UNIVERSITY OF WAIKATO

Hamilton
New Zealand

Attributing Returns and Optimising United States Swaps Portfolios Using an Intertemporally-Consistent and Arbitrage-Free Model of the Yield Curve

Leo Krippner

Department of Economics

Working Paper in Economics 3/05

March 2005

Leo Krippner
AMP Capital Investors
and University of Waikato

Email: leo.krippner@ampcapital.com

Abstract

This paper uses the volatility-adjusted orthonormalised Laguerre polynomial model of the yield curve (the VAO model) from Krippner (2005), an intertemporally-consistent and arbitrage-free version of the popular Nelson and Siegel (1987) model, to develop a multi-dimensional yield-curve-based risk framework for fixed interest portfolios. The VAO model is also used to identify relative value (i.e potential excess returns) from the universe of securities that define the yield curve. In combination, these risk and return elements provide an intuitive framework for attributing portfolio returns ex-post, and for optimising portfolios ex-ante. The empirical applications are to six years of daily United States interest rate swap data. The first application shows that the main sources of fixed interest portfolio risk (i.e unanticipated variability in ex-post returns) are first-order ('duration') effects from stochastic shifts in the level and shape of the yield curve; second-order ('convexity') effects and other contributions are immaterial. The second application shows that fixed interest portfolios optimised ex-ante using the VAO model risk/relative framework significantly outperform a naive evenly-weighted benchmark over time.

Keywords

yield curve
term structure
fixed interest securities
portfolio optimisation
interest rate swaps

JEL Classification

E43, G11, G12

Acknowledgements

I thank Iris Claus, Arthur Grimes, Dimitri Margaritis, and attendees of the New Zealand Finance Colloquium 2004 for suggestions and comments. Any errors and/or omissions remain the sole responsibility of the author.

1 Introduction

This article uses volatility-adjusted orthonormalised Laguerre polynomial model of the yield curve (the VAO model) from Krippner (2005), an intertemporally-consistent and arbitrage-free model of the Nelson and Siegel (1987) model, to develop a framework applicable to measuring risk, attributing returns, and optimising fixed interest portfolios. This formally links and extends two areas of literature that have until now remained independent from each other, i.e: (1) using analytical yield-curve-based frameworks to measure interest rate risk and attribute returns; and (2) using yield or price residuals from yield curve estimation to identify "relative value" (i.e potential excess returns) from the universe of securities that define the yield curve. These are discussed in turn below.

The measurement and immunisation of interest rate risk in fixed interest portfolios has been an active and ongoing area of theoretical and empirical research for many decades. One stream of this literature is the development of "duration" and "convexity" measures, i.e analytical first-order and second-order approximations of the change in portfolio market-value for a given yield curve change.[1] For example, Macauley (1938) and Fisher and Weill (1971) developed the traditional duration measures for parallel changes to the yield curve, while Elton and Gruber (1995) pp. 540-541, and Hull (2000) pp. 112-113 note the convexity measures for parallel yield curve changes. More recently, duration measures have been developed for non-parallel changes to the yield curve, e.g see Chambers, Carleton and McEnally (1988), Reitano (1996), Mann and Ramanlal (1997), and Bowden (1997).

Duration measures have also been extended to multiple dimensions. For example, Willner (1996) and Diebold and Li (2002) use orthonormalised Laguerre polynomial (OLP) models of the yield curve, as originally introduced by Nelson and Siegel (1987), to define duration measures with three components. These measures simultaneously represent the risks associated with three potential ways that the yield curve may change, i.e a level/shift/parallel change, a slope/twist/curve change, and a bow/barbell/butterfly/curvature change, to use some of the intuitive names familiar to fixed interest portfolio managers. A conceptually similar approach is based on principal components analysis, which empirically defines the potential ways in which the yields of different maturity buckets along the yield curve may change relative to each other, e.g see Barber and Copper (1996), Hull (2000) pp. 357-361, and Kopprasch (2004).

The concept of estimating or "fitting" the yield curve with smooth analytical functions and using the resulting yield or price residuals (i.e actual less estimated yield or price) as indications of relative value is used widely by financial market participants, e.g see Brown and Giurda (2003), HSBC Bank (2003) and Malik, Barry and Xiao (2003). Several financial market participants use OLP models to identify over-valued and under-valued bonds in a wide range

[1] Other analytical approaches to the measurement and immunisation of interest rate risk are the generalised M-vector approach of Nawalkha, Soto and Zhang (2003), "gap" management (e.g see Hull (2000) pp. 113-114), key rate durations (Ho 1992), and value-at-risk analysis (e.g see Golub and Tilman (2000) chapter 5). However, these are not explicitly based on an analytical model of the entire yield curve and/or its potential movements.

of sovereign bond markets, e.g see Kacala (1993) and HSBC Bank (2001). In the literature, Sercu and Wu (1997) applies the Vasicek (1977), Cox, Ingersoll and Ross (1985), and polynomial spline models of the yield curve to Belgian government bond data, and finds a significant relationship between the resulting bond price residuals and future excess returns. Ioannides (2003) applies the Sercu and Wu (1997) approach to the UK government bond market models, and obtains the best excess returns from an out-of-sample trading strategy based on the price residuals from the Nelson and Siegel (1987) and Svensson (1994) OLP models.[2]

Apart from combining the risk and relative value elements noted above into an intuitive framework for measuring risk, attributing returns, and optimising fixed interest portfolios, this article also makes several other contributions: (1) the VAO model on which the framework is based is intertemporally-consistent and arbitrage-free model of the yield curve, whereas prior related work has not been;[3] (2) the multi-dimensional analytical risk measures are extended to second-order effects, while prior multi-dimensional frameworks based on OLP models have been limited to first-order effects; and (3) the empirical application is to United States swaps data, which is a new market relative to prior related work on sovereign bond markets.

The outline of the article is as follows: section 2 outlines the key elements and intuition of the VAO model relevant to this article; section 3 develops the risk and return frameworks based on the VAO model, and combines those into the portfolio optimisation framework; and section 4 contains the empirical application to swaps data, including ex-post return attribution and simulated real-time ex-ante portfolio optimisation. Section 5 concludes.

2 The VAO(3) model of the yield curve

2.1 The theoretical VAO(3) model

The volatility-adjusted orthonormalised Laguerre polynomial model of the yield curve (the VAO model) is a generic, intertemporally-consistent, and arbitrage-free version of the Nelson and Siegel (1987) model. The derivation of the generic VAO model, via the Heath, Jarrow and Morton (1992) framework, is detailed in Krippner (2005). The risk/return framework developed in this article uses the $N = 3$ VAO model, or the VAO(3) model for short. In the VAO(3) model, the

[2] In related work not based on yield curve estimation, Ronn (1987) exploits the "mispricing" of fixed interest securities to significantly enhance returns on portfolios of US Treasury securities while meeting future cashflow obligations, and Cornell and Shapiro (1989) provides a case study of an apparent pricing anomaly in the US Treasury market.

[3] The lack of intertemporal consistence in standard OLP models is noted in Björk and Christensen (1999), Filipović (1999*a*), Filipović (1999*b*), and Krippner (2005). The use of the Vasicek (1977) and Cox et al. (1985) models in Sercu and Wu (1997) also lacks intertemporal consistence, as those authors note, because parameters that should remain constant over time are independently re-estimated at each point in time. Also, in this article the VAO(3) model is arbitrage-free, not necessarily the yield curve data it is applied to. As noted in Brandt and Yaron (2002), applications of arbitrage-free models to identically replicate each point on the yield curve (e.g see Hull (2000) pp. 571-577 for background) lack intertemporal consistence, and may even admit arbitrage while purporting to be arbitrage-free.

shape of the yield curve and changes to the shape of the yield curve are essentially represented by three coefficients applied to three underlying components or "modes" (as detailed below). This makes the results comparable to prior uses of the Nelson and Siegel (1987) model (which also has three coefficients), and consistent with the idea that three principal components may be used to adequately capture interest rate risks, as suggested in the work of Litterman and Sheinkman (1991). However, the risk/return framework can be arbitrarily extended to $N > 3$, as might be required or desired by practitioners or researchers, and the vector derivations in section 3 continue to apply generally.

The VAO(3) model of the yield curve is:

$$R(t,m) = [\boldsymbol{\beta}(t)]' \, \mathbf{s}(\phi,m) + \frac{1}{2}\sigma_1\theta_1 m - \mathbf{v}'\mathbf{u}(\phi,m) \tag{1}$$

where $R(t,m)$ is the continuously compounding zero-coupon interest rate curve at time t, as a function of maturity m measured in years; $\boldsymbol{\beta}(t)$ is a 3-vector of the linear coefficients β_n at time t that apply to the corresponding three interest rate modes $s_n(\phi,m)$; $\mathbf{s}(\phi,m)$ is a time-invariant 3-vector function of maturity m containing the three interest rate modes $s_n(\phi,m)$ noted below; and ϕ is a constant parameter that alters the natural curvature of the modes. σ_1 and θ_1 are respectively the volatility and the market price of risk for $\beta_1(t)$, both constant parameters; $\mathbf{v}$ is a constant 3-vector of variance coefficients σ_n^2; and $\mathbf{u}(\phi,m)$ is a time-invariant 3-vector function of maturity m. These are detailed in Krippner (2005), and are not central to this article.

From Krippner (2005), the modes for the VAO(3) model are, respectively:

$$s_1(\phi,m) = 1 \tag{2a}$$

$$s_2(\phi,m) = \frac{1}{\phi m}\left[\exp(-\phi m) - 1\right] \tag{2b}$$

$$s_2(\phi,m) = -\frac{1}{\phi m}\left[2\phi m \exp(-\phi m) + \exp(-\phi m) - 1\right] \tag{2c}$$

To illustrate the intuition behind the VAO(3) model, figure 1 illustrates the first three interest rate modes of the VAO model , which are colloquially named the Level, Slope and Bow modes in reference to their intuitive shapes. Figure 2 illustrates how the shape of the yield curve may be represented by the 3-vector $\boldsymbol{\beta}(t) = (5.00, 2.00, -1.00)\,\%$, comprised of the Level, Slope, and Bow coefficients at time t, applied to the modes in figure 1. Figure 2 also shows how an instantaneous increase of 50 basis points (bps, where 1 bp = 0.01 percentage points) in the Level coefficient represents a parallel upward shift of the yield curve (i.e the interest rates of all maturities rise by 50 bps), and an instantaneous 75 bps increase in the Slope coefficient represents a "steepening" of the yield curve (i.e the short rate moves down by 75 bps, infinite-maturity rates remain unchanged, and intermediate-maturity rates move down in proportion to the magnitude of the Slope mode by maturity). Figure 3 shows how an instantaneous 75 bp increase in the Bow coefficient represents an "up-bowing" of the yield curve (i.e the short rate moves down by 75 bps, infinite-maturity rates remain unchanged, and intermediate-maturity rates move up or down in proportion to the sign and magnitude of the Bow mode by maturity). Figure 3 also contains an example of a simultaneous instantaneous change to the Level, Slope and Bow coefficients, represented by the 3-vector $\boldsymbol{\delta}(t) = (+50, -75, +75)$ bps, resulting in a new yield curve shape represented by the 3-vector $\boldsymbol{\beta}(t) = (5.50, 1.25, -0.25)\,\%$.

[Figure 1 here], [Figure 2 here], [Figure 3 here]

2.2 The VAO(3) model in practice

Appendix C of Krippner (2005) details the method for estimating the VAO(3) model coefficients and parameters from market-quoted data. Anticipating the empirical application and detailed discussion of the data in section 4, figure 4 illustrates the application of the VAO(3) model to a single observation of the US swaps yield curve, i.e 16 market-quoted mid-yields for securities with maturities ranging from overnight to 30-years, all observed at the close-of-market on Monday 16 June 2003. The estimation of the VAO(3) model results in the coefficients $\boldsymbol{\beta}(\text{16-Jun-03}) = (6.16, 9.04, -4.27)\,\%$. This coefficient vector in tandem with the other VAO(3) model parameters noted in figure 4 defines the underlying zero-coupon yield curve that prevailed on that day, which may then be used to reconstruct the fitted market prices and market yields using the cashflows of each security. Those fitted price and yields do not correspond perfectly to the market-quoted prices and yields of the securities that compose the yield curve, and so the estimation also produces 16 price and yield residuals. Table 7 in Appendix B contains a detailed numerical example of the fitted price, the price residual, and the yield residual for the two-year swap.

In general, the VAO(3) estimation of a yield curve defined by K securities at time t will generate K relationships $P_k(t) = P_k[\boldsymbol{\beta}(t)] + \varepsilon_k(t)$, where $P_k(t)$ is the market price (or market value, MV) of security k, $P_k[\boldsymbol{\beta}(t)]$ is the fitted price of security k, determined by the cashflows of security k discounted using the yield curve defined by the VAO(3) model, and $\varepsilon_k(t)$ is the price residual of security k. The price residuals may be equivalently expressed as yield residuals, i.e $\eta_k(t) = -\varepsilon_k(t)/\text{BPV}_k(t)$, where $\text{BPV}_k(t)$ is the "basis point value" (i.e the change in the security price for a single bp change in the yield) of security k at the time the yield curve is estimated.[4]

[Figure 4 here]

Estimating the VAO(3) model for each observation of the yield curve over time produces a time series of yield curve coefficients $\boldsymbol{\beta}(t)$ for the sample period, and an associated time series of price and yield residuals for each security used to define the yield curve. Again anticipating the empirical application in section 4, figure 5 plots the time series of three of the 16 yields used to define the yield curve at each point in time, and figures 5 and 6 summarise the corresponding output from the VAO(3) model; i.e respectively, the time series of Level, Slope, and Bow coefficients, and the time series of yield residuals for three of the 16 swaps data series.

[Figure 5 here], [Figure 6 here], [Figure 7 here]

[4] BPV is sometimes called dV01, or PV01 in portfolio manager jargon.

In practice, changes to the coefficient vectors $\boldsymbol{\beta}(t)$ will be measured over finite periods of time (rather than instantaneously, as assumed in the examples of section 2.1). Denoting a finite time horizon as τ, Krippner (2005) shows that $\boldsymbol{\beta}(t+\tau) - \boldsymbol{\beta}(t)$ will contain both a deterministic (anticipated) component, and a stochastic (unanticipated) component $\boldsymbol{\delta}(t, t+\tau)$. As will be detailed in section 3.1, $\boldsymbol{\delta}(t, t+\tau)$ delivers variable returns to the portfolio and so represents a source of portfolio risk. The deterministic component of $\boldsymbol{\beta}(t+\tau) - \boldsymbol{\beta}(t)$ is a source of interest accrual (i.e expected return, or "running yield" in portfolio manager jargon) to the portfolio. Changes to the price or yield residuals of each security are another potential source of return (and marginal risk) to the portfolio, as will be detailed in section 3.2.

3 Fixed interest portfolio risk, relative value, and optimisation

This section develops a framework for portfolio risk, relative value, and optimisation using the VAO(3) model. For clarity and economy of notation, the explicit time notation for $\boldsymbol{\beta}(t)$ and $\boldsymbol{\delta}(t, t+\tau)$, and the functional dependence of $\mathbf{s}(\phi, m)$ and $\mathbf{u}(\phi, m)$ on ϕ and m are omitted from this point onward. Also, because only $\boldsymbol{\beta}$ is time-varying in the framework developed in this article, equation 1 may be further abbreviated for convenience to $R(t,m) = \boldsymbol{\beta}'\mathbf{s} + Q$, where $Q = Q(m) = \frac{1}{2}\sigma_1\theta_1 m - \mathbf{v}'\mathbf{u}(\phi, m)$.[5]

The outline of section 3 is as follows: section 3.1 discusses interest rate risk, starting from an individual cashflow, to securities with multiple cashflows, and to practical portfolios with multiple securities. Section 3.2 discusses expected returns for fixed interest portfolios, and section 3.3 combines the risk/return elements together to obtain a framework for portfolio optimisation.

3.1 A component framework for yield curve exposure and risk

3.1.1 The present-value and risk of a unit cashflow

The present-value of single unit cashflow is $p(m) = \exp[-R(m) \cdot m]$, by definition. Hence, for a given initial value of $\boldsymbol{\beta}$ and Q, the present-value according to the VAO(3) model (hereafter abbreviated to PV) may be expressed as $p(\boldsymbol{\beta}, m) = \exp\left[-\left(\boldsymbol{\beta}'\mathbf{s} + Q\right) \cdot m\right]$. After a stochastic disturbance $\boldsymbol{\delta}$ over a time horizon τ, the PV of the unit cash-flow will now be $p(\boldsymbol{\beta} + \boldsymbol{\delta}, m - \tau) = \exp\left[-\left([\boldsymbol{\beta} + \boldsymbol{\delta}]'\mathbf{s} + Q\right) \cdot (m - \tau)\right]$. This relationship is non-linear, and so the changes in τ and the components of $\boldsymbol{\delta}$ will result in non-proportional changes

[5] The risks from unanticipated changes to the volatility coefficients are not considered in this article, although it would be important in a portfolio that contained material interest rate optionality (e.g options on interest rates, or mortgage-backed securities). The complete treatment of the effect of changing volatility would require option valuation within the VAO model framework, which is well beyond the scope of this article, and will be investigated in future work by the author. As a first-order approximation, an option on a fixed interest security may be included in the framework by "delta-weighting" (i.e probability-weighting) the cashflows of the underlying security, or equivalently delta-weighting the first-order yield curve exposures of the underlying security.

to the PV. However, the attributions of the change in PV due to τ and the components of $\boldsymbol{\delta}$ may be approximated to the desired degree using a Taylor expansion. As detailed in Appendix A, the second-order Taylor expansion of this expression excluding interest accrual terms (which are components of expected return, as discussed in section 3.2) is:

$$p(\boldsymbol{\beta}+\boldsymbol{\delta}, m-\tau) \simeq p(\boldsymbol{\beta}, m) - m \cdot p(\boldsymbol{\beta}, m)\, \mathbf{s}'\boldsymbol{\delta} + \boldsymbol{\delta}' \left[\frac{1}{2} m^2 \cdot p(\boldsymbol{\beta}, m)\, \mathbf{ss}'\right] \boldsymbol{\delta} \quad (3)$$

where $p(\boldsymbol{\beta}, m)\,\mathbf{s}$ is the first-order yield curve exposure (FOYCE), a column 3-vector; and $m^2 \cdot p(\boldsymbol{\beta}, m)\,\mathbf{ss}'$ is the second-order yield curve exposure (SOYCE), a 3×3 symmetric matrix.[6]

3.1.2 The present-value and risk of a fixed interest security

A unit face-value of fixed interest security k may be defined as a collection of $J[k]$ cashflows, each of amount a_{kj} occurring at time m_{kj}. The PV of security k will therefore initially be $P_k(\boldsymbol{\beta}) = \sum_{j=1}^{J} a_{kj} \cdot p(\boldsymbol{\beta}+\boldsymbol{\delta}, m_{kj})$. Excluding interest accrual terms, the PV to a second-order approximation following a stochastic disturbance $\boldsymbol{\delta}$ is:

$$P_k(\boldsymbol{\beta}+\boldsymbol{\delta}, m-\tau) \simeq P_k(\boldsymbol{\beta}) - \boldsymbol{\lambda}_k'\boldsymbol{\delta} + \boldsymbol{\delta}'\boldsymbol{\Omega}_k\,\boldsymbol{\delta} \quad (4)$$

where $\boldsymbol{\lambda}_k = \sum_{j=1}^{J} -a_{kj} m_{kj} \cdot p(\boldsymbol{\beta}, m_{kj})\,\mathbf{s}$, which represents the FOYCE of security k; and $\boldsymbol{\Omega}_k = \frac{1}{2} \cdot \sum_{j=1}^{J} a_{kj} m_{kj}^2 \cdot p(\boldsymbol{\beta}, m_{kj})\,\mathbf{ss}'$, which represents the SOYCE of security k. Table 7 in Appendix B contains a detailed numerical example of the YCEs (i.e the FOYCE and SOYCE components) for the two-year swap on 16 June 2003.

3.1.3 The present-value and risk of a fixed interest portfolio

A fixed interest portfolio may be defined as a collection of K securities, each with face-value A_k. The PV of the portfolio will therefore initially be $\sum_{k=1}^{K} A_k \cdot P_k(\boldsymbol{\beta})$. Excluding interest accrual terms, the PV to a second-order approximation following a stochastic disturbance $\boldsymbol{\delta}$ is:

[6] The interpretation of equation 3 may be clarified with a simple example, i.e assume an instantaneous parallel shift in the yield curve by Δy. In this case, $\boldsymbol{\delta} = (\Delta y, 0, 0)$ percentage points, equation 3 becomes $p(\boldsymbol{\beta}+\boldsymbol{\delta}, m) \simeq p(\boldsymbol{\beta}, m) - m \cdot p(\boldsymbol{\beta}, m) \cdot \Delta y + \frac{1}{2} m^2 \cdot p(\boldsymbol{\beta}, m) \cdot \Delta y^2$, and rearranging gives $\frac{\Delta p}{p(\boldsymbol{\beta}, m)} \simeq -m \cdot \Delta y + \frac{1}{2} m^2 \cdot \Delta y^2$, where $\Delta p = p(\boldsymbol{\beta}+\boldsymbol{\delta}, m) - p(\boldsymbol{\beta}, m)$. This is the familiar second-order approximation of the relative price sensitivity of a unit cashflow to a level shift in the yield curve, with duration m and convexity m^2. See, for example, Hull (2000) pp 108-114, and substitute a single cashflow.

$$\sum_{k=1}^{K} A_k \cdot P_k(\boldsymbol{\beta}+\boldsymbol{\delta},m-\tau) \simeq \sum_{k=1}^{K} A_k \cdot P_k(\boldsymbol{\beta}) - \left[\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k\right]' \boldsymbol{\delta} + \boldsymbol{\delta}' \left[\sum_{k=1}^{K} A_k \boldsymbol{\Omega}_k\right] \boldsymbol{\delta} \tag{5}$$

where $\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k$ represents the FOYCE of the portfolio, and $\sum_{k=1}^{K} A_k \boldsymbol{\Omega}_k$ represents the SOYCE of the portfolio. Table 8 in Appendix B contains a detailed numerical example of how the YCEs of a portfolio of fixed interest securities are derived from the unit YCEs of the constituent securities as at 16 June 2003.

Note that the measures of risk can be expressed in proportional terms (analogous to traditional duration and convexity), aggregated to a single value-at-risk (VaR) measure, and/or expressed relative to a benchmark portfolio. Also, given a view of how the yield curve might change, the portfolio manager can construct the portfolio to take an active risk on all or selected components of the yield curve. These aspects are not central to this article, but are included in Appendix C for completeness.

3.2 A framework for relative value and expected returns

Section 2.2 introduces the decomposition of the price of a fixed interest security into $P_k(t) = P_k[\boldsymbol{\beta}(t)] + \varepsilon_k(t)$ via the VAO(3) model. The expected return from a fixed interest security over a time horizon τ is therefore (by definition) $E_t[\Delta P_{k,t+\tau}] = E_t[P_{k,t+\tau}(\boldsymbol{\beta}) - P_{k,t}(\boldsymbol{\beta})] + E_t[\Delta\varepsilon_{k,t+\tau}]$, where E_t is the expectations operator applied at time t, $\Delta P_{k,t+\tau} = P_{k,t+\tau} - P_{k,t}$ is the change in the MV, $[P_{k,t+\tau}(\boldsymbol{\beta}) - P_{k,t}(\boldsymbol{\beta})]$ is the change in the PV, and $\Delta\varepsilon_{k,t+\tau} = \varepsilon_{k,t+\tau} - \varepsilon_{k,t}$ is the change in the price residual. The expected return on portfolio of K securities with face-values $A_{k,t}$ is then the summation:

$$\sum_{k=1}^{K} A_{k,t} \cdot E_t[\Delta P_{k,t+\tau}] = \sum_{k=1}^{K} A_{k,t} \cdot E_t[P_{k,t+\tau}(\boldsymbol{\beta}) - P_{k,t}(\boldsymbol{\beta})] + \sum_{k=1}^{K} A_{k,t} \cdot E_t[\Delta\varepsilon_{k,t+\tau}] \tag{6}$$

The first right-hand-side summation of equation 6 simply represents the interest accrual on the portfolio, i.e the aggregation of expected returns from each security due to the fully-anticipated passage of time. This aspect is revisited in section 4.2.

The second right-hand-side summation represents another potential source of expected return if any $E_t[\Delta\varepsilon_{k,t+\tau}] \neq 0$. In general, if $E_t[\Delta\varepsilon_{k,t+\tau}]$ is different for each security, expected portfolio returns will differ according to the weighting of each security held in the portfolio. In other words, a portfolio overweight securities with positive $E_t[\Delta\varepsilon_{k,t+\tau}]$ would offer excess expected returns relative to a portfolio with lower weights of those securities.

Any predictability of $E_t[\Delta\varepsilon_{k,t+\tau}]$ may be captured in a time-series process for the yield residual $\eta_k(t) = -\varepsilon_k(t)/\mathrm{BPV}_k(t)$. The simplest representation, as adopted in this article, is to assume that each $\eta_{k,t+\tau}$ follows an independent and stationary first-order autoregressive process, or AR(1), with identical rates of mean-reversion, i.e:

$$\eta_{k,t+\tau} - \pi_k = \theta \left(\eta_{k,t} - \pi_k\right) + \upsilon_{k,t+\tau} \tag{7}$$

where π_k is a "mean-adjustment", i.e a constant that allows for any persistent deviations of $\eta_{k,t+\tau}$ away from zero due to security-specific factors external to the VAO(3) model framework (e.g liquidity premia and/or preferred habitats, as noted in Elton and Gruber (1995), pp. 513-518); θ is the AR(1) coefficient that is assumed to be $0 < \theta < 1$;[7] and $\upsilon_{k,t+\tau}$ represents unpredictable stochastic noise, which will be distributed $\upsilon_{k,t+\tau} \sim N\left(0, \sigma_\upsilon^2\right)$ for any security k if the VAO(3) model is estimated by minimising squared yield residuals (as in this article). An advantage of assuming this time-series process is the high degree of parsimony imparted to the optimisation framework derived in section 3.3; in particular, it turns out that an estimate of θ is not required.[8]

The expectation of equation 7 is $E_t\left[\eta_{k,t+\tau}\right] - \pi_k = \theta\left(\eta_{k,t} - \pi_k\right)$, which means that $E_t\left[\Delta\eta_{k,t+\tau}\right] = (\theta - 1)\left(\eta_{k,t} - \pi_k\right)$. Hence, a security with positive $\left(\eta_{k,t} - \pi_k\right)$, i.e the yield residual above the typical yield residual, would be expected to contribute positive returns equal to $-(\theta - 1)\left(\eta_{k,t} - \pi_k\right) \cdot \mathrm{BPV}_k(t)$, over the horizon τ, and contribute risk in the order of $\sigma_\upsilon \cdot \mathrm{BPV}_k(t)$. Conversely, a security with $\left(\eta_{k,t} - \pi_k\right)$ negative would be expected to contribute negative returns. For later use, it is convenient to define $\alpha_{k,t} = \eta_{k,t} - \mu_k$ as the "potential yield enhancement" of a unit of security k at time t. This is so-named because the MV of security k could potentially be enhanced by $\alpha_{k,t} \cdot \mathrm{BPV}_k(t)$ before further expected changes to $\Delta\eta_{k,t+\tau}$ become zero.

3.3 Portfolio optimisation

3.3.1 Vector/matrix notation for fixed interest securities and portfolios

To dynamically combine the risks and returns of individual securities into portfolios, it is convenient to re-express the MV and FOYCE components for each security at each point in time in an alternative vector/matrix notation. Specifically, use the following three steps: (1) For each security, "stack" the MV and the three individual components of the FOYCE vector into a column 4-vector

[7] The empirical results are omitted for brevity, but the estimated AR(1) coefficients for each yield residual series were positive, and unit root tests typically rejected the unit root hypothesis. The assumption is also theoretically sound; apart from financial-arbitrage relative to other securities that define the yield curve, it is also a mathematical impossibility for security yields to diverge arbitrarily from the fitted yield curve.

[8] In principle, any stationary time-series process could be assumed for the residuals or estimated from the data (e.g a general vector autoregression), and the resulting expected returns would be used in the optimisation framework developed in section 3.3. However, the complexity of estimation might prove prohibitive in practical applications, and it is well known that improving the in-sample fit of a model is often detrimental to predictability relative to a parsimonious model.

$[P_k, \boldsymbol{\lambda}_{k,1}, \boldsymbol{\lambda}_{k,2}, \boldsymbol{\lambda}_{k,3}]'_t$ denoted as $\boldsymbol{\Lambda}_{k,t}$.[9] (2) Collect the vectors $\boldsymbol{\Lambda}_{k,t}$ of each security that may exist in the portfolio into a $4 \times K$ matrix $[\boldsymbol{\Lambda}_1, \ldots, \boldsymbol{\Lambda}_k, , \ldots, \boldsymbol{\Lambda}_K]_t$, denoted as $\boldsymbol{\Lambda}_t$. (3) Represent the individual face-values of the securities in the portfolio as a column K-vector $[A_{0,1}, \ldots, A_{0,k}, \ldots, A_{0,K}]'_t$, denoted as $\mathbf{A}_{0,t}$.

The MV and the FOYCE components for the portfolio will now be summarised by the column 4-vector $\boldsymbol{\Lambda}_t \mathbf{A}_{0,t}$. Regarding expected returns, collect the $\alpha_{k,t}$ for each security that may exist in the portfolio into a column K-vector $[\alpha_{1,t}, \ldots, \alpha_{k,t} \ldots, \alpha_{K,t}]'$, denoted as $\boldsymbol{\alpha}_t$. Table 8 in Appendix B contains a detailed numerical example of $\boldsymbol{\Lambda}_{k,t}$, $\boldsymbol{\Lambda}_t$, $\mathbf{A}_{0,t}$, $\boldsymbol{\Lambda}_t \mathbf{A}_{0,t}$, and $\boldsymbol{\alpha}_t \mathbf{A}_{0,t}$ for a portfolio as at 16 June 2003.

3.3.2 The optimisation of portfolios of fixed interest securities

The mean/variance approach of Markowitz (1959), as noted in Elton and Gruber (1995), essentially seeks to maximise expected portfolio returns versus the expected standard deviation of those returns while respecting given constraints on individual securities and the overall portfolio. The approach in this article is similar in that it seeks to maximise expected returns while keeping the expected standard deviation unchanged. Specifically, using the notation from section 3.3.1, define a benchmark portfolio by the face-value vector $\mathbf{A}_{0,t}$, and then propose an alternative portfolio defined by the face-value vector $\mathbf{A}_{1,t}$ that has the same expected standard deviation but maximum expected return. This optimisation problem may be summarised as the system:

$$\begin{aligned}
\text{Maximise} \quad &: \quad \sum_{k=1}^{K} A_{1,k,t} \cdot -(\theta - 1) \cdot \alpha_{k,t} \cdot \mathrm{BPV}_k(t) && \text{(8a)} \\
&\quad + \sum_{k=1}^{K} A_{1,k,t} \cdot E_t \left[P_{k,t+\tau}(\boldsymbol{\beta}) - P_{k,t}(\boldsymbol{\beta}) \right] && \text{(8b)} \\
\text{subject to} \quad &: \quad \sum_{k=1}^{K} A_{1,k,t} \cdot P_{k,t} = \sum_{k=1}^{K} A_{0,k,t} \cdot P_{k,t} && \text{(8c)} \\
\text{and} \quad &: \quad \sigma[\mathbf{A}_{1,t}] = \sigma[\mathbf{A}_{0,t}] && \text{(8d)} \\
\text{and} \quad &: \quad A_{1,k,\min} \leq A_{1,k} \leq A_{1,k,\max} && \text{(8e)}
\end{aligned}$$

where $\sigma(\cdot)$ denotes the standard deviation of portfolio returns using or $\mathbf{A}_{0,t}$ or $\mathbf{A}_{1,t}$, and $A_{1,k,\min}$ and $A_{1,k,\max}$ are given minimum and maximum constraints on the face-values of $A_{1,k}$ that may be held in the portfolio (e.g $A_{1,k,\min} = 0$ prohibits negative face-values or "short" positions in the portfolio).

[9] Using the MV anticipates the typical practical constraint that trading be cash-neutral (so that cash injections or withdrawals are not required). The SOYCEs could also be included if required, in which case the six unique individual elements of the SOYCE matrix $\boldsymbol{\Omega}_{k,t}$, i.e $\boldsymbol{\Omega}_{k,11}, \boldsymbol{\Omega}_{k,12}, \boldsymbol{\Omega}_{k,13}, \boldsymbol{\Omega}_{k,22}, \boldsymbol{\Omega}_{k,23}, \boldsymbol{\Omega}_{k,33}$, would also be stacked into $\boldsymbol{\Lambda}_{k,t}$ to capture the second-order effects. The generic VAO model to second-order would have $1 + N + N(N-1)/2$ terms in each vector $\boldsymbol{\Lambda}_k$, where N is the number of modes, and $N(N-1)/2$ is the number of unique components in the $\boldsymbol{\Omega}_k$ matrix.

The equations in system 8 may be simplified using three reasonable assumptions. These are collected here for convenience, including a brief justification (and will later be confirmed empirically), i.e:

1. Assumption 1: $\sum_{k=1}^{K} A_{1,k,t} \cdot E_t \left[P_{k,t+\tau}(\boldsymbol{\beta}) - P_{k,t}(\boldsymbol{\beta})\right]$ will be approximately constant for all feasible alternative portfolios. This follows from that fact that the universe of feasible alternative portfolios must all have the same portfolio MV, as specified by the equality constraint in equation 8b, and the interest accrual returns should therefore be similar.

2. Assumption 2: Scaling each security in the objective function by $1/\mathrm{BPV}_k(t)$ will leave all feasible alternative portfolios with similar contributions to expected portfolio standard deviation from changes to relative value. This follows from the note in section 3.2 that the unpredictable stochastic noise on the yield residual $\upsilon_{k,t+\tau}$ is distributed as $N\left(0, \sigma_\upsilon^2\right)$ for all securities. Hence, the unit expected standard deviation on the price residual for security k will be $\sigma_\upsilon \cdot \mathrm{BPV}_k(t)$, and so scaling by $1/\mathrm{BPV}_k(t)$ will leave the unit expected standard deviation constant.

3. Assumption 3: Feasible portfolios with identical FOYCE components $\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k$ will have very similar expected portfolio standard deviations. This follows from the fact that the distribution of $\boldsymbol{\delta}$ is independent of the portfolio structure. Hence, portfolios with $\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k$ identical will have $\sigma\left\{\left[\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k\right]' \boldsymbol{\delta}\right\}$ identical , and the latter is the first-order contribution to the standard deviation of the portfolio (which follows from the results derived in section 3.1).

Assumption 1 means the second line of the objective function equation 8a may be eliminated, and then the scalar $-(\theta - 1)$ may be eliminated from the first line (being identical for each security). Assumption 2 scales the remainder of the objective function by $1/\mathrm{BPV}_k(t)$. This gives the final objective function: $\sum_{k=1}^{K} A_{1,k,t} \cdot \alpha_{k,t} = \boldsymbol{\alpha}_t' \mathbf{A}_{1,t}$, using the notation of section 3.3.1.

Regarding the constraints, using the vector notation from section 3.3.1, the MV and variance constraints of 8b and c may be replaced by $\boldsymbol{\Lambda}_t \mathbf{A}_{1,t} = \boldsymbol{\Lambda}_t \mathbf{A}_{0,t}$. Specifically, if the first component of the 4-vector $\boldsymbol{\Lambda}_t \mathbf{A}_{1,t}$ equals that of $\boldsymbol{\Lambda}_t \mathbf{A}_{0,t}$, then the MVs of the two portfolios will be identical, if the second to fourth components of $\boldsymbol{\Lambda}_t \mathbf{A}_{1,t}$ equal those of $\boldsymbol{\Lambda}_t \mathbf{A}_{0,t}$, then the FOYCE components will be identical.[10]

The system represented by equations 8a to d therefore reduces to the system:

$$\begin{aligned} \text{Maximise:} \quad & \boldsymbol{\alpha}_t' \mathbf{A}_{1,t} && \text{(9a)} \\ \text{subject to:} \quad & \boldsymbol{\Lambda}_t \mathbf{A}_{1,t} = \boldsymbol{\Lambda}_t \mathbf{A}_{0,t} && \text{(9b)} \\ \text{and:} \quad & A_{1,k,\min} \leq A_{1,k,t} \leq A_{1,k,\max} && \text{(9c)} \end{aligned}$$

[10] And if the six unique SOYCE components were also included, the SOYCE components would be identical.if the fifth to tenth components of $\boldsymbol{\Lambda}_t \mathbf{A}_{1,t}$ equalled those of $\boldsymbol{\Lambda}_t \mathbf{A}_{0,t}$.

which is a linear programme. Compared to the alternative approach (in principle) of maximising expected returns versus standard deviations defined via variances and covariances, the advantages of the linear programming approach are twofold: (1) the optimisation may now be undertaken using the simplex algorithm, a standard and straightforward method of optimisation;[11] (2) the optimisation problem has ready intuition, i.e the portfolio with the highest potential value and MV and FOYCE components identical to the initial/benchmark portfolio will offer the highest expected returns for the same risks.

Note that transactions costs are deliberately not included within the optimisation framework, because they introduce several complexities that are well beyond the scope of this article. This is discussed further in Appendix C. Excluding transaction costs in both the model and the empirical results is also standard in the prior related literature (e.g see Sercu and Wu (1997) and Ioannides (2003)).

4 The empirical application of the VAO(3) risk/return framework

4.1 Description of the data

The empirical analysis is undertaken using United States fixed-for-floating interest rate swaps data. Swaps data are used rather than US Treasury market data for the following reasons: (1) swaps are a new class of security on which to investigate relative value, while the issue of relative pricing in sovereign bond markets has already been addressed previously in Sercu and Wu (1997), Ioannides (2003), and for the US Treasury market in Ronn (1987) and Cornell and Shapiro (1989); (2) the swaps data are quoted for standard maturities making the analysis more straightforward than for sovereign bond markets where the investment universe must be continuously adjusted to allow for maturities and new issuance; and (3) swaps are more standardised and homogeneous than government bonds, so there is less chance of unique market-structure factors influencing the results. This applies especially to the US Treasury market, where the relative prices of securities are influenced dynamically by on-the-run/off-the-run effects, issuance/buyback effects, liquidity considerations, differences in tax treatment, and differences in the effective underlying funding rates.[12]

The data are obtained from Datastream, and are the daily closing mid-rates for the federal funds target rate, and the 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 20, 25, and 30-year fixed-for-floating swaps rates. This gives 16 rates in total, and the sample period is from 1 May 1998 to 22 September 2004 (the beginning of the period is limited by the availability of the 20, 25, and 30-year swaps rates). Note that the federal funds rate is used to provide a time-consistent

[11] See, for example, Murty (1983).

[12] See Fleming (2003) for a discussion of these aspects in the context of measuring market liquidity. Note that the effective funding rate for each US Treasury security is its associated repurchase rate, and these often differ markedly between bonds due to bonds going "special" (i.e being tightly held by a few market participants) in the physical market.

representative short-maturity rate for the swaps yield curve. While a bank-risk short-maturity rate would be more ideal (to be consistent the swaps rates that are also bank-risk), the London interbank offered (LIBOR) rates that are available are fixed in the London morning, which would not be time-consistent with the swaps rates at the US market close. Using the federal funds rate may make a minor impact on the outright attribution of interest accrual returns, as noted in the following section, but it makes no impact on the comparative analysis in section 4.3.

While the sample period is relatively short in chronological time, it should be representative; i.e the data spans 1,599 trading days, it captures a full monetary policy cycle (i.e the 1999 to 2000 sequence of federal funds rate hikes, the 2001 to 2003 sequence of cuts, and the 2004 sequence of hikes to-date), and it captures a full trough-peak-trough cycle in long-maturity rates. The sample also includes the financial market stress events of the Asian/Russian/LTCM crisis, the 11 September 2001 World Trade Centre tragedy, the 1999 30-year Treasury buy-back programme and the subsequent 2001 cessation of issuance, and the deflationary scare of 2003 to 2004. Before beginning the empirical analysis, 24 obvious data anomalies occurring over 11 days of the dataset were corrected,[13] and non-trading days were removed from the dataset. Figure 5 illustrates the time series of three of the 16 data series used in the empirical analysis.

As noted in Hull (2000) pp. 132-133, a fixed-for-floating rate swap agreement is equivalent to a fixed coupon bond funded by a floating rate note liability. A market-quoted swaps rate defines the coupon of a par fixed coupon bond, and the other parameters are defined by agreed market convention; i.e a US swaps rate $S(t,x)$ quoted at date t for maturity x-calendar-years implies notional settlement of the unit face-value (i.e a cashflow of -1) on date $t+2$-working-days, with the first coupon (i.e a cashflow of $+S(t,x)/2$) on date $t+2+6$-calendar-months, subsequent coupons (i.e cashflows of $+S(t,x)/2$) each 6-calendar-months thereafter, and the final coupon payment and notional return of principal (i.e a cashflow of $1+S(t,x)/2$) at the maturity date of $t+2+x$-calendar-years.[14] Figure 1 shows an example of the fixed cashflows implied by the 2-year swap rate quoted on Monday 16 June 2003. The floating rate leg of the swap is a par floating rate note with notional drawdown of the unit face-value on date $t+2$-working-days, subsequent payments of interest at three-monthly intervals based on the 3-month LIBOR rate, and the notional payback at the maturity date of $t+2+x$-calendar-years. However, these floating cashflows make no contribution to the valuation and the interest rate risk of the swap agreement implied by the market-quoted rate, and may therefore be ignored for the analysis in this article.[15]

[13] Specifically, one "big figure error" (i.e an incorrect percentage point for one swap rate) and "stale quotes" indicated by daily changes in yields for individual swap maturities that were 10 to 50 bps inconsistent with the daily changes for swaps rates of similar maturities.

[14] All subject to the modified following business day convention, as noted in Hull (2000) p. 128.

[15] The floating leg of the swap will only contribute valuation and interest rate risk once the first floating rate is set, and therefore becomes a known cashflow. In this article, the swaps are

4.2 The ex-post attribution of portfolio returns

The investigation of ex-post portfolio returns is undertaken using a benchmark portfolio constructed as follows: (1) the benchmark portfolio is established as at 1 May 1998 with zero cash, a \$10 million face-value for each swap maturity (to give a total market value of zero, because the MV of floating leg of the swap equals the MV of the fixed leg); (2) this portfolio is carried over to the following trading day, and the daily return is calculated by revaluing the cash flows of the swaps using the zero-coupon curve "boot-strapped" from the new prevailing yield curve;[16] (3) the face-values in the portfolio are reset to \$10 million; and (4) steps 2 and 3 are repeated for the entire sample. This process gives a time-series of 1,598 independent daily returns for the benchmark portfolio. The cumulative returns for the benchmark portfolio are plotted in figure 8.

Attributing ex-post portfolio returns to the YCEs for a given day firstly requires an ex-post estimate of $\boldsymbol{\delta}$ for that day. This can be calculated as:[17]

$$\boldsymbol{\delta} = \boldsymbol{\beta}(t+\tau) - \boldsymbol{\Phi}(\phi,\tau)\,\boldsymbol{\beta}(t) - \boldsymbol{\mu}(\tau) \tag{10}$$

where:

$$\boldsymbol{\Phi}(\phi,\tau) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \exp(-\phi\tau) & -2\phi\tau\exp(-\phi\tau) \\ 0 & 0 & \exp(-\phi\tau) \end{bmatrix} \tag{11}$$

and $\boldsymbol{\mu}(\tau)$ is a constant 3-vector of term premia coefficients applicable to the horizon τ (one working day in this case). As used in this article, an internally-consistent estimate of $\boldsymbol{\mu}(\tau)$ for the sample may be estimated ex-post as the average of the time series $\boldsymbol{\beta}(t+\tau) - \boldsymbol{\Phi}(\phi,\tau)\,\boldsymbol{\beta}(t)$ calculated for each day of the sample. This ensures that the average of the realised $\boldsymbol{\delta}$ values will identically equal zero (which is the expected value of $\boldsymbol{\delta}$) over the sample. Secondly, the calculations of the vector $\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k$ and the matrix $\sum_{k=1}^{K} A_k \boldsymbol{\Omega}_k$ for the given day are undertaken using the estimated VAO(3) model for the given day, and the cashflows of each of the securities in the benchmark portfolio on that day. Finally, substituting the values of $\boldsymbol{\delta}$, $\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k$, and $\sum_{k=1}^{K} A_k \boldsymbol{\Omega}_k$ into equation 5 gives the returns for that day that are attributable to the individual FOYCE components and the six unique SOYCE components. Repeating this over the entire sample gives the sequence of attributions to the FOYCE and SOYCE components.

Portfolio returns due to changes in the relative value of the portfolio are calculated directly by comparing the relative value of each security to its relative value on the following day. The final attribution is the interest accrual return,

---

effectively terminated via the exchange of cash equal to the market-value of the swap, before the floating leg becomes effective.

[16] Hull (2000) p. 150 discusses the concepts behind this technique. The analysis in this article uses a continuous, stepwise zero-coupon curve based on the linear interpolation of the continuously compounding interest rates at the maturity of each swap.

[17] Equation 10 follows from the result in Krippner (2005) that $E_t[\boldsymbol{\beta}(t+\tau)] = \boldsymbol{\mu}(\tau) + \boldsymbol{\Phi}(\phi,\tau)\,\boldsymbol{\beta}(t)$, and then substituting this result into the definition $\boldsymbol{\delta}(t,t+\tau) = \boldsymbol{\beta}(t+\tau) - E_t[\boldsymbol{\beta}(t+\tau)]$.

which is estimated as the difference between the actual benchmark portfolio returns less the FOYCE, SOYCE, and relative value returns already attributed above.

The ex-post portfolio attribution results are summarised in table 1. This shows that the dispersion of ex-post daily returns (as measured by the standard deviation, minimum, maximum, or the spread between maximum and minimum) are dominated by the FOYCE components, e.g the standard deviation rankings are $\sigma\,(\text{Level FOYCE}) > \sigma\,(\text{Slope FOYCE}) > \sigma\,(\text{Bow FOYCE}) \gg \sigma\,(\text{Relative value}) > \sigma\,(\text{Accrual returns}) > \sigma\,(\text{SOYCEs})$. Table 2 contains the variances and covariances between each of the attribution groups, which shows that the variances and covariances outside of the "FOYCE block" are very small. Specifically, the FOYCE block variance is within 3% of total portfolio variance, and therefore the FOYCE standard deviation would be within 1.5% of the total portfolio standard deviation.

[Table 1 here], [Table 2 here]

These results offer an important insight into ex-ante portfolio risks. That is, $\boldsymbol{\delta}$ is random quantity ex-ante, and so $\left[\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k\right]' \boldsymbol{\delta}$ and $\boldsymbol{\delta}' \left[\sum_{k=1}^{K} A_k \boldsymbol{\Omega}_k\right] \boldsymbol{\delta}$ represent risks to portfolio returns due to unanticipated changes in the Level, Slope, and/or Bow of the yield curve. It is therefore evident that the risks of the portfolio are adequately captured by the FOYCE components. This result accords with assumption 3 in section 3.3.2, and it also suggests that the SOYCE components of the YCEs may be ignored in the practical management of fixed interest portfolios.[18]

Regarding returns, the FOYCE and SOYCE returns simply reflect the aggregation of the changes to the shape of the yield curve that prevailed over the sample period applied to the YCEs of the benchmark portfolios (e.g the attribution to Level FOYCE component is positive because the portfolio had negative Level FOYCE in a falling rate environment). The returns attributed to relative value are relatively small, which suggests that the contributions from relative value tend to average out over time in the benchmark portfolio. The interest accrual returns are positive, which is worthy of note. In a risk-neutral environment, this interest accrual component should be identically zero, because the interest accrual from all cashflows should be identical, and a swap is equivalent to a fixed interest asset exactly offset by a floating rate liability. The positive interest accrual of $3.870 million (which equates to 39 basis points per annum on the constant face-value of $150 million over the 6.57 years of the sample period) therefore reflects the risk-averse environment that would typically be expected in financial markets; i.e the interest accrual is implicitly higher on the cashflows of the longer-maturity fixed interest asset longer than on the floating rate liability. As noted in the previous section, some of these positive returns may be due to the use of the federal funds rate instead of a bank-risk short-

[18] The latter suggestion is consistent with the results of Soto (2001), where it is found that constraints on "level, slope and curvature of term structure shifts are necessary to guarantee a return close to target", while differences in traditional convexity have little impact over horizons of one and two years. However, SOYCE effects will aggregate steadily over time, (because they are effectively the sums of the squared components of the vector $\boldsymbol{\delta}$), which means they will ultimately make material contributions to portfolio returns over long horizons.

maturity rate. However, the 3–month rate LIBOR has averaged only 17.8 bps above the federal funds rate over the sample, and since the overnight LIBOR rate was introduced in January 2001, it has averaged only 7.7 bps above the federal funds rate. Hence, the interest accrual component would still be positive even allowing for adjustments of those magnitudes.[19]

4.3 Simulated real-time ex-ante portfolio optimisation

Simulated real-time (SRT) ex-ante portfolio optimisation is so-named because the portfolio optimisation at each point in time uses only information that would have been available at that point in time, and it seeks to maximise future returns.[20] The SRT ex-ante optimisation in this article uses the VAO(3) parameters previously estimated in Krippner (2005) using monthly data for the government bond curve from October 1986 to January 1994, i.e $\sigma_1 = 0.84\%$, $\theta_1 = 1.62$, $\phi = 0.8040$, and $\mathbf{v} = \left(0.84^2, 1.49^2, 1.17^2\right)\%^2$. These estimates would obviously have been available if the portfolio optimisation had begun in 1 May 1998. Regarding mean-adjustments to the yield residuals to obtain $\boldsymbol{\alpha}_t$ for the optimisation process, two alternatives are tested: optimised portfolio 1 (OP1) uses no mean-adjustment, so $\pi_k = 0$; and OP2 uses SRT mean-adjustments, so π_k is set by recursive estimation using the mean of the yield residuals up to the previous working day, i.e $\pi_k(t) = \frac{1}{t-1}\sum_{i=1}^{t-1} \eta_{k,i}$ (the initial value π_k (1-May-98) is set to zero, given that the yield residual from the previous day would not be available).

Several further optimisations are performed using the in-sample estimates of the VAO(3) parameters and three different calculations of the mean-adjustment, i.e OP3 uses $\pi_k = 0$; OP4 uses the full-sample estimated means for the yield residuals, i.e $\pi_k = \frac{1}{1598}\sum_{i=\text{1-May-98}}^{\text{22-Sep-04}} \eta_{k,i}$; and OP5 uses SRT estimates of π_k as for OP2. These are obviously not genuine SRT ex-ante optimisations, but OP4 provides a direct comparison to the non-optimised benchmark, and the other optimisations are used to investigate the sensitivity of optimisation performance to the VAO(3) parameters and the mean-adjustments used in the optimisations.

The investigation of portfolio optimisation performance is undertaken using an alternative portfolio constructed as follows: (1) the benchmark portfolio is established as at 1 May 1998 with zero cash, and a \$10 million face-value for each swap maturity; (2) the VAO(3) model is estimated using the yield curve data at time t, and this is used to calculate the yield residuals for each swap security and the FOYCEs for the benchmark portfolio; (3) π_k is set according the alternatives discussed for OP1 to OP5 above, and $\boldsymbol{\alpha}_t$ is calculated; (4) the alternative portfolio is optimised using the linear programme in equation 9 (i.e with the alternative portfolio MV and FOYCE components equal to those of

[19] The variability in the interest accrual returns partly reflects the uneven spacing of working days (e.g there will be more interest accrual expected over a weekend or holiday than between adjacent weekdays). Also, being a "remainder", the interest accrual term will implicitly capture third-order and higher effects ignored in the second-order Taylor approximation of section 3, but those should be very small.

[20] The name is adopted from the simulated real-time forecasting of Stock and Watson (2002) in a macroeconomic context, and is also known as out-of-sample testing.

the benchmark portfolio on that day), and the constraint that the face-values of each swap security are maintained between \$0 and \$20 million, and cash is maintained at zero; (5) this optimised portfolio is carried over to the following trading day and the daily return is calculated by revaluing the cash flows of the swaps using the zero-coupon curve "boot-strapped" from the new prevailing yield curve; and (6) steps 2 to 4 are repeated for the entire sample.

This process gives a time series of 1,598 independent daily returns for the optimised portfolios. Figure 8 plots the cumulative returns for OP4. It is evident that the returns for OP4 are higher than for the non-optimised benchmark (by \$15.049 million over the full sample), and those excess returns accrue steadily over the sample period (hence, the excess performance is unlikely to result from one or more fortuitous events). The other optimised portfolios also outperform the benchmark portfolio.

To gauge the source of those excess returns, the OP4 returns are attributed ex-post as for the benchmark portfolio, and those results are shown in table 3. The attributions to the FOYCE components are identical to the benchmark, which occurs by definition because the optimisation process exactly matches the FOYCE components of the optimised and benchmark portfolios. The attributions to the SOYCE components are very similar to those of the benchmark, indicating that leaving the SOYCE components uncontrained makes an immaterial difference to portfolio returns.

The largest difference is in the relative value component, which is \$14.859 million higher in the optimised portfolio. This accords with the premise of the optimisation framework, i.e that the maximisation of relative value in the optimisation process should deliver excess returns over time relative to a non-optimised benchmark portfolio. There is also a slight difference between the optimised and benchmark interest accrual components, but this is several orders of magnitude smaller than the relative value differences. Indeed, the similarity of the interest accrual returns accords with assumption 1 in section 3.3.2 that interest accrual returns do not differ much between feasible portfolios. Specifically, the total interest accrual return of \$4.051 million equates to 41 basis points per annum, compared to the 39 basis points per annum in the benchmark portfolio.

Regarding the dispersion of attributed returns for OP4, the standard deviations in table 3 and the variances and covariances in table 4 are typically identical or very similar to those of the benchmark portfolio. The exception is again in the components related to relative value. While table 3 shows that the standard deviation of the relative value attributions is higher in OP4 than the benchmark portfolio, table 4 shows that this is offset by a negative covariance with the FOYCE components, leaving the overall variance of OP4 approximately equal to the benchmark portfolio. This results accords with assumption 2 in section 3.3.2, and most importantly it also indicates that OP4 is not taking on excess risk relative to the benchmark to achieve the relative excess returns. The other optimised portfolios show similar results relative to the benchmark portfolio.

Figure 9 plots the cumulative returns of the OP2, OP4, and OP5 (i.e the portfolios optimised with full-sample or SRT mean-adjustments) less the cumulative benchmark returns. Each of these series indicate excess returns accruing steadily over the sample period, with similar total excess returns by the end of the sample. Table 5 contains the summary annualised statistics for the excess returns. The information ratios (i.e annualised returns divided by the annualised standard deviations) are extremely high, and the corresponding t-statistics underlying the information ratios are extremely significant (i.e well beyond the 1% threshold).

Figure 10 plots the cumulative returns of the OP1 and OP3 (i.e with $\pi_k = 0$) less the cumulative benchmark returns. While the cumulative excess returns are still positive, the end-of-sample excess returns are much less than for OP2, OP4, and OP5, and the excess returns do not accrue as steadily. Table 5 shows that the information ratio for OP1 is moderate (with the t-statistic only significant to the 10% level), and that for OP3 is small (with an insignificant t-statistic).

Table 6 compares the returns for the optimised portfolios to each other. OP4 is the natural benchmark for optimised portfolio performance, because it uses the ideal parameters for the optimisation (i.e in-sample parameters for the VAO(3) model and in-sample estimates for the mean-adjustments). Within the optimised portfolios that use the in-sample VAO(3) parameters, line 1 of table 6 (i.e OP3 less OP4) shows that the difference in excess returns is significantly negative using no mean-adjustment, but line 2 (i.e OP5 less OP4) shows the difference is insignificant using the SRT mean-adjustment. This suggests that optimisation performance deteriorates materially when inappropriate mean-adjustments are used, but using consistent estimates provided by the SRT mean-adjustments makes little impact. As an aside, the deterioration of the optimisation results using no mean-adjustment tentatively suggests that factors external to the VAO(3) model framework (e.g liquidity premia and/or preferred habitats, as noted in Elton and Gruber (1995) pp. 513-518) may influencing the shape of the US swaps curve over the sample period, although further research would be required to make any firm conclusions on that aspect.

Comparing the returns of optimised portfolios that use pre-sample VAO(3) parameters to OP4, line 3 of table 6 (i.e OP1 less OP4) again shows the material deterioration of optimisation performance with no mean-adjustment, while line 4 of table 6 (i.e OP2 less OP4) indicates that performance is not materially affected by using different VAO(3) parameters when a consistent mean-adjustment is made. This suggests that the optimisation results are much more sensitive to whether consistent mean-adjustments are being made, rather than whether the "correct" VAO(3) parameters are being used. This is confirmed in line 5, where performance materially deteriorates with no mean-adjustment even when the same pre-sample VAO(3) parameters are used.

Finally, lines 6 and 7 in table 6 indicate that the choice of VAO(3) parameters can have a material influence on performance when the mean-adjustment aspect is held constant between the optimisations. However, using "incorrect" VAO(3) parameter estimates is evidently not necessarily detrimental to optimisation performance, because the optimisations using pre-sample VAO(3) parameter estimates show higher returns that with the "correct" VAO(3) parameter estimates.

5 Conclusion

This article uses volatility-adjusted orthonormalised Laguerre polynomial model of the yield curve (the VAO model) from Krippner (2005), an intertemporally-consistent and arbitrage-free version of the Nelson and Siegel (1987) model, to develop a framework applicable to measuring risk, attributing returns, and optimising fixed interest portfolios. In the empirical application using six years of US interest rate swaps data, the ex-post attribution analysis shows that nearly all of the variability in portfolio returns is due to first-order yield curve exposures (i.e FOYCEs, or "duration" effects) from stochastic shifts in the level and shape of the yield curve; second-order ("convexity") effects and other contributions are immaterial. Ex-ante, those yield curve changes are unpredictable, and so represent sources of risk to the portfolio.

The second empirical application shows that portfolios optimised ex-ante using the VAO model risk/return framework significantly outperform a naive evenly-weighted benchmark over time. This provides support for the idea that "relative value" (i.e deviations of actual yields from the yields implied by the VAO model) is a quantifiable concept, and maximising that quantity offers a way of enhancing portfolio returns. That said, consistent with the prior related literature, the analysis presented in this article does not include transactions costs (Appendix C discusses the complexities that transactions costs introduce). Hence, it remains an open question whether the US swaps market offers arbitrage opportunities that can be systematically exploited in practice. This will be explored by the author in future work.

A The second-order Taylor expansion for a unit cash-flow

Using the notation of Greene (1997), the second-order Taylor expansion of $p(\boldsymbol{\beta}+\boldsymbol{\delta}, m-\tau) = \exp\left[-\left([\boldsymbol{\beta}+\boldsymbol{\delta}]'\mathbf{s}+Q\right)\cdot(m-\tau)\right]$ around the column 4-vector $[\beta_1, \beta_2, \beta_3, m]'$ is defined as:

$$\begin{aligned} p(\boldsymbol{\beta}+\boldsymbol{\delta}, m-\tau) \quad \simeq \quad & p(\boldsymbol{\beta}, m) + \left[\frac{\partial p(\boldsymbol{\beta}, m)'}{\partial \boldsymbol{\beta}}, \frac{\partial p(\boldsymbol{\beta}, m)}{\partial m}\right]\left[\begin{array}{c} \boldsymbol{\delta} \\ -\tau \end{array}\right] \\ & + \frac{1}{2}\left[\boldsymbol{\delta}', -\tau\right]\left[\begin{array}{cc} \frac{\partial^2 p(\boldsymbol{\beta},m)}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}'} & \frac{\partial^2 p(\boldsymbol{\beta},m)}{\partial m \partial \boldsymbol{\beta}} \\ \frac{\partial^2 p(\boldsymbol{\beta},m)}{\partial m \partial \boldsymbol{\beta}}' & \frac{\partial p(\boldsymbol{\beta},m)}{\partial m^2} \end{array}\right]\left[\begin{array}{c} \boldsymbol{\delta} \\ -\tau \end{array}\right] \qquad (12) \end{aligned}$$

where, for notational convenience, $\left[\boldsymbol{\delta}', -\tau\right]$ is the row 4-vector $[\delta_1, \delta_2, \delta_3, -\tau]$ partitioned as the row 3-vector $\boldsymbol{\delta}'$ and the scalar τ,[21] and the first-order and second-order components in equation 12 have been partitioned in accordance with this notation. Expanding equation 12 using the given partitioned components gives:

[21] In full, $[\delta_1, \delta_2, \delta_3, -\tau] = [\beta_1+\delta_1, \beta_2+\delta_2, \beta_3+\delta_3, m-\tau] - [\beta_1, \beta_2, \beta_3, m]$, or $[\boldsymbol{\delta}, -\tau] = [\boldsymbol{\beta}+\boldsymbol{\delta}, m-\tau] - [\boldsymbol{\beta}, m]$.

$$p(\boldsymbol{\beta},m)+\left[\frac{\partial p(\boldsymbol{\beta},m)}{\partial\boldsymbol{\beta}}\right]'\boldsymbol{\delta}+\frac{1}{2}\boldsymbol{\delta}'\left[\frac{\partial^2 p(\boldsymbol{\beta},m)}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'}\right]\boldsymbol{\delta}$$
$$-\frac{\partial p(\boldsymbol{\beta},m)}{\partial m}\cdot\tau+\frac{1}{2}\frac{\partial p(\boldsymbol{\beta},m)}{\partial m^2}\cdot\tau^2-\tau\cdot\left[\frac{\partial^2 p(\boldsymbol{\beta},m)}{\partial m\partial\boldsymbol{\beta}}\right]\boldsymbol{\delta} \quad (13)$$

where the first line of equation 13 contains the capital value terms, and the second line contains the interest accrual terms. The partial derivatives in the first line of equation 13 may be calculated directly, i.e:

$$\begin{aligned}
\frac{\partial p(\boldsymbol{\beta},m)}{\partial\boldsymbol{\beta}} &= \frac{\partial\exp\left[-\left(\boldsymbol{\beta}'\mathbf{s}+Q\right)\cdot m\right]}{\partial\boldsymbol{\beta}} && (14a)\\
&= \frac{\partial\exp\left[-\left(\boldsymbol{\beta}'\mathbf{s}+Q\right)\cdot m\right]}{\partial\left(\boldsymbol{\beta}'\mathbf{s}+Q\right)}\frac{\partial\left(\boldsymbol{\beta}'\mathbf{s}+Q\right)}{\partial\boldsymbol{\beta}} && (14b)\\
&= -m\cdot\exp\left[-\left(\boldsymbol{\beta}'\mathbf{s}+Q\right)\cdot m\right]\mathbf{s} && (14c)\\
&= -m\cdot p(\boldsymbol{\beta},m)\,\mathbf{s} && (14d)
\end{aligned}$$

where the second line applies the chain rule of differentiation (in a scalar sense, because $\boldsymbol{\beta}'\mathbf{s}+Q=R(m)$, which is a scalar function of m), and the third line makes the substitution $\boldsymbol{\beta}'\mathbf{s}+\mathbf{Q}=\mathbf{s}'\boldsymbol{\beta}+Q$ (because both expressions are the scalar function $R(m)$) and applies the result from Greene (1997) p. 51 that $\frac{\partial[\mathbf{s}'\boldsymbol{\beta}]}{\partial\boldsymbol{\beta}}=\mathbf{s}$. Using similar techniques, the second partial derivative may be calculated using the result from equation 14, i.e:

$$\begin{aligned}
\frac{\partial^2 p(\boldsymbol{\beta},m)}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'} &= \frac{\partial}{\partial\boldsymbol{\beta}}\left[\frac{\partial p(\boldsymbol{\beta},m)}{\partial\boldsymbol{\beta}'}\right]=\frac{\partial}{\partial\boldsymbol{\beta}}\left[\frac{\partial p(\boldsymbol{\beta},m)}{\partial\boldsymbol{\beta}}\right]' && (15a)\\
&= \frac{\partial\left\{-m\cdot\exp\left[-\left(\boldsymbol{\beta}'\mathbf{s}+Q\right)\cdot m\right]\mathbf{s}'\right\}}{\partial\boldsymbol{\beta}} && (15b)\\
&= -m\cdot\frac{\partial\exp\left[-\left(\boldsymbol{\beta}'\mathbf{s}+Q\right)\cdot m\right]}{\partial\left(\boldsymbol{\beta}'\mathbf{s}+Q\right)}\frac{\partial\left(\boldsymbol{\beta}'\mathbf{s}+Q\right)}{\partial\boldsymbol{\beta}}\mathbf{s}' && (15c)\\
&= -m\cdot\left\{-m\cdot\exp\left[-\left(\boldsymbol{\beta}'\mathbf{s}+Q\right)\cdot m\right]\cdot\mathbf{s}\mathbf{s}'\right\} && (15d)\\
&= m^2\cdot\exp\left[-\left(\boldsymbol{\beta}'\mathbf{s}+Q\right)\cdot m\right]\mathbf{s}\mathbf{s}' && (15e)\\
&= m^2\cdot p(\boldsymbol{\beta},m)\,\mathbf{s}\mathbf{s}' && (15f)
\end{aligned}$$

To illustrate that the elements of the second line of equation 13 represent interest accrual terms, the first term may also be derived directly (which is simplified by writing $R(m)$ as the equivalent scalar function of m), i.e:

$$\begin{aligned}
-\frac{\partial p(\boldsymbol{\beta},m)}{\partial m} &= -\frac{\partial\exp\left[-R(m)\cdot m\right]}{\partial m} && (16a)\\
&= -\frac{\partial\exp\left[-R(m)\cdot m\right]}{\partial\left[R(m)\cdot m\right]}\frac{\partial\left[R(m)\cdot m\right]}{\partial m} && (16b)\\
&= \exp\left[-R(m)\cdot m\right]\cdot f(m) && (16c)\\
&= p(\boldsymbol{\beta},m)\cdot f(m) && (16d)
\end{aligned}$$

where equation 16c uses the result that $\frac{d[R(m)\cdot m]}{dm} = f(m)$, where $f(m)$ is the forward rate as a function of maturity m.[22] $p(\boldsymbol{\beta}, m)\cdot f(m)\cdot\tau$ therefore represents the interest earned on the PV of the unit cashflow over the horizon τ. The calculations for the remaining second-order terms of equation 13 are not shown here for brevity, but in summary $\frac{1}{2}\frac{\partial p(\boldsymbol{\beta},m)}{\partial m^2}\cdot\tau^2 = p(\boldsymbol{\beta}, m)\left\{[f(m)]^2 + \frac{\partial f(m)}{\partial m}\right\}\cdot\frac{1}{2}\tau^2$, which represents "interest on interest" over the horizon τ, and $-\tau\cdot\left[\frac{\partial^2 p(\boldsymbol{\beta},m)}{\partial m\partial\boldsymbol{\beta}}\right]'\boldsymbol{\delta} = -\tau\cdot p(\boldsymbol{\beta}, m)\left[m\cdot f(m)\,\mathbf{s} + m\mathbf{g} + \mathbf{s}\right]'\boldsymbol{\delta}$, which represents "interest on changes in PV" over the time-step τ, and $\mathbf{g}(\phi, m) = \frac{\partial[\mathbf{s}(\phi,m)m]}{\partial m}$.[23]

B Numerical examples, and extensions of the VAO model risk/return framework

B.1 Calculating yield curve exposures and relative value

Table 7 illustrates the calculation of the fitted market price, the YCEs, and the relative value for a 2-year swap. Note that the FOYCE components are expressed as the dollar sensitivity per 1 bp change in the associated coefficient, which is analogous to BPV. For example, the PV of the 2-year security in table 7 would decrease (increase) by \$200.05 for a 1 bp increase (decrease) in the Level coefficient, and the PV would increase (decrease) by \$114.55 for a 1 bp increase (decrease) in the Slope coefficient.

[Table 7 here]

Table 8 illustrates the calculation of the fitted market price, the YCEs, and the relative value for an arbitrary portfolio of swaps. Again, the FOYCE components are expressed as the dollar sensitivity per 1 bp change in the associated coefficient, so the PV of the portfolio in table 8 would decrease (increase) by \$144,600 for a 1 bp increase (decrease) in the Level coefficient, and the PV would increase (decrease) by \$21,053 for a 1 bp increase (decrease) in the Slope coefficient.

[Table 8 here]

B.2 Value-at-Risk (VaR) calculations

Under the typical assumptions of a linear model and multi-variate normal distributions for changes in the underlying variables, as noted in Hull (2000) pp. 345-351, the calculation of Value-at-Risk (VaR) within the VAO(3) model framework is very straightforward. That is, the results in section 3.1 show that the expected variance of the PV of the portfolio to a first-order ap-

[22] This result follows from the definition $R(m) = \frac{1}{m}\int_0^m f(x)\,dx$, and the second fundamental theorem of integral calculus noted, for example, in Thomas and Finney (1984) p. 286, i.e $\frac{d}{dm}[R(m)\cdot m] = \frac{d}{dm}\int_0^m f(x)\,dx = f(m)$.

[23] $\mathbf{g}(\phi, m)$ are the foward rate modes originally used to calculate $\mathbf{s}(\phi, m)$ in Krippner (2005), i.e $\mathbf{s}(\phi, m) = \frac{1}{m}\int_0^m \mathbf{g}(\phi, x)dx$.

proximation is $\operatorname{var}\left\{\left[\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k\right]' \boldsymbol{\delta}\right\}$. This may be expressed equivalently as $\left[\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k\right]' \operatorname{var}(\boldsymbol{\delta}) \left[\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k\right]$, where $\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k$ is a 3-vector containing the FOYCE components for the portfolio, and $\operatorname{var}(\boldsymbol{\delta})$ is a 3×3 variance-covariance matrix for changes in the VAO(3) model coefficients over the required horizon. For example, the calculation of $\operatorname{var}(\boldsymbol{\delta})$ on a daily basis (i.e the variances and covariances of daily changes) over the full sample period noted in the text gives the result:

$$\operatorname{var}(\boldsymbol{\delta}) = \begin{bmatrix} 38.7 & 28.1 & 8.6 \\ 28.1 & 103.4 & -76.8 \\ 8.6 & -76.8 & 103.4 \end{bmatrix} \mathrm{bp}^2 \tag{17}$$

Hence, the standard deviation of changes in the Level, Slope, and Bow coefficients are respectively 6.2, 10.2, and 10.2 bps. It is also evident that changes in the coefficients do not occur independently; i.e there is material positive covariance between changes in the Level coefficient and changes in the Slope coefficient, and substantial negative covariance between changes in the Slope coefficient and changes in the Bow coefficient.

$\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k =$ (-\$144,600, \$21,053, -\$11,061)$'$ for the portfolio in table 8 of the previous section. Hence, the standard deviation calculation for the daily VaR of this portfolio is:

$$\begin{aligned} [\sigma\,(\text{1-day})]^2 &= \sqrt{\left[\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k\right]' \operatorname{var}(\boldsymbol{\delta}) \left[\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k\right]} \\ &= \$871,552 \end{aligned}$$

The daily VaR corresponding to a given threshold level of significance x is $\sigma(\tau) \cdot \Phi^{-1}(x)$, where $\Phi^{-1}(x)$ is the inverse normal distribution. A typical threshold level significance is 1%, and $\Phi^{-1}(0.01) = -2.33$. Hence, the 1% daily VaR for the portfolio in table 8 is $\$871,552 \times -2.33 = -\$2,027,509$; i.e there is a 1% probability of a loss of \$2,027,509 or more in a single day.

This procedure is analogous to the principal components approach noted in Hull (2000) pp. 357-363, except the VAO(3) framework contains non-zero covariances between changes in the modes. Note also that the VAO(3) framework SOYCE components could be included to extend the VaR calculation to a quadratic approximation, as with the model noted in Hull (2000) pp. 352-355.

Of course, one major critique of the typical VaR calculation is the assumption of multi-variate normal distributions; in practice, the tails of the distributions of financial market variables do not often accord closely to those of the normal distribution. However, VaR calculations independent of the multivariate normal distributions can be still undertaken conveniently within the VAO(3) framework. For example, the historical simulation approach noted in Hull (2000) p. 356 would be undertaken using simulations based on the historical values of $\boldsymbol{\delta}$, and then applying those to the FOYCE vector (and the SOYCE matrix in the quadratic approximation) of the current portfolio to build up a distribution of potential changes in portfolio value. Alternatively, samples of $\boldsymbol{\delta}$ could be generated via multi-variate time-series models of the Level, Slope, and Bow coefficients (potentially allowing for generalised time-varying volatility), which would again be applied to the FOYCE vector of the current portfolio to build up a distribution of potential changes in portfolio value.

B.3 Level, Slope, and Bow durations (and convexities)

The risk measures in the VAO(3) framework may be standardised from outright returns to proportional or percentage returns by dividing each risk measure component by the portfolio MV. Hence, $\frac{1}{\mathrm{MV}}\sum_{k=1}^{K} A_k \boldsymbol{\lambda}(1)_k$ is the percentage change in the value of the portfolio for a 1 percentage point change in the Level coefficient (which is analogous to the traditional measure of duration, i.e the percentage change in the portfolio value for a 1 percentage point level shift in the yield curve). $\frac{1}{\mathrm{MV}}\sum_{k=1}^{K} A_k \boldsymbol{\lambda}(2)_k$ and $\frac{1}{\mathrm{MV}}\sum_{k=1}^{K} A_k \boldsymbol{\lambda}(3)_k$ are the Slope and Bow durations respectively, which have the interpretation of the percentage change in the portfolio MV given a percentage point change in the Slope or Bow coefficients. These are analogous to the partial duration measures of Golub and Tilman (2000) pp. 24-25. The VaR may also be expressed in proportional terms.

As an example, the portfolio in figure 8 is composed of \$150 million face-value swaps. This has a MV of zero, but if it were backed by \$150 million of overnight cash, then the MV would be \$150 million, and the FOYCE components would remain essentially unchanged. The Level duration would then be -\$144600 per bp / \$150 million $\times$ 1 percentage point per 100 bps = -9.64 (no unit); i.e approximately 9.6 years of traditional duration. The Slope and Bow durations would respectively be 1.40, and -0.74. The proportional VaR is \$2,027,509 / \$150 million = 1.35 (so there is a 1% probability of a loss of 1.35% of portfolio value or more in a single day).

The second-order terms in the matrix $\sum_{k=1}^{K} A_k \boldsymbol{\Omega}_k$ can also be scaled by $\frac{1}{\mathrm{MV}}$ to make the second-order sensitivities of portfolios with different market values comparable. For example, $\frac{1}{\mathrm{MV}_0}\sum_{k=1}^{K} A_k \boldsymbol{\Omega}(1,1)_k$ is analogous to the traditional measure of convexity, while the remaining diagonal elements would give the Slope and the Bow convexities, and the off-diagonal elements would give the Level-Slope, Level-Bow, and Slope-Bow cross-term convexities. These are analogous to the partial convexity measures of Golub and Tilman (2000) pp. 24-25.

B.4 Portfolio optimisation with active trading

The framework developed in this article is also directly applicable to active yield curve trading; i.e where the portfolio manager deliberately seeks to take on YCEs relative to the initial/benchmark portfolio based on a view of how the yield curve is likely to change. If the view is proven correct, then the PV and hence MV of the portfolio will increase relative to the initial/benchmark portfolio, but a relative loss will occur if the yield moves in the opposite direction.

As background, a minimum of four securities is required to perfectly match any given MV and three FOYCE components. Hence, in principle, any four securities could be transacted to change the MV and three FOYCE components to those desired. Given a set of four securities, the required face-values to transact could be found by straightforward matrix algebra. However, the transaction might not be allowed if it breached any constraints on the amounts of securities allowed in the portfolio, so substantial trial and error on the selection of the four trading securities (from the allowable universe) might be required before an allowable transaction is found.

Conversely, the optimisation framework in this article automatically calculates the optimal feasible solution, if a solution exists. The optimisation problem expressed relative to a benchmark or initial portfolio is a trivial variation on the linear programme of equation 9, i.e:

$$\text{Maximise:} \quad \boldsymbol{\alpha}'\mathbf{A}_1 \tag{18a}$$

$$\text{subject to:} \quad \boldsymbol{\Lambda}\mathbf{A}_1 = \boldsymbol{\Lambda}\mathbf{A}_0 + \boldsymbol{\kappa} \tag{18b}$$

$$\text{and:} \quad A_{1,k,\min} \leq A_{1,k} \leq A_{1,k,\max} \tag{18c}$$

where $\boldsymbol{\kappa}$ represents the desired (or acceptable) differences between the MV and FOYCEs of the alternative portfolio and the initial/benchmark portfolio. For example, a pure relative slope/twist/curve trade may be specified by $\boldsymbol{\kappa} = (0, 0, \$x_2, 0)'$, and a portfolio constructed with $\boldsymbol{\Lambda}\mathbf{A}_1 = \boldsymbol{\Lambda}\mathbf{A}_0 + \boldsymbol{\kappa}$ would return $\$x_2$ relative to the initial/benchmark portfolio for each bp increase in the Slope coefficient; changes in the Level or Bow coefficients would deliver a zero change relative to the initial/benchmark portfolio. Similarly, $\boldsymbol{\kappa} = (0, 0, 0, \$x_3)'$ represents a pure relative bow/barbell/butterfly/curvature trade, and hybrid trades (i.e with several distinct exposures to the yield curve) could be specified using several non-zero entries. Even the element for portfolio MV could be non-zero if cash were being injected or withdrawn relative to the initial/benchmark portfolio. Note that one possible output of the linear programme of equation 18 is "infeasible". This would indicate that the desired MV and FOYCEs cannot be obtained simultaneously given the portfolio constraints, and therefore $\boldsymbol{\kappa}$ would need to be adjusted (or the constraints relaxed, if possible) to obtain a feasible solution.

For active portfolio exposures, the VAO(3) model framework developed in this article is highly desirable relative to a "black box" of variances and covariances, for three reasons: (1) the intended exposure/s to yield curve changes may be visualised using the VAO(3) modes; (2) the intended active risks may be precisely specified and constructed, as noted above; and (3) the optimisation framework determines the portfolio with the highest relative value that achieves the desired exposure/s to yield curve changes. This will act to enhance portfolio returns at the margin, independently of whether the active yield curve trading is successful or not.

C The complexities introduced by including transactions costs

It is firstly worth noting that fixed transactions costs (e.g fixed overheads and/or settlement charges) could be included in the linear programme of equation 9 without changing the nature of the optimisation problem. However, fixed transactions costs are not a realistic description of the total trading costs incurred in practice, given that most of those costs are variable (e.g brokerage costs proportional to the face-values of the traded securities, and/or the proportional cost of half a typical bid-ask spread from each security in the trade, which represents the impact on portfolio MV that occurs when securities are transacted at the bid or the ask rate and are then revalued at mid-rates).

Secondly, variable transactions costs cannot be included by simply subtracting the appropriate cost of the trade that the optimisation model in this article recommends. Rather, variable transactions costs must be included as an additional influence on the system to be optimised, otherwise the system might lead to non-optimal transactions. For example, it might not be optimal to transact a trade with positive relative value if the cost of the transaction outweighs that relative value. On the other hand, it might still be optimal to transact the trade if not trading would leave the YCEs outside of given tolerances.

In general, variable transactions costs are incurred regardless of the direction of the transaction. Hence, these costs would enter into the objective function as an absolute value, and so the objective function would become: Maximise: $\sum_{k=1}^{K} \alpha_k A_k - z_k \left|\Delta A_k\right|$, where ΔA_k represents the change to the face-value of security k, z_k represents the variable cost of transacting security k (e.g half the typical bid-ask spread in the market price), and the absolute value is applied to ensure the transaction cost will always be positive regardless of whether ΔA_k is positive or negative. This makes the optimisation system non-linear, and highly so because the absolute value function is discontinuous in the first derivative. This in itself is not an insurmountable problem; non-linear programming could, in principle, be used to provide a solution.

However, another complication that arises when transactions costs are included in the optimisation is the potential path-dependency of the results. This arises because the initial portfolio and transactions costs will determine the initial optimal portfolio, and that optimal portfolio will then determine the following optimal portfolio, and so on. Hence, a different starting portfolio and/or transactions costs and/or starting date for the optimisation might lead to different outcomes. In this case, there is likely to be some ambiguity about whether the excess returns in the optimised portfolio reflected the optimisation itself or a fortuitous choice of the initial portfolio, transactions costs, and starting date.

Finally, another complication that arises when transactions costs are included is that the optimal transaction is not necessarily to rebalance back to the benchmark. Rather, as noted in Donohue and Yip (2003), if the portfolio is within allowable thresholds (e.g risk tolerances) from the benchmark, then no rebalancing transaction is required. If a portfolio is outside allowable thresholds, then the optimal transaction is that which takes the portfolio back to the allowable threshold. This is relatively easy to operationalise if there is only a threshold in a single dimension (e.g a bond versus equity allocation in a balanced portfolio), but it becomes increasingly complex as the number of dimensions increases. The VAO(3) model has three dimensions, and would therefore require three thresholds relating to allowable tolerances on Level, Slope, and Bow exposure.

Almost needless to say then, but the resolution of the issues noted above is well beyond the scope of this paper, and will therefore be explored by the author in future work

References

Barber, J. and Copper, M. (1996), 'Immunisation using principal components analysis', *Journal of Portfolio Management* **Fall**, 99–105.

Björk, T. and Christensen, B. (1999), 'Interest rate dynamics and consistent forward rate curves', *Mathematical Finance* **9(4)**, 323–348.

Bowden, R. (1997), 'Generalising interest rate duration with directional derivatives: direction X and applications', *Management Science* **43(5)**, 586–595.

Brandt, M. and Yaron, A. (2002), 'Time-consistent no-arbitrage models of the term structure', *Working paper, Wharton School, University of Pennsylvania* .

Brown, R. and Giurda, F. (2003), 'Euro-govvie relative value - overview', *UBS Investment Research* **3 September**.

Chambers, D., Carleton, W. and McEnally, R. (1988), 'Immunizing default-free bond portfolios with a duration vector', *Journal of Financial and Quantitative Analysis* **23(1)**, 89–104.

Cornell, B. and Shapiro, A. (1989), 'The mispricing of U.S Treasury bonds: a case study', *Review of Financial Studies* **2**, 297–310.

Cox, J., Ingersoll, J. and Ross, S. (1985), 'A theory of the term structure of interest rates', *Econometrica* **53**, 385–407.

Diebold, F. and Li, C. (2002), 'Forecasting the term structure of government bond yields', *Working paper, University of Pennsylvania* .

Donohue, C. and Yip, K. (2003), 'Optimal portfolio rebalancing with transactions costs', *Journal of Portfolio Management* **Summer 2003**, 49–63.

Elton, E. and Gruber, M. (1995), *Modern Portfolio Theory and Investment Analysis*, Wiley and Sons.

Filopović, D. (1999*a*), 'Exponential-polynomial families and the term structure of interest rates', *Working paper, ETH Zurich* .

Filopović, D. (1999*b*), 'A note on the Nelson-Siegel family', *Mathematical Finance* **9(4)**, 349–359.

Fisher, L. and Weill, R. (1971), 'Coping with the risk of interest rate fluctuations', *Journal of Business* **44**, 408–431.

Fleming, M. (2003), 'Measuring Treasury market liquidity', *Federal Reserve Bank of New York Bulletin* **9(3)**, 83–108.

Golub, B. and Tilman, L. (2000), *Risk Management Approaches for Fixed Income Markets*, Wiley and Sons.

Greene, W. (1997), *Econometric Analysis, Third Edition*, Prentice Hall.

Heath, D., Jarrow, R. and Morton, A. (1992), 'Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation', *Econometrica* **60(1)**, 77–106.

Ho, T. (1992), 'Key rate durations: measures of interest rate risks', *Journal of Fixed Income* **3**, 29–44.

HSBC Bank (2001), 'Government fair value models', *HSBC Bank Bloomberg site* **18 October**.

HSBC Bank (2003), 'Gilt fair value model', *HSBC Bank Bloomberg site* **24 November**.

Hull, J. (2000), *Options, Futures and Other Derivitives, Fourth Edition*, Prentice Hall.

Ioannides, M. (2003), 'A comparison of yield curve estimation techniques using UK data', *Journal of Banking and Finance* **27**, 1–26.

Kacala, V. (1993), 'Zero coupon curves part 3: fair bond yields', *Research paper, Buttle Wilson* .

Kopprasch, R. (2004), 'A look at a variety of duration measures', *United States Fixed Income Research, Citibank* .

Krippner, L. (2005), 'An intertemporally-consistent and arbitrage-free version of the Nelson and Siegel class of yield curve models', *Working paper, University of Waikato* .

Litterman, R. and Sheinkman, J. (1991), 'Common factors affecting bond returns', *Journal of Fixed Income* **2**, 54–61.

Macauley, F. (1938), *Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields, and Stock Prices in the United States since 1865*, Columbia University Press.

Malik, H., Barry, J. and Xiao, Q. (2003), 'US Treasury relative value and asset swap report', *JP Morgan internet website* **25 November**.

Mann, S. and Ramanlal, P. (1997), 'The relative performance of yield curve strategies', *Journal of Portfolio Management* **Summer**, 64–70.

Markowitz, H. (1959), *Portfolio Selection: Efficient Diversification of Investments*, Wiley and Sons.

Murty, K. (1983), *Linear Programming*, Wiley and Sons.

Nawalkha, S., Soto, G. and Zhang, J. (2003), 'Generalised M-vector models for hedging interest rate risk', *Journal of Banking and Finance* **27**, 1581–1604.

Nelson, C. and Siegel, A. (1987), 'Parsimonious modelling of yield curves', *Journal of Business* **October**, 473–489.

Reitano, R. (1996), 'Non-parallel yield curve shifts and stochastic immunisation', *Journal of Portfolio Management* **Winter**, 71–105.

Ronn, E. (1987), 'A new linear programming approach to bond portfolio management', *Journal of Financial and Quantitative Analysis* **22(4)**, 439–466.

Sercu, P. and Wu, X. (1997), 'The information content in bond model residuals: an empirical study on the Belgian bond market', *Journal of Banking and Finance* **21**, 685–720.

Soto, G. (2001), 'Immunization derived from a polynomial duration vector in the Spanish bond market', *Journal of Banking and Finance* **25**, 1037–1057.

Stock, J. and Watson, M. (2002), 'Macroeconomic forecasting using diffusion indexes', *Journal of Business and Economic Statistics* **20(2)**, 147–162.

Svensson, L. (1994), 'Estimating and interpreting forward interest rates: Sweden 1992-4', *Discussion paper, Centre for Economic Policy Research* **1051**.

Thomas, G. and Finney, R. (1984), *Calculus and Analytic Geometry, Sixth Edition*, Addison Wesley.

Vasicek, O. (1977), 'An equilibrium characterisation of the term structure', *Journal of Financial Economics* **5**, 177–188.

Willner, R. (1996), 'A new tool for portfolio managers: level, slope, and curvature durations', *Journal of Fixed Income* **2**, 48–59.

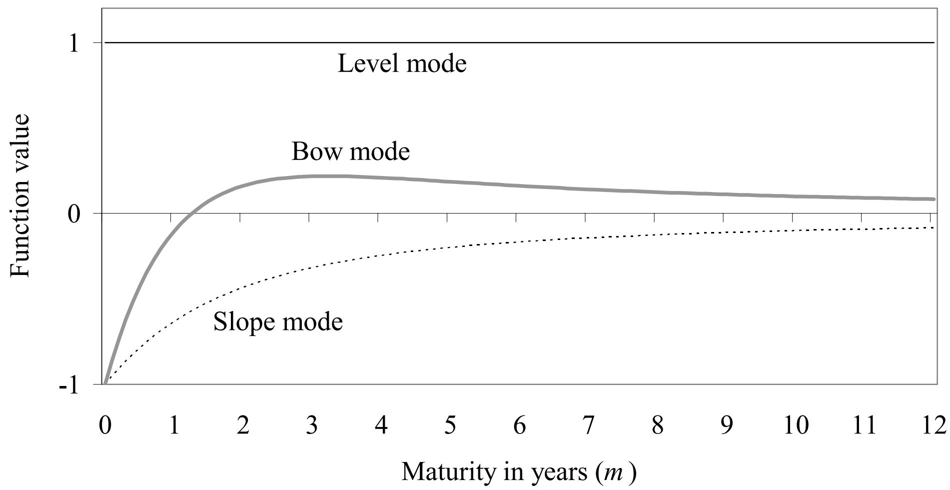

Figure 1: The VAO(3) model interest rate modes. $\phi = 1$ for this illustration.

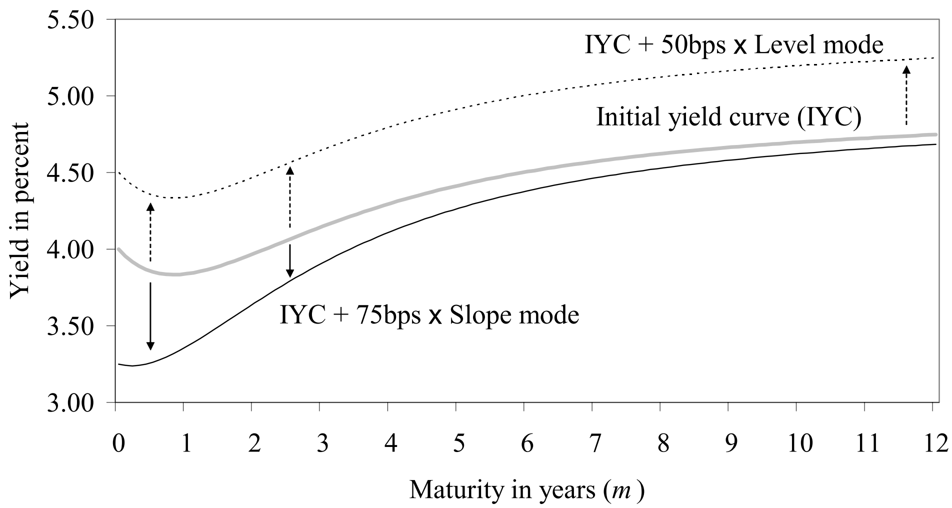

Figure 2: Example of an initial yield curve (IYC), and changes to the Level and Slope coefficients. The IYC is $\boldsymbol{\beta}(t) = (5.00, 2.00, -1.00)\,\%$, IYC + 50bps $\times$ Level mode is $\boldsymbol{\beta}(t) = (5.50, 2.00, -1.00)\,\%$, and IYC + 75bps $\times$ Slope mode is $\boldsymbol{\beta}(t) = (5.00, 2.75, -1.00)\,\%$. For this illustration, $\phi = 1$ and all other parameters have been set to zero.

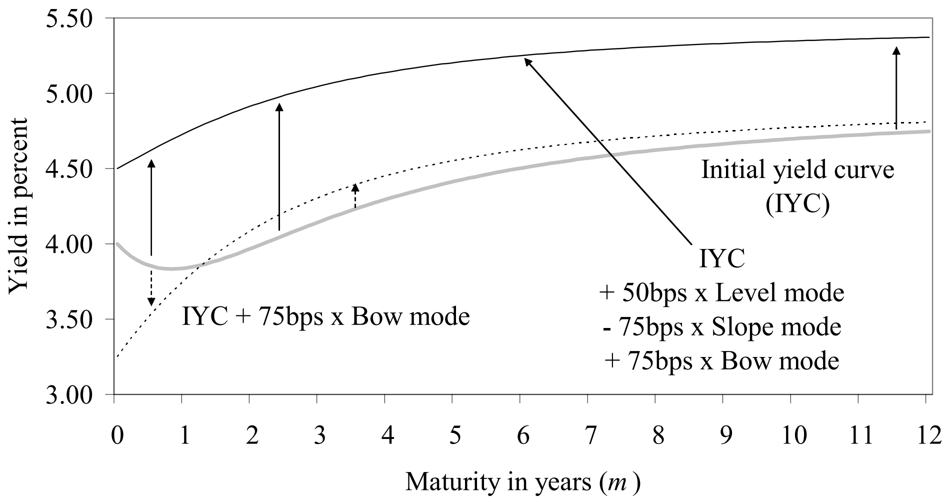

Figure 3: Example of an initial yield curve (IYC), a change to the Bow coefficient, and a simultaneous change to all coefficients. The IYC is $\boldsymbol{\beta}(t) = (5.00, 2.00, -1.00)\,\%$, IYC + 75bps × Bow mode is $\boldsymbol{\beta}(t) = (5.00, 2.00, -0.25)\,\%$, and IYC + the combined Level, Slope, and Bow mode shifts is $\boldsymbol{\beta}(t) = (5.50, 1.25, -0.25)\,\%$. For this illustration, $\phi = 1$ and all other parameters have been set to zero.

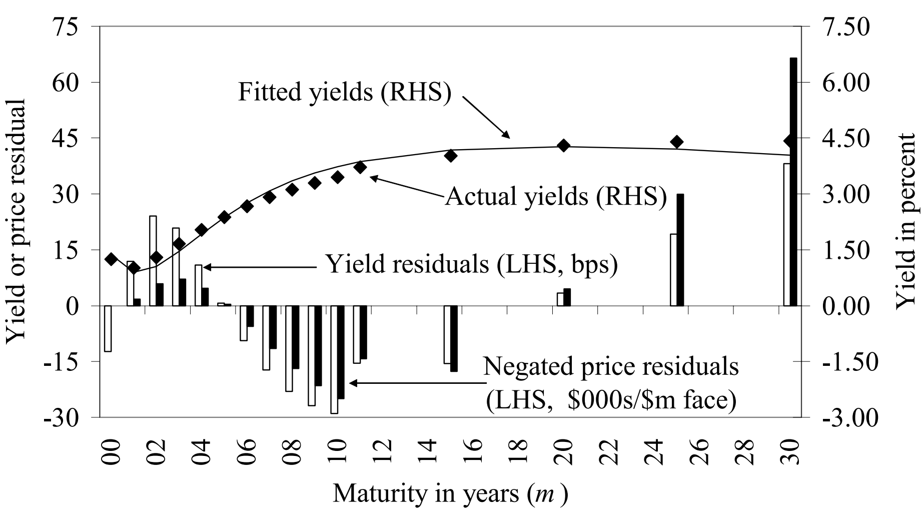

Figure 4: The actual and estimated US swaps curve on Monday 16 June 2003, and the associated yield residuals and negated price residuals. The estimated coefficients and parameters are $\boldsymbol{\beta}(\text{16-Jun-2003}) = (6.16, 9.04, -4.27)\,\%$, $\phi = 0.6173$, $\theta_1 = 0.8825\%$, $\sigma_1 = 1.03\%$, and $\mathbf{v} = \left(1.03^2, 1.65^2, 1.59^2\right)\%^2$.

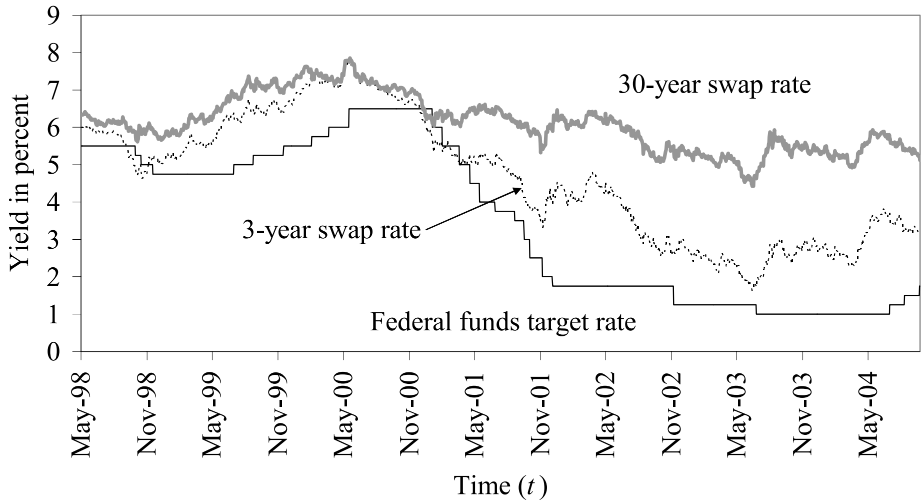

Figure 5: The time series for three of the 16 rates used to define the US swaps yield curve over the sample period.

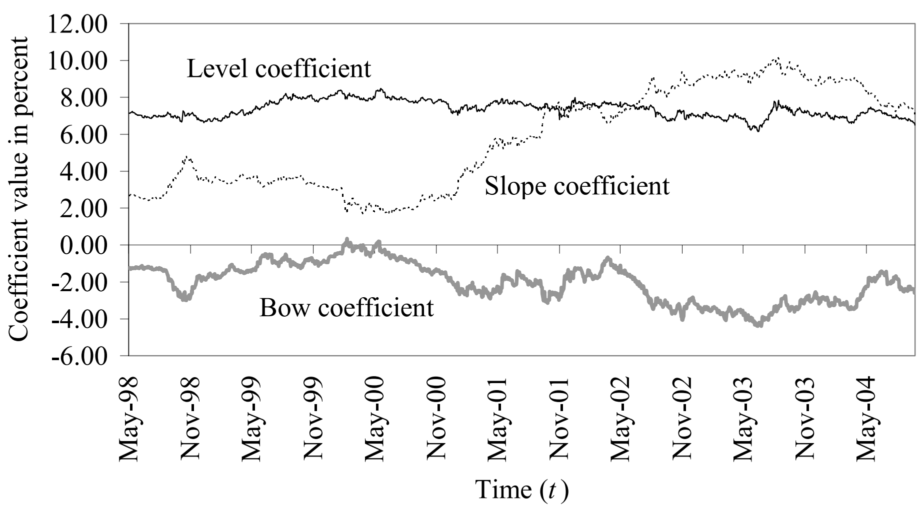

Figure 6: The time series of the estimated Level, Slope, and Bow coefficients over the full sample. The estimated parameters are $\phi = 0.6173$, $\theta_1 = 0.8825\%$, $\sigma_1 = 1.03\%$, and $\mathbf{v} = \left(1.03^2, 1.65^2, 1.59^2\right)\%^2$.

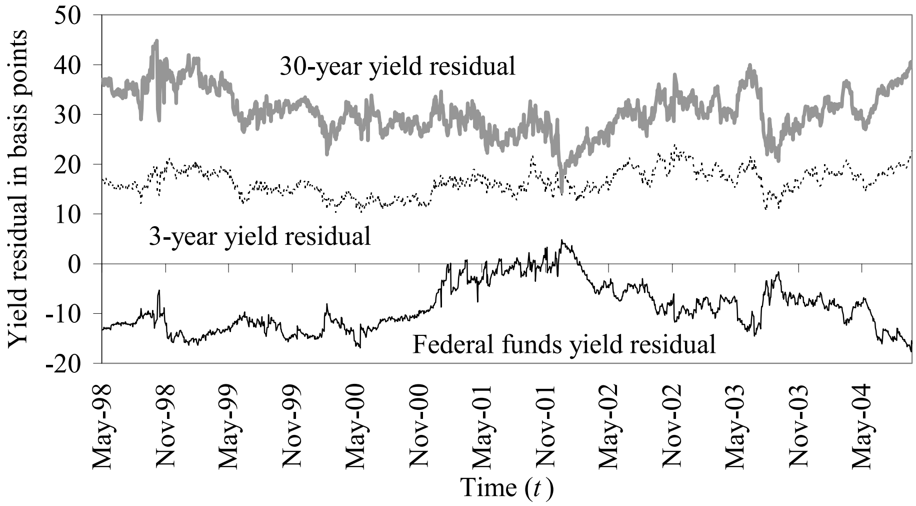

Figure 7: The time series of estimated yield residuals for three of the 16 rates used to define the yield curve. The yield residuals are the actual yields less the VAO(3) model fitted yields using the estimated coefficients and the parameters in figure 6.

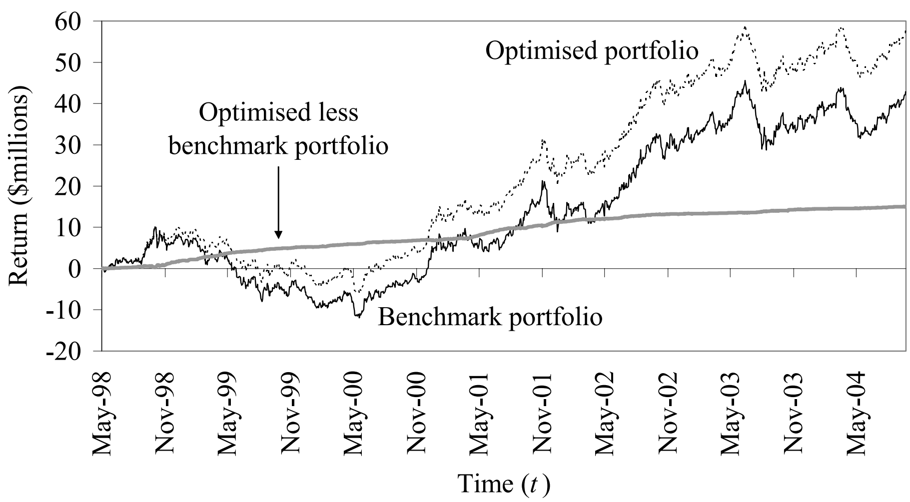

Figure 8: Cumulative returns for the benchmark portfolio and the optimised portfolio 4 (OP4). As detailed in section 4.3, OP4 uses in-sample estimates of the parameters ϕ, θ_1, $\mathbf{v}$, and in-sample estimates of π_k.

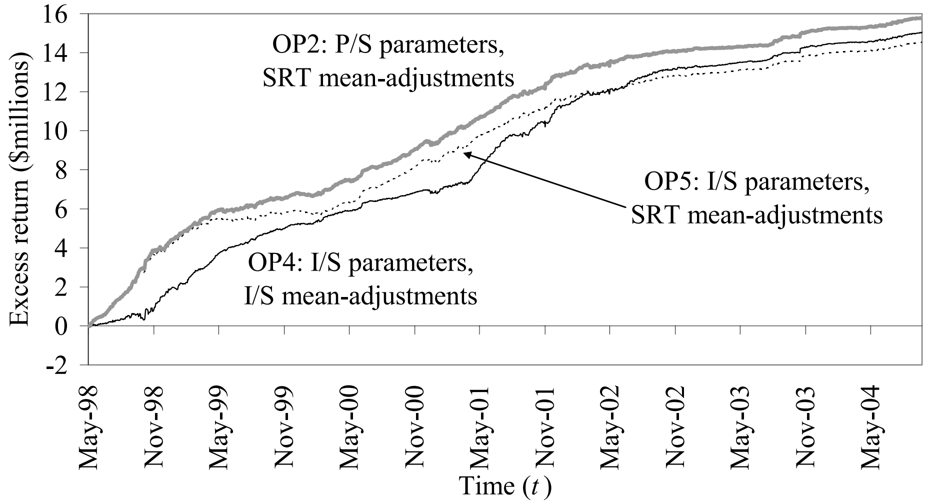

Figure 9: Cumulative returns for the optimised portfolios OP2, OP4, and OP5. "I/S parameters" means in-sample estimates of ϕ, θ_1, and $\mathbf{v}$, "P/S parameters" means pre-sample estimates of ϕ, θ_1, and $\mathbf{v}$. "SRT mean-adjustments" means simulated real time calculations of π_k, and "I/S mean-adjustments" means in-sample calculations of π_k. Details are contained in section 4.3.

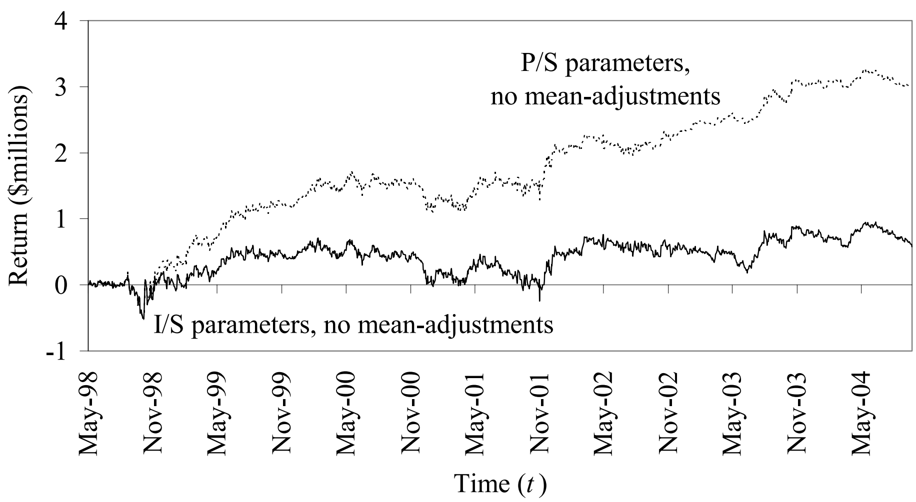

Figure 10: Cumulative returns for the optimised portfolios OP1 and OP3. These have no mean-adjustments, so $\pi_k = 0$. "I/S parameters" means in-sample estimates of ϕ, θ_1, and $\mathbf{v}$, "P/S parameters" means pre-sample estimates of ϕ, θ_1, and $\mathbf{v}$. Details are contained in section 4.3.

Attribution	Sum	Mean	Std dev.	Min.	Max.	Max. less min.
Actual return	42.852	0.027	0.659	-2.852	3.107	5.959
λ(1) FOYCE	18.652	0.012	0.660	-3.629	3.825	7.453
λ(2) FOYCE	8.308	0.005	0.213	-0.972	1.354	2.326
λ(3) FOYCE	6.301	0.004	0.140	-0.776	0.636	1.412
Total FOYCE	33.261	0.021	0.668	-2.936	3.086	6.021
Ω(1,1) SOYCE	4.048	0.003	0.006	0.000	0.085	0.085
Ω(1,2) SOYCE	-0.738	0.000	0.002	-0.036	0.005	0.042
Ω(1,3) SOYCE	0.203	0.000	0.001	-0.014	0.010	0.025
Ω(2,2) SOYCE	0.256	0.000	0.000	0.000	0.006	0.006
Ω(2,3) SOYCE	0.272	0.000	0.000	-0.001	0.005	0.006
Ω(3,3) SOYCE	0.164	0.000	0.000	0.000	0.003	0.003
Total SOYCE	4.206	0.003	0.005	0.000	0.057	0.057
Relative value	1.516	0.001	0.006	-0.040	0.029	0.069
Interest accrual	3.870	0.002	0.007	-0.032	0.033	0.065

Table 1: Statistical summary of benchmark portfolio returns ($millions) and ex-post attributions of those returns to the 11 components noted in section 4.2.

	Total FOYCE	Total SOYCE	Relative value	Interest accrual
Total FOYCE	0.4467	-0.0005	-0.0017	-0.0044
Total SOYCE	-0.0005	0.0000	0.0000	0.0000
Relative value	-0.0017	0.0000	0.0000	0.0000
Interest accrual	-0.0044	0.0000	0.0000	0.0000
	Total	0.4336	FOYCE/Total	1.0300

Table 2: Variances and covariances of the benchmark portfolio attributed returns ("Total FOYCE" and "Total SOYCE" are aggregates of the individual components contained in table 1).

Attribution	Sum	Mean	Std dev.	Min.	Max.	Max. less min.
Actual return	57.902	0.036	0.658	-3.035	3.323	6.357
$\lambda(1)$ FOYCE	18.652	0.012	0.660	-3.629	3.825	7.453
$\lambda(2)$ FOYCE	8.308	0.005	0.213	-0.972	1.354	2.326
$\lambda(3)$ FOYCE	6.301	0.004	0.140	-0.776	0.636	1.412
Total FOYCE	33.261	0.021	0.668	-2.936	3.086	6.021
$\Omega(1,1)$ SOYCE	4.058	0.003	0.006	0.000	0.093	0.093
$\Omega(1,2)$ SOYCE	-0.738	0.000	0.002	-0.036	0.005	0.042
$\Omega(1,3)$ SOYCE	0.203	0.000	0.001	-0.014	0.011	0.025
$\Omega(2,2)$ SOYCE	0.256	0.000	0.000	0.000	0.006	0.006
$\Omega(2,3)$ SOYCE	0.272	0.000	0.000	-0.001	0.005	0.006
$\Omega(3,3)$ SOYCE	0.163	0.000	0.000	0.000	0.003	0.003
Total SOYCE	4.215	0.003	0.005	0.000	0.066	0.066
Relative value	16.375	0.010	0.035	-0.325	0.242	0.567
Interest accrual	4.051	0.003	0.007	-0.042	0.040	0.082

Table 3: Statistical summary of the returns ($millions) of optimised portfolio OP4 and ex-post attributions of those returns to the 11 components noted in section 4.2. OP4 uses in-sample estimates of the parameters ϕ, θ_1, $\mathbf{v}$, and in-sample estimates of π_k, as detailed in section 4.3.

	Total FOYCE	Total SOYCE	Relative value	Interest accrual
Total FOYCE	0.4467	-0.0006	-0.0027	-0.0043
Total SOYCE	-0.0006	0.0000	0.0000	0.0000
Relative value	-0.0027	0.0000	0.0012	0.0001
Interest accrual	-0.0043	0.0000	0.0001	0.0000
	Total	0.4328	FOYCE/Total	1.0319

Table 4: Variances and covariances of the optimised portfolio OP4 attributed returns ("Total FOYCE" and "Total SOYCE" are aggregates of the individual components contained in table 3).

Optimised portfolio relative to the benchmark portfolio	Annual-ised return ($million)	Annual-ised standard deviation ($million)	Infor-mation ratio	t-statistic
OP1 (P/S parameters, no M/A)	0.47	0.65	0.72	1.82 *
OP2 (P/S parameters, SRT M/A)	2.47	0.50	4.90	12.37 ***
OP3 (I/S parameters, no M/A)	0.09	0.69	0.13	0.33
OP4 (I/S parameters, I/S M/A)	2.35	0.55	4.26	10.78 ***
OP5 (I/S parameters, SRT M/A)	2.28	0.51	4.47	11.30 ***

Table 5: Statistical summary of the returns of the optimised portfolios (as detailed in section 4.3) relative to the returns of the benchmark portfolio (as detailed in section 4.2). "I/S" is in-sample, "P/S" is pre-sample, "M/A" is mean-adjustment, and "SRT" is simulated real time. Details are in section 4.3.

Specified relative returns between optimised portfolios	Annual-ised return ($million)	Annual-ised standard deviation ($million)	Infor-mation ratio	t-statistic
OP3 (I/S parameters, no M/A) less OP4 (I/S parameters, I/S M/A)	-2.26	0.75	-3.02	-7.63 ***
OP5 (I/S parameters, SRT M/A) less OP4 (I/S parameters, I/S M/A)	-0.08	0.48	-0.16	-0.40
OP1 (P/S parameters, no M/A) less OP4 (I/S parameters, I/S M/A)	-1.89	0.68	-2.76	-6.97 ***
OP2 (P/S parameters, SRT M/A) less OP4 (I/S parameters, I/S M/A)	0.12	0.48	0.24	0.61
OP1 (P/S parameters, no M/A) less OP2 (P/S parameters, SRT M/A)	-2.00	0.87	-2.30	-5.82 ***
OP1 (P/S parameters, no M/A) less OP3 (I/S parameters, no M/A)	0.38	0.21	1.78	4.49 ***
OP2 (P/S parameters, SRT M/A) less OP5 (I/S parameters, SRT M/A)	0.19	0.19	1.02	2.58 ***

Table 6: Statistical summary of the returns of the optimised portfolios relative to each other (as discussed in section 4.3). "I/S" is in-sample, "P/S" is pre-sample, "M/A" is mean-adjustment, and "SRT" is simulated real time. Details are contained in section 4.3.

Cashflow number	CF1	CF2	CF3	CF4	CF5	Total
Cashflow date	Wed. 18-Jun-03	Thu. 18-Dec-03	Fri. 18-Jun-03	Mon. 20-Dec-04	Fri. 20-Jun-05	
Cashflow maturity (m)	0.01	0.51	1.01	1.52	2.01	
Cashflow magnitude	-1	0.0065	0.0065	0.0065	1.0065	
Level mode value at m	1	1	1	1	1	
Slope mode value at m	-0.9983	-0.8587	-0.7444	-0.6496	-0.5724	
Bow mode value at m	-0.9949	-0.6040	-0.3289	-0.1354	-0.0046	
Total volatility adjustment	0.000	0.002	0.007	0.014	0.020	
Total risk adjustment	0.000	0.000	0.002	0.005	0.007	
$R(t,m)$ in percent	1.39	0.98	0.83	0.86	1.00	
Unit present-value	0.9999	0.9950	0.9916	0.9870	0.9802	
Cashflow present-value	-0.9999	0.0064	0.0064	0.0064	0.9865	0.0058
Unit market-value						0
Unit price residual						-0.0058
Unit yield residual						24.1

Unit λ vector	CF1	CF2	CF3	CF4	CF5	Total
$\lambda(1)$	-0.0055	-0.5043	-0.9998	-1.4954	-1.9737	-2.0005
$\lambda(2)$	0.0055	0.4331	0.7443	0.9714	1.1297	1.1455
$\lambda(3)$	0.0055	0.3046	0.3288	0.2025	0.0091	0.0091

Unit $\boldsymbol{\Omega}$ matrix elements	CF1	CF2	CF3	CF4	CF5	Total
$\boldsymbol{\Omega}(1,1)$	0.0000	0.1278	0.5040	1.1328	1.9873	2.0115
$\boldsymbol{\Omega}(1,2)$	0.0000	-0.1097	-0.3752	-0.7358	-1.1374	-1.1527
$\boldsymbol{\Omega}(1,3)$	0.0000	-0.0772	-0.1657	-0.1534	-0.0091	-0.0118
$\boldsymbol{\Omega}(2,2)$	0.0000	0.0942	0.2793	0.4780	0.6510	0.6607
$\boldsymbol{\Omega}(2,3)$	0.0000	0.0663	0.1234	0.0996	0.0052	0.0071
$\boldsymbol{\Omega}(3,3)$	0.0000	0.0466	0.0545	0.0208	0.0000	0.0008

λ vector	Values
Level FOYCE	-200.05
Slope FOYCE	114.55
Bow FOYCE	0.91

$\boldsymbol{\Omega}$ matrix	Level	Slope	Bow
Level	201.15	-115.27	-1.18
Slope	-115.27	66.07	0.71
Bow	-1.18	0.71	0.08

Table 7: An example of the fixed cashflows of the 2-year swap (1.295% quote on Monday 16 June 2003), and the calculation of the relative value and YCEs using the 16 June 2003 VAO(3) coefficients and parameters, i.e $\boldsymbol{\beta}(t) = (6.16, 9.04, -4.27)\,\%$, $\phi = 0.6173$, $\theta_1 = 0.8825\%$, and $\mathbf{v} = \left(1.03^2, 1.65^2, 1.59^2\right)\%^2$.

Security k	1	2	3	4		
Security name	2-year swap	5-year swap	10-year swap	30-year swap		Portfolio potential value
Face-value ($\mathbf{A}'$ vector)	70	10	5	65		
Price residual vector ε'	-0.006	0.000	0.025	-0.066	$\varepsilon'\mathbf{A}$	-4.612
Yield residual vector η'	24.1	0.7	-28.9	38.1	$\eta'\mathbf{A}$	4024

Λ matrix	$\Lambda(1)$	$\Lambda(2)$	$\Lambda(3)$	$\Lambda(4)$	Portfolio $\Lambda\mathbf{A}$ vector
Market-value	0	0	0	0	0
$\lambda(1)$	-200	-475	-828	-1872	-144600
$\lambda(2)$	115	150	150	166	21053
$\lambda(3)$	1	-101	-132	-145	-11061
$\Omega(1,1)$	201	1168	3895	22484	1506681
$\Omega(1,2)$	-115	-365	-663	-1508	-113071
$\Omega(1,3)$	-1	252	624	1475	101435
$\Omega(2,2)$	66	115	119	130	14840
$\Omega(2,3)$	1	-78	-106	-117	-8891
$\Omega(3,3)$	0	55	101	113	8435

Table 8: An example of an arbitrary portfolio composed of 2, 5, 10, and 30-year swaps as at Monday 16 June 2003. The 16 June 2003 VAO(3) coefficients and parameters are $\boldsymbol{\beta}(t) = (6.16, 9.04, -4.27)\,\%$, $\phi = 0.6173$, $\theta_1 = 0.8825\%$, and $\mathbf{v} = \left(1.03^2, 1.65^2, 1.59^2\right)\%^2$.