Construction Of Good Rank-1 Lattice Rules Based On The Weighted Star Discrepancy

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Summary. The 'goodness' of a set of quadrature points in $[0,1]^d$ may be measured by the weighted star discrepancy. If the weights for the weighted star discrepancy are summable, then we show that for n prime there exist n-point rank-1 lattice rules whose weighted star discrepancy is $O\left(n^{-1+\delta}\right)$ for any $\delta > 0$, where the implied constant depends on δ and the weights, but is independent of d and n. Further, we show that the generating vector $\mathbf{z}$ for such lattice rules may be obtained using a component-by-component construction. The results given here for the weighted star discrepancy are used to derive corresponding results for a weighted L_p discrepancy.

1 Introduction

Integrals over the d-dimensional unit cube given by

$$I_d(f) = \int_{[0,1]^d} f(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

may be approximated using n-point rank-1 lattice rules. These are quadrature rules of the form

$$Q_{n,d}(f) = \frac{1}{n}\sum_{k=0}^{n-1} f\left(\left\{\frac{k\mathbf{z}}{n}\right\}\right) ,$$

where $\mathbf{z} \in \mathbb{Z}^d$ is the 'generating vector' with no factor in common with n, and the braces around a vector indicate that we take the fractional part of each component of the vector. For our purposes, it is convenient to assume that $\gcd(z_j, n) = 1$ for $1 \le j \le d$, where z_j is the j-th component of $\mathbf{z}$.

The star discrepancy of the point set $P_n(\mathbf{z}) := \{\{k\mathbf{z}/n\},\ 0 \le k \le n-1\}$ is defined by

$$D^*(P_n(\mathbf{z})) = D_n^*(\mathbf{z}) := \sup_{\mathbf{x}\in[0,1)^d} |\mathrm{discr}(\mathbf{x}, P_n)| ,$$

where $\mathrm{discr}(\mathbf{x}, P_n)$ is the 'local discrepancy' defined by

$$\mathrm{discr}(\mathbf{x}, P_n) := \frac{|P_n(\mathbf{z}) \cap [\mathbf{0}, \mathbf{x})|}{n} - \mathrm{Vol}([\mathbf{0}, \mathbf{x})) \ . \tag{1}$$

The star discrepancy occurs in the well-known Koksma-Hlawka inequality. Further details may be found in [3] and [19] or in more general works such as [11].

It is known (see [10] or [11]) that there exist d-dimensional rank-1 lattice rules whose star discrepancy is $O(n^{-1}(\ln(n))^d)$ with the implied constant depending on only d. For n prime it was shown in [4] that such rules may be obtained by constructing their generating vectors component-by-component. In this paper we extend these results to the case of a weighted star discrepancy.

Such component-by-component constructions first appeared in [16], but there the integrands were in a periodic setting. Since then, there has been much work done in the L_2 case both in the periodic setting of weighted Korobov spaces and in the non-periodic setting of weighted Sobolev spaces (for example, see [7], [8], [9], [14], and [15]). Here we consider the weighted star discrepancy, since, as we shall see later, we are able to derive corresponding results for the weighted L_p discrepancy.

In order to introduce the weighted star discrepancy, let $\mathfrak{u}$ be any subset of $\mathcal{D} := \{1, 2, \ldots, d-1, d\}$ with cardinality $|\mathfrak{u}|$. For the vector $\mathbf{x} \in [0,1]^d$, let $\mathbf{x}_\mathfrak{u}$ denote the vector from $[0,1]^{|\mathfrak{u}|}$ containing the components of $\mathbf{x}$ whose indices belong to $\mathfrak{u}$. By $(\mathbf{x}_\mathfrak{u}, \mathbf{1})$ we mean the vector from $[0,1]^d$ whose j-th component is x_j if $j \in \mathfrak{u}$ and 1 if $j \notin \mathfrak{u}$. From Zaremba's identity (see [17] or [19]) we have

$$Q_{n,d}(f) - I_d(f) = \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} (-1)^{|\mathfrak{u}|} \int_{[0,1]^{|\mathfrak{u}|}} \mathrm{discr}\left((\mathbf{x}_\mathfrak{u}, \mathbf{1}), P_n\right) \frac{\partial^{|\mathfrak{u}|}}{\partial \mathbf{x}_\mathfrak{u}} f(\mathbf{x}_\mathfrak{u}, \mathbf{1}) \, \mathrm{d}\mathbf{x}_\mathfrak{u} \ . \tag{2}$$

Now let us introduce a sequence of positive weights $\{\gamma_j\}_{j=1}^\infty$ and set

$$\boldsymbol{\gamma}_\mathfrak{u} = \prod_{j \in \mathfrak{u}} \gamma_j \quad \text{with} \quad \boldsymbol{\gamma}_\emptyset := 1 \ . \tag{3}$$

Then we can write

$$\begin{aligned} &Q_{n,d}(f) - I_d(f) \\ &= \sum_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} (-1)^{|\mathfrak{u}|} \boldsymbol{\gamma}_\mathfrak{u} \int_{[0,1]^{|\mathfrak{u}|}} \mathrm{discr}\left((\mathbf{x}_\mathfrak{u}, \mathbf{1}), P_n\right) \boldsymbol{\gamma}_\mathfrak{u}^{-1} \frac{\partial^{|\mathfrak{u}|}}{\partial \mathbf{x}_\mathfrak{u}} f(\mathbf{x}_\mathfrak{u}, \mathbf{1}) \, \mathrm{d}\mathbf{x}_\mathfrak{u} \ . \end{aligned}$$

Applying Hölder's inequality for integrals and sums we obtain

$$\begin{aligned} |Q_{n,d}(f) - I_d(f)| \le &\left(\sup_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \sup_{\mathbf{x}_\mathfrak{u} \in [0,1]^{|u|}} \boldsymbol{\gamma}_\mathfrak{u} \left|\mathrm{discr}\left((\mathbf{x}_\mathfrak{u}, \mathbf{1}), P_n\right)\right| \right) \\ &\times \left(\sum_{\mathfrak{u} \subseteq \mathcal{D}} \boldsymbol{\gamma}_\mathfrak{u}^{-1} \int_{[0,1]^{|u|}} \left| \frac{\partial^{|\mathfrak{u}|}}{\partial \mathbf{x}_\mathfrak{u}} f(\mathbf{x}_\mathfrak{u}, \mathbf{1}) \right| \mathrm{d}\mathbf{x}_\mathfrak{u} \right) \ . \end{aligned}$$

Then we can define a weighted star discrepancy $D^*_{n,\boldsymbol{\gamma}}(\mathbf{z})$ by

$$D^*_{n,\boldsymbol{\gamma}}(\mathbf{z}) := \sup_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{D}} \boldsymbol{\gamma}_{\mathfrak{u}} \sup_{\mathbf{x}_{\mathfrak{u}} \in [0,1]^{|u|}} \left| \operatorname{discr}\left((\mathbf{x}_{\mathfrak{u}}, \mathbf{1}), P_n\right) \right| . \tag{4}$$

In Section 2 we use an averaging argument to show that if the weights γ_j are summable, there exist rank-1 lattice rules whose weighted star discrepancy is $O(n^{-1+\delta})$ for any $\delta > 0$, where the implied constant depends on δ and the weights. A more specific averaging argument is applied to lattice rules of the Korobov form, namely those for which $\mathbf{z} = (1, a, \ldots, a^{d-1}) \pmod n$, $1 \leq a \leq n-1$, to show there exist lattice rules of the Korobov form having $O(n^{-1+\delta})$ weighted star discrepancy.

Besides existence results we are interested in how to find such lattice rules. One way, of course, is to find an appropriate a in the Korobov form. However, such rules are not extensible in dimension; a value of a that is good for one value of the dimension d may not be good for a different value of the dimension. In Section 3 we present results showing that, alternatively, the generating vectors $\mathbf{z}$ for such lattice rules may be constructed a component at a time resulting in a $\mathbf{z}$ which is extensible in dimension. The cost of this component-by-component construction is $O(n^2 d^2)$ operations, but it may be reduced to $O(n^2 d)$ operations at the extra cost of $O(n)$ storage. It may be reduced even further to $O(n \ln(n) d)$ operations by making use of the approach proposed by Nuyens and Cools in [12]. We remark that constructions for polynomial lattice rules having small weighted star discrepancy have recently been proposed in [1]. As here, they consider a Korobov construction and a component-by-component construction.

The weighted star discrepancy considered here may be viewed as the L_∞ version of a weighted L_p discrepancy for $p \geq 1$. Weighted L_p discrepancies have been considered in works such as [2] and [17]. In Section 4 we use the results obtained in Sections 2 and 3 for the weighted star discrepancy to derive corresponding results for the weighted L_p discrepancy. Unlike the earlier results in the L_2 setting, the results presented here do not require the lattice points to be shifted.

2 Rank-1 Lattice Rules Having Certain Weighted Star Discrepancy Bounds

It follows from (4) that the weighted star discrepancy satisfies

$$D^*_{n,\boldsymbol{\gamma}}(\mathbf{z}) \leq \sum_{\mathfrak{u} \subseteq \mathcal{D}} \boldsymbol{\gamma}_{\mathfrak{u}} \sup_{\mathbf{x}_{\mathfrak{u}} \in [0,1]^{|u|}} \left| \operatorname{discr}\left((\mathbf{x}_{\mathfrak{u}}, \mathbf{1}), P_n\right) \right| . \tag{5}$$

Moreover, it follows from [11, Theorem 3.10 and Theorem 5.6] (see also [2]) that

$$\sup_{\mathbf{x}_{\mathfrak{u}} \in [0,1]^{|u|}} \left| \operatorname{discr}\left((\mathbf{x}_{\mathfrak{u}}, \mathbf{1}), P_n\right) \right| \leq 1 - (1 - 1/n)^{|\mathfrak{u}|} + \frac{R_n(\mathbf{z}, \mathfrak{u})}{2} ,$$

where

$$R_n(\mathbf{z},\mathfrak{u}) = \sum_{\substack{\mathbf{h}\cdot\mathbf{z}_\mathfrak{u}\equiv 0 \pmod{n} \\ \mathbf{h}\in C^*_{n,|\mathfrak{u}|}}} \prod_{j=1}^{|\mathfrak{u}|} \frac{1}{\max(1,|h_j|)} \, .$$

Here $\mathbf{z}_\mathfrak{u}$ is the vector consisting of the components of $\mathbf{z}$ whose indices belong to $\mathfrak{u}$ and

$$C^*_{n,|\mathfrak{u}|} = \{\mathbf{h}\in\mathbb{Z}^{|\mathfrak{u}|},\ \mathbf{h}\neq\mathbf{0} : -n/2 < h_j \le n/2,\ 1\le j\le|\mathfrak{u}|\} \, .$$

We then obtain

$$D^*_{n,\boldsymbol{\gamma}}(\mathbf{z}) \le \sum_{\mathfrak{u}\subseteq\mathcal{D}} \boldsymbol{\gamma}_\mathfrak{u} \left(1-(1-1/n)^{|\mathfrak{u}|} + \frac{R_n(\mathbf{z},\mathfrak{u})}{2}\right) \, . \tag{6}$$

Under the assumption that $\gcd(z_j,n)=1$ for $1\le j\le d$, then $\mathbf{z}_\mathfrak{u}$ is the generating vector for a $|\mathfrak{u}|$-dimensional rank-1 lattice rule having n points. It then follows from the error theory of lattice rules (for example, see [11, Chapter 5] or [13, Chapter 4]) that we may write $R_n(\mathbf{z},\mathfrak{u})$ as

$$R_n(\mathbf{z},\mathfrak{u}) = \frac{1}{n}\sum_{k=0}^{n-1}\prod_{j\in\mathfrak{u}}\left(1+\sum_{-n/2<h\le n/2}{}' \frac{e^{2\pi \mathrm{i} hkz_j/n}}{|h|}\right) - 1 \, , \tag{7}$$

where the $'$ on the sum indicates that we omit the $h=0$ term.

Bounds on the weighted star discrepancy $D^*_{n,\boldsymbol{\gamma}}(\mathbf{z})$ may be obtained by making use of (6). We first consider $\sum_{\mathfrak{u}\subseteq\mathcal{D}} \boldsymbol{\gamma}_\mathfrak{u}\left(1-(1-1/n)^{|\mathfrak{u}|}\right)$.

Lemma 1. *Suppose the weights γ_j are summable, that is, $\sum_{j=1}^{\infty}\gamma_j<\infty$. Then*

$$\sum_{\mathfrak{u}\subseteq\mathcal{D}} \boldsymbol{\gamma}_\mathfrak{u}\left(1-(1-1/n)^{|\mathfrak{u}|}\right) \le \frac{\max(1,\Gamma)}{n}\prod_{j=1}^{\infty}(1+\gamma_j) \le \frac{\max(1,\Gamma)\, e^{\sum_{j=1}^{\infty}\gamma_j}}{n} \, ,$$

where $\Gamma := \sum_{j=1}^{\infty}\gamma_j/(1+\gamma_j)<\infty$.

Proof. We may write

$$\begin{aligned}\sum_{\mathfrak{u}\subseteq\mathcal{D}} \boldsymbol{\gamma}_\mathfrak{u}\left(1-(1-1/n)^{|\mathfrak{u}|}\right) &= \prod_{j=1}^{d}(1+\gamma_j) - \prod_{j=1}^{d}(1+\gamma_j(1-1/n)) \\ &= \prod_{j=1}^{d}(1+\gamma_j)\left[1-\prod_{j=1}^{d}\left(1-\frac{\gamma_j}{n(1+\gamma_j)}\right)\right] \, .\end{aligned}$$

According to [2] we have

$$\ln\left(\prod_{j=1}^{d}\left(1-\frac{\gamma_j}{n(1+\gamma_j)}\right)\right)\geq \ln(1-1/n)\sum_{j=1}^{d}\frac{\gamma_j}{1+\gamma_j}\ ,$$

which leads to

$$\sum_{\mathfrak{u}\subseteq\mathcal{D}}\boldsymbol{\gamma}_{\mathfrak{u}}\left(1-(1-1/n)^{|\mathfrak{u}|}\right)\leq\prod_{j=1}^{d}(1+\gamma_j)\left[1-\left(1-\frac{1}{n}\right)^{\sum_{j=1}^{d}\gamma_j/(1+\gamma_j)}\right]\ . \quad (8)$$

Since $0<\gamma_j/(1+\gamma_j)<\gamma_j$, we see that since the γ_j are summable, then so are the $\gamma_j/(1+\gamma_j)$, that is, $\mathit{\Gamma}<\infty$.

If $\mathit{\Gamma}\leq 1$, then we have $(1-1/n)^{\mathit{\Gamma}}\geq 1-1/n$ and hence

$$1-\left(1-\frac{1}{n}\right)^{\mathit{\Gamma}}\leq\frac{1}{n}\ .$$

Now suppose $\mathit{\Gamma}>1$ and set $v(x)=(1+x)^{\mathit{\Gamma}}-\mathit{\Gamma}x-1$ for $x>-1$. Then it is easily verified that $v'(0)=0$. Moreover, $v''(0)=\mathit{\Gamma}^2-\mathit{\Gamma}$ which is positive for $\mathit{\Gamma}>1$. Since $v'(x)<0$ for $-1<x<0$ and $v'(x)>0$ for $x>0$, we deduce that if $\mathit{\Gamma}>1$, then $v(x)\geq v(0)=0$ or $(1+x)^{\mathit{\Gamma}}\geq\mathit{\Gamma}x+1$ for $x>-1$. With $x=-1/n$ we thus obtain

$$\left(1-\frac{1}{n}\right)^{\mathit{\Gamma}}\geq-\frac{\mathit{\Gamma}}{n}+1\quad\text{and so}\quad 1-\left(1-\frac{1}{n}\right)^{\mathit{\Gamma}}\leq\frac{\mathit{\Gamma}}{n}\ .$$

It then follows from (8) that

$$\begin{aligned}\sum_{\mathfrak{u}\subseteq\mathcal{D}}\boldsymbol{\gamma}_{\mathfrak{u}}\left(1-(1-1/n)^{|\mathfrak{u}|}\right)&\leq\prod_{j=1}^{d}(1+\gamma_j)\left[1-\left(1-\frac{1}{n}\right)^{\mathit{\Gamma}}\right]\\&\leq\frac{\max(1,\mathit{\Gamma})}{n}\prod_{j=1}^{d}(1+\gamma_j)\leq\frac{\max(1,\mathit{\Gamma})}{n}\prod_{j=1}^{\infty}(1+\gamma_j)\\&=\frac{\max(1,\mathit{\Gamma})}{n}e^{\sum_{j=1}^{\infty}\ln(1+\gamma_j)}\leq\frac{\max(1,\mathit{\Gamma})\,e^{\sum_{j=1}^{\infty}\gamma_j}}{n}\ ,\end{aligned}$$

where we have used $\ln(1+x)\leq x$ for $x\geq 0$. □

With $\boldsymbol{\gamma}_{\emptyset}=1$, we make use of (3) and (7) to next consider

$$\begin{aligned}R_{n,\boldsymbol{\gamma}}(\mathbf{z})&:=\sum_{\mathfrak{u}\subseteq\mathcal{D}}\boldsymbol{\gamma}_{\mathfrak{u}}R_n(\mathbf{z},\mathfrak{u})\\&=\sum_{\mathfrak{u}\subseteq\mathcal{D}}\boldsymbol{\gamma}_{\mathfrak{u}}\left[\frac{1}{n}\sum_{k=0}^{n-1}\prod_{j\in\mathfrak{u}}\left(1+\sideset{}{'}\sum_{-n/2<h\leq n/2}\frac{e^{2\pi\mathrm{i}hkz_j/n}}{|h|}\right)-1\right]\end{aligned}$$

$$= \sum_{\mathfrak{u}\subseteq\mathcal{D}} \left[\frac{1}{n}\sum_{k=0}^{n-1}\prod_{j\in\mathfrak{u}}\gamma_j \left(1 + {\sum_{-n/2<h\le n/2}}' \frac{e^{2\pi \mathrm{i} hkz_j/n}}{|h|}\right) - \prod_{j\in\mathfrak{u}}\gamma_j \right]$$
$$= \sum_{\mathfrak{u}\subseteq\mathcal{D}} \frac{1}{n}\sum_{k=0}^{n-1}\prod_{j\in\mathfrak{u}}\gamma_j \left(1 + {\sum_{-n/2<h\le n/2}}' \frac{e^{2\pi \mathrm{i} hkz_j/n}}{|h|}\right) - \prod_{j=1}^{d}(1+\gamma_j) \,.$$

By interchanging the first two sums, we obtain

$$R_{n,\boldsymbol{\gamma}}(\mathbf{z}) = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{\mathfrak{u}\subseteq\mathcal{D}}\prod_{j\in\mathfrak{u}}\gamma_j \left(1 + {\sum_{-n/2<h\le n/2}}' \frac{e^{2\pi \mathrm{i} hkz_j/n}}{|h|}\right) - \prod_{j=1}^{d}(1+\gamma_j)$$
$$= \frac{1}{n}\sum_{k=0}^{n-1}\prod_{j=1}^{d} \left(1 + \gamma_j + \gamma_j {\sum_{-n/2<h\le n/2}}' \frac{e^{2\pi \mathrm{i} hkz_j/n}}{|h|}\right) - \prod_{j=1}^{d}(1+\gamma_j) \,.$$

Setting $\beta_j = 1+\gamma_j$, we then see that

$$R_{n,\boldsymbol{\gamma}}(\mathbf{z}) = \frac{1}{n}\sum_{k=0}^{n-1}\prod_{j=1}^{d} \left(\beta_j + \gamma_j {\sum_{-n/2<h\le n/2}}' \frac{e^{2\pi \mathrm{i} hkz_j/n}}{|h|}\right) - \prod_{j=1}^{d}\beta_j \,. \tag{9}$$

In the case $d=1$, it is not hard to verify that $R_{n,\boldsymbol{\gamma}}(\mathbf{z}) = 0$. We also see from this expression that for given dimension d, calculation of $R_{n,\boldsymbol{\gamma}}(\mathbf{z})$ would require $O(n^2 d)$ operations. However, the asymptotic expansion techniques found in [5] may be used to reduce this to $O(nd)$ operations. Further details may be found in Appendix A.

We shall obtain bounds on $R_{n,\boldsymbol{\gamma}}(\mathbf{z})$ for the case in which n is prime by obtaining an expression for the mean value of $R_{n,\boldsymbol{\gamma}}(\mathbf{z})$ taken over all integer vectors $\mathbf{z}\in\mathcal{Z}_n^d$, where $\mathcal{Z}_n = \{1,2,\ldots,n-1\}$. Thus the mean $M_{n,d,\boldsymbol{\gamma}}$ is defined by

$$M_{n,d,\boldsymbol{\gamma}} := \frac{1}{(n-1)^d}\sum_{\mathbf{z}\in\mathcal{Z}_n^d} R_{n,\boldsymbol{\gamma}}(\mathbf{z}) \,.$$

Theorem 1. *Let n be a prime number. Then*

$$M_{n,d,\boldsymbol{\gamma}} = \frac{1}{n}\prod_{j=1}^{d}(\beta_j + \gamma_j S_n) + \frac{n-1}{n}\prod_{j=1}^{d}\left(\beta_j - \gamma_j\frac{S_n}{n-1}\right) - \prod_{j=1}^{d}\beta_j \,,$$

where

$$S_n = {\sum_{-n/2<h\le n/2}}' \frac{1}{|h|} \,.$$

Proof. In (9) we can take out the $k=0$ term which is independent of $\mathbf{z}$ to obtain

$$\begin{aligned} &M_{n,d,\boldsymbol{\gamma}} \\ &= \frac{1}{n}\prod_{j=1}^{d}\left(\beta_j+\gamma_j S_n\right) \\ &\quad + \frac{1}{n}\sum_{k=1}^{n-1}\prod_{j=1}^{d}\left[\frac{1}{n-1}\sum_{z_j=1}^{n-1}\left(\beta_j+\gamma_j \sideset{}{'}\sum_{-n/2<h\le n/2}\frac{e^{2\pi \mathrm{i} h k z_j/n}}{|h|}\right)\right] - \prod_{j=1}^{d}\beta_j \,. \end{aligned}$$

Now define

$$T_n(k) := \frac{1}{n-1}\sum_{z=1}^{n-1}\sideset{}{'}\sum_{-n/2<h\le n/2}\frac{e^{2\pi \mathrm{i} h k z/n}}{|h|}\,, \qquad 0\le k\le n-1\,. \tag{10}$$

When $k=0$, $T_n(0)$ is simply S_n. For n prime and $1\le k\le n-1$ we see that k cannot be a multiple of n, and nor can h in the situation when $-n/2<h\le n/2$ with $h\neq 0$. Hence $hk\not\equiv 0 \pmod{n}$ and we have

$$\begin{aligned} T_n(k) &= \frac{1}{n-1}\sideset{}{'}\sum_{-n/2<h\le n/2}\sum_{z=1}^{n-1}\frac{e^{2\pi \mathrm{i} h k z/n}}{|h|} \\ &= \frac{1}{n-1}\sideset{}{'}\sum_{-n/2<h\le n/2}\frac{1}{|h|}\left(\sum_{z=0}^{n-1}\left(e^{2\pi \mathrm{i} h k/n}\right)^z - 1\right) = \frac{-S_n}{n-1}\,, \end{aligned} \tag{11}$$

which we note is independent of k. It then follows that

$$M_{n,d,\boldsymbol{\gamma}} = \frac{1}{n}\prod_{j=1}^{d}\left(\beta_j+\gamma_j S_n\right) + \frac{1}{n}\sum_{k=1}^{n-1}\prod_{j=1}^{d}\left(\beta_j+\gamma_j\frac{-S_n}{n-1}\right) - \prod_{j=1}^{d}\beta_j\,,$$

which leads to the desired result. □

In the case $d=1$, the expression for $M_{n,1,\gamma_1}$ simplifies to 0, which is as expected, since for $d=1$ the values of $R_{n,\gamma_1}(z_1)$ are all zero.

Since $\beta_j = 1+\gamma_j > \gamma_j$ and $S_n \le n-1$, we have $\beta_j > \beta_j - \gamma_j S_n/(n-1) \ge 1$ and so

$$\frac{n-1}{n}\prod_{j=1}^{d}\left(\beta_j - \gamma_j\frac{S_n}{n-1}\right) - \prod_{j=1}^{d}\beta_j < 0\,.$$

Moreover, we have from [10, Lemmas 1 and 2] that $S_n < 2\ln(n) + 1/n^2 - 0.2319$. So for $n\ge 3$ we have

$$S_n < 2\ln(n) \tag{12}$$

and direct calculation shows this holds for $n=2$ also. We then obtain the following corollary.

Corollary 1. *Let n be a prime number. Then there exists a generating vector $\mathbf{z}$ such that*

$$R_{n,\boldsymbol{\gamma}}(\mathbf{z}) \leq \frac{1}{n} \prod_{j=1}^{d} (1 + \gamma_j + \gamma_j S_n) \leq \frac{1}{n} \prod_{j=1}^{d} (1 + \gamma_j + 2\gamma_j \ln(n)) \ .$$

Now recall from (6) and the definition of $R_{n,\boldsymbol{\gamma}}(\mathbf{z})$ that

$$D^*_{n,\boldsymbol{\gamma}}(\mathbf{z}) \leq \sum_{\mathfrak{u} \subseteq \mathcal{D}} \gamma_{\mathfrak{u}} \left(1 - (1 - 1/n)^{|\mathfrak{u}|}\right) + \frac{R_{n,\boldsymbol{\gamma}}(\mathbf{z})}{2} \ . \tag{13}$$

This equation together with Lemma 1 and Corollary 1 show that if the γ_j are summable, then there exists a generating vector $\mathbf{z}$ such that

$$D^*_{n,\boldsymbol{\gamma}}(\mathbf{z}) \leq O\left(n^{-1}\right) + \frac{1}{2n} \prod_{j=1}^{d} (1 + \gamma_j + 2\gamma_j \ln(n)) \ ,$$

where the implied constant depends on the weights, but is independent of d. This bound for $D^*_{n,\boldsymbol{\gamma}}(\mathbf{z})$ has a $\ln(n)$ dependency. In order to obtain a bound without this $\ln(n)$ dependency, we can make use of the next lemma (stated and proved in [2]) and conclude that there exists a generating vector $\mathbf{z}$ such that

$$D^*_{n,\boldsymbol{\gamma}}(\mathbf{z}) = O\left(n^{-1+\delta}\right) \ ,$$

for any $\delta > 0$, where the implied constant depends on δ and the weights, but is independent of d and n.

Lemma 2. *Let $\tilde{\gamma}_j = 2\gamma_j/(1 + \gamma_j)$ and suppose that the γ_j are summable so that*

$$\sum_{j=1}^{\infty} \tilde{\gamma}_j < \infty \ .$$

Then for any $\delta > 0$, there exists $C(\tilde{\boldsymbol{\gamma}}, \delta)$, independent of d and n, such that

$$\prod_{j=1}^{d} (1 + \gamma_j + 2\gamma_j \ln(n)) \leq C(\tilde{\boldsymbol{\gamma}}, \delta) n^{\delta} \prod_{j=1}^{\infty} (1 + \gamma_j) \leq C(\tilde{\boldsymbol{\gamma}}, \delta) n^{\delta} e^{\sum_{j=1}^{\infty} \gamma_j} \ .$$

We recall from Section 1 that lattice rules of the Korobov form are those for which $\mathbf{z} = (1, a, \ldots, a^{d-1}) \pmod{n}$ for some a satisfying $1 \leq a \leq n - 1$. Writing such generating vectors as $\mathbf{z}(a)$, we now define the mean

$$\mu_{n,d,\boldsymbol{\gamma}} := \frac{1}{n-1} \sum_{a=1}^{n-1} R_{n,\boldsymbol{\gamma}}(\mathbf{z}(a)) \ .$$

The next result shows that $\mu_{n,d,\boldsymbol{\gamma}}$ satisfies a bound of the same order as the one given in Corollary 1. Hence there exist lattice rules of the Korobov form which have $O\left(n^{-1+\delta}\right)$ weighted star discrepancy.

Theorem 2. *Let n be a prime number. Then*

$$\mu_{n,d,\boldsymbol{\gamma}} \le \frac{d}{n-1} \prod_{j=1}^{d} (1+\gamma_j+\gamma_j S_n) \ .$$

Proof. The proof we present is similar to the proof of Theorem 1 in [18]. We see from (9) that $R_{n,\boldsymbol{\gamma}}(\mathbf{z}(a))$ is the error from applying the lattice rule to the function

$$f(\mathbf{x}) = \sum_{\mathbf{h}\in C^*_{n,d}\cup\{\mathbf{0}\}} \frac{e^{2\pi i \mathbf{h}\cdot\mathbf{x}}}{\prod_{j=1}^{d} r(\gamma_j,h_j)} \ ,$$

where

$$r(\gamma,h) = \begin{cases} 1+\gamma, & h=0 \ , \\ |h|/\gamma, & h\neq 0 \ . \end{cases}$$

It then follows from the theory of lattice rules that we may write

$$R_{n,\boldsymbol{\gamma}}(\mathbf{z}(a)) = \sum_{\mathbf{h}\in C^*_{n,d}} \frac{\delta_n(\mathbf{h}\cdot\mathbf{z}(a))}{\prod_{j=1}^{d} r(\gamma_j,h_j)} \ ,$$

where $\delta_n(m)$ denotes one or zero depending on whether $m \equiv 0 \pmod n$ or not.

From the definition of $\mu_{n,d,\boldsymbol{\gamma}}$, it follows that we have

$$\mu_{n,d,\boldsymbol{\gamma}} = \frac{1}{n-1} \sum_{\mathbf{h}\in C^*_{n,d}} \prod_{j=1}^{d} \frac{1}{r(\gamma_j,h_j)} \sum_{a=1}^{n-1} \delta_n(\mathbf{h}\cdot\mathbf{z}(a)) \ . \tag{14}$$

Since $\mathbf{h}\cdot\mathbf{z}(a) = h_1 + h_2 a + \cdots + h_d a^{d-1}$, we see this last sum is just the number of solutions of the congruence $h_1 + h_2 a + \cdots + h_d a^{d-1} \equiv 0 \pmod n$. Now because n is prime and $\mathbf{h}\in C^*_{n,d}$, then the greatest common divisor of the numbers $h_1, h_2, \ldots, h_d$ cannot be a multiple of n. It then follows from a well-known result in number theory (for example, see [6]) that the last sum in (14) is bounded by $d-1$. We then have

$$\begin{aligned} \mu_{n,d,\boldsymbol{\gamma}} &\le \frac{d}{n-1} \sum_{\mathbf{h}\in C^*_{n,d}} \prod_{j=1}^{d} \frac{1}{r(\gamma_j,h_j)} \\ &< \frac{d}{n-1} \prod_{j=1}^{d} \left(1+\gamma_j+\gamma_j \sideset{}{'}\sum_{-n/2<h\le n/2} \frac{1}{|h|} \right) \ , \end{aligned}$$

which leads to the desired bound. □

3 A Component-By-Component Construction

We shall now prove that for n prime we can construct $\mathbf{z}$ component-by-component such that

$$R_{n,\boldsymbol{\gamma}}(\mathbf{z}) \leq \frac{1}{n-1} \prod_{j=1}^{d} (\beta_j + \gamma_j S_n) \ ,$$

where we recall that $\beta_j = 1 + \gamma_j$.

Theorem 3. *Let n be a prime number. Suppose there exists a $\mathbf{z} \in \mathcal{Z}_n^d$ such that*

$$R_{n,\boldsymbol{\gamma}}(\mathbf{z}) \leq \frac{1}{n-1} \prod_{j=1}^{d} (\beta_j + \gamma_j S_n) \ , \quad \textit{where} \quad S_n = \sideset{}{'}\sum_{-n/2<h\leq n/2} \frac{1}{|h|} \ .$$

Then there exists $z_{d+1} \in \mathcal{Z}_n$ such that

$$R_{n,\boldsymbol{\gamma}}(\mathbf{z}, z_{d+1}) \leq \frac{1}{n-1} \prod_{j=1}^{d+1} (\beta_j + \gamma_j S_n) \ .$$

Such a z_{d+1} can be found by minimizing $R_{n,\boldsymbol{\gamma}}(\mathbf{z}, z_{d+1})$ over the set $\mathcal{Z}_n$.

Proof. For any $z_{d+1} \in \mathcal{Z}_n$ we have from (9) that

$$\begin{aligned} R_{n,\boldsymbol{\gamma}}(\mathbf{z}, z_{d+1}) = \beta_{d+1} R_{n,\boldsymbol{\gamma}}(\mathbf{z}) \\ &+ \frac{\gamma_{d+1}}{n} \sum_{k=0}^{n-1} \left[\prod_{j=1}^{d} \left(\beta_j + \gamma_j \sideset{}{'}\sum_{-n/2<h\leq n/2} \frac{e^{2\pi \mathrm{i} hkz_j/n}}{|h|} \right) \right] \\ &\times \left[\sideset{}{'}\sum_{-n/2<h\leq n/2} \frac{e^{2\pi \mathrm{i} hkz_{d+1}/n}}{|h|} \right] \ . \end{aligned} \tag{15}$$

Next we average over the possible $n-1$ values of z_{d+1} in the last term to form

$$\frac{1}{n-1} \sum_{z_{d+1}=1}^{n-1} \sideset{}{'}\sum_{-n/2<h\leq n/2} \frac{e^{2\pi \mathrm{i} hkz_{d+1}/n}}{|h|} \ , \qquad 0 \leq k \leq n-1 \ .$$

However, this is just the quantity $T_n(k)$ defined previously in (10).

It then follows from (15) by separating out the $k = 0$ term that there exists a $z_{d+1} \in \mathcal{Z}_n$ such that

$$
\begin{aligned}
R_{n,\boldsymbol{\gamma}}(\mathbf{z}, z_{d+1}) \le\ & \beta_{d+1} R_{n,\boldsymbol{\gamma}}(\mathbf{z}) + \frac{\gamma_{d+1}}{n} \prod_{j=1}^{d} (\beta_j + \gamma_j S_n)\, S_n \\
& + \frac{\gamma_{d+1}}{n} \sum_{k=1}^{n-1} \prod_{j=1}^{d} \left(\beta_j + \gamma_j \sum_{-n/2<h\le n/2}{}' \frac{e^{2\pi \mathrm{i} hkz_j/n}}{|h|} \right) T_n(k) \\
=\ & \beta_{d+1} R_{n,\boldsymbol{\gamma}}(\mathbf{z}) + \frac{\gamma_{d+1}}{n} \prod_{j=1}^{d} (\beta_j + \gamma_j S_n)\, S_n \\
& + \frac{\gamma_{d+1}}{n} \sum_{k=1}^{n-1} \prod_{j=1}^{d} \left(\beta_j + \gamma_j \sum_{-n/2<h\le n/2}{}' \frac{e^{2\pi \mathrm{i} hkz_j/n}}{|h|} \right) \left(\frac{-S_n}{n-1} \right) \\
=\ & \beta_{d+1} R_{n,\boldsymbol{\gamma}}(\mathbf{z}) + \frac{\gamma_{d+1}}{n} \prod_{j=1}^{d} (\beta_j + \gamma_j S_n)\, S_n \\
& + \frac{\gamma_{d+1} S_n}{n-1} \left(-\frac{1}{n} \sum_{k=0}^{n-1} \prod_{j=1}^{d} \left(\beta_j + \gamma_j \sum_{-n/2<h\le n/2}{}' \frac{e^{2\pi \mathrm{i} hkz_j/n}}{|h|} \right) \right. \\
& \qquad \left. + \ \frac{1}{n} \prod_{j=1}^{d} (\beta_j + \gamma_j S_n) \right) \ ,
\end{aligned}
$$

where we have made use of (11) and in the last equation, subtracted and added in the $k = 0$ term. By using (9) we find that for this z_{d+1} we have

$$
\begin{aligned}
R_{n,\boldsymbol{\gamma}}(\mathbf{z}, z_{d+1}) \le\ & \beta_{d+1} R_{n,\boldsymbol{\gamma}}(\mathbf{z}) + \frac{\gamma_{d+1}}{n} \prod_{j=1}^{d} (\beta_j + \gamma_j S_n)\, S_n \\
& + \frac{\gamma_{d+1} S_n}{n-1} \left(-R_{n,\boldsymbol{\gamma}}(\mathbf{z}) - \prod_{j=1}^{d} \beta_j + \frac{1}{n} \prod_{j=1}^{d} (\beta_j + \gamma_j S_n) \right) \\
\le\ & \beta_{d+1} R_{n,\boldsymbol{\gamma}}(\mathbf{z}) + \frac{\gamma_{d+1} S_n}{n} \left(\prod_{j=1}^{d} (\beta_j + \gamma_j S_n) \right) \left(1 + \frac{1}{n-1} \right) \\
=\ & \beta_{d+1} R_{n,\boldsymbol{\gamma}}(\mathbf{z}) + \frac{\gamma_{d+1} S_n}{n-1} \prod_{j=1}^{d} (\beta_j + \gamma_j S_n) \\
\le\ & \frac{1}{n-1} \left(\prod_{j=1}^{d} (\beta_j + \gamma_j S_n) \right) (\beta_{d+1} + \gamma_{d+1} S_n) \\
=\ & \frac{1}{n-1} \prod_{j=1}^{d+1} (\beta_j + \gamma_j S_n) \ ,
\end{aligned}
$$

where we have made use of the fact that $R_{n,\boldsymbol{\gamma}}(\mathbf{z})$ satisfies the assumed bound. This completes the proof. $\square$

Recalling that for $d = 1$ we have $R_{n,\gamma_1}(z_1) = 0$, the previous theorem leads to the following corollary.

Corollary 2. *Let n be a prime number. We can construct $\mathbf{z} \in \mathcal{Z}_n^d$ component-by-component such that for all $s = 1, \ldots, d$,*

$$R_{n,\boldsymbol{\gamma}}(z_1, \ldots, z_s) \leq \frac{1}{n-1} \prod_{j=1}^{s} (\beta_j + \gamma_j S_n) \ .$$

We can set $z_1 = 1$, and for $2 \leq s \leq d$, each z_s can be found by minimizing $R_{n,\boldsymbol{\gamma}}(z_1, \ldots, z_s)$ over the set $\mathcal{Z}_n$.

Since $1/(n-1) \leq 2/n$ for $n \geq 2$, this corollary together with (12) and (13) show that for n a prime number, we can construct $\mathbf{z}$ component-by-component such that

$$D_{n,\boldsymbol{\gamma}}^*(\mathbf{z}) \leq \sum_{\mathfrak{u} \subseteq \mathcal{D}} \boldsymbol{\gamma}_{\mathfrak{u}} \left(1 - (1 - 1/n)^{|\mathfrak{u}|}\right) + \frac{1}{n} \prod_{j=1}^{d} (1 + \gamma_j + 2\gamma_j \ln(n)) \ .$$

If the γ_j are summable we then see from Lemma 1 and Lemma 2 that the rank-1 lattice rule constructed in this manner is such that

$$D_{n,\boldsymbol{\gamma}}^*(\mathbf{z}) = O(n^{-1+\delta}) \ , \quad \delta > 0 \ ,$$

where the implied constant depends on δ and the weights, but is independent of d and n.

Appendix A shows that $R_{n,\boldsymbol{\gamma}}(\mathbf{z})$ may be calculated using asymptotic expansion techniques in $O(nd)$ operations. This together with Corollary 2 then shows that the cost of constructing the integer vector $\mathbf{z}$ up to dimension d is $O(n^2 d^2)$ operations. This can be reduced to $O(n^2 d)$ operations if we store the products during the construction, but this would be at the expense of $O(n)$ storage. We remark that in [12], Nuyens and Cools proposed a more efficient implementation of the component-by-component construction. Their construction of $\mathbf{z}$ was based on minimizing a function of the form

$$\frac{1}{n} \sum_{k=0}^{n-1} \prod_{j=1}^{d} \left(1 + \gamma_j \omega \left(\left\{\frac{k z_j}{n}\right\}\right)\right) - 1 \ ,$$

where ω was a certain function. Now we see from (9) that $R_{n,\boldsymbol{\gamma}}(\mathbf{z})$ may be written in a similar form since

$$R_{n,\boldsymbol{\gamma}}(\mathbf{z}) = \frac{1}{n} \sum_{k=0}^{n-1} \prod_{j=1}^{d} \left(\beta_j + \gamma_j \omega \left(\left\{\frac{k z_j}{n}\right\}\right)\right) - \prod_{j=1}^{d} \beta_j \ ,$$

where

$$\omega(x) = \sum_{-n/2<h\le n/2}{}^{'} \frac{e^{2\pi i h x}}{|h|} , \quad x \in [0,1] .$$

With some minor modifications, the approach of Nuyens and Cools may then be used to similarly speed up the component-by-component construction proposed here so that only $O(n \ln(n) d)$ operations are required.

4 Results For The Weighted L_p Discrepancy

In this section we apply the results of the previous two sections to obtain corresponding results for the weighted L_p discrepancy which we define below. From Zaremba's identity given in (2) one can apply Hölder's inequality for integrals and sums to obtain

$$|Q_{n,d}(f) - I_d(f)| \le D_{n,\boldsymbol{\gamma},p}(\mathbf{z}) \left(\sum_{\mathfrak{u}\subseteq\mathcal{D}} \boldsymbol{\gamma}_{\mathfrak{u}}^{-q} \int_{[0,1]^{|\mathfrak{u}|}} \left| \frac{\partial^{|\mathfrak{u}|}}{\partial \mathbf{x}_{\mathfrak{u}}} f(\mathbf{x}_{\mathfrak{u}}, \mathbf{1}) \right|^q \mathrm{d}\mathbf{x}_{\mathfrak{u}} \right)^{1/q} ,$$

where $D_{n,\boldsymbol{\gamma},p}(\mathbf{z})$, the weighted L_p discrepancy, is defined by

$$D_{n,\boldsymbol{\gamma},p}(\mathbf{z}) := \left(\sum_{\emptyset\ne\mathfrak{u}\subseteq\mathcal{D}} \boldsymbol{\gamma}_{\mathfrak{u}}^{p} \int_{[0,1]^{|u|}} |\mathrm{discr}\left((\mathbf{x}_{\mathfrak{u}}, \mathbf{1}), P_n\right)|^p \, \mathrm{d}\mathbf{x}_{\mathfrak{u}} \right)^{1/p} ,$$

with the local discrepancy $\mathrm{discr}\,(\mathbf{x}, P_n)$ for any $\mathbf{x} \in [0,1]^d$ being defined in (1) and $1/p + 1/q = 1$, p, $q \ge 1$. Then we see that we have

$$D_{n,\boldsymbol{\gamma},p}(\mathbf{z}) \le \left[\sum_{\mathfrak{u}\subseteq\mathcal{D}} \left(\boldsymbol{\gamma}_{\mathfrak{u}} \sup_{\mathbf{x}_{\mathfrak{u}}\in[0,1]^{|\mathfrak{u}|}} |\mathrm{discr}\left((\mathbf{x}_{\mathfrak{u}}, \mathbf{1}), P_n\right)| \right)^p \right]^{1/p} .$$

Now Jensen's inequality shows that for $\lambda \ge 1$,

$$\left(\sum a_i^{\lambda}\right)^{1/\lambda} \le \sum a_i ,$$

where the a_i are arbitrary non-negative numbers. So for $p \ge 1$ we can take $\lambda = p$ and hence obtain

$$D_{n,\boldsymbol{\gamma},p}(\mathbf{z}) \le \sum_{\mathfrak{u}\subseteq\mathcal{D}} \boldsymbol{\gamma}_{\mathfrak{u}} \sup_{\mathbf{x}\in[0,1]^{|\mathfrak{u}|}} |\mathrm{discr}\left((\mathbf{x}_{\mathfrak{u}}, \mathbf{1}), P_n\right)| .$$

The bound on the right-hand side of this expression is the bound analyzed in Section 2 (for example, see (5) and (6)). Hence the results from that section and Section 3 hold. Suppose we apply the component-by-component algorithm implied in Corollary 2. Then, under the assumption that the weights are summable, the generating vector $\mathbf{z}$ constructed yields a point set that not only has a weighted star discrepancy of $O(n^{-1+\delta})$, $\delta > 0$, but also has a weighted L_p discrepancy of the same order.

In the case $p = 2$, Kuo [7] showed that the component-by-component algorithm achieves the optimal $O(n^{-1+\delta})$ rate for the weighted L_2 discrepancy if the sum of the square roots of the weights is finite. Since the weights used in [7] are the squares of the weights considered in this paper, the condition in [7] is equivalent to the condition here that the weights are summable. Moreover, the proof of the result in [7] was in a randomized setting, that is, it applied only to randomly shifted lattice rules. In contrast, the previous paragraph indicates that under the same condition on the weights, the component-by-component contruction presented here yields a purely deterministic point set whose weighted L_2 discrepancy is $O(n^{-1+\delta})$.

A Calculation of $R_{n,\boldsymbol{\gamma}}(\mathbf{z})$

Here we provide details of how

$$R_{n,\boldsymbol{\gamma}}(\mathbf{z}) = \frac{1}{n}\sum_{k=0}^{n-1}\prod_{j=1}^{d}\left(\beta_j + \gamma_j \sum_{-n/2<h\le n/2}{}' \frac{e^{2\pi \mathrm{i} hkz_j/n}}{|h|}\right) - \prod_{j=1}^{d}\beta_j$$

may be calculated in $O(nd)$ operations. We see that because $\{kz_j/n\} = m/n$ for some m satisfying $0 \le m \le n-1$, then to calculate $R_{n,\boldsymbol{\gamma}}(\mathbf{z})$ we need to have the values of

$$\beta_j + \gamma_j \sum_{-n/2<h\le n/2}{}' \frac{e^{2\pi \mathrm{i} hm/n}}{|h|}\,, \quad 0 \le m \le n-1\,.$$

However, if

$$f_n(x) = 1 + \sum_{-n/2<h\le n/2}{}' \frac{e^{2\pi \mathrm{i} hx}}{|h|}\,, \quad x \in [0,1]\,, \tag{16}$$

then

$$\beta_j + \gamma_j \sum_{-n/2<h\le n/2}{}' \frac{e^{2\pi \mathrm{i} hm/n}}{|h|} = \beta_j + \gamma_j(f_n(m/n) - 1)\,.$$

Since $\overline{f_n(1-x)} = f_n(x)$ for $x \in [0,1]$, then to calculate $R_{n,\boldsymbol{\gamma}}(\mathbf{z})$ we need to have the values of $f_n(m/n)$ for $0 \le m \le \lfloor n/2 \rfloor$. These $\lfloor n/2 \rfloor + 1$ values may be calculated once and then stored.

Suppose we wish to calculate $f_n(m/n)$ with an absolute error of at most ε. Then the results in [5] show that if ℓ and L are positive integers satisfying

$$2 \le \ell \le \left(\frac{6n^2}{\pi^2}\right)^{1/3} \quad \text{and} \quad \frac{4(L+1)!}{(\ell-1)^{L+2}\pi^{L+2}} \le \varepsilon\,, \tag{17}$$

then we should calculate $f_n(m/n)$ directly using (16) for $0 \le m < \ell$. For $\ell \le m \le \lfloor n/2 \rfloor$ we use the approximation $G(m/n)$, where for n odd,

$$G(x) = 1 - 2\ln(2|\sin(\pi x)|) - 2\sum_{i=0}^{L} b_i(x)\cos(\pi[(n+i)x + (i+1)/2]) \, .$$

In this expression, $b_0(x) = 1/\left[(n+1)|\sin(\pi x)|\right]$ and

$$b_{i+1}(x) = \frac{-(i+1)}{(n+2i+3)|\sin(\pi x)|} b_i(x) \, .$$

Similar expressions for G and the b_i are available when n is an even number.

When $\varepsilon = 2.0 \times 10^{-16}$, then for $n \ge 115$, (17) is satisfied with the choices $\ell = 20$ and $L = 14$. As another example, if $\varepsilon = 1.0 \times 10^{-18}$, then for $n \ge 161$, we can take $\ell = 25$ and $L = 15$. So we see that the $\lfloor n/2 \rfloor + 1$ values of $f_n(m/n)$ required may be obtained with an absolute error of at most ε in $O(\ell n) + O(L) \times (\lfloor n/2 \rfloor + 1 - \ell) = O(n)$ operations which means that even if n is large, $R_{n,\boldsymbol{\gamma}}(\mathbf{z})$ may be calculated in $O(nd)$ operations.

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