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An Intertemporally-Consistent and Arbitrage-Free Version of the Nelson and Siegel Class of Yield Curve Models

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Abstract

This article derives a generic, intertemporally-consistent, and arbitrage-free version of the popular class of yield curve models originally introduced by Nelson and Siegel (1987). The derived model has a theoretical foundation (conferred via the Heath, Jarrow and Morton (1992) framework) that allows it to be used in applications that involve an implicit or explicit time-series context. As an example of the potential application of the model, the intertemporal consistency is exploited to derive a theoretical time-series process that may be used to forecast the yield curve. The empirical application of the forecasting framework to United States data results in out-of-sample forecasts that outperform the random walk over a sample period of almost 50 years, for forecast horizons ranging from six months to three years.

Keywords

yield curve
term structure of interest rates
Nelson and Siegel model
Heath-Jarrow-Morton framework

JEL Classification

E43, C22, G12

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1 Introduction

This article derives a generic, intertemporally-consistent, and arbitrage-free version of a popular class of yield curve models originally introduced by Nelson and Siegel (1987). The Nelson and Siegel (1987) model, and subsequent extentions and respecifications in Svensson (1994), Hunt (1995), Bliss (1997), Mansi and Phillips (2001), Diebold and Li (2002), and Krippner (2003*a*), may be classified as exponential-polynomial or orthonormalised Laguerre polynomial (OLP) models of the yield curve, based on the functions of maturity used to represent the underlying forward rate curve at each point in time. OLP models have been shown to perform favourably in comparison with other approaches to modelling the yield curve, being straightforward to estimate with output that is sensible and intuitive, and providing empirical results comparable to more complex and customised models.[1] Not surprisingly then, OLP models are used frequently by researchers and practitioners in a wide variety of markets and applications, such as (1) forecasting the yield curve; (2) analysing relative values of fixed interest securities; (3) deriving monetary policy expectations; (4) managing fixed interest portfolio risk; (5) investigating macroeconomic time-series data; (6) studying interest rate swap spreads; and (7) providing estimates of zero-coupon yields as a direct valuation exercise or for subsequent empirical analysis.[2]

Notwithstanding their popularity and success in empirical applications, there are two theoretical shortcomings of OLP models that leave some researchers apprehensive about their application where a time-series context is involved, which is implicit or explicit in several of the applications noted above. The first shortcoming is that OLP models cannot be intertemporally consistent (i.e the yield curves specified by the OLP model at different points in time cannot be mapped to each other via an underlying stochastic time-series process), as identified in Björk and Christensen (1999), Filipović (1999*a*), and Filipović (1999*b*). The second shortcoming is that OLP models have not been established as being arbitrage-free models of the yield curve, thus falling short of what is effectively a minimum benchmark in the contemporary literature.[3] In addition, a practical shortcoming of the OLP models currently available in the literature is the absence of a generic specification, and so the user is not free to choose the trade-off between parsimony of the model versus the precision of fit to the

[1] See Dahlquist and Svensson (1996), Seppala and Viertio (1996), Bliss (1997), Fergusson and Raymar (1998), Subramanian (2001), Ioannides (2003), and Jordan and Mansi (2003) for comparisons to other yield curve modelling approaches.

[2] Examples of published work within each category are, respectively: (1) Diebold and Li (2002); (2) Kacala (1993), and Ioannides (2003); (3) Söderlind and Svensson (1997), Monetary Authority of Singapore (1999), and Bank for International Settlements (1999) contains sub-articles and further references regarding ten central banks (of twelve surveyed) that use OLP models; (4) Barrett, Gosnell and Heuson (1995), Willner (1996), and Diebold and Li (2002); (5) Diebold, Rudebusch and Aruoba (2003); (6) Brooks and Yong Yan (1999), and Fang and Muljono (2003); and (7) Diaz and Skinner (2001), Soto (2001), Schmid and Kalemanova (2002), and Steeley (2004).

[3] An arbitrage-free model will not necessarily be intertemporally-consistent. For example, Brandt and Yaron (2002) notes that arbitrage-free models are often applied in an intertemporally-inconsistent manner, because parameters assumed to be constant over time are recalibrated at each point in time without regard to historical data.

observed yield curve that is most suitable for their application.

The primary motivation of this article is therefore to develop a generic, intertemporally-consistent, and arbitrage-free model of the yield curve that is based on the OLP approach, and that therefore retains the empirical simplicity and intuition that have made OLP models so popular. The resulting volatility-adjusted OLP (VAO) model of the yield curve may then be used in applications that involve a time-series context. Indeed, the theoretical foundation underlying the VAO model (conferred via the Heath, Jarrow and Morton (1992) framework) may offer valuable insights into the issue being investigated. In addition, the generic form of the VAO model allows the model to be extended arbitrarily, as might be required to suit the user's required parsimony versus precision trade-off.

The practical application of the VAO model in this article is to forecasting the yield curve. The intertemporal consistency of the VAO model framework is exploited to derive a convenient vector autoregressive (VAR) process for the VAO model coefficients, and that VAR process is then used to make out-of-sample forecasts of the United States (US) yield curve. This complements and extends related work in Diebold and Li (2002) that uses an OLP model in conjunction with atheoretical univariate time-series analysis to forecast the US yield curve, and it offers an alternative to the more complex frameworks presented in Brandt and Yaron (2002), Duffee (2002), and Ang and Piazzesi (2003) that have also been used for forecasting the US yield curve.

The article proceeds as follows: section 2 introduces a generic OLP model of the forward rate curve as a link to the existing literature, and then proceeds to derive the corresponding generic VAO model of the forward rate curve. Section 3 derives the VAR process to be used for forecasting the yield curve. Section 4 applies the VAO model in tandem with the VAR process to make out-of-sample forecasts of the United States yield curve over the period 1954 to 2004, and section 5 concludes. The proofs are contained in the appendices.

2 The volatility-adjusted orthonormalised Laguerre polynomial model of the forward rate curve

2.1 A generic specification for OLP models of the yield curve

Most OLP models that have been specified in the literature and applied empirically may be nested within the following generic forward rate representation:

$$f(t,m) = \sum_{n=1}^{N} \beta_n(t) \cdot g_n(\phi,m) \tag{1}$$

where $f(t,m)$ is the (continuously-compounding instantaneous) forward rate curve at time t as a function of maturity m (≥ 0); N represents the number of components used to represent the forward rate curve; and $\beta_n(t)$ are the linear coefficients estimated at time t that are associated with the OLP forward rate modes $g_n(\phi,m)$. The latter are time-invariant functions of maturity defined as $g_1(\phi,m) = 1$, and for $n > 1$:

$$g_n(\phi, m) = \exp(-\phi m) \cdot \sum_{k=0}^{n-2} \frac{(-1)^k (n-2)!(2\phi m)^k}{(k!)^2 (n-2-k)!} \tag{2}$$

where ϕ is a fixed positive constant that governs the rate of exponential decay.[4] For example, the OLP(3) model (i.e the $N = 3$ specification of the OLP model) has $g_2(\phi, m) = -\exp(-\phi m)$, and $g_3(\phi, m) = -\exp(-\phi m)(-2\phi m+1)$, making equation 1 linearly equivalent to the models of Nelson and Siegel (1987), Hunt (1995), and Diebold and Li (2002).[5] The first three OLP forward rate modes are illustrated in figure 1, and are named the Level, Slope, and Bow modes based on their shapes.

[Figure 1 here]

The (continuously-compounding zero-coupon) interest rate curve associated with the generic OLP forward rate curve has a similar functional form except the $\beta_n(t)$ coefficients correspond to interest rate modes, i.e: $R(t, m) = \sum_{n=1}^{N} \beta_n(t) \cdot s_n(\phi, m)$, where $s_n(\phi, m) = \frac{1}{m} \int_0^m g_n(\phi, m)\, dm$. This facilitates the estimation of the $\beta_n(t)$ coefficients directly from cross-sectional yield curve data (i.e the market-observed yields and/or prices of a group of similar fixed interest securities with a span of maturities, all observed at time t). The cross-sectional fit is sensible and typically close, and the estimated coefficients also have an intuitive interpretation.[6]

The natural temptation is then to treat the OLP model coefficients as state variables of the yield curve, implicitly with stochastic components to allow for unanticipated changes to the shape of the yield curve as time evolves. Unfortunately, OLP models were only ever proposed as a convenient framework for modelling cross-sections of yield curve data, and so extensions into the time-series context lack a sound theoretical foundation. Indeed, Björk and Christensen (1999), Filopović (1999*a*), and Filopović (1999*b*) have established that OLP models cannot be intertemporally-consistent, i.e the cross-sectional yield curves specified by the OLP model at different points in time cannot be mapped to each other via an underlying stochastic time-series process. This undermines the validity of using OLP models for applications that involve a time-series context, and leads Filopović (1999*a*) to conclude that OLP models should not be used for modelling the yield curve.

However, an alternative to completely abandoning the OLP approach to modelling the yield curve is to specify a related version of the OLP model

[4] Courant and Hilbert (1953) pp. 93-97, or Rainville and Bedient (1981) pp. 395-396, contain more information on OLP modes. They are a series of solutions to the second-order differential equation noted in Courant and Hilbert (1953) pp. 328-331. Members of such solution sets are commonly referred to as modes, hence the terminology adopted in this article.

[5] The OLP models of Svensson (1994), Bliss (1997), and Mansi and Phillips (2001) are analogous to the generic OLP specification in equation 1, but also contain additional exponential terms with a different decay rate. The derivations in this article may also be applied directly to those models, if required.

[6] Dahlquist and Svensson (1996), for example, details a typical estimation method based on market-observed data, and Diebold and Li (2002), for example, discusses the intuition behind each coefficient.

where intertemporal consistency is assured by construction. Such a model is derived in the following subsection.

2.2 The volatility-adjusted OLP model of the forward rate curve

The derivation of the generic volatility-adjusted OLP (VAO) model of the forward rate curve is based on the framework provided by Heath, Jarrow and Morton (1992) (hereafter HJM). At each point in time, the HJM framework specifies an intertemporally-consistent and arbitrage-free relationship between: (1) the forward rate curve; (2) the expected path of the short rate; (3) the volatility structure that dictates how the entire forward rate curve can potentially change due to random factors; and (4) the market prices of risk. Defining functional forms for the latter three components therefore defines a functional form for the forward rate curve.

Proposition 1 outlines the generic VAO model and the essential assumptions, definitions, and notation involved in its construction. The proof of Proposition 1 is relegated to Appendix A, leaving the remainder of this section to discuss the intuition of the model from an economic and financial perspective.

Proposition 1 ***The generic VAO model of the forward rate curve***

Assumption 1: At time t and as a function of future time $t+m$ $(m \geq 0)$, the expected path of the short rate $E_t\left[r(t+m)\right]$ under the physical measure is defined as:

$$E_t\left[r(t+m)\right] = \sum_{n=1}^{N} \lambda_n(t) \cdot g_n(\phi, m) \tag{3}$$

where E_t is the expectations operator as at time t; $\lambda_n(t)$ are time-varying coefficients, and $g_n(\phi, m)$ are the modes defined in section 2.1.

Assumption 2: Potential stochastic changes to the expected path of the short rate $d\left\{E_t\left[r(t+m)\right]\right\}_{Stoc.}$ are defined as:

$$d\left\{E_t\left[r(t+m)\right]\right\}_{Stoc.} = \sum_{n=1}^{N} \sigma_n \cdot g_n(\phi, m) \cdot dW_n(t+m) \tag{4}$$

where σ_n are constant standard deviations, and $dW_n(t+m)$ are Wiener increments under the physical measure.

Assumption 3: The expected market prices of risk associated with each mode, i.e θ_n, are constants over time.

Then, at time t as a function of maturity m, the forward rate curve $f(t,m)$ under the physical measure will have the following functional form:

$$f(t,m) = \sigma_1\theta_1 m + \sum_{n=1}^{N} \beta_n(t) \cdot g_n(\phi, m) - \sum_{n=1}^{N} \sigma_n^2 \cdot h_n(\phi, m) \tag{5}$$

where $\beta_n(t) = \gamma_n + \lambda_n(t)$, γ_n are constant parameters each expressible as linear combinations of $\sigma_1\theta_1, \sigma_2\theta_2, \ldots, \sigma_N\theta_N$, and $h_n(\phi, m)$ are time-invariant *functions of maturity that may be derived as:*

$$h_n(\phi, m) = \frac{1}{2\phi^2} \cdot \sum_{k=0}^{n-2} \frac{(-2)^k (n-2)!}{(k!)^2 (n-2-k)!} \cdot (k! - \Gamma[1+k, \phi m])^2 \tag{6}$$

where $\Gamma[\cdot,\cdot]$ *is the incomplete Gamma function.*

Proof. In Appendix A. ■

2.3 Discussion of the VAO model assumptions

2.3.1 The expected path of the short rate, $E_t[r(t+m)]$

The full and formal justification for representing $E_t[r(t+m)]$ with OLP modes is noted in Krippner (2003*b*). Briefly, $E_t[r(t+m)]$ represents the market's assessment, given all currently available information, of the future path of the short rate. This will contain two components: (1) the assessment of the long-run nominal equilibrium interest rate (i.e the long-run neutral real rate plus long-run inflation), which is a constant by maturity and is therefore represented by the constant function $\lambda_1(t) \cdot g_1(\phi, m)$; and (2) the expected deviation of the path of the short rate from the long-run rate over the short- to medium-term. This second component corresponds to the current and expected state of the economic cycle and the associated stance of monetary policy, and is represented by the remaining modes in the OLP representation, e.g by the Slope and Bow components $\lambda_2(t) \cdot g_2(\phi, m)$ and $\lambda_3(t) \cdot g_3(\phi, m)$ in the VAO(3) model.

2.3.2 The stochastic component of changes to the expected path of the short rate, $d\{E_t[r(t+m)]\}_{\mathbf{Stoc.}}$

The expected path of the short rate will continuously be subjected to unanticipated changes as time evolves, as the market incorporates new information relevant to the assessment of the long-run equilibrium rate and the anticipated profile of the stance of monetary policy. Equation 4 represents these unanticipated changes as stochastic changes to each component of the initial expected path of the short rate. Because $d\{E_t[r(t+m)]\}_{\text{Stoc.}}$ is represented with the same OLP modes as $E_t[r(t+m)]$, then $E_t[r(t+m)] + d\{E_t[r(t+m)]\}_{\text{Stoc.}} = [\lambda_n(t) + \sigma_n dW_n(t)] \cdot g_n(\phi, m)$. Hence, $d\{E_t[r(t+m)]\}_{\text{Stoc.}}$ will be realised as stochastic changes to the values of $\lambda_n(t)$, and therefore $\beta_n(t)$, as time evolves.

2.3.3 The intertemporal specification of volatility, $\sigma_n d\tilde{W}_n(t)$

Proposition 1 implies that potential stochastic changes to each $\beta_n(t)$ coefficient are expected to be homoskedastic and independent over time (i.e the volatility in each coefficient is expected to be constant, and the variance-covariance matrix is expected to be diagonal). This specification results in the most tractable and intuitive VAO model, and will typically be reasonable in practice, as discussed below.

Regarding homoskedasticity, this assumption will be appropriate unless there is strong evidence at the time of estimation that future volatilities are likely to have large and/or persistent time-varying components that will prevail over the medium-term. Indeed, even if there have been structural changes to volatilities in the past due to changes in economic and/or monetary policy regimes, it is reasonable to assume that volatilities will remain at the levels realised in the current regime so long as future potential changes to the regime cannot be readily anticipated and/or clearly identified.

Regarding independence, the appropriateness of this assumption is largely an empirical issue. Krippner (2003*b*) notes that US interest rate data supports the assumption that changes to the Level coefficient have been independent to changes in the Slope and Bow coefficients from 1954 to the present, but that changes to the Slope and Bow coefficients have shown significant negative correlation. However, given that $h_n(\phi, m)$ for the Slope and Bow modes are several orders of magnitude smaller than for the Level mode (see figure 2), the practical implications of assuming independence between the Slope and Bow coefficients will be negligible relative to a more complex specification where changes are constructed to be orthogonal (e.g based on principle components analysis of the covariance matrix of changes to the Level, Slope, and Bow coefficients).

2.3.4 The market prices of risk, θ_n

In the "real world" (i.e under the physical measure), investors require extra returns to compensate them for bearing risks relative to the risk-free investment, i.e a rolling investment in the short rate. Securities on the yield curve bear interest rate risk associated with unanticipated changes to the shape of the yield curve, which arise from the potential stochastic changes to the expected path of the short rate noted previously. The compensation for those risks are specified via the market prices of risk for each component of $E_t\left[r(t+m)\right]$, and Proposition 1 assumes that these market prices of risk are expected to be constant over time. This specification results in the most tractable and intuitive VAO model, and will typically be reasonable in practice for the same reasons as discussed for the homoskedasticity assumption in the previous section. Note that the market prices of risk would be zero in the risk-neutral version of the VAO model (i.e investors are indifferent to risk under the risk-neutral measure).

2.4 Discussion of the VAO model of the forward rate curve

The VAO(3) model is developed as a specific example in the discussion that follows because it is convenient to compare and contrast it with the OLP(3) model noted in section 2.1. However, this comparison should not be interpreted as a guide to choosing between the OLP(3) and VAO(3) models; intertemporal and arbitrage-free consistency should be regarded as essential requirements for any application, not an aspect that can be traded off for empirical convenience.

Setting $N = 3$ in equations 5 and 1, the first point of similarity is that the VAO(3) model retains a cross-sectional functional form that is primarily based on a linear combination of the Level, Slope, and Bow modes. Most importantly, the Level, Slope, and Bow coefficients remain as the only coefficients to estimate for each cross-section of the yield curve; ϕ remains as a constant parameter, and σ_1, σ_2, σ_3 and θ_1 are additional constant parameters as discussed below. The second point of similarity is that the VAO(3) model retains a relatively simple cross-sectional functional form, and so the Level, Slope, and Bow coefficients may still be estimated directly from a cross-section of yield curve data. The final point of similarity is that the VAO(3) model nests the OLP(3) model. Specifically, the VAO(3) model is identical to the OLP(3) model if all volatilities and the market prices of risk are set to zero, i.e a risk-neutral and deterministic model of the yield curve. However, this connection clearly exposes the theoretical short-comings of OLP models; practical yield curve data will not accord with these assumptions, and so the OLP(3) model will therefore be inadequate as a practical model of the yield curve.

The main contrast is that the cross-sectional functional form of VAO(3) model is more complex than the OLP(3) model, i.e as a result of imposing intertemporal consistency the VAO(3) model has several time-invariant functions of maturity that appear as adjustments to the OLP(3) model.

The first series of adjustments relates to the explicit modelling of expected volatility in the HJM framework (hence the name "volatility-adjusted OLP model"), and these adjustments are of the form $\sum_{n=1}^{N} \sigma_n^2 \cdot h_n(\phi, m)$. For the VAO(3) model $h_1(\phi, m) = \frac{1}{2}m^2$, $h_2(\phi, m) = \frac{1}{2\phi^2}\left[1 - \exp\left(-\phi m\right)\right]^2$, and $h_3(\phi, m) = \frac{1}{2\phi^2}\left[1 - \exp\left(-\phi m\right)\right]^2 - \frac{1}{\phi^2}\left[1 - \exp\left(-\phi m\right) - \phi m \exp\left(-\phi m\right)\right]^2$.[7] These functions are illustrated in figure 2, and may be interpreted as the effects on the shape of the forward rate curve per unit of variance in the stochastic component of each $\beta_n(t)$ coefficient.

[Figure 2 here]

The second series of adjustments relates to the explicit modelling of the expected market prices of risk applied to the expected volatility structure within the HJM framework. As detailed in Appendix A.3, the market price of risk for the Level mode leads to an adjustment of non-OLP form, i.e $\sigma_1\theta_1 m$, but the remaining adjustments are expressible using OLP modes, i.e as $\sum_{n=1}^{3} \gamma_n \cdot g_n(\phi, m)$ for the VAO(3) model. $\sigma_1\theta_1 m + \sum_{n=1}^{3} \gamma_n \cdot g_n(\phi, m)$ has the intuitive interpretation as a term premium function (by maturity) that is expected to be time-invariant. Note that $\sigma_1\theta_1$ and the constants γ_n would all be zero in the risk-neutral version of the VAO model, which would correspond to a term premium function of zero for all maturities.

Empirically, the adjustments are easy to incorporate as modifications to the OLP(3) model; i.e the additional functions $\gamma_n \cdot g_n(\phi, m)$ are subsumed directly into $\beta_n(t) \cdot g_n(\phi, m)$, and $\sigma_1\theta_1 m$ and $\sigma_n^2 \cdot h_n(\phi, m)$ are simple functions of maturity that are easy to allow for in the estimation process. Most importantly, the adjustments relative to the OLP(3) model do not represent extra degrees of freedom or additional flexibility to model each individual cross-section of the yield curve (which is the context implied in Diebold and Li (2002) and

[7] These volatilty-adjustment terms are the drift terms in the HJM framework under the risk-neutral measure. The results for $h_1(\phi, m)$ and $h_2(\phi, m)$ have been reported in the literature; see HJM pp. 90-92, or de La Grandville (2001) pp. 368-372. Note that HJM uses the Slope volatility function $\exp\left(-\phi m/2\right)$, so there is a scalar difference between the HJM result and $h_2(\phi, m)$ as presented in this article.

Diebold et al. (2003) in reference to the work of Björk and Christensen (1999)), because they are time-invariant functions with constant parameters estimated from historical yield curve data (as discussed in section 4 and Appendix C). In this context, the adjustments actually impose a constraint on the estimation of each individual cross-section of the yield curve, and so the VAO(3) model should typically produce an inferior fit to the cross-sectional yield curve data than the OLP(3) model.[8]

Finally, it is worth discussing two aspects of the adjustments that might seem problematic; i.e the unbounded nature of $\sigma_1\theta_1 m$ and $h_1(\phi, m)$, both associated with the Level mode. Firstly, $\sigma_1\theta_1 m$ and $h_1(\phi, m)$ do not cause any empirical problems because practical yield curve data typically only ranges up to 30 years, which is well short of the maturities where the unbounded nature of $h_1(\phi, m)$ and $\sigma_1\theta_1 m$ might potentially start to dominate the estimation process. Secondly, terms analogous to $h_1(\phi, m)$ occur in other models of the forward rate that incorporate constant forward rate volatility for all maturities.[9]

2.5 Further observations about the VAO model

From the perspective of the wider literature, there are four additional points about the VAO model that are worthy of note. Firstly, the derivation of the VAO model provides independent verification of the results from Björk and Christensen (1999), Filopović (1999*a*), and Filopović (1999*b*), i.e that forward rate curves specified with OLP functions cannot be intertemporally consistent when the yield curve evolves with a stochastic component. Specifically, the stochastic component will lead to functions with form $\frac{1}{2}m^2$ and $\exp(-2\phi m) \cdot (4\phi m)^n$ that are not expressible within the original OLP specification.

Secondly, the addition of each $h_n(\phi, m)$ function within the VAO model may be seen as a "manifold expansion" (i.e the addition of appropriate functions of maturity) analogous to that suggested by Björk and Christensen (1999) pp. 338-339 to make the Nelson and Siegel (1987) model consistent with the Hull and White (1990) model.

Thirdly, the VAO model is arbitrage-free as a consequence of its construction via the HJM framework. Arbitrage-free models of the yield curve are usually estimated to provide a precise fit to each cross-section of yield curve data. However, such applications are typically intertemporally-inconsistent (e.g see Brandt and Yaron (2002)), may admit arbitrage while appearing to be arbitrage-free (e.g see Backus, Foresi and Zin (1998)), and may overfit the measurement errors and anomalies that exist in the cross-sectional data[10] to the detriment of pricing associated securities (e.g see Brandt and Yaron (2002), and Bliss (1997)). The

[8] This prediction is consistent with the results reported in Krippner (2003*b*) for US data.

[9] For example, the Ho and Lee (1986) model (as noted in Hull 2000, pp. 108 and 572-574), the examples in HJM pp. 90-92, and the Vasicek (1977) model with zero mean-reversion (as noted in Hull 2000 p. 567). The latter suggests that bounded versions of the VAO model could easily be obtained by using $g_1(\kappa, m) = -\exp(-\kappa m)$ with $0 < \kappa \ll \phi$, or alternatively the Level mode can be seen as the limiting function of a bounded mode associated with very low mean-reversion.

[10] For example, due to bid-ask bounce, stale quotes, and genuine temporary distortions in the yield curve due to large market flows and/or market scarcity in particular physical securities.

VAO model differs from such arbitrage-free models because it is not intended to give a precise fit to each cross-section of the yield curve. In this respect, the VAO model is conceptually similar to the time-consistent no-arbitrage approach by Brandt and Yaron (2002), which uses a parsimonious representation in conjunction with some well-behaved yield or price residuals, both for tractability and to allow for the fact that market-observed data will inevitably contain measurement errors and anomalies. The advantage of the generic specification of the VAO model is that the user may easily adjust the number of modes to obtain their required trade-off between parsimony and cross-sectional fit. Of course, analogous to the suggestion of Brandt and Yaron (2002), if a precise fit to market-observed data is specifically required, then the number of modes in the VAO model may in principle be increased to equal the number of securities used to define the yield curve.

Finally, the VAO model has a term structure of volatility (i.e the volatility of expected short rates and forward rates as a function of maturity) that is of OLP form. This naturally allows for the humped shape of forward rate volatilities that is typically observed in the market, as noted in Hull (2000) pp. 541-542.

3 The VAO model coefficients over time

The intertemporal consistency of the generic VAO model is already implicit by its construction via the HJM framework, but it is useful to translate that intertemporal consistency into an explicit stochastic time-series process for the VAO model coefficients. This shows that the VAO model coefficients can be interpreted as state variables of the yield curve, and the resulting vector autoregressive (VAR) process provides a convenient model for forecasting the yield curve.

For this article, the derivation of the VAR process is undertaken for the VAO(3) model used in the empirical work of section 4, but analogous results apply generally for the VAO(N) model. Proposition 2 specifies the VAR process for the VAO(3) coefficients and the associated notation. The proof of Proposition 2 is relegated to Appendix B, leaving the remainder of this section to discuss the economic intuition and the practical interpretation of the model.

Proposition 2 ***The VAR process for the VAO*(3) *model coefficients***
The intertemporal consistency of the VAO(3) *process translates into the following VAR process for the VAO*(3) *model coefficients:*

$$\boldsymbol{\beta}(t+\tau) = \boldsymbol{\mu} + \boldsymbol{\Phi}(\phi,\tau)\,\boldsymbol{\beta}(t) + \boldsymbol{\varepsilon}(t+\tau) \tag{7}$$

where $\boldsymbol{\beta}(t) = \{\beta_1(t), \beta_2(t), \beta_3(t)\}'$*, a (column)* 3*-vector containing the VAO*(3) *model coefficients at time* t*;* $\boldsymbol{\beta}(t+\tau)$ *is the vector of VAO*(3) *model coefficients at time* $t+\tau$*;* τ (> 0) *is a parameter representing an arbitrary future horizon from time* t*;* $\boldsymbol{\mu}$ *is a vector of constants;* $\boldsymbol{\Phi}(\phi,\tau)$ *is a time-invariant* 3×3 *matrix:*

$$\boldsymbol{\Phi}(\phi,\tau) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \exp(-\phi\tau) & -2\phi\tau\exp(-\phi\tau) \\ 0 & 0 & \exp(-\phi\tau) \end{bmatrix} \tag{8}$$

and $\boldsymbol{\varepsilon}(t+\tau)$ is a vector of independent random variables.

Proof. In Appendix B. ■

The intuition underlying equation 7 is essentially the expectations hypothesis of the yield curve with an allowance for term premia, i.e after allowing for term premia, the maturity τ rate from the forward rate curve at time t implies an expectation of the short rate at time $t+\tau$ in the future. Within the HJM framework, this concept carries over to the entire forward rate curve and the underlying expected path of the short rate, i.e the expected path of the short rate today $E_t[r(t+m)]$ implies the entire expected path of the short rate in the future $E_t[r(t+\tau+m)]$ (as detailed in Appendix B.1). As shown in Appendix B.2, if those current and future expected paths of the short rate are both represented using OLP modes, the expectations hypothesis within the HJM framework condenses into simple time-series processes for the Level, Slope, and Bow coefficients. That is, the coefficients at time t imply an expectation of the coefficients at time $t+\tau$, and those coefficients summarise the entire expected path of the short rate at time $t+\tau$.

It is most convenient to represent the time-series processes in vector form, and so $\boldsymbol{\beta}(t)$ and $\boldsymbol{\beta}(t+\tau)$ are vectors containing the Level, Slope, and Bow coefficients at times t and $t+\tau$ respectively. Equation 7 is a first-order VAR process, where the entries in $\boldsymbol{\Phi}(\phi,\tau)$ relate the current values of the coefficient to their expected future values.[11] Algebraically, $\boldsymbol{\beta}(t)=\boldsymbol{\gamma}+\boldsymbol{\lambda}(t)$, and the vector $\boldsymbol{\lambda}(t)$ represents genuine market expectations for the future path of the short rate, while the vector $\boldsymbol{\gamma}$ represents the term premia that exist in the various securities that define the yield curve. These term premia result in the constant vector $\boldsymbol{\mu}$ in equation 7, which would be zero in the absence of term premia.

$\boldsymbol{\varepsilon}(t+\tau)$ is the realised forecast error, i.e $\boldsymbol{\beta}(t+\tau)-[\boldsymbol{\mu}+\boldsymbol{\Phi}(\phi,\tau)\boldsymbol{\beta}(t)]$, which represents the fact that the expectation of $\boldsymbol{\beta}(t+\tau)$ as at time t will inevitably differ to that realised at time $t+\tau$ due to the collection of new information that arrives between these times. As time evolves, the current $\boldsymbol{\beta}(t)$ will always reflect the up-to-date expectations embedded in the yield curve, which will be a combination of past expectations and the accumulation of forecast errors.

The relationship between the expected path of the short rate and the forward rate curve from equation 5 provides the link to forecasting the forward rate curve (and hence the yield curve) from the current yield curve, i.e $E_t[\boldsymbol{\beta}(t+\tau)]=\boldsymbol{\mu}+\boldsymbol{\Phi}(\phi,\tau)\boldsymbol{\beta}(t)$, and so:

$$E_t[f(t+\tau,m)]=\sigma_1\theta_1 m+[\boldsymbol{\mu}+\boldsymbol{\Phi}(\phi,\tau)\boldsymbol{\beta}(t)]'\,\mathbf{g}(\phi,m)-\mathbf{v}'\mathbf{h}(\phi,m) \qquad (9)$$

where $\mathbf{g}(\phi,m)=\{g_1(\phi,m),g_2(\phi,m),g_3(\phi,m)\}'$, $\mathbf{v}=\{\sigma_1^2,\sigma_2^2,\sigma_3^2\}'$ and $\mathbf{h}(\phi,m)=\{h_1(\phi,m),h_2(\phi,m),h_3(\phi,m)\}'$ for the VAO(3) model.

[11] As an aside, the eigenvalues of $\boldsymbol{\Phi}(\phi,\tau)$ for the VAO(3) model are $\{1,\exp(-\phi\tau),\exp(-\phi\tau)\}$, implying a unit root process for the Level coefficient, and mean-reverting processes for the Slope and Bow coefficients. This prediction is consistent with the results reported in Diebold and Li (2002) and Krippner (2003*b*) for US data.

4 Forecasting the yield curve with the VAO(3) model

The empirical application of the VAO model in this article is to forecasting the yield curve, i.e forecasting yields on the future yield curve and also the spreads between yields of different maturities on that future yield curve. The VAO(3) framework, i.e the VAO(3) model in conjunction with the VAR process derived for the VAO(3) model coefficients, is used as a basis for the forecasting.[12]

The interest rate data used are monthly averages of constant maturity bond rates obtained from the online Federal Reserve Bank of St Louis economic database. Monthly averages are acceptable for this article because the sample size is large and the forecasting horizons investigated range from three months to three years, and so the results (i.e the forecast errors) should not be unduly influenced by the averaging of the daily data and the absence of precisely specified cashflows for each observation of each point on the yield curve.[13]

The data series used are the federal funds rate (FF, quoted on a simple interest basis), the 3-month Treasury bill rate (TB3, quoted on a discount basis), and the yields-to-maturity of the 1-year, 3-year, 5-year, 10-year, and 20-year or 30-year constant maturity bonds (GS1, GS3, GS5, GS10, and GS20 or GS30 respectively, all quoted on a semi-annual basis and with semi-annual coupons).[14] The sample period is July 1954 (the first month FF data is available) to February 2004 (the last month available at the time of the analysis), giving 593 monthly observations of the yield curve. Figure 3 illustrates the FF and GS10 data, the longest and shortest maturity rates available for the entire data period, and the FF/GS10 spread measure used in the empirical analysis is the difference between these two rates.

[Figure 3 here]

The sample period spans four distinct monetary policy regimes, as specified in Gordon (1990) and Walsh (1998) and identified in figure 3, which are used for sub-sample analysis. The regimes are the Bretton Woods / gold price target (start-of-sample to December 1971), the federal funds rate target (January 1972 to September 1979), the non-borrowed reserves target (October 1979 to October 1982), and the borrowed reserves / federal funds rate target (November 1982 to end-of-sample).

The method used to estimate the VAO(3) model coefficients for each cross-sections of yield curve data follows the existing literature, and is detailed in Appendix C for completeness. Figure 4 illustrates the results of this estimation process for the yield curve data from February 2004. Each monthly observation

[12] The empirical results regarding the cross-sectional fit of the VAO(3) model to the yield curve data, and the time-series properties of the VAO(3) model coefficients over the entire sample have been omitted from this article due to space constraints. Readers interested in further detail and discussion on these aspects are referred to Krippner (2003*b*).

[13] Monthly or quarterly averages would also be acceptable for applications of the VAO model in conjunction with economic data. However, most financial applications of the model would require point-in-time quotes for individual securities in conjunction with the precise cashflows of those securities, as in the application to US swaps data in Krippner (2004).

[14] GS20 data is unavailable from January 1987 to September 1993, and so GS30 data is used during this period (with a 30-year maturity in the estimation process noted in the following section).

of yield curve data will give an associated estimate of the Level, Slope, and Bow coefficients for that month, and hence the full sample will provide a time-series of 593 monthly observations of the 3-vector $\boldsymbol{\beta}(t)$. The cross-section estimation process also requires estimates of the VAO(3) model parameters ϕ, θ_1, and $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$. These are estimated using the appropriate time-series of historical yield curve data, as outlined in the following two sections.

4.1 Forecasting the yield curve without a term premium function

The first application to forecasting the yield curve uses the VAO(3) framework with no term premia, which is obtained by setting θ_1 to zero in equation 5 and $\boldsymbol{\mu} = \{0, 0, 0\}'$ in equation 7. This exercise is undertaken as an initial gauge of the importance of term premia with respect to forecasting the yield curve.

All of the forecasting is out-of-sample and uses recursive estimation of the VAO(3) model parameters. Specifically, the first three years of data (July 1954 to June 1957) are used to determine the initial estimates of the parameter ϕ (using a bisection search to minimise the sum of squared yield residuals from all past cross-sectional estimations of the yield curve), and the volatility vector $\boldsymbol{\sigma}$ (using the usual definition of annualised variance $\sigma_n^2 = \frac{12}{T}\sum_{i=1}^{T}\left[\Delta\beta_n(i)\right]^2$, as noted in Hull (2000) pp. 368-369, where T is the number of data points [36 initially], and 12 annualises the monthly data).[15]

The following steps are then used to obtain yield curve forecasts from the July 1957 yield curve data: (1) as detailed in Appendix C, $\boldsymbol{\beta}$(Jul-57) is estimated using the July 1957 yield curve data and the initial estimates of ϕ and $\boldsymbol{\sigma}$; (2) $\boldsymbol{\beta}$(Jul-57) is used to obtain the forecasts of $\boldsymbol{\beta}(\text{Jul-57} + \tau)$ for the horizons of 3 months, 6 months, 1 year, 1.5 years, 2 years, and 3 years (i.e using equation 7 with $\boldsymbol{\mu} = \{0, 0, 0\}'$ and $\tau = 0.25$, 0.5, 1, 1.5, 2, and 3 respectively); (3) the forecasts of $\boldsymbol{\beta}(\text{Jul-57} + \tau)$ are used in equation 9 with the initial estimates of ϕ and $\boldsymbol{\sigma}$ to obtain forecasts of the forward rate curve and hence the zero-coupon interest rate curve at times Jul-57+τ; (4) the forecast rates or yields-to-maturity for FF, TB3, GS1, GS3, GS5, GS10 at times Jul-57+τ are re-constructed using the forecast zero-coupon curve;[16] (5) the forecast FF/GS10 spread is calculated as the GS10 forecast less the FF forecast; and (6) the estimates of ϕ and $\boldsymbol{\sigma}$ are updated using the methods noted in the previous paragraph and all of the

[15] An initial estimate of the $\boldsymbol{\beta}(t)$ coefficients to use for this volatility calculation may be obtained by firstly assuming zero volatility, i.e $\sigma_n^2 = 0$, in equation 5. This creates a two-step process that may be iterated to convergence, but the volatility estimates obtained from the initial estimation of the $\boldsymbol{\beta}(t)$ coefficients are immaterially different from the subsequent estimates. Alternatively, σ_n could potentially be calibrated from data for options on interest rate securities observed at the same time as the yield curve data, if such data are available.

[16] For the bonds, this reconstruction obtains the coupon rate that corresponds to a par bond (i.e a principal of 1) using the forecast zero-coupon curve to provide the discount factors. While this process is more complex than simply using zero-coupon yield data, it is worthwhile because it avoids any model-induced bias in the forecast error analysis; i.e the forecast yields are compared directly to the original yield curve data rather than to model-generated or pre-processed data.

historical data up to the current month.[17] These six steps are repeated for each subsequent observation of the yield curve from August 1954 to February 2004, producing six series of forecast yields and the series of forecast FF/GS10 spreads for each of the six forecasting horizons τ. The corresponding forecast errors are calculated as the actual data at time $t+\tau$ less the corresponding forecasts made at time t for horizon τ.

Table 3 contains the root-mean-squared forecast errors (RMSEs) for the VAO(3) framework forecasts. To save space, only the results for FF, GS10, and FF/GS10 are shown; the results for intermediate maturities generally fall between these sets of results. The RMSEs broadly show an increase by horizon, as expected because the yield curve will be subject to greater amounts of new information from the time of forecast. The magnitudes of the RMSEs in each regime broadly follow the interest rate volatilities within those regimes that is apparent from figure 3. This result is also as expected because higher interest rate volatility will tend to result in larger forecast errors.

Table 4 contains the RMSEs for the VAO(3) framework forecasts less the RMSEs for the random walk forecast (which is the typical naive benchmark used to assess forecasting performance). A negative entry (non-shaded) therefore indicates an outperformance of the random walk model, and the statistical significance of each entry is estimated using the Diebold and Mariano (1995) method with the bandwidth set to one less than the forecast horizon in months.[18] Over the whole sample, the FF and FF/GS10 forecasts outperform those of the random walk, and the magnitude and significance of the outperformance tends to rise by forecast horizon. However, the GS10 forecasts consistently underperform the random walk over the full sample for all horizons.

The sub-sample results offers some insight into the GS10 results; the general outperformance of the VAO(3) framework during the Bretton Woods, federal funds rate target, and non-borrowed reserves regimes is more than counterbalanced by the significant underperformance during the borrowed reserves target regime. Another interesting aspect during the latter regime is that the FF forecasts move from an outperformance for shorter horizons, to an increasing underperformance for longer horizons. At the same time, the FF/GS10 forecasts generally show an outperformance over all regimes. This suggests that the poor forecasting performance in the borrowed reserves regime lies with the forecast levels of yields rather than their relativities to each other, which in turn suggests that term premia become relatively more important in this regime with respect to forecasting the yield curve. Further investigation confirmed that with no allowance for term premia the VAO(3) framework has a strong bias to

[17] This is the most naive method of recursive estimation, and avoids any hint of data mining by using a favourable moving-average window. The recursively-estimated values of ϕ ranged from 0.83 to 1.33. This compares to an estimate of 0.73 in Diebold and Li (2002), which is based on zero-coupon data from January 1970 to December 1997 with a maximum maturity of 10 years.

[18] This is the procedure suggested in Diebold and Mariano (1995) and used in Diebold and Li (2002), because it allows for overlapping forecast errors due to the frequency of the data being greater than the forecast horizons. Note that the small size of the non-borrowed reserves sub-sample means that statistical significance cannot be ascertained using the Diebold and Mariano (1995) test, so no indications are given.

over-forecast yields during the borrowed reserves regime for all of the horizons investigated, even if the ϕ and $\boldsymbol{\sigma}$ parameters are re-estimated using just the data from this regime alone. Hence, the forecasting exercise is repeated for the borrowed reserves regime using the VAO(3) framework with an estimated term premium function.

4.2 Forecasting the yield curve including a term premium function

The procedure for estimating and forecasting with the VAO(3) framework including a term premium function is similar to that outlined in section 4.1, except that estimates of the parameters θ_1 and $\boldsymbol{\mu}$ are required in addition to estimates of ϕ and $\boldsymbol{\sigma}$. This makes the estimation process more complex (as noted below), and so a single estimation is undertaken for these parameters over an appropriate period of history rather than using recursive estimation. Specifically, the period October 1986 to January 1994 (88 months) is chosen as the parameter estimation period because it spans the first full monetary policy cycle (i.e a trough-to-trough cycle in the federal funds rate, and a similar cycle in long–maturity yields) following the period of substantial financial and economic change up to the mid-1980s.[19] The estimation of θ_1 and ϕ is undertaken simultaneously using a grid search to minimise the sum of squared yield residuals from all cross-sectional estimations of the yield curve over the parameter estimation period (the point estimates are $\theta_1 = 1.62$ percentage points and $\phi = 0.804$), and $\boldsymbol{\sigma}$ is estimated over this period using the calculation noted in section 4.1 (the point estimate is $\sigma = \{0.84, 1.49, 1.17\}'$ percentage points). $\boldsymbol{\mu}$ is estimated using the mean realised forecast errors over the parameter estimation period, i.e $\boldsymbol{\mu} = \frac{1}{(88-X)} \sum_{t=\text{Oct-86}+X}^{\text{Jan-94}} \{\boldsymbol{\beta}(t+\tau) - \boldsymbol{\Phi}(\phi,\tau)\,\boldsymbol{\beta}(t)\}$, where $X = 12\tau$ is the number of months at the beginning of the parameter estimation period where no forecasts of $\boldsymbol{\beta}(t+\tau)$ are available to compare to the realised $\boldsymbol{\beta}(t+\tau)$. The point estimates of $\boldsymbol{\mu}$ for each horizon are contained in table 3, and these all result in term premium functions that are positive (i.e forecast yields overstate realised yields) for all horizons.

Using these estimated parameters, the out-of-sample forecasting exercise proceeds as outlined in section 4.1 (but without the parameter updating step) from February 1994 to February 2004. The resulting RMSEs from this process less the RMSEs from the random walk forecasts over the same period are contained in table 4. Negative entries (non-shaded) again indicate an outperformance of the VAO(3) framework, and the Diebold and Mariano (1995) method provides the indicated levels of statistical significance. The main point to note from table 4 is that the VAO(3) framework outperforms the random walk for all maturities over all horizons, except for the forecasts of long-maturity yields over short horizons. The magnitude and significance of the outperformance

[19] For example, financial deregulation (see Gordon (1990) p. 101 and pp. 504-508), the simplification of reserve requirements (see Gordon (1990) p. 536), and the re-establishment of price stability and inflation credibility following the late-1970s to early-1980s period of disinflation under Chairman Volcker (see Walsh (1998) pp. 418-422).

again tends to rise by forecast horizon, although the smaller sample size results in less instances of significance than in table 2.

The forecasting performance of the VAO(3) framework is similar to the out-of-sample results of Diebold and Li (2002) that uses an OLP(3) model in conjunction with a univariate time-series model for each OLP(3) coefficient. For example, Diebold and Li (2002) table 7 notes a 90.4 basis point RMSE for the 10-year zero-coupon rate on a one-year horizon over 1994 to 1997, and the VAO(3) framework produces a RMSE of 90.0 for the 10-year coupon bond over same period and horizon. Also, the VAO(3) framework forecast results are comparable to the substantially more complex framework of Duffee (2002), which offers a 7.7 basis point RMSE improvement over the random walk for the 10-year zero-coupon rate on a one-year horizon over 1995 to 1998, compared to 9.9 for the VAO(3) framework over the same period and horizon.[20]

5 Conclusion

Since being introduced in Nelson and Siegel (1987), OLP models of the yield curve have proved popular with researchers and practitioners alike. The VAO model of the yield curve derived in this article is based on the OLP approach and therefore continues the tradition of the OLP model, i.e the VAO model is straightforward to estimate from cross-sections of yield curve data and it provides output that is sensible and intuitive.

However, the VAO model makes three important extensions relative to the OLP model: (1) the VAO model is intertemporally-consistent, and so may be used in applications that involve an explicit or implicit time series context; (2) the VAO model is an arbitrage-free model of the yield curve, thus meeting what is effectively a minimum benchmark in the literature; and (3) the VAO model is specified in a generic form, so users may easily adjust the model to obtain their required trade-off between parsimony (of coefficients and parameters) versus precision (of fitting the yield curve data) for their particular application. In summary, researchers and practitioners who require a simple yet theoretically-robust model of the yield curve should find the VAO model a useful tool.

[20] Brandt and Yaron (2002) and Ang and Piazzesi (2003) also provide out-of-sample forecast results based on intertemporally consistent models that are more complex than the VAO(3) framework, but the results are not directly comparable; i.e Brandt and Yaron (2002) quotes only mean-absolute errors (of 70 to 80 basis points for a one-year forecast horizon), and Ang and Piazzesi (2003) only forecasts for the one-month horizon.

A Proof of Proposition 1

The proof of Proposition 1 proceeds in four sections: (1) outlining the details of the HJM framework relevant to the derivation of the VAO model; (2) calculating the effect of volatility in the VAO model coefficients; (3) calculating the effect of the market prices of risk for the VAO model; and (4) combining the results together to obtain the generic VAO model of the forward rate curve.

A.1 The HJM framework

HJM specifies the relationship between the instantaneous forward rate curve and the instantaneous short rate under the physical measure as:[21]

$$\begin{aligned} r\left(t+m\right) &= f\left(t,m\right)+\sum_{n=1}^{N}\int_{0}^{m}\sigma_{n}\left(s,m\right)\left[\int_{s}^{m}\sigma_{n}\left(s,u\right)du\right]ds \\ &\quad -\sum_{n=1}^{N}\int_{0}^{m}\sigma_{n}\left(s,m\right)\theta_{n}ds+\sum_{n=1}^{N}\int_{t}^{t+m}\sigma_{n}\left(s,m\right)dW_{n}\left(s\right) \quad (10) \end{aligned}$$

where $r\left(t+m\right)$ is the short rate at time $t+m$; $f\left(t,m\right)$ is the forward rate curve at time t, as a function of maturity m ($m\geq 0$); N is the number of independent stochastic processes that impart instantaneous random changes to the forward rate curve and the short rate; $\sigma_{n}\left(s,m\right)$ is the volatility function for the forward rate curve/short rate process n; $dW_{n}\left(s\right)$ are independent Wiener variables under the physical measure; and u and s are dummy integration variables. Note that the first two integrals in equation 10 have been written with limits 0 and m (i.e independent of t) because Proposition 1 assumes that the volatility functions are not functions of time, and the market prices of risk are constant. The third integral retains time dependence via the path of the Wiener process.

Applying E_{t}, i.e the expectations operator as at time t, to equation 10 and re-arranging provides a relationship that will hold at any point in time, i.e:

$$\begin{aligned} f\left(t,m\right) &= E_{t}\left[r\left(t+m\right)\right]-\sum_{n=1}^{N}\int_{0}^{m}\sigma_{n}\left(s,m\right)\left[\int_{s}^{m}\sigma_{n}\left(s,u\right)du\right]ds \\ &\quad +\sum_{n=1}^{N}\int_{0}^{m}\sigma_{n}\left(s,m\right)\theta_{n}ds \quad (11) \end{aligned}$$

where $E_{t}\left[r\left(t+m\right)\right]$ is the expected path of the short rate at time t as a function of future time m, and the expectation of the stochastic term in equation 10 is zero (see Ross (1997) pp. 541-542). The functional form for $E_{t}\left[r\left(t+m\right)\right]$ has already been specified in section 2.2, and it remains to calculate the integral terms using the definitions and assumptions noted in Proposition 1.

[21] From HJM equation 5 with the substitution of HJM equation 18. Or see equation 26.16 from Chiarella (2003).

A.2 The volatility structure in the VAO model

The volatility integral term for the Level mode has already been reported in the literature as $\frac{1}{2}m^2$ (see HJM pp. 90-92). The volatility integral term for the remaining modes may be calculated using a generic approach. Firstly, define an exponential-polynomial volatility function as $\sigma(t, t+m) = \sigma \cdot \exp(-\phi m)(\phi m)^a$, where a (≥ 0) is an integer, and the volatility functions are dependent on maturity m only.[22] Following the HJM approach, $\int_s^m \sigma_n(s,u)\,du$ is calculated as:

$$\begin{aligned} & \sigma \cdot \int_s^m \exp(-\phi[m-u])(\phi[m-u])^a\,du && \text{(12a)} \\ = & \ \sigma \cdot \left[-\frac{1}{\phi}\Gamma[1+a, \phi[m-u]]\right]_s^m && \text{(12b)} \\ = & \ \frac{\sigma}{\phi} \cdot (-\Gamma[1+a, \phi(m-s)] + \Gamma[1+a, 0]) && \text{(12c)} \end{aligned}$$

where $\Gamma[\cdot,\cdot]$ is the incomplete Gamma function.[23] Note that $\Gamma[1+a, 0] = a!$, the factorial definition, and these expressions are used interchangeably below. Substituting equation 12c into $\int_0^m \sigma_n(s,m)\left[\int_s^m \sigma_n(s,u)\,du\right] ds$ gives:

$$\begin{aligned} & \frac{\sigma^2}{\phi} \cdot \int_0^m \left[\begin{array}{c} \exp(-\phi[m-s])(\phi[m-s])^a \\ \times(-\Gamma[1+a, \phi(m-s)] + a!) \end{array}\right] ds && \text{(13a)} \\ = & \ \frac{\sigma^2}{2\phi^2}\left[2a!\Gamma[1+a, \phi(m-s)] - (\Gamma[1+a, \phi(m-s)])^2\right]_0^m && \text{(13b)} \\ = & \ \frac{\sigma^2}{2\phi^2}\left[2(a!)^2 - (a!)^2 - 2a! \cdot \Gamma[1+a, \phi m] + (\Gamma[1+a, \phi m])^2\right] && \text{(13c)} \\ = & \ \frac{\sigma^2}{2\phi^2}(a! - \Gamma[1+a, \phi m])^2 && \text{(13d)} \end{aligned}$$

To calculate $h_n(\phi, m)$ for $n > 1$, write the generic OLP volatility function as a summation of exponential-polynomial terms, i.e $\sigma_n(m) = \sigma_n \cdot g_n(\phi, m) = \sigma_n \cdot \exp(-\phi m) \cdot \sum_{k=0}^{n-2} \frac{(-1)^k (n-2)!(2\phi m)^k}{(k!)^2 (n-2-k)!} = \sigma_n \cdot \sum_{k=0}^{n-2} \frac{(-2)^k (n-2)!}{(k!)^2 (n-2-k)!} \exp(-\phi m)(\phi m)^k$, and apply the corresponding results from equation 13d. This gives the result in equation 6 (which is premultiplied by the σ_n^2 that appears in equation 5).

A.3 The market prices of risk in the VAO model

The constant market prices of risk have a physical realisation on the shape of the forward rate curve via the integrals $\sum_{n=1}^N \int_0^m \sigma_n \cdot g_n(\phi, m) \cdot \theta_n dm$. This expression may be calculated directly for each mode as required. For the Level mode (i.e $n = 1$) $\int_0^m \sigma_1 \cdot g_1(\phi, s) \cdot \theta_1 ds = \sigma_1 \theta_1 [s]_0^m = \sigma_1 \theta_1 m$. For the Slope mode (i.e $n = 2$) $\int_0^m \sigma_2 \cdot g_2(\phi, m) \cdot \theta_2 ds$ is calculated as:

[22] In the HJM notation using time t and time of maturity T, this would be written $\sigma(t, T) = \sigma \cdot \exp(-\phi[T-t])(\phi[T-t])^a$, so $T = t + m$.

[23] I.e $\Gamma[1+a, z] = \int_z^\infty x^a \exp(-x)\,dx$; see Wolfram (1996) p. 740.

$$
\begin{aligned}
& \sigma_2\theta_2 \cdot \left[\frac{1}{\phi}\exp\left(-\phi s\right)\right]_0^m && \text{(14a)}\\
= \; & \frac{\sigma_2\theta_2}{\phi} \cdot \exp\left(-\phi m\right) - \frac{\sigma_2\theta_2}{\phi} && \text{(14b)}\\
= \; & -\frac{\sigma_2\theta_2}{\phi} \cdot g_2\left(\phi, m\right) - \frac{\sigma_2\theta_2}{\phi} \cdot g_1\left(\phi, m\right) && \text{(14c)}
\end{aligned}
$$

For the Bow mode (i.e $n = 3$) $\int_0^m \sigma_3 \cdot g_3\left(\phi, m\right) \cdot \theta_3 ds$ is calculated as:

$$
\begin{aligned}
& \sigma_3\theta_3 \cdot \left[\frac{1}{\phi}\exp\left(-\phi s\right)\left(-2s\phi - 1\right)\right]_0^m && \text{(15a)}\\
= \; & \frac{\sigma_3\theta_3}{\phi} \cdot \exp\left(-\phi m\right)\left(-2m\phi - 1\right) + \frac{\sigma_3\theta_3}{\phi} && \text{(15b)}\\
= \; & \frac{\sigma_3\theta_3}{\phi} \cdot \exp\left(-\phi m\right)\left(-2m\phi + 1\right) - \frac{2\sigma_3\theta_3}{\phi} \cdot \exp\left(-\phi m\right) + \frac{\sigma_3\theta_3}{\phi} && \text{(15c)}\\
= \; & \frac{\sigma_3\theta_3}{\phi} \cdot g_3\left(\phi, m\right) - \frac{2\sigma_3\theta_3}{\phi} \cdot g_2\left(\phi, m\right) + \frac{\sigma_3\theta_3}{\phi} \cdot g_1\left(\phi, m\right) && \text{(15d)}
\end{aligned}
$$

For the VAO(3) model then, $\sum_{n=1}^{3}\int_0^m \sigma_n\left(s, m\right)\theta_n ds = \sum_{n=1}^{3}\gamma_n \cdot g_n\left(\phi, m\right)$, where $\gamma_1 = \frac{1}{\phi}\left(-\sigma_2\theta_2 + \sigma_3\theta_3\right)$, $\gamma_2 = \frac{1}{\phi}\left(-\sigma_2\theta_2 - 2\sigma_3\theta_3\right)$, $\gamma_3 = \frac{1}{\phi}\sigma_3\theta_3$. Analogous results will hold for the VAO(N) model, i.e $\sum_{n=1}^{N}\int_0^m \sigma_n \cdot g_n(\phi, m) \cdot \theta_n dm = \sigma_1\theta_1 m + \sum_{n=1}^{N}\gamma_n \cdot g_n\left(\phi, m\right)$, which follows from Proposition 3.

Proposition 3 *$\int_0^m \sigma_n \cdot g_n\left(\phi, s\right) \cdot \theta_n ds$ for the modes $n > 1$ will be a linear expression of the modes $g_n\left(\phi, m\right), g_{n-1}\left(\phi, m\right), \ldots, g_1\left(\phi, m\right)$.*

Proof. *$g_n\left(\phi, s\right) = -\exp\left(-\phi s\right) \cdot \sum_{k=0}^{n-2}\frac{(-1)^k(n-2)!(2\phi s)^k}{(k!)^2(n-2-k)!}$. Write $u\left(s\right) = \sum_{k=0}^{n-2} p_{n,k} \cdot s^k$ so $du\left(s\right) = \sum_{k=0}^{n-2} q_{n,k} \cdot s^{k-1}ds$ where $p_{n,k}$ and $q_{n,k}$ capture all of the associated constants, and $dv\left(s\right) = -\exp\left(-\phi s\right)ds$ so $v\left(s\right) = \frac{1}{\phi}\exp\left(-\phi s\right)$. Integration by parts, i.e $\int u\left(s\right)dv\left(s\right) = u\left(s\right)v\left(s\right) - \int v\left(s\right)du\left(s\right)$, will result in the indefinite integral $\exp\left(-\phi s\right) \cdot \sum_{k=0}^{n-2} w_{n,k} \cdot s^k - \int \exp\left(-\phi s\right) \cdot \sum_{k=0}^{n-2} x_{n,k} \cdot s^{k-1}ds$, where $w_{n,k}$ and $x_{n,k}$ capture all of the associated constants, and the maximum order of the polynomial term in the new integration term has been reduced by 1. Hence, the repeated application of integration by parts will ultimately result in a finite sequence of exponential-polynomial functions with a maximum order polynomial term of $n - 2$, and a minimum order 0. This may be evaluated at the limits of integration 0 and m, and the resulting series of exponential-polynomial functions (with maximum order $n - 2$ and a minimum order 0) may be re-arranged into an equivalent sequence of OLP functions plus a constant.* ■

A.4 The VAO model forward rate curve

Substituting the results from sections A.2 and A.3 into equation 11 gives the generic VAO model of the forward rate curve, i.e:

$$f(t,m) = \sigma_1\theta_1 m + \sum_{n=1}^{N}[\gamma_n + \lambda_n(t)] \cdot g_n(\phi,m) - \sum_{n=1}^{N}\sigma_n^2 \cdot h_n(\phi,m) \qquad (16)$$

which is equivalent to equation 5 with the substitution $\beta_n = \gamma_n + \lambda_n(t)$.

B Proof of Proposition 2

The proof of Proposition 2 proceeds in two sections: (1) deriving the intertemporal relationship between expected paths of the short rate as time evolves within the HJM framework; and (2) substituting expected paths of the short rate as defined within the VAO model into the result from the HJM framework.

B.1 The expected path of the short rate within the HJM framework

Define $\alpha_n(s,m) = \sum_{n=1}^{N}\sigma_n(s,m)\left[-\theta_n + \int_0^m \sigma_n(s,u)\,du\right]$ in equation 11, so that $f(t,m) = E_t[r(t+m)] - \sum_{n=1}^{N}\int_0^m \alpha_n(s,m)\,ds$. Hence, given a finite time-increment τ, $f(t,\tau+m) = E_t[r(t+\tau+m)] - \sum_{n=1}^{N}\int_0^{\tau+m}\alpha_n(s,m)\,ds$; and $f(t+\tau,m) = E_{t+\tau}[r(t+\tau+m)] - \sum_{n=1}^{N}\int_\tau^{\tau+m}\alpha_n(s,m)\,ds$.

Substituting these expressions into equation 4 from HJM, i.e $f(t+\tau,m) = f(t,\tau+m) + \sum_{n=1}^{N}\int_0^\tau \alpha_n(s,m)\,ds + \sum_{n=1}^{N}\int_t^{t+\tau}\sigma_n(s,m)\,dW_n(s)$, gives the equality $E_{t+\tau}[r(t+\tau+m)] - \sum_{n=1}^{N}\int_\tau^{\tau+m}\alpha_n(s,m)\,ds = E_t[r(t+\tau+m)] - \sum_{n=1}^{N}\int_0^{\tau+m}\alpha_n(s,m)\,ds + \sum_{n=1}^{N}\int_0^\tau \alpha_n(s,m)\,ds + \sum_{n=1}^{N}\int_t^{t+\tau}\sigma_n(s,m)\,dW_n(s)$.[24] The right-hand side of this equality contains two identical integrals with different upper limits of integration. These may be combined into a single integral with a new lower limit of integration, i.e $-\sum_{n=1}^{N}\int_0^{\tau+m}\alpha_n(s,m)\,ds + \sum_{n=1}^{N}\int_0^\tau \alpha_n(s,m)\,ds = -\sum_{n=1}^{N}\int_\tau^{\tau+m}\alpha_n(s,m)\,ds$. The latter integral identically cancels with the same term on the left-hand side of the equality, giving the result:

$$E_{t+\tau}[r(t+\tau+m)] = E_t[r(t+\tau+m)] + \sum_{n=1}^{N}\int_t^{t+\tau}\sigma_n(s,m)\,dW_n(s) \qquad (17)$$

This is intuitive: the expected path of the short rate would be realised if not for the impact of new and unpredictable information represented by the summation of stochastic integrals $\sum_{n=1}^{N}\int_t^{t+\tau}\sigma_n(s,m)\,dW_n(s)$. These stochastic integrals do not have closed form solutions but $E_t\left[\int_t^{t+\tau}\sigma_n(s,m)\,dW_n(s)\right] = 0$ (see Ross (1997) pp. 541-542), and with time-invariant $\sigma_n(s,m)$ (as assumed for the VAO model), each integral will be a summation of infinitesimal $\sigma_n(s,m)\,dW_n(s)$ increments expressible as $\varepsilon_n(t+\tau)\cdot\sigma_n(s,m)$.

[24] Note that m on the left-hand side of the equality and $\tau+m$ on the right-hand side of the equality refer to the same future point in time, which is denoted by T (the time of maturity) in HJM.

B.2 The expected path of the short rate within the VAO(3) model

Using the definitions from Proposition 1 and the vector notation introduced in Proposition 2, the expected path of the short rate at times $t+\tau$ and t from equation 17 may be expressed respectively as $E_{t+\tau}\left[r\left(t+\tau+m\right)\right] = \left[\boldsymbol{\lambda}\left(t+\tau\right)\right]'\mathbf{g}\left(\phi,m\right)$, and $E_t\left[r\left(t+\tau+m\right)\right] = \left[\boldsymbol{\lambda}\left(t\right)\right]'\mathbf{g}\left(\phi,\tau+m\right)$. Each stochastic term $\int_t^{t+\tau}\sigma_n\left(s,m\right)dW_n\left(s\right)$ may be written as $\varepsilon_n\left(t+\tau\right)\cdot g_n\left(\phi,m\right)$ where $\varepsilon\left(t+\tau\right)$ has an expected value of zero, and so $\sum_{n=1}^{N}\int_t^{t+\tau}\sigma_n\left(s,m\right)dW_n\left(s\right)$ may be written in vector form as $\left[\boldsymbol{\varepsilon}\left(t+\tau\right)\right]'\mathbf{g}\left(\phi,m\right)$. Substituting these expressions into equation 17 gives the equality $\left[\boldsymbol{\lambda}\left(t+\tau\right)\right]'\mathbf{g}\left(\phi,m\right) = \left[\boldsymbol{\lambda}\left(t\right)\right]'\mathbf{g}\left(\phi,\tau+m\right) + \left[\boldsymbol{\varepsilon}\left(t+\tau\right)\right]'\mathbf{g}\left(\phi,m\right)$.

For the VAO(3) model $\mathbf{g}\left(\phi,\tau+m\right) = \left[\boldsymbol{\Phi}\left(\phi,\tau\right)\right]'\mathbf{g}\left(\phi,m\right)$, where $\mathbf{g}(\phi,\tau+m) = \left\{g_1\left(\phi,\tau+m\right),g_2\left(\phi,\tau+m\right),g_3\left(\phi,\tau+m\right)\right\}'$, $\left[\boldsymbol{\Phi}\left(\phi,\tau\right)\right]'$ is the transpose of equation 8, and $\mathbf{g}(\phi,m) = \left\{g_1\left(\phi,m\right),g_2\left(\phi,m\right),g_3\left(\phi,m\right)\right\}'$.[25] This enables the equality for the expected path of the short rate at times $t+\tau$ and t to be written as $\left[\boldsymbol{\lambda}\left(t+\tau\right)\right]'\mathbf{g}\left(\phi,m\right) = \left[\boldsymbol{\lambda}\left(t\right)\right]'\left[\boldsymbol{\Phi}\left(\phi,\tau\right)\right]'\mathbf{g}\left(\phi,m\right)+\left[\boldsymbol{\varepsilon}\left(t+\tau\right)\right]'\mathbf{g}\left(\phi,m\right)$. Adding the term premium function $\boldsymbol{\gamma}'\mathbf{g}\left(\phi,m\right)$ to both sides of the equality enables the left-hand side to be written as $\left[\boldsymbol{\beta}\left(t+\tau\right)\right]'\mathbf{g}\left(\phi,m\right) = \boldsymbol{\gamma}'\mathbf{g}\left(\phi,m\right) + \left[\boldsymbol{\lambda}\left(t\right)\right]'\left[\boldsymbol{\Phi}\left(\phi,\tau\right)\right]'\mathbf{g}\left(\phi,m\right)+\left[\boldsymbol{\varepsilon}\left(t+\tau\right)\right]'\mathbf{g}\left(\phi,m\right)\boldsymbol{\beta}\left(t+\tau\right)$, because $\boldsymbol{\beta}\left(t+\tau\right) = \boldsymbol{\gamma}+\boldsymbol{\lambda}\left(t+\tau\right)$. Factoring out the common term $\mathbf{g}(\phi,m)$, and then taking the transpose gives the following result (in column-vector form):

$$\boldsymbol{\beta}\left(t+\tau\right) = \boldsymbol{\gamma}+\boldsymbol{\Phi}\left(\phi,\tau\right)\boldsymbol{\lambda}\left(t\right)+\boldsymbol{\varepsilon}\left(t+\tau\right) \tag{18}$$

and this may be rewritten in terms of $\boldsymbol{\beta}\left(t\right)$ as follows:

$$\begin{aligned}
\boldsymbol{\beta}\left(t+\tau\right) &= \left[\mathbf{I}-\boldsymbol{\Phi}\left(\phi,\tau\right)\right]\boldsymbol{\gamma}+\boldsymbol{\Phi}\left(\phi,\tau\right)\left[\boldsymbol{\gamma}+\boldsymbol{\lambda}\left(t\right)\right]+\boldsymbol{\varepsilon}\left(t+\tau\right) && \text{(19a)}\\
&= \boldsymbol{\mu}+\boldsymbol{\Phi}\left(\phi,\tau\right)\boldsymbol{\beta}\left(t\right)+\boldsymbol{\varepsilon}\left(t+\tau\right) && \text{(19b)}
\end{aligned}$$

where $\boldsymbol{\beta}\left(t\right) = \boldsymbol{\gamma}+\boldsymbol{\lambda}\left(t\right)$, and $\boldsymbol{\mu} = \left[\mathbf{I}-\boldsymbol{\Phi}\left(\phi,\tau\right)\right]\boldsymbol{\gamma}$.

C The empirical application of the VAO model to market-quoted interest rate data

At each point in time, the VAO(3) model coefficients β_n are estimated using the cashflows and market-quoted data for the fixed interest securities that represent the yield curve at that time. Those securities are typically coupon-bearing, and so an allowance for multiple cashflows (each with a different zero-coupon discount rate corresponding to the timing of the cashflow) is required, i.e:[26]

[25] This may be verified directly by substitution of the functions $g_n\left(\phi,\tau+m\right)$ and $g_n\left(\phi,m\right)$, matrix multipication, and simplification. The general proof for the VAO(N) model (available from the author on request) revolves around a step analogous to this, but the required notation and proof that the factoristion $\mathbf{g}\left(\phi,\tau+m\right) = \left[\boldsymbol{\Phi}\left(\phi,\tau\right)\right]'\mathbf{g}\left(\phi,m\right)$ always exists is more involved than for the VAO(3) example.

[26] This is the most widely used approach for estimating OLP model coefficients directly from market-quoted data, and is outlined in Söderlind and Svensson (1997) and the articles

$$\text{Minimise} \quad : \quad \sum_{k=1}^{K} (w_k \cdot \varepsilon_k)^2 \tag{20a}$$

$$\text{where} \quad : \quad \varepsilon_k = \sum_{j=1}^{J[k]} a_{jk} \cdot \exp\left[-m_{jk} \cdot R\left(m_{jk}\right)\right] \tag{20b}$$

$$\text{and} \quad : \quad R\left(m\right) = \frac{\sigma_1 \theta_1 m}{2} + \sum_{n=1}^{3} \beta_n \cdot s_n(\phi, m) - \sum_{n=1}^{3} \sigma_n^2 \cdot u_n(\phi, m) \tag{20c}$$

where K is the number of fixed interest securities used to define the yield curve; w_k is a weighting factor, which is set to the inverse of the "basis point value" (i.e the price change of the security for a yield change of a single basis point) to obtain a minimisation of yield residuals; $J\left[k\right]$ is the number of cashflows for security k; a_{jk} is the magnitude of the cashflow j for security k (defined to be negative for the settlement price, and positive for all cashflows beyond settlement); m_{jk} is the maturity of the cashflow j of security k; and $R\left(m_{jk}\right)$ is the zero-coupon interest rate. The zero-coupon interest rates in equation 20c are $R\left(m\right) = \frac{1}{m}\int_0^m f(m)dm$, so $s_n(\phi, m) = \frac{1}{m}\int_0^m g_n(\phi, m)dm$, and $u_n(\phi, m) = \frac{1}{m}\int_0^m h_n(\phi, m)dm$. The relevant results for $s_n(\phi, m)$ and $u_n(\phi, m)$ in the VAO(3) model are, respectively:

$$s_1(\phi, m) = 1 \tag{21a}$$

$$s_2(\phi, m) = \frac{1}{\phi m}\left[\exp(-\phi m) - 1\right] \tag{21b}$$

$$s_2(\phi, m) = -\frac{1}{\phi m}\left[2\phi m \exp(-\phi m) + \exp(-\phi m) - 1\right] \tag{21c}$$

$$u_1(\phi, m) = \frac{1}{6}m^2 \tag{22a}$$

$$u_2(\phi, m) = \frac{1}{4\phi^3 m}\left[4\exp\left(-\phi m\right) - 3 + 2\phi m - \exp\left(-2\phi m\right)\right] \tag{22b}$$

$$u_3(\phi, m) = \frac{1}{2\phi^3 m}\left[\begin{array}{c} 6\exp\left(-\phi m\right) - 3\phi m \exp\left(-2\phi m\right) \\ +4\phi m \exp\left(-\phi m\right) - 4 + \phi m \\ -\phi^2 m^2 \exp\left(-2\phi m\right) - 2\exp\left(-2\phi m\right) \end{array}\right] \tag{22c}$$

Because the bond yield data in this article are monthly averages of constant-maturity yields, they do not correspond to precisely defined cashflows. Hence, the bond yield data is assumed to correspond to a par bond for the specified maturity (i.e the cashflows are a settlement price of -1, a principal of 1 for the given maturity, and semi-annual coupons between settlement and maturity

in the Bank for International Settlements (1999). Zero-coupon interest rate data, as used in Diebold and Li (2002), can also be used within this set-up by specifying just two cashflows for each security. However, that zero-coupon data is originally derived from market-quoted data anyway, and so the direct estimation method is more efficient.

equal to half the yield). Note that ϕ, θ_1, and the volatility coefficients σ_n in equation 20c are parameters that are estimated using historical data, as discussed in the text. With the cashflows and parameters defined, expression 20 may be estimated using the Newton-Raphson technique.

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Forecast horizon (years)	Yield or spread forecast	Monetary policy regime				
		Full sample	Bretton-Woods / gold price target	Federal funds rate target	Non-borrowed reserves target	Borrowed reserves / federal funds rate target
0.25	FF	122	68	148	367	56
	GS10	61	31	35	139	66
	FF/GS10	106	55	139	279	73
0.5	FF	169	108	216	446	90
	GS10	88	45	46	150	98
	FF/GS10	133	81	190	323	92
1	FF	220	148	272	557	154
	GS10	130	63	64	240	150
	FF/GS10	152	107	223	353	110
1.5	FF	259	157	296	505	221
	GS10	160	75	87	284	186
	FF/GS10	158	115	229	254	124
2	FF	285	151	278	362	274
	GS10	178	83	96	299	207
	FF/GS10	161	113	207	141	136
3	FF	316	155	169	n/a	338
	GS10	206	105	95	n/a	235
	FF/GS10	170	114	168	n/a	153

Table 1: Root-mean-squared forecast errors for the VAO(3) framework, by horizon and monetary policy regime.

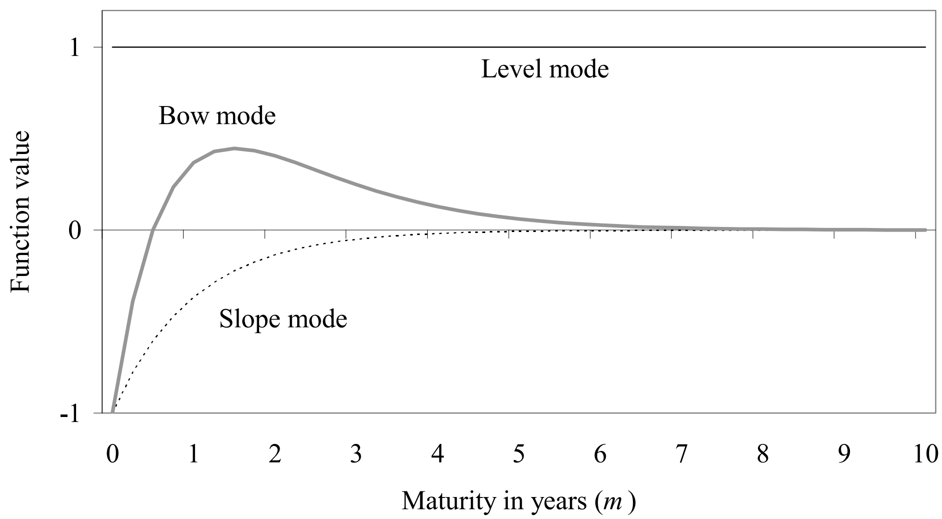

Figure 1: The first three OLP modes, $g_n(\phi, m)$, with $\phi = 1$.

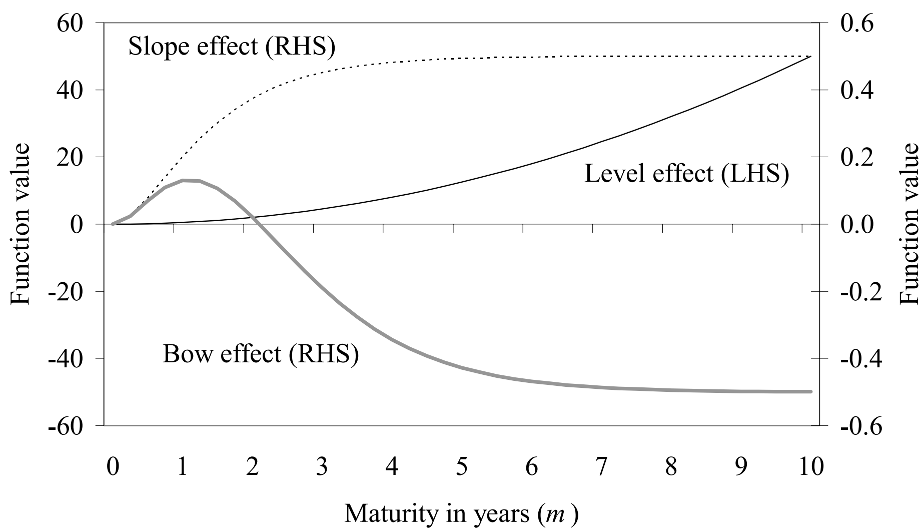

Figure 2: The first three volatility-adjustment functions, $h_n(\phi, m)$, with $\phi = 1$.

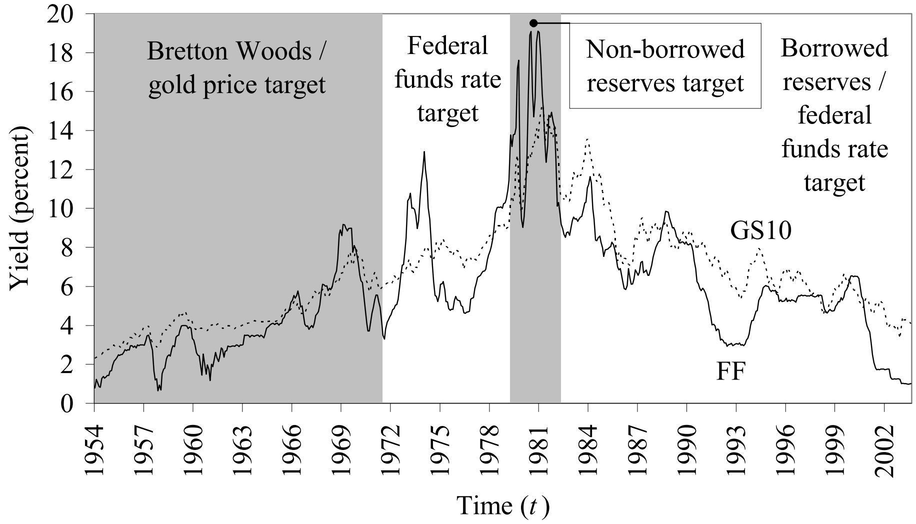

Figure 3: The time-series data for the federal funds rate (FF) and the 10-year government bond yield (GS10). The shading indicates the four different monetary policy regimes that prevailed over the sample.

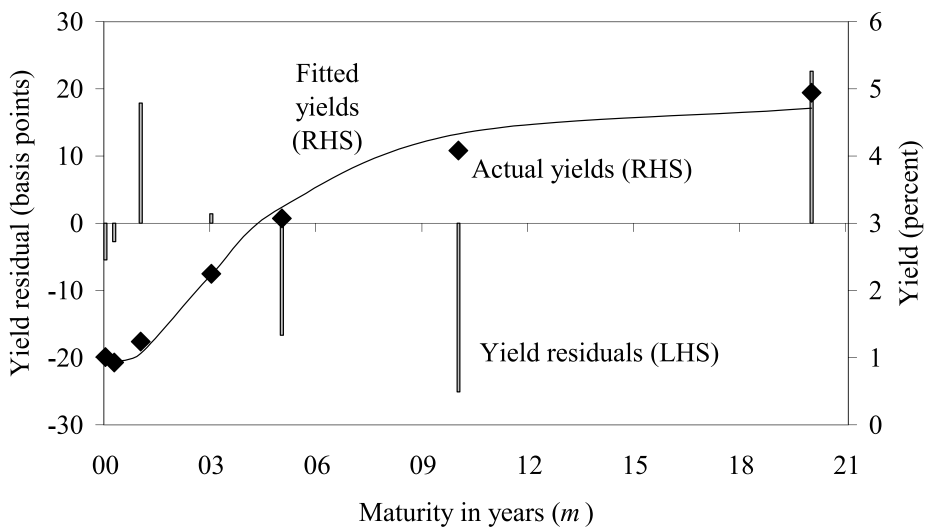

Figure 4: The cross-sectional yield curve data for February 2004, and the estimated yields using the VAO(3) model. The coefficients are $\boldsymbol{\beta}\,$(Feb-04) $= \{5.91, 8.05, -3.21\}'$ percentage points, and the parameters are $\phi = 0.804$, $\theta_1 = 1.62$ percentage points, and $\boldsymbol{\sigma} = \{0.84, 1.49, 1.17\}'$ percentage points.

Forecast horizon (years)	Yield or spread forecast	Monetary policy regime				
		Full sample	Bretton-Woods / gold price target	Federal funds rate target	Non-borrowed reserves target	Borrowed reserves / federal funds rate target
0.25	FF	1	5	24	-24	-3
	GS10	4 **	-2 **	0	2	9 ***
	FF/GS10	1	2	28	-48	12
0.5	FF	3	3	12	-3	-8
	GS10	6 **	-3 *	-2	-1	12 **
	FF/GS10	-2	-1	7	-29	-1
1	FF	-8	-9	-36	24	-5
	GS10	10 *	-4 **	-7	-1	21 **
	FF/GS10	-32 *	-12	-60	-56	-31 *
1.5	FF	-24	-29	-118 ***	80	15
	GS10	13 *	-5 *	-12	2	31 ***
	FF/GS10	-59 **	-22	-146 **	-100	-40 ***
2	FF	-36	-37	-165 ***	122	29
	GS10	15 *	-4	-15	-3	39 ***
	FF/GS10	-75 **	-25	-191 ***	-151	-40 ***
3	FF	-44	-9	-225 ***	n/a	47 *
	GS10	18	-4 *	-15	n/a	53 ***
	FF/GS10	-87 **	19	-200 ***	n/a	-49 ***

Table 2: Root-mean-squared forecast errors (RMSEs) for the VAO(3) framework less RMSEs for the random walk, by horizon and monetary policy regime. A negative entry (non-shaded) indicates VAO model outperformance, and ***, **, * respectively represent 1, 5, and 10 percent two-tailed levels of significance using the Diebold and Mariano (1995) method.

μ component	Forecast horizon (years)					
	0.25	0.5	1	1.5	2	3
$\mu(1)$	-3.9	-8.8	-22.8	-34.2	-44.9	-58.0
$\mu(2)$	28.5	64.2	150.4	232.8	304.5	466.1
$\mu(3)$	-33.9	-62.2	-111.2	-144.5	-167.3	-205.0

Table 3: Estimates of the three components of the vector $\boldsymbol{\mu}$ over the parameter estimation period October 1986 to January 1994, by forecast horizon.

Yield or spread	Forecast horizon (years)					
	0.25	0.5	1	1.5	2	3
FF	-15 **	-25	-33	-26	-32	-65
TB3	-7	-16	-21	-20	-31	-71
GS1	-1	-3	-6	-11	-27	-66
GS3	0	-1	-4	-10	-25	-64
GS5	-1	-3	-8	-14	-26	-62
GS10	5	1	-6	-11 *	-22 ***	-53 ***
GS20	5	4	1	-4	-9	-29 **
FF/GS10	-5	-26 *	-53 *	-40	-25	-32

Table 4: Root-mean-squared forecast errors (RMSEs) for the VAO(3) framework less RMSEs for the random walk for the period February 1994 to February 2004, by horizon. A negative entry (non-shaded) indicates VAO model outperformance, and ***, **, * respectively represent 1, 5, and 10 percent two-tailed levels of significance using the Diebold and Mariano (1995) method.