the other elements of the ith column are zero. Further, suppose M' is the $(s-1) \times (s-1)$ upper triangular matrix obtained from M by removing the ith row and the ith column. Then*

$$\begin{array}{lll} \textit{when } m_{ii} \neq 0, & & \bar{\rho}_p(M') = \bar{\rho}_p(M), \\ \textit{when } m_{ii} = 0, & \textit{either} & \bar{\rho}_p(M') = \bar{\rho}_p(M) \\ & \textit{or} & \bar{\rho}_p(M') = \bar{\rho}_p(M) - 1. \end{array}$$

PROOF. When $m_{ii} \neq 0$, the ith row is independent of all other rows. Removing this row reduces the rank by one. The resulting $(s-1) \times s$ matrix has a zero ith column which may be removed without affecting the rank. Thus the $(s-1) \times (s-1)$ matrix M' obtained is of dimension one less than M and of rank one less than M and so has the same rank deficiency.

When $m_{ii} = 0$, the ith column is zero and may be removed without affecting the rank. Then removing the ith row may reduce the rank by 1 or may not affect the rank. Thus the rank deficiency may be unaltered or reduced by 1. □

LEMMA 4.2. *Lemma 4.1 is valid when the words "column" and "row" are interchanged.*

These last two lemmas may be used iteratively to establish the next lemma. First we introduce the following definition.

DEFINITION 4.1. *A* diagonal block submatrix *of an $\ell \times \ell$ matrix $\widetilde{M}$ is an $n \times n$ submatrix $\widetilde{M}'$ whose diagonal elements coincide with n* adjacent *diagonal elements of $\widetilde{M}$.*

LEMMA 4.3. *Let $\widetilde{M}$ be an upper triangular $\ell \times \ell$ integer matrix. Let $\widetilde{M}'$ be a diagonal block submatrix of $\widetilde{M}$. Then, for all prime p,*

$$\bar{\rho}_p(\widetilde{M}) \geq \bar{\rho}_p(\widetilde{M}') \geq \bar{\rho}_p(\widetilde{M}) - \nu, \tag{4.1}$$

where ν is the number of zero diagonal elements of $\widetilde{M}$ (modulo p) lying outside $\widetilde{M}'$.

Colloquially: If one notices a diagonal block submatrix of any upper triangular matrix that has rank deficiency $\bar{\rho}$, then the rank deficiency of the matrix cannot be less than $\bar{\rho}$.

Comment: All three lemmas are valid with subscript p removed. One can see this by taking p sufficiently large.

5 $\widetilde{B}$, the Essential Part of B, and the Rank Recognition Matrices $\widetilde{B}'_p$

Given a general $s \times s$ generator matrix B in utlf, it is usually a straightforward but tedious problem to obtain the lattice rank by using (2.8). In this section we develop techniques that allow the problem to be somewhat simplified. For example, given B in utlf, we shall see in Theorem 5.1 that rows and columns of B containing a 1 on the diagonal may be removed without affecting the lattice rank. Further size reductions of a similar nature can be made until the final rank recognition depends on a set of smaller matrices; these are termed rank recognition matrices.

DEFINITION 5.1. *The* essential part *$\widetilde{B}$ of an $s \times s$ matrix B in utlf is the matrix obtained from B by removing all rows and columns whose diagonal element is 1.*

Thus the essential part $\widetilde{B}$ is an $\ell \times \ell$ matrix in utlf all of whose diagonal elements exceed 1, with $\ell \leq s$.

When B is a generator matrix in utlf, every unit diagonal element $b_{ii} = 1$ gives rise to an ith column having only this nonzero element, so satisfying the conditions of Lemma 4.1. We may apply this lemma to each unit diagonal element in turn to obtain the following result.

THEOREM 5.1. *Let B be in utlf, and let $\widetilde{B}$ be its essential part. Then*

$$\bar{\rho}_p(B) = \bar{\rho}_p(\widetilde{B}) \text{ for all prime } p$$

and

$$r(B) = r(\widetilde{B}).$$

The theorem shows that we may replace the calculation of the lattice rank of the $s \times s$ matrix B by that of the (smaller) $\ell \times \ell$ matrix $\widetilde{B}$. The lattice rank $r(B)$ depends only on the elements in $\widetilde{B}$. Since the rank deficiency of any matrix cannot exceed its dimension, we have the following corollary.

COROLLARY 5.2. *When B is in utlf, then $r(B)$, the lattice rank of B, cannot exceed the number ℓ of nonunit diagonal elements of B.*

However, this also follows directly from Corollary 3.3.

To continue, we look more closely at the $\ell \times \ell$ matrix $\widetilde{B}$, all of whose integer diagonal elements exceed 1. In view of Theorem 2.1, we have $r(B) = \max_{p \in \mathcal{P}} \bar{\rho}_p(\widetilde{B})$; we are interested in calculating $\bar{\rho}_p(\widetilde{B})$ for various prime p. This coincides with $\bar{\rho}_p(\widetilde{B}_p)$, where $\widetilde{B}_p = \widetilde{B} \bmod p$. If $\widetilde{B}_p$ has no zero diagonal elements, then p is not a factor of N $(= |\det B|)$ and $\bar{\rho}_p(\widetilde{B}_p) = 0$. If $\widetilde{B}_p$ has precisely one zero diagonal element, then p is a factor of N. In this case, the other $\ell - 1$ diagonal elements are nonzero, and $\bar{\rho}_p(\widetilde{B}_p) = 1$. (The determinant of $\widetilde{B}_p$ is zero, and $\widetilde{B}_p$ has $\ell - 1$ independent rows.)

When $\widetilde{B}_p$ has more than one zero diagonal element, the situation with respect to its rank deficiency is less straightforward. This cannot exceed the number of zero diagonal elements and is at least 1. To clarify the situation, we introduce a submatrix.

DEFINITION 5.2. *Given $\widetilde{B}_p$, its rank recognition matrix $\widetilde{B}'_p$ is the smallest diagonal block submatrix that contains all the zero diagonal elements of $\widetilde{B}_p$.*

This is illustrated in (5.1). Here, d stands for a diagonal element that is not zero (modulo p), δ stands for a diagonal element that may or may not be zero, and y stands for a nondiagonal element that may or may not be zero. The rank recognition matrix $\widetilde{B}'_p$ is the 4×4 matrix within the dark thick lines.

$$\widetilde{B}_p = \left[\begin{array}{ccc|cccc|cc} d & y & y & y & y & y & y & y & y \\ 0 & d & y & y & y & y & y & y & y \\ 0 & 0 & d & y & y & y & y & y & y \\ \hline 0 & 0 & 0 & 0 & y & y & y & y & y \\ 0 & 0 & 0 & 0 & \delta & y & y & y & y \\ 0 & 0 & 0 & 0 & 0 & \delta & y & y & y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & y \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \end{array}\right] \tag{5.1}$$

THEOREM 5.3. *Let $\widetilde{B}_p$ have at least one zero diagonal element. Let its rank recognition matrix $\widetilde{B}'_p$ be as given in Definition 5.2. Then*

$$\bar{\rho}_p(\widetilde{B}_p) = \bar{\rho}_p(\widetilde{B}'_p).$$

PROOF. This is a direct corollary of Lemma 4.3. The rank recognition matrix is specifically constructed so that there are no zero diagonal elements of $\widetilde{B}_p$ outside $\widetilde{B}'_p$; thus $\nu = 0$ in (4.1). □

The previous theorem requires $\widetilde{B}_p = \widetilde{B} \bmod p$ to have at least one zero entry on the diagonal. As mentioned earlier, when $\widetilde{B}_p$ has no zero diagonal elements $\bar{\rho}_p(\widetilde{B}_p) = 0$.

The theorems and results in this section take the theory of rank recognition to a convenient plateau. In some applications, results of this nature can be used to significantly abbreviate a particular search. Nevertheless, in a modern computational context, to what extent it is worthwhile to reduce any search at the expense of making its organization more complicated is an early and critical question to be faced by the individual investigator. In this paper, various possibilities are presented, ranging from the trivial bound given in (2.2) to the sophisticated approach of this section.

6 Example

To illustrate the overall results in this paper, we treat the following example. What is the lattice rank of the five-dimensional $Q(\Lambda)$ when $\Lambda^{\perp}$ is generated by

$$B = \begin{bmatrix} 7\cdot 2^5 & 0 & a_1 & a_2 & a_3 \\ 0 & 1 & b_1 & b_2 & b_3 \\ 0 & 0 & 5\cdot 2 & c_2 & c_3 \\ 0 & 0 & 0 & 5^2 & d_3 \\ 0 & 0 & 0 & 0 & 2\cdot 5\cdot 3^2 \end{bmatrix}?$$

Thus $N = 2^7 \cdot 3^2 \cdot 5^4 \cdot 7$, and the classical result (2.2) gives $r(B) \leq 7$.

Moving on to Section 3, we find that Theorem 3.2 allows us to replace B by $\bar{B}$ given by

$$\bar{B} = \begin{bmatrix} 7\cdot 2 & 0 & a_1 & a_2 & a_3 \\ 0 & 1 & b_1 & b_2 & b_3 \\ 0 & 0 & 5\cdot 2 & c_2 & c_3 \\ 0 & 0 & 0 & 5 & d_3 \\ 0 & 0 & 0 & 0 & 2\cdot 5\cdot 3 \end{bmatrix}.$$

The number of elements b_{ii} of B containing the prime $p = j$ is denoted by σ_j, which takes the values

$$\sigma_2 = 3, \quad \sigma_3 = 1, \quad \sigma_5 = 3, \quad \sigma_7 = 1.$$

Corollary 3.3 then gives $r(B) = r(\bar{B}) \leq \max_j \sigma_j = 3$.

To obtain the essential part (as described in Definition 5.1), since $\bar{b}_{22} = 1$, we remove row 2 and column 2 to obtain

$$\widetilde{B} = \begin{bmatrix} 7\cdot 2 & a_1 & a_2 & a_3 \\ 0 & 5\cdot 2 & c_2 & c_3 \\ 0 & 0 & 5 & d_3 \\ 0 & 0 & 0 & 2\cdot 5\cdot 3 \end{bmatrix}.$$

Then Theorem 5.1 shows $r(B) = r(\widetilde{B})$, confirming that this rank is independent of the values of b_1, b_2, and b_3 in B.

At this stage we look at $\widetilde{B}_p = \widetilde{B} \bmod p$ corresponding to $p = 2, 3, 5, 7$. These matrices are

$$\widetilde{B}_2 = \begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & c_2 & c_3 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \widetilde{B}_3 = \begin{bmatrix} 2 & a_1 & a_2 & a_3 \\ 0 & 1 & c_2 & c_3 \\ 0 & 0 & 2 & d_3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{B}_5 = \begin{bmatrix} 4 & a_1 & a_2 & a_3 \\ 0 & 0 & c_2 & c_3 \\ 0 & 0 & 0 & d_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \widetilde{B}_7 = \begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 3 & c_2 & c_3 \\ 0 & 0 & 5 & d_3 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

We have included $\widetilde{B}_3$ and $\widetilde{B}_7$ for illustration only. Since $\sigma_3 = \sigma_7 = 1$, they each have rank deficiency 1 and this can be confirmed by inspection. The numerical values of the diagonal elements are significant only to the extent that they are zero or nonzero modulo p. Altering the value of any nonzero diagonal element to 1 in any of these four matrices has no effect on its rank deficiency.

The rank recognition matrices are the smallest diagonal blocks that include all zero diagonal elements. These are

(6.1)
$$\widetilde{B}'_2 = \widetilde{B}_2 = \begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & c_2 & c_3 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \widetilde{B}'_3 = \begin{bmatrix} 0 \end{bmatrix}, \ \widetilde{B}'_5 = \begin{bmatrix} 0 & c_2 & c_3 \\ 0 & 0 & d_3 \\ 0 & 0 & 0 \end{bmatrix}, \ \widetilde{B}'_7 = \begin{bmatrix} 0 \end{bmatrix}.$$

Clearly $\bar{\rho}_3 = \bar{\rho}_7 = 1$. Both $\widetilde{B}'_2$ and $\widetilde{B}'_5$ have a row of zeros, and so $\bar{\rho}_2$ and $\bar{\rho}_5 \geq 1$. The lattice rank of Q is $\max(\bar{\rho}_2, \bar{\rho}_3, \bar{\rho}_5, \bar{\rho}_7)$. It remains to evaluate $\bar{\rho}_2$ and $\bar{\rho}_5$, and these will depend on the remaining six parameters.

This is as far as the theory of Section 5 takes us. We have

$$r(B) = \max(\bar{\rho}_2, \bar{\rho}_3, \bar{\rho}_5, \bar{\rho}_7).$$

Both $\bar{\rho}_2$ and $\bar{\rho}_5$ are integers 1, 2, or 3. These are the rank deficiencies modulo p of $\widetilde{B}'_p$ given in (6.1) above. One can find $\bar{\rho}_5$ almost by inspection. One sees that

$$\begin{aligned} \bar{\rho}_5 &= 3 \quad \text{when } c_2 = c_3 = d_3 = 0 \bmod 5, \\ \text{otherwise } \bar{\rho}_5 &= 2 \quad \text{when } c_2 d_3 = 0 \bmod 5, \\ \text{otherwise } \bar{\rho}_5 &= 1. \end{aligned}$$

Determining $\bar{\rho}_2$ is more complicated. In view of the zero row and column in $\widetilde{B}_2'$ (see (6.1) above), one finds

$$\bar{\rho}_2 = \bar{\rho}_2(\widetilde{B}_2') = 1 + \bar{\rho}_2 \left(\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & c_2 & c_3 \\ 0 & 1 & d_3 \end{bmatrix} \right).$$

Consideration of the determinant of the latter matrix shows that when $a_1(c_2d_3 - c_3) \neq 0 \bmod 2$, then the matrix is nonsingular, and hence $\bar{\rho}_2 = 1$. For $\bar{\rho}_2 = 3$, we need $a_1 = 0 \bmod 2$ and the last column of the latter matrix to be either equal to the second column (mod 2) or a column of zeros (mod 2). Otherwise, $\bar{\rho}_2 = 2$.

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