Odd multiperfect numbers of abundancy four

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1. INTRODUCTION

We say a number is multiperfect of abundancy k (or k-perfect) if $\sigma(N) = kN$. No k-perfect odd numbers are known for any $k \geq 2$, and it is believed that none exist. For a survey of known results see [6] or [3] and the references given there. For example, if N is odd and 4-perfect then N has at least 22 distinct prime factors. If it is also not divisible by 3 then it has at least 142 prime factors.

In this paper we consider properties of classes of odd numbers which must be satisfied if they are to be 4-perfect. Conversely, properties of classes which can never be 4-perfect. In a number of cases theorems follow, with some changes, in the pattern of corresponding results for 2-perfect numbers. However, mostly because of the number of primes involved, some of those techniques, from the theory of 2-perfect numbers, are not so readily available.

We show that Euler's structure theorem, that every odd 2-perfect number has the shape $N = q^e p_1^{2\alpha_1} \cdots p_m^{2\alpha_m}$, where $q \equiv e \equiv 1 \bmod 4$, has an extension to odd 4-perfect numbers, and then to odd 2^k-perfect numbers. For 4-perfect numbers there are three possible shapes like Euler's form, (A) with 2 q's instead of 1, (B) with $q \equiv 3 \bmod 8$ and $e \equiv 1 \bmod 4$, and (C) with $q \equiv 1 \bmod 4$ and $e \equiv 3 \bmod 8$. An immediate corollary is that no square or square free number is 4-perfect.

For 2^k-perfect numbers we need to derive a fact, which could be of independent interest. For $j \geq 1$, odd primes p and odd e, we have $2^j || \sigma(p^e)$ if and only if $2^{j+1} || (p+1)(e+1)$.

We include negative results (i.e. shapes which no odd 4-perfect number can have) for odd 4-perfect cubes, numbers with 9 being the maximum power of 3 dividing N, numbers with each of the p_i occurring to the power 2, and a positive result on the power of 3 dividing any odd 2^k-perfect number.

2. RESTRICTED FORMS FOR AN ODD PERFECT NUMBER OF ABUNDANCY 4

We begin with two lemmas, summarizing well known results.

LEMMA 2.1. *Let d and n be whole numbers and p a prime number. If $d+1 \mid n+1$ then $\sigma(p^d) \mid \sigma(p^n)$.*

LEMMA 2.2. *(Congruences modulo 3)*
Let p be an odd prime with $p \neq 3$.
(1) Let j be any natural number and $p > 3$ an odd prime. Then $\sigma(p^{6j}) \equiv 1 \bmod 3$.
(2) Let the whole number j be odd. If $p \equiv 1 \bmod 3$ then $\sigma(p^{3j}) \equiv 1 \bmod 3$. If $p \equiv 2 \bmod 3$ then $\sigma(p^{3j}) \equiv 0 \bmod 3$.
(3) Let j be a natural number. If $p \equiv 1 \bmod 3$ then $\sigma(p^{1+3j}) \equiv 2 \bmod 3$. If $p \equiv 2 \bmod 3$ then $\sigma(p^{1+3j}) \equiv 0 \bmod 3$ if j is even and $\sigma(p^{1+3j}) \equiv 1 \bmod 3$ if j is odd.
(4) Let j be any natural number. If $p \equiv 1 \bmod 3$ then $\sigma(p^{2+3j}) \equiv 0 \bmod 3$. If $p \equiv 2 \bmod 3$ then $\sigma(p^{2+3j}) \equiv 1 \bmod 3$ if j is even and $\sigma(p^{2+3j}) \equiv 0 \bmod 3$ if j is odd.

This set of results is best summarized in a table, with the rows corresponding to values of p modulo 3, in the first column, and the columns the values of $\sigma(p^e) \bmod 3$ for values of e modulo 6 which are in the first row:

p/e	0	1	2	3	4	5
1	1	2	0	1	2	0
2	1	0	1	0	1	0

THEOREM 2.1. *(Euler equivalent)*
Let N be an odd 4-perfect number. Then N has one of the following forms, where the α_i are whole numbers and the p_i odd primes:
(A) $N = q_1^{e_1} q_2^{e_2} p_1^{2\alpha_1} \cdots p_m^{2\alpha_m}$ for primes q_i and whole numbers e_i with $q_i \equiv e_i \equiv 1 \bmod 4$.
In the remaining types $N = q^e p_1^{2\alpha_1} \cdots p_m^{2\alpha_m}$ where:

(B) $q \equiv 1 \bmod 4$ *and* $e \equiv 3 \bmod 8$ *or*
(C) $q \equiv 3 \bmod 8$ *and* $e \equiv 1 \bmod 4$.

Proof. (1) Let $N = p_1^{\beta_1} \cdots p_m^{\beta_m}$ where the p_i are odd primes and the β_i whole numbers. Then $\sigma(N) = 4N$ implies $2^2 || \sigma(p_1^{\beta_1}) \cdots \sigma(p_m^{\beta_m})$ so either 2^1 is the maximum power of two dividing two distinct terms in the product and the remaining terms are odd, or 2^2 is the maximum power dividing one term and the remaining terms are odd. So type (A) is the former shape and (B) and (C) the latter. Therefore we need only consider primes q and powers α such that $2^2 || \sigma(q^\alpha)$.

(2) Claim: If $q \equiv 1 \bmod 4$ and $\alpha \equiv 3 \bmod 8$ then $4 \mid \sigma(q^\alpha)$. To see this let $\alpha = 3 + 8e$ and $q = 1 + 4x$ then (where f and y are integers)

$$\begin{aligned}\sigma(q^\alpha) &= \frac{(1+4x)^{4+8e} - 1}{4x} \\ &= \frac{(1+4x)^{4f} - 1}{4x} \\ &= \frac{1}{4x}(4x \cdot 4f + \binom{4f}{2}(4x)^2 + (4x)^3 y) \\ &= 4g\end{aligned}$$

so $4 \mid \sigma(q^\alpha)$.

(3) In the same situation as in (2), $8 \nmid \sigma(q^\alpha)$: Write

$$\sigma(q^\alpha) = 1 + q + q^2 + q^3 + \cdots + q^{3+8e},$$

group the $4 + 8e$ terms in $1 + 2e$ sets of 4 terms, so that

$$\sigma(q^\alpha) \equiv (1 + q + q^2 + q^3)(1 + 2e) \bmod 8,$$

where we have used $q^4 \equiv 1 \bmod 8$. Replacing q by $1 + 4x$ and reducing modulo 8 we get $\sigma(q^\alpha) \equiv 4 \cdot (1 + 2e) \bmod 8$, which is non-zero, so $8 \nmid \sigma(q^\alpha)$.

(4) Claim: If $q \equiv 3 \bmod 8$ and $\alpha \equiv 1 \bmod 4$ then $4 \mid \sigma(q^\alpha)$. Let $\alpha = 1 + 4e$ and $q = 3 + 8x$ then (where f, y, z and w are integers)

$$\begin{aligned}\sigma(q^\alpha) &= \frac{(3+8x)^{2f} - 1}{2+8x} \quad \text{where } f \text{ is odd} \\ &= \frac{(1+2y)^{2f} - 1}{2y} \quad \text{where } y \text{ is odd} \\ &= \frac{1}{2y}(2y \cdot 2f + \binom{2f}{2}(2y)^2 + \cdots) \\ &= 2f + 2f(2f-1)y + 4z \\ &= 4w\end{aligned}$$

so $4 \mid \sigma(q^\alpha)$.

(5) In the same situation as in (4) $8 \nmid \sigma(q^\alpha)$: write

$$\sigma(q^\alpha) = \frac{q^{2+4e}-1}{2+8x} \equiv \frac{q^2-1}{2} \equiv \frac{(3+8x)^2-1}{2} \equiv 4 \bmod 8.$$

so, again $8 \nmid \sigma(q^\alpha)$.

(6) The remainder of the proof consists in showing the above cases constitute the only possibilities by examining in turn the 12 possible additional values of $\{q, e\}$ modulo 8. In summary, using the notation q^e for the values of q and e modulo 8, and using the same techniques as used in parts (2), (3) and (4) of the proof, the cases $1^1, 1^5, 5^1, 5^5$ give $4 \nmid \sigma(q^e)$. The cases $1^7, 3^3, 3^7, 5^7, 7^1, 7^3, 7^5, 7^7$ give $8 \mid \sigma(p^e)$, so cannot occur. The remaining cases $5^3, 3^5$ are covered by (B) and (C). ∎

Corollary 2.1. *No square or square free number is odd and 4-perfect.*

Proof. Since the exponents of the leading primes are odd, and one of the three forms is always present, the first part of the claim is immediate. For the second part we need only consider the special forms $N = q_1 q_2$ and $N = q_1$, where the q_i are odd primes to see that $m \neq 0$, so no odd 4-perfect number is square free. ∎

It might be of interest to speculate, on the basis of Euler's theorem and the above, on the general form for division of $\sigma(p^\alpha)$ by powers of 2. However for powers 2^3, and beyond, the situation appears to be well structured but mysterious.

For example, in the following each pair corresponds to the classes modulo 2^4 of an odd prime and odd exponent (p, e) such that $2^3 || \sigma(p^e)$. The list appears to be complete for this power of 2:

$$\begin{gathered}(1,7),(3,3),(3,11),(5,7),(7,1),(7,5),\\(7,9),(7,13),(9,7),(11,3),(11,11),(13,7).\end{gathered}$$

Note that in each case $2^4 || (p+1)(e+1)$. It is a beautiful fact that this is true in general for all powers of 2.

Theorem 2.2. *(P^+E^+) For all odd primes p, powers $j \geq 1$ and odd exponents $e > 0$ we have*

$$2^j || \sigma(p^e) \iff 2^{j+1} || (p+1)(e+1).$$

Proof. (1) Let $2^j || \sigma(p^e)$. First expand p to base 2:

$$p = 1 + e_1 2^1 + e_2 2^2 + \cdots + 2^{j+1}\eta,$$

where $\eta \in \{0\} \cup \mathbb{N}$ and $e_i \in \{0,1\}$. There exits a minimum i with $1 \leq i \leq j$ so that

$$p = 1 + 2^1 + 2^2 + \cdots + 2^{i-1} + 0.2^i + \cdots + 2^{j+1}\eta$$

since otherwise

$$p = 1 + 2^1 + \cdots + 2^j + 2^{j+1}\eta \equiv -1 \bmod 2^{j+1}$$

so

$$\begin{aligned} \sigma(p^e) &= 1 + p + p^2 + \cdots + p^e \\ &\equiv 1 - 1 + 1 \cdots - 1 \equiv 0 \bmod 2^{j+1} \end{aligned}$$

so $2^{j+1} | \sigma(p^e)$ which is impossible. Hence we can write

$$p = 2^i - 1 + 2^{i+1}\beta, \beta \in \{0\} \cup \mathbb{N}.$$

Therefore $p + 1 = 2^i \cdot o$ where here, and in what follows, "o" represents a generic odd integer, with not necessarily the same value in a given expression.

Since $e + 1$ is even, there exists an $l \geq 1$ such that $e + 1 = 2^l \cdot o$. Since $2^j || \sigma(p^e)$ we have

$$\frac{p^{2^l \cdot o} - 1}{p - 1} = 2^j \cdot o$$

and therefore $(2^i \cdot o - 1)^{2^l \cdot o} - 1 = 2^j \cdot o \cdot (2^i \cdot o - 2)$. Call this equation (1).

(1a) If $i > 1$ examine both sides of equation (1) in base 2 and equate the lowest powers of 2. This leads to $i + l = j + 1$ since $2^i \cdot o - 2 = 2 \cdot o$. Therefore $l = j - i + 1$.

(1b) If $i = 1$ write $p + 1 = 2.o$ so $p - 1 = 2^k \cdot o$ for some $k \geq 2$. Hence, because $2^j || \sigma(p^e)$,

$$\begin{aligned} p^{2^l \cdot o} - 1 &= 2^j \cdot 2^k \cdot o \\ (1 + 2^k \cdot o)^{2^l \cdot o} - 1 &= 2^{j+k} \cdot o \end{aligned}$$

so, again comparing the lowest powers of 2 on both sides, $k + l = j + k$ so $l = j = j - 1 + 1$. Hence, for all $i \geq 1$, $l = j - i + 1$ and we can write

$$\begin{aligned} p &= 2^i - 1 + 2^{i+1} \cdot x \\ e &= 2^{j-i+1} - 1 + 2^{j+1-i+1} \cdot y \end{aligned}$$

where x, y are integers. Hence $(p+1)(e+1) = 2^{j+1}(1+2x)(1+2y)$ so $2^{j+1}||(p+1)(e+1)$.

(2) Conversely, let $2^{j+1}||(p+1)(e+1)$ so for some $i > 0$, $2^i||p+1$ and $2^{j+1-i}||e+1$. We now consider two cases, depending on the values if i and j.

(2a) Let $i = 1$ and $j = 1$. (This is really Euler's theorem). In this case $p+1 = 2\cdot o = 2(2x+1)$ so $p = 4x+1$ and $e+1 = 2\cdot o$. Therefore

$$\begin{aligned}\sigma(p^e) &= \frac{p^{2\cdot o}-1}{p-1} = \frac{p^o-1}{p-1}(p^o+1)\\ &= (1+p+\cdots+p^{o-1})((4x+1)^o+1)\\ &= o\cdot(4y+2) = 2\cdot o\end{aligned}$$

so $2^1||\sigma(p^e)$.

(2b) Let $i = 1$ and $j > 1$. Again $p = 4x+1$. The inductive hypothesis is that for all $j' < j$, $2^{j'}||(p^{2^{j'}\cdot o}-1)/(p-1)$. Then

$$\begin{aligned}\sigma(p^e) &= \frac{p^{2^{j-1}\cdot o}-1}{p-1}(p^{2^{j-1}\cdot o}+1)\\ &= 2^{j-1}\cdot o((4x+1)^{2^{j-1}\cdot o}+1)\\ &= 2^{j-1}\cdot o(4y+2)\\ &= 2^j\cdot o\end{aligned}$$

so in this case also $2^j||\sigma(p^e)$.

(3) First we make some preliminary polynomial constructions where all polynomials are in $\mathbb{Z}[x]$. For $n \in \mathbb{N}$ define f_n, q_n, s_n, r_n by

$$\begin{aligned}f_n(x) &= (1+x)^n - 1 = xq_n(x)\\ s_n(x) &= (1+x)^n + 1 = (x+2)r_n(x) \text{ for } n \text{ odd.}\end{aligned}$$

Then

$$f_{2\cdot o} = ((1+x)^o-1)((1+x)^o+1) = x\cdot r_o(x)\cdot(x+2)\cdot q_o(x),$$

and for $l \geq 1$

$$\begin{aligned}f_{2^l\cdot o} &= f_{2^{l-1}\cdot o}(x)\cdot s_{2^{l-1}\cdot o}(x)\\ &= s_{2^{l-1}\cdot o}(x)\cdot s_{2^{l-2}\cdot o}(x)\cdots s_{2\cdot o}(x)x(x+2)\cdot r_o(x)\cdot q_o(x)\\ s_{2^l\cdot o} &= (((1+x)^{2^l})^o-(-1))\\ &= ((1+x)^{2^l}-(-1))(((1+x)^{2^l})^{o-1}+\cdots+1)\\ &= ((1+x)^{2^l}+1)(\cdots)\end{aligned}$$

Since $i > 1$, $x = 2^i \cdot o - 2 = 2 \cdot o$. Hence $x + 1 = 2 \cdot o + 1 = o$ and

$$s_{2^l \cdot o} = (o^{2^l} + 1)(\text{an even number of odd terms } + 1) = 2 \cdot o \cdot o = 2 \cdot o,$$

and $x + 2 = 2^i \cdot o$. Note also that $r_o(x) = ((1 + 2y)^o - 1)/(2y) = o + 2z = o$ and $q_o(x) = ((1 + x)^o)/(x + 2) = (o^{o-1} - o^{o-2} \cdots + 1) = o$. Therefore, with this value of x

$$\frac{2^l \cdot o}{x} = 2^{l-1} \cdot o \cdot 2^i \cdot o \cdot o \cdot o = 2^{l+i-1} \cdot o.$$

Now, at last, we can complete the proof. Let $x = p - 1 = 2^i \cdot o - 2$ and $l = j + 1 - i$. Then

$$\begin{aligned} \sigma(p^e) &= \frac{p^{e+1} - 1}{p - 1} = \frac{(1 + x)^{2^l \cdot o} - 1}{x} \\ &= f_{2^l \cdot o}(x) = 2^{l+i-1} \cdot o = 2^j \cdot o \end{aligned}$$

so $2^j || \sigma(p^e)$. ▮

COROLLARY 2.2. *Let M_q be a Mersenne prime and e odd with $e \geq 1$. If $2^j || \sigma(M_q^e)$ then $j \geq q$.*

From the theorem we also get the following corollary, which is an extension of Euler's theorem to perfect numbers of abundancy 2^k.

COROLLARY 2.3. *Let N be odd and 2^k-perfect. Then there exists a partition of k, $k = k_1 + \cdots + k_n$, with $k_i \geq 1$, such that*

$$N = \prod_{i=1}^{n} p_i^{e_i} \prod_{j=1}^{m} q_j^{2f_j}$$

where the e_i are odd, the p_i, q_j odd primes, and for each i with $1 \leq i \leq n$ there exist positive integers l_i and m_i such that $2^{l_i} || p_i + 1$, $2^{m_i} || e_i + 1$ and $l_i + m_i = k_i + 1$.

THEOREM 2.3. *(Cubes)*
Let N be an odd cube with $3 \nmid N$.
(A) If N has shape $N = q_1^{1+4e_1} \cdot q_2^{1+4e_2} \cdot p_1^{2\alpha_1} \cdots p_m^{2\alpha_m}$ and $q_1 \equiv 5 \bmod 12$ (i.e. $q_1 \equiv 1 \bmod 4$ and $2 \bmod 3$) and $q_2 \equiv 1 \bmod 4$, then N is not a 4-perfect number.

(B) If N has shape $N = q^{1+4e}p_1^{2\alpha_1}\cdots p_m^{2\alpha_m}$ and $q \equiv 11 \bmod 24$ (i.e. $q \equiv 3 \bmod 8$ and $2 \bmod 3$), then N is not a 4-perfect number.

(C) If N has shape $N = q^{3+8e}p_1^{2\alpha_1}\cdots p_m^{2\alpha_m}$ and $q \equiv 5 \bmod 12$ (i.e. $q \equiv 1 \bmod 4$ and $2 \bmod 3$), then N is not a 4-perfect number.

Proof. (C) Let N be an odd and 4-perfect cube with $q \equiv 2 \bmod 3$. Then we can write:

$$\sigma(N) = \sigma(q^{3+24e})\sigma(p_1^{6\alpha_1})\cdots\sigma(p_m^{6\alpha_m}).$$

By Lemma 2.2, the first factor on the right is congruent to 0 modulo 3, so $\sigma(N) \equiv 0 \bmod 3$. Since $\sigma(N) = 4N$,

$$0 \equiv q^{3+24e}p_1^{6\alpha_1}\cdots p_m^{6\alpha_m} \bmod 3,$$

but each factor on the right hand side is non-zero modulo three. Hence N is not 4-perfect.

In parts (A) and (B) the result also follows since $\sigma(q^{3+6f}) \equiv 0 \bmod 3$. ▮

THEOREM 2.4. *(Nine is the maximum power of three dividing N)*

If N is a whole number with $3^2||N$ and such that if 13, 61 and 97 appear in the prime factorization of N, they do so to powers congruent to 2 modulo 6. Then N is not an odd 4-perfect number.

Proof. Let the hypotheses of the theorem hold for N, but let it also be odd and 4-perfect. Then $3^2||N$ implies $\sigma(3^2) = 13 \mid N$. So 13 must appear, and by the argument given below, 61 and 97 must also appear.

Now, by Lemma 2.1, for all primes p and natural numbers e, $\sigma(p^2) \mid \sigma(p^{2+6e})$. So $3 \cdot 61 = \sigma(13^2) \mid \sigma(13^{2\alpha}) \mid N$, for some $\alpha \geq 1$, which implies $3 \cdot 61 \mid N$. Again $3 \cdot 13 \cdot 97 = \sigma(61^2) \mid \sigma(61^{2\beta}) \mid N$, for some $\beta \geq 1$, which implies $3 \cdot 13 \cdot 97 \mid N$. Finally $3 \cdot 3169 = \sigma(97^2) \mid \sigma(97^{2\gamma}) \mid N$, for some $\gamma \geq 1$, so $3 \mid N$.

Now if 13, 61 or 97 appear, even though each is congruent to 1 modulo 4, their powers, being congruent to 2 modulo 6, are even, so must appear amongst the p_i in each of the three shapes given in Theorem 2.1. Therefore $3^3 \mid N$, which is a contradiction. Therefore N is not 4-perfect. ▮

The following result uses techniques similar to those developed for 2-perfect numbers by Steuerwald in [8].

THEOREM 2.5. *(Small powers)*

(1) If N is odd, $3 \mid N$ and N has the shape either (1a) $N = q_1^{1+4e_1} \cdot q_2^{1+4e_2} \cdot 3^2 \cdot p_1^2 \cdots p_m^2$ or (1b) $N = q_1^{3+8e_1} \cdot 3^2 \cdot p_1^2 \cdots p_m^2$ where, in either case, $q_i \equiv 1 \bmod 4$, or (1c) $q_1^{1+4e} \cdot 3^2 \cdot p_1^2 \cdots p_m^2$, where $q_1 \equiv 3 \bmod 8$, where the primes are distinct, then N is not an odd 4-perfect number.

(2) If N is odd, $3 \nmid N$ and N has the shape either (2a) $N = q^{3+8e} \cdot p_1^2 \cdots p_m^2$ with $q \equiv 1 \bmod 4$ or (2b) $N = q^{1+4e} \cdot p_1^2 \cdots p_m^2$ with $q \equiv 3 \bmod 8$, or (2c) $N = q_1^{1+4e_1} \cdot q_2^{1+4e_2} \cdot p_1^2 \cdots p_m^2$, with $q_i \equiv 1 \bmod 4$, then N is not a 4-perfect number.

Proof. (1) Let N satisfy $\sigma(N) = 4N$. Then $\sigma(3^2) = 13 \mid N$. In case (1c), q_1 is *not* in the set $\{13, 61, 97\}$. Assume first that the q_i are not in this set in cases (1a) and (1b). (Below we consider the situation which arises when a q_i *is* in this set.)

Under this assumption we obtain the chain:

$$\sigma(13^2) = 3 \cdot 61, \quad \sigma(61^2) = 3 \cdot 13 \cdot 97, \quad \sigma(97^2) = 3 \cdot 3169,$$

so $3^3 \mid N$, which is false. Hence N is not 4-perfect.

Since the exponent of each q_i is odd, for $q = q_1$ or q_2, $e = e_1$ or e_2, $q + 1 \mid \sigma(q^e)$.

If $q = 13$, since $q + 1 \mid N$ we obtain the chain:

$$\sigma(7^2) = 3 \cdot 19 \mid N, \sigma(19^2) = 3 \cdot 127 \mid N, \sigma(127^2) = 3 \cdot 5419 \mid N,$$

giving $3^3 \mid N$, which is false.

If $q = 61$ we can assume also $\sigma(13^2) = 3 \cdot 61 \mid N$. Again, since $q + 1 \mid N$ We obtain the chain:

$$\sigma(31^2) = 3 \cdot 331 \mid N, \sigma(331^2) = 3 \cdot 7 \cdot 5233 \mid N, \sigma(127^2) = 3 \cdot 5419 \mid N,$$

again giving $3^3 \mid N$, which is false.

If $q = 97$ then $(q + 1)/2 = 7^2 \mid N$ and the same chain as in the $q = 13$ case can be derived with the same conclusion. Thus our assumption that no q_i is in the set $\{13, 61, 97\}$ is valid and the proof is complete.

(2a) and (2b): Let N satisfy $\sigma(N) = 4N$ and $3 \nmid N$, with shape

$$N = q^f \cdot p_1^2 \cdots p_m^2,$$

where $3 < p_1 < \cdots < p_m$ and f is odd.

Since, for each i, $\sigma(p_i^2) = 1 + p_i + p_i^2$ and $3 \nmid N$, we must have $p_i \equiv 2 \bmod 3$.

By Theorem 2.1, q is congruent to 1 modulo 4 or 3 modulo 8. Because f is odd, $q + 1 \mid \sigma(q^f) \mid N$ and since also $3 \nmid N$ we cannot have $q \equiv 2 \bmod 3$, so must have $q \equiv 1 \bmod 3$.

Since $\sigma(p_1^2) < (p_1 + 1)^2 < p_2^2$, $\sigma(p_1^2)$ is divisible by at most one p_i. Therefore either (a) $\sigma(p_1^2) = q^g$ with $1 \leq g$ or (b) $\sigma(p_1^2) = q^g \cdot p_i$ for some i. Case (b) is impossible, since it is invalid modulo 3. In case (a), [1, Lemma 1] shows the only possibility is $g = 1$.

Let $x = (q+1)/2$. Then $x \equiv 1 \bmod 3$. Since x is too small to include a power of at least two q's, it must be a product of the p_i. We cannot have $x = p_i$ since $p_i \equiv 2 \bmod 3$, so it must have at least 2 prime factors, with the smallest factor being less than or equal to $\sqrt{x}$, and therefore $p_i \leq \sqrt{x}$ for some i. But then

$$q = 1 + p_1 + p_1^2 \leq 1 + p_i + p_i^2 \leq 1 + \sqrt{x} + x \leq \frac{q+3}{2} + \sqrt{\frac{q+1}{2}}$$

so $q = 5$ or $q = 7$. Each of these is impossible since $q \equiv 1 \bmod 4$ and $1 \bmod 3$ or $q \equiv 3 \bmod 8$.

(2c): Now let $N = q_1^{1+4e_1} q_2^{1+4e_2} p_1^2 \cdots p_m^2$, be odd and 4-perfect with $3 \nmid N$. Since $\sigma(N) = 4N$ we can write:

$$\sigma(q_1^{1+4e_1})\sigma(q_2^{1+4e_2})(1+p_1+p_1^2)(\cdots)(1+p_m+p_m^2) = 4q_1^{1+4e_1} q_2^{1+4e_2} p_1^2 \cdots p_m^2.$$

Considering this equation modulo 3 shows each $p_i \equiv 2 \bmod 3$ and then $\sigma(q_1^{1+4e_1})\sigma(q_2^{1+4e_2}) \equiv q_1^{1+4e_1} q_2^{1+4e_2} \bmod 3$. But $q_i \equiv 2 \bmod 3$ implies, by Lemma 2.2, $3 \mid \sigma(q_i^{1+4e_i})$, which is impossible. This means $q_1 \equiv 1 \bmod 3$, $q_2 \equiv 1 \bmod 3$.

(Now we modify the argument of Steuerwald, and find that the Lemma of Brauer is not needed.) Since $\sigma(p_1^2) < p_2^2$, $\sigma(p_1^2)$ is divisible by at most one of the p_i, so we can write

$$\sigma(p_1^2) = q_1^{g_1} q_2^{g_2} p_i \text{ or } \sigma(p_1^2) = q_1^{g_1} q_2^{g_2} \text{ or } \sigma(p_1^2) = q_1^{g_1} \text{ or } \sigma(p_1^2) = q_2^{g_2},$$

where $q_1 < q_2$, $g_i \geq 1$ except in the first case where $g_i \geq 0$. Consideration of these possibilities modulo 3 shows that the first case cannot occur.

Since e_1 is odd, by Lemma 2.1, $x = \frac{q_1+1}{2} \mid N$ and $x \equiv 1 \bmod 3$. Now x is too small to include a q_i in its prime factorization, so must be a product of the p_i. We cannot have $x = p_i$ (consider modulo 3 again), so there must be two or more of the p_i in the factorization of x, so there exists an i with $p_i \leq \sqrt{x}$. But then, in all remaining cases,

$$q_1 \leq 1 + p_1 + p_1^2 \leq 1 + p_i + p_i^2 \leq 1 + \sqrt{x} + x = 1 + \sqrt{\frac{q_1+1}{2}} + \frac{q_1+1}{2},$$

so $q_1 \leq 1 + \sqrt{\frac{q_1+1}{2}} + \frac{q_1+1}{2}$. But this means q_1 must be $2, 3, 5$ or 7. Each of these is impossible, since $q \equiv 1 \bmod 4$ and $1 \bmod 3$. This contradiction verifies our conclusion (that no such 4-perfect number exists) in this final case. ∎

If we call the leading prime(s) to odd power(s) with special shape the "Euler part" and the rest the "squared part", then the previous result says that "no odd 4-perfect number exists with squared part a square of a square free number".

The following result is based on the technique of Starni [7] whose theorem, for 2-perfect numbers, had uniform powers for the p_i. This, in turn depended on a result of McDaniel [4] (incorrectly cited), where the powers are not uniform.

THEOREM 2.6.
Let $N = \Pi 3^{2\beta} \prod_{i=1}^{M} p_i^{2\alpha_i}$ be odd and 2^k-perfect, where the p_i are distinct odd primes with $p_i > 3$, $\beta > 0$, the Euler factor Π has any of the forms given by Theorem 2.1, and, for all i $\alpha_i \not\equiv 1 \bmod 3$. Then $3^{2\beta} \mid \sigma(\Pi)$.

Proof. Firstly $(\sigma(3^{2\beta}), 3^{2\beta}) = 1$. Since $\alpha_i \not\equiv 1 \bmod 3$, $1+2\alpha_i \equiv 1, 5 \bmod 6$.

Since $p_i \equiv 1, -1 \bmod 6$, $\sigma(p_i^{2\alpha_i}) \equiv 1 \bmod 6$ if $p_i \equiv -1 \bmod 6$ or $\sigma(p_i^{2\alpha_i}) \equiv 1+2\alpha_i \bmod 6$ if $p_i \equiv 1 \bmod 6$. But then, subject maybe to some reordering, there exists an $m \geq 0$ with

$$\begin{aligned} P := \prod_{i=1}^{M} \sigma(p_i^{2\alpha_i}) &\equiv \prod_{i=1}^{m} 1 + 2\alpha_i \bmod 6 \\ &\equiv \prod_{i=1}^{m} 1 + 2\alpha_i \bmod 3. \end{aligned}$$

By the given assumption, $1 + 2\alpha_i \not\equiv 0 \bmod 3$, so $P \not\equiv 0 \bmod 3$, and thus $(P, 3^{2\beta}) = 1$.

But for some whole number k, $\sigma(N) = 2^k \cdot N$ so therefore

$$\sigma(\Pi)\sigma(3^{2\beta})P = 2^k \Pi 3^{2\beta} \prod_{i=1}^{M} p_i^{2\alpha_i}.$$

Therefore $3^{2\beta} \mid \sigma(\Pi)$. ∎

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