

Symmetry and separation of variables for the Hamilton–Jacobi equation $W_t^2 - W_x^2 - W_y^2 = 0$

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We present a detailed group theoretical study of the problem of separation of variables for the characteristic equation of the wave equation in one time and two space dimensions. Using the well-known Lie algebra isomorphism between canonical vector fields under the Lie bracket operation and functions (modulo constants) under Poisson brackets, we associate, with each R -separable coordinate system of the equation, an orbit of commuting constants of the motion which are quadratic members of the universal enveloping algebra of the symmetry algebra $\mathfrak{o}(3,2)$. In this, the first of two papers, we essentially restrict ourselves to those orbits where one of the constants of the motion can be split off, giving rise to a reduced equation with a nontrivial symmetry algebra. Our analysis includes several of the better known two-body problems, including the harmonic oscillator and Kepler problems, as special cases.

INTRODUCTION

This is the first of two papers in which we study the problem of separation of variables for solutions of

$$W_t^2 - W_x^2 - W_y^2 = 0, \quad (0.1)$$

the characteristic equation of the wave equation

$$\Psi_{tt} - \Psi_{xx} - \Psi_{yy} = 0. \quad (0.2)$$

As is well known^{1,2} the symmetry algebra of (0.1) defined in terms of operators acting on t, x, y is $\mathfrak{o}(3,2)$. Furthermore, there is an isomorphism between the symmetry algebras of (0.1) and (0.2).

In Refs. 3–5 it was shown that every R -separable coordinate system for (0.2) is characterized by a pair of commuting second order symmetric differential operators in the enveloping algebra of $\mathfrak{o}(3,2)$. Furthermore, coordinates whose operators lie on the same orbit under the adjoint action of $\mathfrak{O}(3,2)$ can be considered as equivalent. The separated special function solutions are eigenfunctions of the commuting symmetry operators and this relationship is a powerful tool for the derivation of special function identities.

In Ref. 2 it was shown that the symmetry algebra of the Hamilton–Jacobi equation for the free particle

$$S_t + S_x^2 = 0 \quad (0.3)$$

(acting in t, x, S space) was also isomorphic to $\mathfrak{o}(3,2)$. The problem of additive R -separation of variables for (0.3) was studied in Ref. 6. There all solutions of the form

$$S = R(u, v) + U(u) + V(v) \quad (0.4)$$

were classified where u, v is a new coordinate system, either $R \equiv 0$ or $R \neq 0$ and is not expressible as a sum of a function of u alone and a function of v alone, and U, V are arbitrary solutions of first order ordinary differen-

tial equations. It was shown that the separable coordinates agree exactly with the separable coordinates for the free-particle Schrödinger equation

$$i\Psi_t + \Psi_{xx} = 0 \quad (0.5)$$

as derived in Ref. 7. These coordinates are all associated with the Schrödinger subalgebra of $\mathfrak{o}(3,2)$. Furthermore, the other elements of $\mathfrak{o}(3,2)$ lead to symmetry adapted solutions which do not separate additively as in (0.4) but in some more complicated fashion. In exact analogy with the linear results in Ref. 7 it was also shown that the Hamilton–Jacobi equations for the harmonic oscillator, repulsive oscillator, and linear potential are equivalent to (0.3). Finally, it was pointed out that all these equations are equivalent to (0.1) in the sense that (0.1) is the equation of the graph of each of the considered Hamilton–Jacobi equations.

Here, we look for additive R -separable solutions of (0.1) in the form

$$W = R(u_k) + \sum_{j=1}^3 U_j(u_j). \quad (0.6)$$

We show by example that the separable systems are the same as those derived in Refs. 3–5 for (0.2) and, properly interpreted the Lie algebraic characterizations of the systems are the same. The proper interpretation of the symmetry algebra of (0.1) is that it is an algebra of functions in the six-dimensional phase space. $\{t, x, y; p_0 = W_t, p_1 = W_x, p_2 = W_y\}$, linear in the p_j , where the commutator is the Poisson bracket. The second order symmetries are formed by taking linear combinations of products of these functions. For the linear case, separable solutions are eigenfunctions of commuting second order differential operators. For the Hamilton–Jacobi case, separable solutions are those for which the corresponding second order functions, commuting under the Poisson bracket operation,

take constant values. The orbit analysis for the classification of separable systems is identical to that in the linear case.

We follow closely the procedure of Ref. 3 and concentrate here on those systems where one coordinate can be split off by diagonalizing a first order symmetry, leaving a reduced equation in two variables. The additively separable coordinates for the reduced equations (Hamilton—Jacobi equations for the free particle, harmonic oscillator, repulsive oscillator, linear potential, the equation of geometrical optics, etc.) correspond to proper subalgebras of $\mathfrak{o}(3,2)$. However, utilizing the full symmetry algebra, we find many solutions of these reduced equations which separate nonadditively. On one hand each reduced equation is a special case of (0.1), but also (0.1) is the equation of the graph of the reduced equation. Thus each reduced equation is equivalent to (0.1) and admits the symmetry algebra $\mathfrak{o}(3,2)$.

We show that the passage from a Hamilton—Jacobi equation to the associated Hamiltonian system provides us with the analogy of a momentum space model in the linear theory. We also indicate how the results of Ref. 4 concerning cyclidic R -separable coordinates for (0.2), in which it is impossible to split off one variable at a time, carry over directly to (0.1). In the second paper we will provide complete proofs concerning the identity of separable coordinates for these two equations. Finally, due to the fact that Lie algebra computations are much easier (though equivalent) for (0.1) than for (0.2), we have been able to find and correct some computational errors in Refs. 3 and 8.

Although this paper concerns only the nonlinear equation (0.1), it should be obvious to the reader that our Lie algebraic procedure can be applied with little change to more general Hamilton—Jacobi equations. Indeed there has been a recent revival of interest in separation of variables^{9,10} for general Hamilton—Jacobi equations owing to its usefulness as a solution technique for the Einstein and Einstein—Maxwell equations.¹¹ (For the classical literature see Ref. 12, and the book by Hagihara,¹³ where many applications to celestial mechanics are given.) Of the recent literature dealing with separation of variables for the Hamilton—Jacobi and related second order differential equations in general Riemannian (and pseudo-Riemannian) spaces we mention the works of Havas,⁹ Dietz,¹⁰ and Woodhouse.¹⁰ Havas⁹ has given the general form of the metric tensor for coordinates which admit complete or partial separation. He also gives the general form of linear and quadratic integrals of motion. Dietz¹⁰ and Woodhouse¹⁰ consider a much more restrictive definition of separation of variables; they additively split off a single variable at a time. In this way one cannot obtain the more general type separable coordinates.¹⁴ None of the above authors allows nontrivial R separation or considers nonadditive separation such as appears here. Furthermore, this and our subsequent paper are the only ones to associate with separable systems orbits of second order members of the enveloping algebra of the symmetry algebra as the correspondings integrals of the motions, thus allowing us to give explicit lists of separable coordinates classified in equivalence classes.

It is our hope that this treatment of a most interesting example will aid in the establishment of more general results for Hamilton—Jacobi equations.

1. BASIC PRINCIPLES

We begin with the equation

$$W_x^2 - W_y^2 - W_z^2 = 0, \quad W_{x\mu} = \partial_{x\mu} W(x) \quad (1.1)$$

for the characteristics of the wave equation. As is well known^{1,2} the space—time symmetry algebra of (1.1) is $\mathfrak{o}(3,2)$. That is, the set of Lie derivatives

$$L = \sum_{\mu=0}^2 a_\mu(x) \partial_{x\mu}$$

such that LW is a solution of (1.1) whenever W is a solution forms the Lie algebra $\mathfrak{o}(3,2)$ under the operations of addition of Lie derivatives and commutator bracket. A basis for $\mathfrak{o}(3,2)$ is provided by the elements

$$\begin{aligned} M_{\mu\nu} &= x_\mu \partial_{x\nu} - x_\nu \partial_{x\mu}, \quad \mu \neq \nu, \\ P_\mu &= \partial_{x\mu}, \quad D = x \cdot \partial_x \equiv x^\mu \partial_{x\mu}, \\ K_\mu &= 2x_\mu x \cdot \partial_x - x^2 \partial_{x\mu}, \quad 0 \leq \mu, \nu \leq 2, \end{aligned} \quad (1.2)$$

where $x_0 = x^0$, $x_j = -x^j$ for $j=1,2$, $x^2 = x^\mu x_\mu$ and the Einstein summation convention for repeated indices is adopted. The commutation relations are

$$\begin{aligned} [M_{\mu\nu}, M_{\sigma\rho}] &= g_{\mu\rho} M_{\nu\sigma} + g_{\nu\sigma} M_{\mu\rho} - g_{\mu\sigma} M_{\nu\rho} - g_{\nu\rho} M_{\mu\sigma}, \\ [M_{\mu\nu}, P_\sigma] &= g_{\nu\sigma} P_\mu - g_{\mu\sigma} P_\nu, \\ [P_\mu, P_\nu] &= [K_\mu, K_\nu] = [M_{\mu\nu}, D] = 0, \\ [D, P_\mu] &= -P_\mu, \quad [D, K_\mu] = K_\mu, \\ [M_{\mu\nu}, K_\sigma] &= g_{\nu\sigma} K_\mu - g_{\mu\sigma} K_\nu, \\ [K_\mu, P_\nu] &= -2(M_{\mu\nu} + g_{\mu\nu} D), \end{aligned} \quad (1.3)$$

where $g_{00} = -g_{jj} = 1$, $g_{\mu\nu} = 0$ for $\mu \neq \nu$.

These operators can be exponentiated to yield a local Lie transformation group² of symmetries of (1.1). Indeed the operator $M_{\mu\nu}$, P_μ generate the Poincaré symmetry group

$$W(x) \rightarrow W(\Lambda^{-1}(x-a)), \quad a = (a_0, a_1, a_2), \quad \Lambda \in \text{SO}(1,2), \quad (1.4)$$

the dilatation operator generates the symmetry

$$(\exp \lambda D) W(x) = X(e^\lambda x) \quad (1.5)$$

and the K_μ generate the special conformal transformations

$$\exp(a^\mu K_\mu) W(x_\nu) = W\left(\frac{x_\nu - a_\nu x^2}{1 - 2a \cdot x + a^2 x^2}\right). \quad (1.6)$$

We shall also consider the inversion, space reflection, and time reflection symmetries of (1.1),

$$IW(x) = W(-x/x^2), \quad RW(x) = W(x_0, x_1, -x_2),$$

$$TW(x) = W(-x_0, x_1, x_2),$$

which are not generated by the Lie derivatives (1.2).

In Ref. 6 the full infinite-dimensional symmetry algebra of an arbitrary first order partial differential equation was computed and shown that this algebra splits into symmetries which are contact transforma-

tions and an ideal of characteristics. Now there is a well-known Lie algebra isomorphism between canonical vector fields on phase space with the usual Lie brackets and functions on phase space (modulo constants) with the Poisson brackets.¹ Explicitly, given phase space with coordinates (x^μ, p_μ) we have the canonical 2-form $\omega = dp_\mu \wedge dx^\mu$. Then with each vector field X on phase space which leaves ω invariant, we can associate a function $F(x^\mu, p_\nu)$ such that

$$X \lrcorner \omega = dF, \quad (1.7)$$

where \lrcorner denotes the inner product between vector fields and forms. To the Lie brackets for vector fields there correspond the Poisson brackets

$$\{F(x, p), G(x, p)\} = \Sigma \left(\frac{\partial G}{\partial x^\mu} \frac{\partial F}{\partial p_\mu} - \frac{\partial F}{\partial x^\mu} \frac{\partial G}{\partial p_\mu} \right) \quad (1.8)$$

for functions. Explicitly for the Lie derivatives (1.2) we have, using (1.7),

$$\begin{aligned} M_{\mu\nu} &= x_\mu p_\nu - x_\nu p_\mu, & P_\mu &= p_\mu, & D &= x^\mu p_\mu, \\ K_\mu &= 2x_\mu (x^\nu p_\nu) - x^2 p_\mu. \end{aligned} \quad (1.9)$$

One can easily check that the basis functions (1.9) satisfy relations (1.3) under the Poisson bracket operation. From the point of view of separation of variables of (1.1), the Lie algebraic characterization (1.9) in terms of functions on phase space is superior to that of Lie derivatives.

By taking all possible products of operators (1.2) we can generate an enveloping algebra¹⁵ of $so(3, 2)$. Furthermore, we can identify the subspace \mathcal{S}'_k of homogeneous symmetric k th-order elements in the enveloping algebra with the space \mathcal{S}_k of k th-order polynomials in the basis functions (1.9). That is, the two subspaces are isomorphic as vector spaces and the adjoint action of $so(3, 2)$ on \mathcal{S}'_k induced by the commutator $[\cdot, \cdot]$ agrees with the adjoint action on \mathcal{S}_k induced by the Poisson bracket. In particular, \mathcal{S}'_2 is spanned by elements of the form $[L_1, L_2]_* \equiv L_1 L_2 + L_2 L_1$, where the L_j are Lie derivatives belonging to the symmetry algebra. Let L_1, L_2 be the corresponding functions in the Lie algebra (1.9). Then the correspondence

$$[L_1, L_2]_* \rightarrow 2L_1 L_2, \quad (1.10)$$

extended by linearity provides the stated isomorphism between \mathcal{S}'_2 and \mathcal{S}_2 .

As is well known there is an intimate relationship between a first-order partial differential equation

$$H(x^\mu, p_\mu) = 0, \quad p_\mu = \frac{\partial W}{\partial x^\mu}, \quad 0 \leq \mu \leq n, \quad (1.11)$$

and the Hamiltonian system of ordinary differential equations¹⁶

$$\frac{\partial p_\mu}{\partial \tau} = -H_{x^\mu}, \quad \frac{dx^\mu}{d\tau} = H_{p_\mu}, \quad 0 \leq \mu \leq n. \quad (1.12)$$

Indeed, consider the n -dimensional surface $x_\mu = x_\mu(t_1, \dots, t_n)$ and prescribe initial data on this surface:

$$W = W(t_1, \dots, t_n), \quad p_\mu = p_\mu(t_1, \dots, t_n), \quad (1.13)$$

subject to the requirements

$$\begin{aligned} \frac{\partial W}{\partial t_j} &= p_\mu \frac{\partial x^\mu}{\partial t_j}, \quad j = 1, \dots, n, \\ H(x^\mu(t_1, \dots, t_n), p_\mu(t_1, \dots, t_n)) &= 0. \end{aligned}$$

Then, provided

$$\det \begin{bmatrix} H_{p_0} & \dots & H_{p_n} \\ \frac{\partial x_0}{\partial t_1} & \dots & \frac{\partial x_n}{\partial t_1} \\ \dots & \dots & \dots \\ \frac{\partial x_0}{\partial t_n} & \dots & \frac{\partial x_n}{\partial t_n} \end{bmatrix} \neq 0$$

on the surface, the solutions of (1.12) with initial data (1.13) generate a local solution of (1.11). The function W can be obtained either from the equation

$$\frac{dW}{d\tau} = p_\mu H_{p_\mu} \quad (1.14)$$

or the defining relations $p_\nu = W_{x^\nu}$.

Conversely, let

$$W = f(x^\mu, a_1, \dots, a_n) + a_0 \quad (1.15)$$

be a complete integral of (1.11), i. e., W is a solution of (1.11) for each choice of the $n+1$ real constants a_μ and the $n \times (n+1)$ matrix

$$(\partial_{a_j} \partial_{x^\mu} f) = (f_{a_j x^\mu})$$

has rank n . Then (1.15) and relations

$$\begin{aligned} f_{a_j}(x^\mu, a_1) &= \lambda_j, \quad j = 1, \dots, n \\ p_\nu &= f_{x^\nu}(x^\mu, a_1), \quad \nu = 0, \dots, n \end{aligned} \quad (1.16)$$

with $a_0, \dots, a_n, \lambda_1, \dots, \lambda_n$ fixed, define a solution of the characteristic system (1.12).

It is also well known that the canonical transformation generated by (1.12) preserves Poisson brackets.¹ Thus, if

$$F_j^\tau(x^\mu, p_\nu) = F_j(x^\mu(\tau), p_\nu(\tau)), \quad j = 1, 2,$$

where

$$x^\mu(\tau) = x^\mu(\tau, x^{\mu'}, p_{\nu'}), \quad p_\nu(\tau) = p_\nu(\tau, x^{\mu'}, p_{\nu'}), \quad (1.17)$$

are solutions of (1.12) such that $x^\mu(0) = x^\mu, p_\nu(0) = p_\nu$, then

$$\{F_1^\tau, F_2^\tau\} = \{F_1, F_2\}. \quad (1.18)$$

Furthermore,

$$\frac{d}{d\tau} F^\tau = F_{x^\mu} H_{p_\mu} - F_{p_\mu} H_{x^\mu} = \{H^\tau, F^\tau\}$$

so that $F^\tau \equiv F$ if F commutes with H .

Applying this theory to Eq. (1.1), we find

$$H = p_0^2 - p_1^2 - p_2^2, \quad (1.19)$$

so that the associated Hamiltonian system is

$$\frac{dp_\mu}{d\tau} = 0, \quad \frac{dx^\mu}{d\tau} = \frac{1}{2} g^{\mu\nu} p_\nu, \quad 0 \leq \mu, \nu \leq 2. \quad (1.20)$$

Thus we can obtain a solution of (1.1) by prescribing initial data for W, x^μ , and p_μ on a two-dimensional

surface in x -space and solving Eqs. (1.20). For some of our computations we shall choose this surface and data in the special form

$$\begin{aligned} x^0 &= 0, \quad p_0 = (p_1^2 + p_2^2)^{1/2}, \quad x^1 = t_1, \quad x^2 = t_2, \\ p_1 &= p_1(t_1, t_2), \quad p_2 = p_2(t_1, t_2). \end{aligned} \quad (1.21)$$

Note that the basis functions (1.8) restricted to this surface become

$$\begin{aligned} p_0 &= (p_1^2 + p_2^2)^{1/2} = p_0, \quad p_j = p_j, \quad M_{ij} = x_i p_j - x_j p_i, \\ M_{0j} &= x^j p_0, \quad K_0 = [(x_1)^2 + (x_2)^2] p_0, \quad D = x^i p_i, \\ K_j &= 2x_j (x^i p_i) + [(x_1)^2 + (x_2)^2] p_j, \quad i, j = 1, 2. \end{aligned} \quad (1.22)$$

Model (1.22) and its relationship to (1.1) via integration of the Hamiltonian system (1.20) is an analogy of the Fourier transform model for the solution space of the wave equation (0.2) which was treated in Ref. 3.

We now introduce another basis for the symmetry algebra which makes explicit the isomorphism with the usual matrix realization of $\mathfrak{o}(3, 2)$. The matrix algebra $\mathfrak{o}(3, 2)$ is usually defined as the ten-dimensional Lie algebra of 5×5 matrices A such that $AG + GA^t = 0$, where 0 is the zero matrix and

$$G = \begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & & & -1 & \\ & 0 & & & -1 \end{pmatrix}. \quad (1.23)$$

Let \mathcal{E}_{ij} be the 5×5 matrix with a 1 in row i , column j and zeros elsewhere. Then the matrices

$$\begin{aligned} \Pi_{ab} &= \mathcal{E}_{ab} - \mathcal{E}_{ba} = -\Pi_{ba}, \quad a \neq b, \\ \Pi_{aB} &= \mathcal{E}_{aB} + \mathcal{E}_{Ba} = -\Pi_{Ba}, \quad 1 \leq a, b \leq 3, \\ \Pi_{45} &= \mathcal{E}_{54} - \mathcal{E}_{45} = -\Pi_{54}, \quad B = 4, 5 \end{aligned} \quad (1.24)$$

form a basis for $\mathfrak{o}(3, 2)$ with commutation relations

$$[\Pi_{\alpha\beta}, \Pi_{\gamma\epsilon}] = G_{\beta\gamma} \Pi_{\alpha\epsilon} + G_{\alpha\epsilon} \Pi_{\beta\gamma} + G_{\gamma\alpha} \Pi_{\epsilon\beta} + G_{\epsilon\beta} \Pi_{\gamma\alpha}. \quad (1.25)$$

This basis is related to our basis (1.2) by the identifications

$$\begin{aligned} P_0 &= \Gamma_{14} + \Gamma_{45}, \quad P_1 = \Gamma_{25} + \Gamma_{12}, \quad P_2 = \Gamma_{35} + \Gamma_{13}, \\ K_0 &= \Gamma_{45} - \Gamma_{14}, \quad K_1 = \Gamma_{25} - \Gamma_{12}, \quad K_2 = \Gamma_{35} - \Gamma_{13}, \\ M_{12} &= \Gamma_{32}, \quad M_{01} = \Gamma_{24}, \quad M_{02} = \Gamma_{34}, \quad D = \Gamma_{51}. \end{aligned} \quad (1.26)$$

Here and hereafter we denote by $\Gamma(\Gamma)$ the vector field (function) which corresponds to the matrix Π .

Returning to our Poisson bracket model (1.9), we note that if we impose the relation

$$P_0^2 - P_1^2 - P_2^2 = 0 \quad (1.27)$$

to obtain solutions of (1.1), we introduce linear dependencies among the elements of \mathcal{S}_2 . Although there are formally 35 independent terms $L_i L_j$ where the L_i run over a basis for $\mathfrak{o}(3, 2)$, among the explicit functions (1.9) subject to (1.27) there are 20 independent relations obeyed by the $L_i L_j$. Hence, if \mathcal{Q}_2 is the subspace of \mathcal{S}_2 which is mapped to zero under this identification, then \mathcal{Q}_2 is actually an ideal under the adjoint action of $\mathfrak{O}(3, 2)$ and the factor space $\mathcal{J}_2 \equiv \mathcal{S}_2 / \mathcal{Q}_2$ is 15-dimension-

al. In particular, on \mathcal{J}_2 we have the relations

$$\begin{aligned} K_0^2 - K_1^2 - K_2^2 &= 0, \quad \Gamma_{12}^2 + \Gamma_{13}^2 + \Gamma_{23}^2 = \Gamma_{45}^2, \\ M_{12}^2 - M_{01}^2 - M_{02}^2 &= -D^2, \quad \Gamma_{45}^2 - \Gamma_{41}^2 - \Gamma_{51}^2 = \Gamma_{23}^2. \end{aligned} \quad (1.28)$$

[Of course, these relations also hold in the model (1.22).]

We shall now classify all separable solutions of (1.1) which are analogous to the separable solutions of the wave equation (1.23), studied in Refs. 3–5. In these references each (R -) separable solution of (1.23) was characterized by the fact that the corresponding separated solutions were common eigenfunctions of a commuting pair S_1, S_2 of second-order differential symmetry operators from the enveloping algebra of $\mathfrak{so}(3, 2)$. Analogously we will characterize separated solutions W of (1.1) by the conditions $F_1(x^\mu, p_\nu) = \lambda$, $F_2(x^\mu, p_\nu) = \mu$, where $F_1, F_2 \in \mathcal{J}_2$ and $\{F_1, F_2\} = 0$. Here $p_\nu = W_{x^\nu}$ and λ, μ are the separation constants. For all cases treated in this paper we shall see that coordinates which yield separation for (1.23), corresponding to commuting operators S_1, S_2 , also yield separation for (1.1) corresponding to commuting functions $F_1, F_2 \in \mathcal{J}_2$. Here, S_j and F_j are related by (1.10). We also mention, although we will not make use of it, that corresponding to each commuting pair (F_1, F_2) there is via (1.7) a commuting pair of vector fields (X_1, X_2) on phase space and hence a two-parameter local Abelian group of symmetries.

In analogy with Ref. 3 we shall begin by studying separable systems which can be characterized by commuting functions F_1, F_2 such that $F_1 = A^2$ for some $A \in \mathfrak{so}(3, 2)$ with $\{A, F_2\} = 0$ —and where A has a non-trivial centralizer.

In Ref. 3 the corresponding systems were called “semi-subgroup” coordinates. However, as has been pointed out by Winternitz, this name is not appropriate because many coordinate systems which correspond to the restriction of $\mathfrak{so}(3, 2)$ to a subalgebra are not “semi-subgroup” systems. Thus, we shall now call these systems “semi-split” coordinates.

In the following seven sections we shall study systems of the form A^2, F_2 by first splitting off the coordinate associated with A to obtain a reduced equation (\dagger) such that F_2 belongs to the space of second-order elements in the enveloping algebra of the symmetry algebra (centralizer of A) of (\dagger). We shall then investigate the coordinate systems in which (\dagger) separates. In the last section we shall give an example of a completely non-splitting type coordinate. In the study of separable coordinates it is common to associate with each coordinate system $x^\mu = x^\mu(y_\nu)$ a metric tensor $g_{\mu\nu}$ and a quadratic differential form

$$ds^2 = dx_0^2 - dx_1^2 - dx_2^2 = g_{\mu\nu} dy^\mu dy^\nu \quad (1.29)$$

then the Hamilton–Jacobi equation (1.1) takes the form

$$g^{\mu\nu} W_{y^\mu} W_{y^\nu} = 0 \quad (1.30)$$

in the new coordinate system, where $g^{\mu\nu}$ is the inverse tensor to $g_{\mu\nu}$ given by

$$g^{\mu\nu} = \frac{(-1)^{\mu+\nu} \text{cof } g_{\mu\nu}}{\det g_{\mu\nu}}. \quad (1.31)$$

We also mention for convenience that for orthogonal coordinates (1.29) takes the form

$$ds^2 = \sum_i h_i^2 (d\xi^i)^2 \quad (1.32)$$

and the relation between the canonical moment in p_μ in Cartesian coordinates and arbitrary curvilinear coordinates ξ^i is

$$p^\mu = \sum_i h_i^{-2} \frac{\partial x^\mu}{\partial \xi^i} p_{\xi^i}. \quad (1.33)$$

The above relations will be used throughout our computations without further mention.

2. THE SPHERE

For our case we consider the function $A = \Gamma_{45} = \frac{1}{2}(P_0 + K_0)$. Setting $\Gamma_{45} = \lambda$, λ const, we see from (1.29) that (1.1) reduces to

$$\Gamma_{12}^2 + \Gamma_{13}^2 + \Gamma_{23}^2 = \lambda^2. \quad (2.1)$$

Since the centralizer of Γ_{45} in $\mathfrak{o}(3, 2)$ is $\{\Gamma_{45}\} \oplus \mathfrak{o}(3)$, where $\mathfrak{o}(3)$ is the subalgebra with basis $\Gamma_{12}, \Gamma_{13}, \Gamma_{23}$, it follows that $\mathfrak{o}(3)$ is a symmetry algebra for (2.1). We call it the *reduced* symmetry algebra.

Equation (2.1) can be viewed as the result of separating off one variable ψ in W . Indeed, we choose new coordinates such that $\Gamma_{45} = -\partial_\psi$. Standard Lie theory gives

$$\begin{aligned} x^0 &= \frac{\sin\psi}{y_1 - \cos\psi}, & x^1 &= \frac{y_2}{y_1 - \cos\psi}, & x^2 &= \frac{y_3}{y_1 - \cos\psi}, \\ y_1^2 + y_2^2 + y_3^2 &= 1. \end{aligned} \quad (2.2)$$

Thus, choosing any parametrization $y_j(\sigma, \alpha)$ of the unit sphere S_2 , we obtain a new set of coordinates for space time. In these coordinates we have $\Gamma_{23} = y_2 \partial_{y_3} - y_3 \partial_{y_2}$,

$$\Gamma_{12} = y_1 \partial_{y_2} - y_2 \partial_{y_1}, \quad \Gamma_{13} = y_1 \partial_{y_3} - y_3 \partial_{y_1}.$$

The equation $\Gamma_{45} = \lambda$ or, what is the same thing, $\Gamma_{45} W = \lambda$ implies

$$W = -\lambda\psi + S(\sigma, \alpha). \quad (2.3)$$

Substituting this expression into (1.1), we obtain the reduced Eq. (2.1) for S .

As is well known¹⁷ the space of second order symmetry operators in $\mathfrak{o}(3)$, modulo the invariant $\Gamma_{12}^2 + \Gamma_{13}^2 + \Gamma_{23}^2$, splits into exactly two orbit types under the adjoint action of $\mathfrak{o}(3)$. A representative on each orbit type is given by the assignment

- (1) $\Gamma_{45}^2, \Gamma_{23}^2$,
- (2) $\Gamma_{45}^2, \Gamma_{12}^2 + a^2 \Gamma_{13}^2, \quad 1 > a > 0$.

For the orbit of type (1) we introduce spherical coordinates on S_2

$$y_1 = \cos\sigma, \quad y_2 = \sin\sigma \cos\alpha, \quad y_3 = \sin\sigma \sin\alpha. \quad (2.4a)$$

Then (2.1) becomes

$$\csc^2\sigma p_\alpha^2 + p_\sigma^2 = \lambda^2, \quad p_\alpha = \frac{\partial S}{\partial \alpha}, \quad p_\sigma = \frac{\partial S}{\partial \sigma}, \quad (2.4b)$$

and the requirement $\Gamma_{23} = p_\alpha = m$ yields the separated

solution

$$\begin{aligned} S &= m\alpha + \int (m^2 \csc^2\sigma - \lambda^2)^{1/2} d\sigma, \\ &= m\alpha + \frac{im}{2} \ln\left(\frac{\Delta + \cos\sigma}{\Delta - \cos\sigma}\right) - i\lambda \ln\left(\frac{\lambda}{m} \cos\sigma + \Delta\right) + c, \\ \Delta &= [1 - (\lambda^2/m^2) \sin^2\sigma]^{1/2}. \end{aligned} \quad (2.4c)$$

For the orbit of type (2) we introduce elliptic coordinates on S_2 :

$$\begin{aligned} y_1 &= k'^{-1} \operatorname{dn}(\alpha, k) \operatorname{dn}(\sigma, a), & y_2 &= ikk'^{-1} \operatorname{cn}(\alpha, k) \operatorname{cn}(\sigma, k), \\ y_3 &= k \operatorname{sn}(\alpha, k) \operatorname{sn}(\sigma, k), \end{aligned} \quad (2.5a)$$

where $k' = (1 - k^2)^{1/2}$ and $\operatorname{dn}(\alpha), \operatorname{cn}(\alpha), \operatorname{sn}(\alpha)$ are Jacobi elliptic functions.¹⁸ Then (2.1) becomes

$$p_\alpha^2 - p_\sigma^2 = -\lambda^2 k^2 (\operatorname{sn}^2\alpha - \operatorname{sn}^2\sigma)$$

and the condition

$$\Gamma_{12}^2 + k^2 \Gamma_{13}^2 = (\operatorname{sn}^2\alpha - \operatorname{sn}^2\sigma)^{-1} (\operatorname{sn}^2\alpha p_\alpha^2 - \operatorname{sn}^2\sigma p_\sigma^2) = \mu \quad (2.5b)$$

leads to the separated solution

$$S = \int (-\lambda^2 a^2 \operatorname{sn}^2\alpha + \mu)^{1/2} d\alpha + \int (-\lambda^2 a^2 \operatorname{sn}^2\sigma + \mu)^{1/2} d\sigma. \quad (2.5c)$$

There is a close relationship between our own study of Γ_{45} and the Hamilton–Jacobi equation for the Kepler problem with closed orbits in two-dimensional space. Indeed, on the surface (1.21) the condition $\Gamma_{45} = \lambda$ for a solution of (1.1) becomes

$$(x^1)^2 + (x^2)^2 - 2\lambda/p_0 = -1, \quad p_0 = (p_1^2 + p_2^2)^{1/2}. \quad (2.6)$$

Performing the canonical transformation $p_j \rightarrow x^j, x^j \rightarrow -p_j$, which preserves Poisson brackets, we transform (2.6) to the Hamilton–Jacobi equation for the Kepler problem with energy normalized to -1 (bound orbits), viz.,

$$p_1^2 + p_2^2 - 2\lambda/r = -1, \quad r = (x_1^2 + x_2^2)^{1/2}. \quad (2.7)$$

Moreover, the $\mathfrak{o}(3)$ symmetry algebra for (2.6) generated by $\Gamma_{12}, \Gamma_{13}, \Gamma_{23}$ is mapped to an $\mathfrak{o}(3)$ symmetry algebra for (2.7). If $\psi(\xi_1, \xi_2)$ is a solution of (2.7) (with $x^1 = \xi_1, x^2 = \xi_2$), then by prescribing the initial data $x^j = \psi_{\xi_j}, p_j = \xi_j$ on the surface (1.21) and integrating along characteristics, we find a solution of (1.1) with $\Gamma_{45} = \lambda$. Conversely, if $W(x^\mu)$ is a solution of (1.1) with $\Gamma_{45} = \lambda$, then a function $\psi(\xi_1, \xi_2)$ such that

$$x^j - \psi_{\xi_j}, \quad W_{x^l}(0, x^l) = \xi_j, \quad j, l = 1, 2 \quad (2.8)$$

with $\det(W_{x^j x^l}) \neq 0$ is a solution of (2.7). This relationship is a classical analogy of Fock's treatment of the quantum mechanical hydrogen atom,¹⁹ and underlies the group theoretical approach to the Kepler problem.²⁰

We have obtained the reduced equation (2.1) from (1.1) by additively separating off dependence on the variable ψ . However, (1.1) can also be viewed as the equation for the graph of (2.1). Indeed, set $\lambda = 1$ for simplicity and let $S(\sigma, \alpha)$ be a solution of (2.1). A graph of S is a function $W(\sigma, \alpha, S)$ such that $W(\sigma, \alpha, S(\sigma, \alpha)) = 0$. Since $W_\sigma + W_S S_\sigma = 0, W_\alpha + W_S S_\alpha = 0$ it follows from (2.2)–(2.4) that W satisfies Eq. (1.1) with ψ replaced by S . In this sense Eqs. (1.1) and (2.1) are equivalent.

Since (1.1) admits the symmetry algebra $\mathfrak{o}(3,2)$, it induces an action of $\mathfrak{o}(3,2)$ as a symmetry algebra^{2,6} of (2.1). Now, however, $\mathfrak{o}(3,2)$ acts not only on σ, α but also on S . We have used only the $\mathfrak{o}(3)$ subgroup of $\mathfrak{o}(3,2)$ to explain the two systems in which (2.1) admits an additive separation of variables. (We believe that there are only two such systems and will settle this point in a future publication.) However, we can use commuting pairs of second-order elements in the enveloping algebra of $\mathfrak{o}(3,2)$ to distinguish many other symmetry adapted solutions of (2.1). [For example, some other solutions may correspond to a product separation in (2.1).] Indeed, every separable solution of (1.1) corresponds via the graph to some symmetry adapted solution of (2.1). Our restriction to the subalgebra $\mathfrak{o}(3)$ merely picks out those solutions which are additively separable for (2.1).

3. THE EQUATION OF GEOMETRICAL OPTICS

In this section we consider the equation obtained from (1.1) by partial separation via the operator P_0 , i. e., we treat the usual equation of geometrical optics obtained from (1.1) by putting $P_0^2 = \lambda^2$, viz.,

$$W = \lambda x^0 + S(x^1, x^2), \quad (3.1a)$$

$$S_{x^1}^2 + S_{x^2}^2 = P_1^2 + P_2^2 = \lambda^2. \quad (3.1b)$$

It is easy to check that the centralizer of P_0 in $\mathfrak{o}(3,2)$ is $\{P_0\} \oplus \mathfrak{e}(2)$, where $\mathfrak{e}(2)$ is generated by $\{M_{12}, P_1, P_2\}$. However, in this case P_0 has a normalizer which is bigger than its centralizer, and for the purpose of separation of variables it is convenient to classify orbits using the full normalizer group $D \otimes E(2)$, where D is the one-parameter group of dilatations generated by D . For (3.1b) there are four separable orthogonal coordinate systems corresponding precisely to the four orbit types²¹ of the quadratic members of the universal enveloping algebra $\mathfrak{e}(2)$ (modulo $P_1^2 + P_2^2$) under the adjoint action of $D \otimes E(2)$. The list of pairs of orbit representatives is

$$(3) P_0^2, P_1^2, \text{ Cartesian,}$$

$$(4) P_0^2, M_{12}^2, \text{ polar,}$$

$$(5) P_0^2, M_{12}P_2, \text{ parabolic,}$$

$$(6) P_0^2, M_{12}^2 + P_2^2, \text{ elliptic.}$$

The coordinate systems and corresponding solutions (3.1a) are now given for each of the above cases:

(3) *Cartesian*: The coordinates are the usual Cartesian coordinates

$$x^0 = t, \quad x^1 = x, \quad x^2 = y \quad (3.2a)$$

with the solution

$$S = \mu x + (\lambda^2 - \mu)^{1/2} y, \quad (3.2b)$$

whose separation constant is

$$P_1^2 = p_x^2 = \mu^2. \quad (3.2c)$$

(4) *Polar*: The coordinates are

$$x^0 = t, \quad x^1 = r \cos \theta, \quad x^2 = r \sin \theta \quad (3.3a)$$

and the well-known solution

$$S = \mu \theta + \int dr (\lambda^2 - \mu^2/r^2)^{1/2} \quad (3.3b)$$

with the constant of the motion

$$M_{12}^2 = p_\theta^2 = \mu^2. \quad (3.3c)$$

(5) *Parabolic*: The coordinates are

$$x^0 = t, \quad x^1 = (\xi^2 - \eta^2)/2, \quad x^2 = \xi \eta \quad (3.4a)$$

with $-\infty < \xi < \infty$, $0 \leq \eta < \infty$. The solution is

$$S = \int (\lambda^2 \xi^2 - \mu)^{1/2} d\xi + \int (\lambda^2 \eta^2 + \mu)^{1/2} d\eta \quad (3.4b)$$

with separation constant

$$2M_{12}P_2 = (\xi^2 + \eta^2)^{-1} (\xi^2 p_\eta^2 - \eta^2 p_\xi^2) = \mu.$$

(6) *Elliptic*: The coordinates are

$$x^0 = t, \quad x^1 = \cosh \rho \cos \sigma, \quad x^2 = \sinh \rho \sin \sigma, \quad (3.5a)$$

with $-\infty < \rho < \infty$, $0 \leq \sigma < 2\pi$. The solution is

$$S = \int (\lambda^2 \cosh^2 \rho - \mu)^{1/2} d\rho + \int (\mu - \lambda^2 \cos^2 \sigma)^{1/2} d\sigma \quad (3.5b)$$

with constant of the motion

$$M_{12}^2 + P_1^2 = (\cosh^2 \rho - \cos^2 \sigma)^{-1} (\cos^2 \sigma p_\rho^2 + \cosh^2 \rho p_\sigma^2) = \mu. \quad (3.5c)$$

Just as in the previous section, we can also interpret (1.1) as the equation for the graph of a solution of (3.1b). In this sense (3.1b) admits the full symmetry algebra and any separable system for (1.1) gives rise to a symmetry adapted solution of (3.1b).

4. THE FREE RELATIVISTIC PARTICLE

By separating off one space variable, say x^2 , via the operator P_2 , Eq. (1.1) reduces to the equation for a free relativistic particle in one space and one time dimension. Explicitly, putting $P_2^2 = \lambda^2$, we find

$$W = \lambda x^2 + S(x^1, x^0), \quad (4.1a)$$

$$(S_{x^0})^2 - (S_{x^1})^2 = P_0^2 - P_1^2 = \lambda^2. \quad (4.1b)$$

Again a straightforward calculation shows that the centralizer of P_2 in $\mathfrak{o}(3,2)$ is $\{P_2\} \oplus \mathfrak{e}(1,1)$. A basis for the subalgebra $\mathfrak{e}(1,1)$ is given by $\{M_{01}, P_0, P_1\}$. The orbit analysis of the quadratic members of the universal enveloping algebra of $\mathfrak{e}(1,1)$ (modulo $P_0^2 - P_1^2$) under the adjoint action of the normalizer group $D \otimes E(1,1)$ extended by certain discrete transformations was given in Ref. 22. There it was also seen that there is a nonuniqueness for the orbit corresponding to the separation of the Klein-Gordon equation in the usual Cartesian coordinates. This nonuniqueness was resolved by considering nonorthogonal coordinates. Here, however, only orthogonal coordinates are considered. Furthermore, there is one orbit for which the Klein-Gordon equation (Laplace operator) does not admit a separation of variables. The question as to whether there is also no separable coordinate system for (3.6b) corresponding to this orbit representative $[M_{01}(P_0 + P_1)]$ will be answered in Paper II.

The list of orbits representatives is

- (3') P_2^2, P_0P_1 , Cartesian,
- (7) P_2^2, M_{01}^2 , polar,
- (8) $P_2^2, 2M_{01}P_1$, parabolic—type 1,
- (9) $P_2^2, M_{01}^2 - (P_0 + P_1)^2$, hyperbolic—type 3,
- (10) $P_2^2, M_{01}^2 + (P_0 + P_1)^2$, hyperbolic—type 2,
- (11) $P_2^2, 2M_{01}(P_0 + P_1) + (P_0 - P_1)^2$, parabolic—type 2,
- (12) $P_2^2, M_{01}^2 - P_0P_2$, hyperbolic—type 1,
- (13) $P_2^2, M_{01}^2 + P_1^2$, elliptic—type 1,
- (14) $P_2^2, M_{01}^2 - P_1^2$, elliptic—type 2.

From the point of view of separable coordinates (3') is equivalent to (3). The remainder of the coordinates and solutions are:

(7) *Polar*: The coordinates for the region $(x^0)^2 - (x^1)^2 > 0$ are

$$x^0 = \pm r \cosh \eta, \quad x^1 = r \sinh \eta, \quad (4.2a)$$

$0 \leq r < \infty, -\infty < \eta < \infty$. The solutions are

$$S = \pm \mu \eta + \int dr (\lambda^2 + \mu^2/r^2)^{1/2}. \quad (4.2b)$$

For the region $(x^0)^2 - (x^1)^2 < 0$, interchange x^0 and x^1 and find (4.2b) with $\lambda^2 \rightarrow -\lambda^2$. The separation constant is

$$M_{01}^2 = P_1^2 = \mu^2. \quad (4.2c)$$

(8) *Parabolic—type 1*: The coordinates are

$$x^0 = \pm \frac{1}{2}(\xi^2 + \eta^2), \quad x^1 = \xi \eta, \quad (4.3a)$$

with $-\infty < \xi < \infty, 0 \leq \eta < \infty$. This parametrizes the region $(x^0)^2 - (x^1)^2 > 0$. The solutions are

$$S = \int d\xi (\mu + \lambda^2 \xi^2)^{1/2} + \int d\eta (\mu + \lambda^2 \eta^2)^{1/2} \quad (4.3b)$$

with separation constant

$$\pm 2M_{01}P_1 = (\xi^2 - \eta^2)^{-1}(\xi^2 p_\xi^2 - \eta^2 p_\eta^2) = \mu. \quad (4.3c)$$

The spacelike region $(x^0)^2 - (x^1)^2 < 0$ can be parametrized by interchanging x^0 and x^1 ; however, the orbit representative then changes (see Ref. 22 for further discussion).

(9) *Hyperbolic—type 3*: The coordinates are

$$x^0 = \cosh(\eta - \zeta) + \exp(\eta + \zeta), \quad x^1 = \cosh(\eta - \zeta) - \exp(\eta + \zeta) \quad (4.4a)$$

with $-\infty < \eta, \zeta < \infty$. The solutions are

$$S = \int d\eta (\mu + 2\alpha\lambda^2 e^{2\eta})^{1/2} + \int d\zeta (\mu + 2\alpha\lambda^2 e^{2\zeta})^{1/2} \quad (4.4b)$$

and separation constant

$$M_{01}^2 - (P_0 + P_1)^2 = (e^{2\eta} - e^{2\zeta})^{-1}(e^{2\eta} p_\eta^2 - e^{2\zeta} p_\zeta^2) = \mu. \quad (4.4c)$$

The coordinates (4.4a) parametrize the region $x^0 + x^1 > 2, x^0 - x^1 > 0$. In this case we can introduce similar parametrizations for all region except the strip $|x^0 + x^1| < 2$. For this strip there appear to be no separable coordinates of this type for (4.1b).

(10) *Hyperbolic—type 2*: The coordinates are

$$x^0 = \sinh(\eta - \zeta) + \exp(\eta + \zeta) \quad x^1 = \sinh(\eta - \zeta) - \exp(\eta + \zeta), \quad (4.5a)$$

where $-\infty < \eta, \zeta < \infty$. These coordinates parametrize the region $x^0 - x^1 > 0$. By interchanging x^0 and x^1 we can parametrize the region $x^0 - x^1 < 0$. The solutions are

$$S = \int d\eta (\mu + 2\lambda^2 e^{2\eta})^{1/2} + \int d\zeta (\mu - 2\lambda^2 e^{2\zeta})^{1/2} \quad (4.5b)$$

with constant of the motion

$$M_{01}^2 + (P_0 + P_1)^2 = (e^{2\eta} + e^{2\zeta})^{-1}(e^{2\eta} p_\eta^2 + e^{2\zeta} p_\zeta^2) = \mu. \quad (4.5c)$$

(11) *Parabolic—type 2*: The coordinates are

$$x^0 = \frac{1}{2}(\eta - \xi)^2 + (\eta + \xi), \quad x^1 = \frac{1}{2}(\eta - \xi)^2 - (\eta + \xi) \quad (4.6a)$$

with $-\infty < \xi < \infty$ and $0 \leq \eta < \infty$. These coordinates cover the half-plane $x^0 + x^1 > 0$. Similarly we can parametrize the remaining half-plane. The solutions are

$$S = \int d\eta (\mu + 4\lambda\eta)^{1/2} + \int (\mu + 4\lambda\xi)^{1/2} d\xi \quad (4.6b)$$

with separation constant

$$2M_{01}(P_0 + P_1) + (P_0 - P_1)^2 = (\eta - \xi)^{-1}(\eta p_\xi^2 - \xi p_\eta^2) = \mu. \quad (4.6c)$$

(12) *Hyperbolic—type 1*:

$$x^0 = \frac{1}{2}[\cosh \frac{1}{2}(\eta - \zeta) + \sinh \frac{1}{2}(\eta + \zeta)], \quad (4.7a)$$

$$x^1 = \frac{1}{2}[\cosh \frac{1}{2}(\eta - \zeta) - \sinh \frac{1}{2}(\eta + \zeta)],$$

where $0 \leq \eta < \infty, -\infty < \zeta < \infty$ and (4.7a) parametrizes the half-plane $x^0 + x^1 \geq 1$. By taking $(x^0, x^1) \rightarrow (x^0, \mp x^1)$ we can parametrize the remaining portion. The solutions are

$$S = \frac{1}{2} \int d\eta (\mu + \frac{1}{2}\lambda^2 \sinh \eta)^{1/2} + \frac{1}{2} \int d\zeta (\mu + \frac{1}{2}\lambda^2 \sinh \zeta)^{1/2} \quad (4.7b)$$

with separation constant

$$M_{01}^2 - 4P_0P_1 = 4(\sinh \eta - \sinh \zeta)^{-1}(\sinh \eta p_\eta^2 - \sinh \zeta p_\zeta^2) = \mu. \quad (4.7c)$$

(13) *Elliptic—type 1*: The coordinates are

$$x^0 = \sinh \rho \cosh \sigma, \quad x^1 = \cosh \rho \sinh \sigma, \quad (4.8a)$$

where the full plane is parametrized with $-\infty < \rho, \sigma < \infty$. The solutions are

$$S = \int d\rho (\mu + \lambda^2 \cosh^2 \rho)^{1/2} + \int d\sigma (\mu - \lambda^2 \sinh^2 \sigma)^{1/2} \quad (4.8b)$$

with separation constant

$$M_{01}^2 + P_1^2 = (\cosh^2 \rho + \sinh^2 \sigma)^{-1}(\sinh^2 \sigma p_\sigma^2 + \cosh^2 \rho p_\rho^2) = \mu. \quad (4.8c)$$

(14) *Elliptic—type 2*: For the last case we have different coordinates for different regions of Minkowski space, viz.,

$$x^0 = \cosh \rho \cosh \sigma, \quad x^1 = \sinh \rho \sinh \sigma, \quad (4.9a)$$

with $-\infty < \rho < \infty, 0 \leq \sigma < \infty$ for the region $x^0 \geq 1$. If we let $(x^0, x^1) \rightarrow (-x^0, x^1)$, we can treat the region $x^0 \leq -1$. However, for the region $-1 \leq x^0 \leq 1$ we have

$$x^0 = \cos \phi \cos \psi, \quad x^1 = \sin \phi \sin \psi \quad (4.9b)$$

with $0 \leq \phi < 2\pi, 0 \leq \psi < \pi$. With (4.9a) and (4.9b) we still miss the region $-1 < x^0 < 1, 1 < |x^1|$. This region can be handled by interchanging x^0 and x^1 ; hence, we can cover the full plane with elliptic—type 2 coordinates.

The solution corresponding to (4.9a) is

$$S = \int d\rho(\mu + \lambda^2 \sinh^2 \rho)^{1/2} + \int d\sigma(\mu + \lambda^2 \sinh^2 \sigma)^{1/2} \quad (4.9c)$$

while for (4.9b)

$$S = \int d\phi(\lambda^2 \sin^2 \phi - \mu)^{1/2} + \int d\psi(\lambda^2 \sin^2 \psi - \mu)^{1/2} \quad (4.9d)$$

with constants of the motion in the corresponding regions

$$M_{01}^2 - P_1^2 = (\cosh^2 \rho - \cosh^2 \sigma)^{-1} (\sinh^2 \rho p_\sigma^2 - \sinh^2 \sigma p_\rho^2) = \mu, \quad (4.9e)$$

$$M_{01}^2 - P_1^2 = (\cos^2 \psi - \cos^2 \phi)^{-1} (\sin^2 \psi p_\phi^2 - \sin^2 \phi p_\psi^2) = \mu.$$

5. THE HYPERBOLOIDS (DOUBLE AND SINGLE SHEETED)

In this case we consider the function D given in (1.9). Putting $D = \lambda$ and using (1.28), we see that (1.1) reduces to

$$M_{01}^2 + M_{02}^2 - M_{12}^2 = \lambda^2. \quad (5.1)$$

Now the centralizer of D in $\mathfrak{o}(3, 2)$ is $\{D\} \oplus \mathfrak{o}(2, 1)$, where $\mathfrak{o}(2, 1)$ is the subalgebra of $\mathfrak{o}(3, 2)$ with basis M_{01} , M_{02} , M_{12} ; hence, $\mathfrak{o}(2, 1)$ is the reduced symmetry algebra for (5.1). Introducing the real variable $0 < \rho = (x \cdot x)^{1/2}$, $x \cdot x > 0$, for which $D = \rho P_\rho$, we obtain

$$\begin{aligned} x_0 = \rho y_0, \quad x_1 = \rho y_1, \quad x_2 = \rho y_2, \\ y_0^2 - y_1^2 - y_2^2 = 1. \end{aligned} \quad (5.2)$$

Thus we have separated off one variable in such a way that we are left with a double-sheeted hyperboloid. We will hereafter restrict ourselves to the upper sheet ($y_0 > 0$). Furthermore, for the region $x \cdot x < 0$ with $\rho = (-x \cdot x)^{1/2} > 0$, we obtain the single-sheeted hyperboloid, viz.,

$$\begin{aligned} x_0 = \rho y_0, \quad x_1 = \rho y_1, \quad x_2 = \rho y_2, \\ y_0^2 - y_1^2 - y_2^2 = -1. \end{aligned} \quad (5.3)$$

We note that, on the light-cone $x \cdot x = 0$, (5.1) reduces to an ordinary differential equation and the problem of separation does not exist.

Now using the coordinates (5.2) or (5.3) it is straightforward to show that $M_{\mu\nu} = y_\mu \partial_\nu - y_\nu \partial_\mu$; thus by diagonalizing \mathbf{D} , i. e., $\mathbf{D}W = \lambda W$ we have

$$W = \lambda \ln \rho + S(y^\mu) \quad (5.4)$$

and (1.1) reduces to (4.1) where the M^3 's in (5.1) are interpreted now as $M_{\mu\nu} S$, i. e., $M_{\mu\nu} \rightarrow M_{\mu\nu} S$.

It was shown by Winternitz, Lukać, and Smorodinskiĭ²³ that the space of second order members of the universal enveloping algebra of $\mathfrak{o}(2, 1)$ modulo its center splits into nine orbits under the adjoint action of $\mathbf{O}(2, 1)$. Furthermore, the connection with the separation of variables for the Laplace-Beltrami operator was established. Here, we list the pairs of commuting functions which separate variables in (1.1) keeping with the notation used previously^{3, 23}

$$(15) D^2, M_{12}^2, \text{ spherical,}$$

$$(16) D^2, M_{02}^2, \text{ equidistant,}$$

$$(7') D^2, (M_{12} - M_{02})^2, \text{ horocycle,}$$

$$(17) D^2, M_{12}^2 + a^2 M_{01}^2, \text{ elliptic, } 0 < a < 1,$$

$$(18) D^2, M_{01}^2 - a^2 M_{12}^2, \text{ hyperbolic, } 0 < a < 1,$$

$$(19) D^2, 2aM_{01}^2 - M_{02}M_{12}, \text{ semihyperbolic, } 0 < a < \infty,$$

$$(20) D^2, (M_{01} + M_{12})^2 + aM_{02}^2, \text{ elliptic-parabolic,} \\ 0 < a < \infty,$$

$$(21) D^2, (M_{01} + M_{12})^2 - aM_{02}^2, \text{ hyperbolic-parabolic,} \\ 0 < a < \infty,$$

$$(22) D^2, \frac{1}{4}M_{02}(M_{01} - M_{12}), \text{ semicircular-parabolic.}$$

We now give the solutions for the above orbit representatives on the upper sheet of the double-sheeted hyperboloid. The coordinates on the single-sheeted hyperboloid can be obtained by $y^\mu \rightarrow iy^\mu$ with the appropriate change of parametrization; however, different parametrizations are sometimes needed for different regions. There is also no guarantee that Eq. (5.1) is separable in all regions (see Ref. 24). Each of the separated solutions is found by solving (4.1) (i. e., $D^2 = \lambda$) and taking one of the above orbit representatives as constant of the motion.

(15) *Spherical*: The coordinates are

$$y^0 = \cosh \eta, \quad y^1 = \sinh \eta \cos \phi, \quad y^2 = \sinh \eta \sin \phi, \quad (5.5a)$$

with $0 < \eta < \infty$ and $0 \leq \phi < 2\pi$ and the separated solution is

$$S = \mu \phi + \int (\lambda^2 - \mu^2 / \sinh^2 \eta)^{1/2} d\eta \quad (5.5b)$$

with separation constant

$$M_{12}^2 = p_\phi^2 = \mu^2. \quad (5.5c)$$

(16) *Equidistant*: The coordinates are

$$y^0 = \cosh \rho \cosh \sigma, \quad y^1 = \sinh \rho, \quad y^2 = \cosh \rho \sinh \sigma \quad (5.6a)$$

with $-\infty < \rho, \sigma < \infty$, and separated solution

$$S = \mu \sigma + \int (\alpha^2 - \mu^2 / \cosh^2 \rho)^{1/2} d\rho \quad (5.6b)$$

with separation constant

$$M_{02}^2 = p_\sigma^2 = \mu^2. \quad (5.6c)$$

(7') is the same as (7).

(17) *Elliptic*: The coordinates are

$$\begin{aligned} y^0 = (k')^{-1} dnu \, dnv, \quad y^1 = k(k')^{-1} cnu \, cnv, \\ y^2 = -ik \, snu \, snv \end{aligned} \quad (5.7a)$$

with $u \in (0, 4K)$ and $v \in (0, iK')$. The separated solution of (4.1) is

$$S = \int du(\mu - \lambda^2 k^2 \text{cn}^2 u)^{1/2} + \int dv(\mu - \lambda^2 k^2 \text{cn}^2 v)^{1/2} \quad (5.7b)$$

and separation constant is

$$k^2 M_{12}^2 + k^2 M_{01}^2 = (\text{cn}^2 u - \text{cn}^2 v)^{-1} (\text{cn}^2 u p_v^2 - \text{cn}^2 v p_u^2) = \mu. \quad (5.7c)$$

(18) *Hyperbolic*: The coordinates are

$$\begin{aligned} y^0 = ik(k')^{-1} cnu \, cnv, \quad y^1 = ik \, snu \, snv, \\ y^2 = i(k')^{-1} dnu \, dnv \end{aligned} \quad (5.8a)$$

with $u \in (iK', iK + 2K)$ and $v \in (-iK', iK)$. The separated solutions are

$$S = (1/k^2) \int du (\lambda^2 k^2 \operatorname{sn}^2 u - \mu)^{1/2} + (1/k^2) \int dv (\lambda^2 k^2 \operatorname{sn}^2 v - \mu)^{1/2} \quad (5.8b)$$

with constant of the motion

$$M_{01}^2 - k^2 M_{12}^2 = k^4 (\operatorname{sn}^2 u - \operatorname{sn}^2 v)^{-1} (\operatorname{sn}^2 v p_u^2 - \operatorname{sn}^2 u p_v^2) = \mu. \quad (5.8c)$$

(19) *Semihyperbolic*: The coordinates are

$$y_1^2 = \frac{-\nu\eta}{\alpha^2 + \beta^2}, \quad y_2^2 = \frac{\nu\eta}{2(\alpha^2 + \beta^2)} - \frac{1}{2} + \frac{1}{2\beta} \left[\frac{((\nu - \alpha)^2 + \beta^2)((\eta - \alpha)^2 + \beta^2)}{\alpha^2 + \beta^2} \right]^{1/2}, \quad (5.9a)$$

$$y_0^2 = \frac{-\nu\eta}{2(\alpha^2 + \beta^2)} + \frac{1}{2} + \frac{1}{2\beta} \left[\frac{((\nu - \alpha)^2 + \beta^2)((\eta - \alpha)^2 + \beta^2)}{\alpha^2 + \beta^2} \right]^{1/2}$$

with $0 \leq \eta < \infty$, $-\infty < \nu < 0$, and α, β real. The separated solutions are

$$S = \int d\nu (\mu + \frac{1}{4}\lambda^2\nu)^{1/2} \nu^{-1/2} [(\nu - \alpha)^2 + \beta^2]^{-1/2} + \int d\eta (\mu + \frac{1}{4}\lambda^2\eta)^{1/2} \eta^{-1/2} [(\eta - \alpha)^2 + \beta^2]^{1/2} \quad (5.9b)$$

with separation constant

$$\alpha M_{01}^2 - 2\beta M_{12} M_{02} = \eta\nu(\nu - \eta)^{-1} [((\eta - \alpha)^2 + \beta^2)p_\eta^2 - ((\nu - \alpha)^2 + \beta^2)p_\nu^2] = \mu. \quad (5.9c)$$

(20) *Elliptic-parabolic*: For simplicity we consider the nondegenerate point $a = 1$; the coordinates are

$$y^0 = \frac{1}{2} \left(\frac{\cosh^2 \rho + \cos^2 \theta}{\cosh \rho \cos \theta} \right), \quad y^2 = \frac{1}{2} \left(\frac{\sin^2 \theta - \cosh^2 \rho}{\cosh \rho \cos \theta} \right), \quad y^1 = -\tanh \rho \tan \theta \quad (5.10a)$$

with $-\infty < \rho < \infty$ and $-\pi/2 < \theta \leq \pi/2$. The solutions are

$$S = \int d\rho (\mu + \lambda^2/\cosh^2 \rho)^{1/2} + \int d\theta (-\mu - \lambda^2/\cos^2 \theta)^{1/2} \quad (5.10b)$$

with constant of the motion

$$M_{02}^2 + (M_{01} + M_{12})^2 = (\cosh^2 \rho - \cos^2 \theta)^{-1} (\cosh^2 \rho p_\rho^2 + \cos^2 \theta p_\theta^2) = \mu. \quad (5.10c)$$

(21) *Hyperbolic-parabolic*: Again for $a = 1$ the coordinates are

$$y^0 = \frac{1}{2} \left(\frac{\cosh^2 \rho + \cos^2 \theta}{\sinh \rho \sin \theta} \right), \quad y^2 = \frac{1}{2} \left(\frac{\sin^2 \theta - \sinh^2 \rho}{\sinh \rho \sin \theta} \right), \quad y^1 = \coth \rho \cot \theta \quad (5.11a)$$

with $0 \leq \rho < \infty$ and $0 \leq \theta < \pi$. The separated solutions are

$$S = \int d\rho (\mu - \lambda^2/\sinh^2 \rho)^{1/2} + \int d\theta (-\mu - \lambda^2/\sin^2 \theta)^{1/2} \quad (5.11b)$$

with separation constant

$$-(M_{01} + M_{12})^2 + M_{02}^2 = (\sinh^2 \rho + \sin^2 \theta)^{-1} (\sinh^2 \rho p_\rho^2 - \sin^2 \theta p_\theta^2) = \mu. \quad (5.11c)$$

(22) *Semicircular-parabolic*: The coordinates are

$$y^0 = \frac{(\xi^2 + \eta^2)^2 + 4}{8\xi\eta}, \quad y^1 = \frac{\eta^2 - \xi^2}{2\xi\eta}, \quad y^2 = \frac{(\xi^2 + \eta^2)^2 - 4}{8\xi\eta} \quad (5.12a)$$

with $0 \leq \xi, \eta < \infty$. The solutions are

$$S = \int d\eta (\mu + \lambda^2/\eta^2)^{1/2} + \int d\xi (\lambda^2/\xi^2 - \mu)^{1/2} \quad (5.12b)$$

with the constant of the motion

$$\frac{1}{4} M_{02} (M_{01} - M_{12}) = (\xi^2 + \eta^2)^{-1} (\eta^2 p_\eta^2 - \xi^2 p_\xi^2) = \mu. \quad (5.12c)$$

As with the sphere in Sec. 2 there is a close relationship between our model on the hyperboloid and the Hamilton-Jacobi equation for the Kepler problem with unbounded orbits (positive energies). Indeed the operator $D = \Gamma_{15}$ is conformally equivalent to $\Gamma_{14} = \frac{1}{2}(\mathbf{P}_0 - \mathbf{K}_0)$. Explicitly, $\operatorname{Ad} \exp(\frac{1}{2}\pi \Gamma_{45}) \Gamma_{15} = \Gamma_{14}$. Thus $\Gamma_{14} = \lambda$, and on the surface (1.21) we have

$$x_1^2 + x_2^2 + 2\lambda/p_0 = 1, \quad p_0 = (p_1^2 + p_2^2)^{1/2}. \quad (5.13)$$

Again implementing the canonical transformation $p_j \rightarrow x^j$, $x^j \rightarrow -p_j$, we obtain

$$p_1^2 + p_2^2 + 2\lambda/r = 1 \quad r = (x_1^2 + x_2^2)^{1/2}, \quad (5.14)$$

i. e., the positive energy Kepler problem. It is clear that under the above canonical transformation the reduced symmetry algebra $\mathfrak{o}(2, 1)$ is preserved.

Again, we emphasize that Eq. (1.1) can also be interpreted as the equation for the graph of a solution $S(\eta, \phi)$ of (5.1), here parametrized by the spherical coordinates (5.5a). Thus $\mathfrak{o}(3, 2)$ is the full symmetry group of (5.1) and every separated solution of (1.1) gives rise via the graph to a symmetry adapted solution of (5.1).

6. THE NONRELATIVISTIC FREE PARTICLE

We now look at the only partial separation of (1.1) which involves nonorthogonal coordinates. Since this case was already treated in detail in Ref. 6, we will be brief here. Considering the reduced equation corresponding to the operator $\mathbf{P}_0 + \mathbf{P}_1$, we set $P_0 + P_1 = \lambda$; then (1.1) reduces to

$$\lambda S_t - S_y^2 = 0, \quad (6.1a)$$

where

$$t = (x^0 - x^1), \quad y = x^2 \quad (6.1b)$$

and

$$W = \lambda(x^0 + x^1) + S(t, y). \quad (6.1c)$$

Clearly (6.1) is equivalent to the equation studied in Ref. 6 (take $t \rightarrow -\lambda^{-1}t$ and $y \rightarrow x$). Its reduced symmetry algebra is the Schrödinger algebra s_1 generated by $\{P_0 - P_1, K_0 + K_1, D + M_{01}, P_2, M_{02} - M_{12}, P_0 + P_1\}$. Notice that in this case we no longer have a Lie algebra direct sum of the operator corresponding to the partial separation (here $\epsilon = P_0 + P_1$) and its centralizer. However, since ϵ is in the center of s_1 , we can consider the factor algebra s_1/ϵ . Because the partial separation in this case involves nonorthogonal coordinates, the R -separable coordinates of the reduced equation (6.1a) are non-orthogonal and are characterized by orbits in the factor

algebra s_1/ϵ . The list of representatives is:

- (3'') $P_0 + P_1, P_2$, free particle,
- (23) $P_0 + P_1, P_0 - P_1 - \frac{1}{4}(K_0 + K_1)$, attractive oscillator,
- (24) $P_0 + P_1, P_0 - P_1 \pm (M_{12} - M_{02})$, free fall (linear potential),
- (25) $P_0 + P_1, D + M_{01}$, repulsive oscillator.

In Ref. 6 orbits in s_1 were classified by equivalence under the full conformal group. It is easy to see from those results that the orbits in s_1/ϵ under the conformal group are precisely those listed above. Moreover, because all of the above are members of the Lie algebra, one can construct constants of the motion via functions on phase space as done here or equivalently construct relative invariants of vector fields as done in Ref. 6. One can easily check by using the Lie algebra isomorphism (1.7) that the two methods are indeed equivalent, keeping in mind that orbits of relative invariants in s_1 considered as vector fields correspond to orbits in s_1/ϵ considered as functions to be set equal to constants.

Again as shown in Refs. 2 and 6, $o(3,2)$ is the full symmetry algebra of (6.1a) corresponding to the fact that Eq. (1.1) can be interpreted as the equation for the graph of a solution of (6.1a). Thus all separated solutions of (1.1) give rise to symmetry adapted solutions of (6.1a). Indeed all those corresponding to first order operators have been given, up to equivalence, in Ref. 6.

7. A NONLINEAR EPD EQUATION

Now we look for coordinate systems yielding separation of variables in (1.1) such that $A = \Gamma_{32} = m$, m constant. Setting $x^0 = t, x = r \cos \phi, x^2 = r \sin \phi$, we have $\Gamma_{32} = -p_\phi$ so

$$W = -m\phi + S(t, r), \quad (7.1a)$$

where

$$S_t^2 - S_r^2 - m^2/r^2 = 0 \quad (7.1b)$$

or, from (1.29),

$$\Gamma_{45}^2 - \Gamma_{14}^2 - \Gamma_{15}^2 = \Gamma_{32}^2 = m^2. \quad (7.2)$$

Since the centralizer of Γ_{32} in $o(3,2)$ is $\{\Gamma_{32}\} \oplus o(2,1)$, where $o(2,1)$ is the subalgebra with basis $\Gamma_{45}, \Gamma_{14}, \Gamma_{15}$, we see that $o(2,1)$ is a symmetry algebra for the reduced equation (6.1). Here,

$$\begin{aligned} \Gamma_{45} &= \frac{1}{2}(1 + t^2 + r^2)p_t + trp_r, \\ \Gamma_{14} &= \frac{1}{2}(1 - t^2 - r^2)p_t - trp_r, \end{aligned} \quad (7.3)$$

$$\Gamma_{15} = -tp_t - rp_r, \quad p_t = S_t, \quad p_r = S_r.$$

It is well known²³ that the space of second order symmetry operators in $o(2,1)$, modulo the invariant $\Gamma_{45}^2 - \Gamma_{14}^2 - \Gamma_{15}^2$, splits into exactly nine orbit types under the adjoint action of $o(2,1)$. A representative of each orbit type is given by the assignment

- (1') $\Gamma_{32}^2, \Gamma_{45}^2$,
- (4') $\Gamma_{32}^2, (\Gamma_{45} + \Gamma_{14})^2$,
- (15') $\Gamma_{32}^2, \Gamma_{15}^2$,
- (26) $\Gamma_{32}^2, \Gamma_{14}^2 + \Gamma_{14}\Gamma_{45} - a\Gamma_{15}^2, \quad a > -\frac{1}{2}$,

$$(27) \Gamma_{32}^2, \Gamma_{45}^2 + \Gamma_{14}\Gamma_{45} + a\Gamma_{15}^2, \quad a > -\frac{1}{2},$$

$$(28) \Gamma_{32}^2, a\Gamma_{15}^2 + \Gamma_{14}\Gamma_{45},$$

$$(29) \Gamma_{32}^2, \Gamma_{15}^2 + a^2\Gamma_{45}^2, \quad 0 < a < 1,$$

$$(30) \Gamma_{32}^2, \Gamma_{15}^2 - a^2\Gamma_{45}^2, \quad 0 < a < 1,$$

$$(31) \Gamma_{32}^2, (\Gamma_{14} + \Gamma_{45})\Gamma_{15}.$$

We shall show explicitly that (1.1), hence (7.1), admits an additive separation of variables corresponding to each of these orbits. The separable coordinate systems are exactly those studied in Ref. 8.

Orbits (1'), (4'), and (15') have been treated above.

(26) We consider for simplicity the nondegenerate point $a = 0$ and

$$t = \cos \theta \cos \alpha, \quad r = \sin \theta \sin \alpha. \quad (7.4a)$$

Then (7.1) yields the separated solutions

$$\begin{aligned} S(\theta, \alpha) &= \int (2\mu^2 - m^2 \cot^2 \theta)^{1/2} d\theta \\ &\quad + \int (2\mu^2 - m^2 \cot^2 \alpha)^{1/2} d\alpha \end{aligned} \quad (7.4b)$$

with constant of the motion

$$\begin{aligned} \Gamma_{14}^2 + \Gamma_{14}\Gamma_{45} &= \frac{1}{2}(\sin^2 \alpha - \sin^2 \theta)^{-1}(\cos^2 \theta \sin^2 \alpha p_\alpha^2 - \sin^2 \theta \cos^2 \alpha p_\theta^2) \\ &= -\mu^2. \end{aligned} \quad (7.4c)$$

The coordinates θ, α are valid only for $|t| + r \leq 1$. As shown in Refs. 4 and 8 there are similar separable parametrizations for $|t| \geq r + 1$ and $|t| \leq r - 1$, but not all regions of the $r-t$ plane with $r > 0$ are covered with parametrizations which permit separation of variables.

(27) With $a = 0$ the separable coordinates are

$$t = \cosh \theta \sinh \alpha, \quad r = \sinh \theta \cosh \alpha \quad (7.5a)$$

and the solutions have a form similar to (7.4b).

Solutions corresponding to orbits (28)–(30) are rather similar. For (28) the separable coordinates are

$$\begin{aligned} t &= 2(kk')^{1/2}(k + ik') \operatorname{sn}(\theta, l) \operatorname{sn}(\alpha, l)/R, \\ r &= 2(kk')^{1/2}/R, \\ R &= (k - ik') \operatorname{dn}(\theta, l) \operatorname{dn}(\alpha, l) + (k + ik') \operatorname{cn}(\theta, l) \operatorname{cn}(\alpha, l), \\ 4a &= k'/k - k/k', \quad k' = (1 - k^2)^{1/2}, \\ l &= (k + ik')/(k - ik'), \end{aligned} \quad (7.6)$$

for (29) the coordinates are

$$\begin{aligned} t &= \operatorname{dn}(\theta, a) \operatorname{dn}(\alpha, a)/a'R, \quad r = 1/R, \\ R &= a \operatorname{sn}(\theta, a) \operatorname{sn}(\alpha, a) + a \operatorname{cn}(\theta, a) \operatorname{cn}(\alpha, a)/a', \\ a' &= (1 - a^2)^{1/2}, \end{aligned} \quad (7.7)$$

and for (30) they are

$$\begin{aligned} t &= k \operatorname{sn}(\theta, k) \operatorname{sn}(\alpha, k)/R, \quad r = 1/R, \\ R &= (k')^{-1} \operatorname{dn}(\theta, k) \operatorname{dn}(\alpha, k) + (k/k') \operatorname{cn}(\theta, k) \operatorname{cn}(\alpha, k), \\ a &= k', \quad k' = (1 - k^2)^{1/2}. \end{aligned} \quad (7.8)$$

(For a discussion of the ranges of the variables θ, α , see Ref. 8.)

As an example of the form of the solutions we insert coordinates (7.6) in (7.1) to obtain

$$p_\theta^2 - p_\alpha^2 = m^2 k^2 (\text{sn}^2 \alpha - \text{sn}^2 \theta) \quad (7.9a)$$

with separated solutions

$$S = \int (\mu^2 + m^2 \text{dn}^2 \theta)^{1/2} d\theta + \int (\mu^2 + m^2 k^2 \text{dn}^2 \alpha)^{1/2} d\alpha. \quad (7.9b)$$

Here, $\Gamma_{15}^2 - (k')^2 \Gamma_{45}^2 = \mu^2$. (Some errors in the corresponding list of elliptic coordinates for the EPD equation, contained in Ref. 8, have been corrected here.)

(31) For this orbit we set

$$t = \pm (\theta^2 + \alpha^2), \quad r = 2\theta\alpha, \quad t \geq |r|, \quad (7.10a)$$

in which case (7.1) becomes

$$p_\theta^2 - p_\alpha^2 - m^2(1/\alpha^2 - 1/\theta^2) = 0. \quad (7.10b)$$

The condition

$$-(\Gamma_{14} + \Gamma_{45})\Gamma_{15} = \frac{1}{4}(\theta^2 - \alpha^2)^{-1}(\theta^2 p_\theta^2 - \alpha^2 p_\alpha^2) = \mu^2 \quad (7.10c)$$

yields the separated solution

$$S(\theta, \alpha) = \int (4\mu^2 - m^2/\theta^2)^{1/2} d\theta + \int (4\mu^2 - m^2/\alpha^2)^{1/2} d\alpha. \quad (7.10d)$$

(Contrary to the statement in Ref. 8, variables do not separate for $r \geq |t|$.)

In analogy with the previous reduced equations it is easy to show that (1.1) is the equation of the graph of (7.1). Thus, $\mathfrak{o}(3, 2)$ is the full symmetry algebra of (7.1).

8. THE SYMMETRY $\Gamma_{23} - \Gamma_{45}$

We next separate a variable from (1.1) by requiring $L = \frac{1}{2}(\Gamma_{23} - \Gamma_{45}) = K$. In terms of the coordinates (2.2), (2.4) with $\beta = \alpha + \psi$, $\phi = \alpha - \psi$ we have $L = p_\beta$, so

$$W = K\beta + S(\phi, \sigma),$$

where

$$\cot^2 \sigma S_\phi^2 + 2K(\csc^2 \sigma + 1)S_\phi + S_\sigma^2 + K^2 \cot^2 \sigma = 0. \quad (8.1)$$

The centralizer of L in $\mathfrak{o}(3, 2)$ is $\{L\} \oplus \mathfrak{o}(2, 1)$, where $\mathfrak{o}(2, 1)$ is the subalgebra with basis A, B, C such that

$$A = \frac{1}{2}(\Gamma_{23} + \Gamma_{45}), \quad B = \frac{1}{2}(\Gamma_{24} + \Gamma_{35}), \quad C = \frac{1}{2}(\Gamma_{25} - \Gamma_{34}), \quad (8.2)$$

$$[A, B] = C, \quad [C, A] = B, \quad [C, B] = A.$$

Thus $\mathfrak{o}(2, 1)$ is a symmetry algebra for the reduced equation (8.1). Here

$$\begin{aligned} A &= S_\phi, \quad -B = \sin \phi S_\sigma + \coth z \cos \phi S_\psi + K \cos \phi / \sinh z \\ C &= -\cos \phi S_\sigma + \coth z \sin \phi S_\psi + K \sin \phi / \sinh z, \quad (8.3) \\ \sin \sigma &= \tanh(z/2), \end{aligned}$$

and in terms of these symmetries equation (8.1) reads

$$A^2 - B^2 - C^2 = 0. \quad (8.4)$$

Note: The simple computation leading to this identity shows that the corresponding identity for the wave equation as given in Refs. 3 and 5 is in error. The correct

result for the wave equation is

$$A^2 - B^2 - C^2 = \frac{3}{16}. \quad (8.5)$$

Indeed, the eigenvalues of $-iL$ are $\lambda_s = \frac{1}{2}(s + \frac{1}{2})$, $s = 0, 1, 2, \dots$. The eigenspace V_s corresponding to eigenvalue λ_s is irreducible under $\mathfrak{o}(2, 1)$ and transforms according to the unitary representation D_l^+ where $l = -[(-1)^s + 2]/4$.

As usual we try to associate separable coordinates for (8.4) with the nine orbits of second order symmetries in the enveloping algebra for $\mathfrak{o}(2, 1)$. It is guaranteed that there are separable coordinates corresponding to the three orbits which correspond to squares of first order symmetries:

$$(1') L^2, A^2,$$

$$(32) L^2, C^2,$$

$$(33) L^2, (A - B)^2.$$

In particular, (1') is equivalent to (1). To obtain the remaining systems, we note that for $K = 0$ the operators and coordinates (8.3) agree with system (15) on the hyperboloid, i. e., coordinates (5.5a). Since the separable coordinates for (8.4) must be independent of K it follows that a separable system for (8.4) must be one of the systems (7'), (15)–(22). However, one of the latter systems need not necessarily yield separation for (8.4).

We are guaranteed success for systems (32) and (33). For (32) we set $\cosh z = \cosh \xi \cosh \eta$, $\tan \phi = \tanh \xi / \sinh \eta$ to obtain

$$\begin{aligned} &\frac{1}{\cosh^2 \xi} S_\eta^2 + S_\xi^2 + \frac{1}{(1 - \cosh^2 \xi \cosh^2 \eta)} \\ &\times (2K \cosh \xi \sinh \eta \cosh \eta S_\xi - 2K \sinh \xi \cosh^2 \eta S_\eta + K^2) = 0. \end{aligned} \quad (8.6a)$$

The condition

$$C = -S_\eta + K \sinh \xi / (\cosh^2 \xi \cosh^2 \eta - 1) = \mu \quad (8.6b)$$

yields the R -separated solution

$$\begin{aligned} S &= -K \tan^{-1}(\sinh \xi \coth \eta) - m\eta \\ &+ \int [(K^2 - \mu^2 - 2\mu K \sinh \xi)^{1/2} / \cosh \xi] d\xi. \end{aligned} \quad (8.6c)$$

For (33) we set $\cosh z = \frac{1}{2}[e^{-\xi} + (\eta^2 + 1)e^\xi]$, $\tan \phi = -2\eta e^\xi / [e^{-\xi} + (\eta^2 - 1)e^\xi]$ and use the condition

$$\begin{aligned} A - B &= S_\eta - 2K[e^{-\xi} + (\eta^2 - 1)e^\xi] / \{[e^{-\xi} + (\eta^2 + 1)e^\xi]^2 - 4\} \\ &= \mu \end{aligned} \quad (8.7a)$$

to obtain the R -separated solution

$$\begin{aligned} S &= -K \tan^{-1} \left[\frac{2\eta}{e^\xi(1 + \eta^2) - e^{-\xi}} \right] + m\eta \\ &+ \int [-2Ke^{-\xi} m + m^2 e^{-2\xi}]^{1/2} d\xi. \end{aligned} \quad (8.7b)$$

We have carefully studied the coordinates corresponding to system (17) and have found that they do not lead to R separation of variables for (8.4). It appears that only the subgroups systems (15'), (32) and (33) yield additive variable separation for this equation, although we have not explicitly checked this for all systems (15)–(22).

Just as for the other reduced equations it is easy to show that (1.1) is the equation of the graph of (8.4). Thus $o(3,2)$ is the full symmetry algebra of (8.4) and all of the additively separated solutions of (1.1) lead to symmetry adapted solutions of (8.4).

9. NONSPLIT COORDINATES

In analogy with Ref. 3 for the wave equation, we have listed, with the exception of some degenerate non-orthogonal systems,⁵ all separable systems for (1.1) in which it is possible to additively split off one variable. In Ref. 4 a classification of all orthogonal R -separable coordinates systems for the wave equation was given for which the coordinate surfaces were families of confocal cyclides. 53 such systems were found, and, except for degenerate cases, it was shown that the variables intertwine in such a complicated fashion that it is necessary to separate them simultaneously, i. e., it is not possible to split off a single variable. Each such system was shown to be characterized by a commuting pair of second order symmetric operators in the enveloping algebra of $o(3,2)$.

The results of Ref. 4 can be applied directly to obtain orthogonal separable coordinates for (1.1) simply by interpreting the Lie algebra of differential operators $o(3,2)$ as a Lie algebra of functions under the Poisson bracket.

For example, the system [311] (i) of Ref. 4 leads to coordinates

$$x^0 = -\frac{1}{2}(\cos^2\alpha + \cos^2\beta + \cos^2\gamma), \quad x^1 = \sin\alpha \sin\beta \sin\gamma, \\ x^2 = \cos\alpha \cos\beta \cos\gamma. \quad (9.1)$$

In these coordinates (1.1) reduces to

$$(\sin^2\beta - \sin^2\gamma)p_\alpha^2 + (\sin^2\gamma - \sin^2\alpha)p_\beta^2 + (\sin^2\alpha - \sin^2\beta)p_\gamma^2 = 0. \quad (9.2)$$

It is not possible to additively separate one of these variables from the other two. However, use of the defining symmetry elements

$$2(P_0 - P_1)M_{02} + P_0^2 + P_1^2 \\ = \frac{\sin^2\alpha p_\alpha^2}{(\sin^2\alpha - \sin^2\beta)(\sin^2\alpha - \sin^2\gamma)} \\ + \frac{\sin^2\beta p_\beta^2}{(\sin^2\beta - \sin^2\alpha)(\sin^2\beta - \sin^2\gamma)} \\ + \frac{\sin^2\gamma p_\gamma^2}{(\sin^2\gamma - \sin^2\alpha)(\sin^2\gamma - \sin^2\beta)} = \mu \quad (9.3a)$$

$$2P_2M_{02} - M_{12}^2 + P_2^2 \\ = \frac{\sin^2\beta \sin^2\alpha p_\alpha^2}{(\sin^2\alpha - \sin^2\beta)(\sin^2\alpha - \sin^2\gamma)} \\ + \frac{\sin^2\alpha \sin^2\gamma p_\beta^2}{(\sin^2\beta - \sin^2\alpha)(\sin^2\beta - \sin^2\gamma)} \\ + \frac{\sin^2\alpha \sin^2\beta p_\gamma^2}{(\sin^2\gamma - \sin^2\alpha)(\sin^2\gamma - \sin^2\beta)} = \nu \quad (9.3b)$$

leads to the separated solution

$$W = \int (\mu \sin^2\alpha + \nu)^{1/2} d\alpha + \int (\mu \sin^2\beta + \nu)^{1/2} \\ d\beta + \int (\mu \sin^2\gamma + \nu)^{1/2} d\gamma. \quad (9.4)$$

In a similar fashion each of the orthogonal R -separable coordinate systems for the wave equation is additively separable for (1.1). In Paper II we shall examine the relationship between the wave equation and (1.1) more closely and provide proofs of our own assertions concerning variable separation.

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