Lie theory and separation of variables. 7. The harmonic oscillator in elliptic coordinates and Ince polynomials

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As a continuation of Paper 6 we study the separable basis eigenfunctions and their relationships for the harmonic oscillator Hamiltonian in two space variables with special emphasis on products of Ince polynomials, the eigenfunctions obtained when one separates variables in elliptic coordinates. The overlaps connecting this basis to the polar and Cartesian coordinate bases are obtained by computing in a simpler Bargmann Hilbert space model of the problem. We also show that Ince polynomials are intimately connected with the representation theory of $SU(2)$, the group responsible for the eigenvalue degeneracy of the oscillator Hamiltonian.

INTRODUCTION

In Ref. 1 (hereafter referred to as 6) the authors gave a detailed investigation of the nine-parameter symmetry group G (the Schrödinger group) of the equation

$$iU_t + \Delta_2 U = 0. \qquad (*)$$

It was found that (*) separates in 26 coordinate systems and that with each coordinate system is associated an orbit under the action of the Galilean subgroup $G(2) \subset G$ consisting of a pair of commuting operators (K, S), where $K \in \mathcal{G}$ the Lie algebra of G and S is a second-order element in the universal enveloping algebra of $\mathcal{G}$. It was further shown that in all except five cases (which are subgroup coordinates) the first-order symmetry operator K corresponds to an orbit which can be associated with one of four types of potentials: the free particle, the attractive and repulsive harmonic oscillator, and the linear potential.

The Schrödinger equation for the attractive harmonic oscillator in two space variables separates in exactly three orthogonal coordinate systems: Cartesian, polar, and elliptic. The corresponding eigenfunctions in the three systems are a product of two Hermite polynomials, a Laguerre polynomial times an exponential function, and a product of two Ince polynomials, respectively. In this paper we examine these bases and compute the overlap functions relating different bases, with special emphasis on the Ince polynomial case. Due to the equivalence of the free particle Schrödinger equation (*) and the (time dependent) harmonic oscillator equation we have chosen to present our eigenfunctions as solutions of (*). However, all our results translate immediately to the harmonic oscillator problem.

It can be seen from Table II of 6 that to each type of potential and corresponding symmetry S *except the attractive harmonic oscillator* there correspond two coordinate systems equivalent under G though not under $G(2)$. In one of these equivalent coordinate systems labeled by superscript (1), the eigenfunctions and corresponding calculations are quite simple, while the other system affords the close connection with one of the physical potentials mentioned above. It was the existence of the "simple" systems which made the computations in 6 so easy. Now it is a remarkable fact that for the attractive harmonic oscillator, the analog of the coordinate systems of type (1) is the realization of the harmonic oscillator given several years ago by Bargmann.[2] Note that although the Bargmann transform is not a member of G, it is a member of the complexification G^c of G.[3] It is the purpose of this work to explore fully this analogy, especially in the case of elliptic coordinates where almost all of the developments presented are new.

It is well known that the eigenvalues of the harmonic oscillator Hamiltonian are degenerate and that the group responsible for the degeneracy is $SU(2)$. In Sec. 3 we discuss the relationship between this group and the elliptic basis, developing the connection between Ince polynomials and the representation theory of $SU(2)$ in analogy to the connection between Lamé polynomials and $SU(2)$ as discussed in Ref. 4.

1. PRELIMINARIES

First we give explicitly the Lie algebra $\mathcal{G}$ of the symmetry group G, as well as the spectral resolutions of the pairs corresponding to the oscillator coordinates mentioned above. For further details the reader is referred to 6. The real Lie algebra $\mathcal{G}$ is spanned by the differential operators

$$K_2 = -t^2\partial_t - t(x_1\partial_{x_1} + x_2\partial_{x_2}) - t + \tfrac{1}{4}i(x_1^2 + x_2^2), \quad K_{-2} = \partial_t,$$
$$P_i = \partial_{xi}, B_i = -t\partial_{xi} + ix_i/2, \quad i = 1, 2, \quad M = x_1\partial_{x_2} - x_2\partial_{x_1},$$
$$D = x_1\partial_{x_1} + x_2\partial_{x_2} + 2t\partial_t + 1, \quad E = i. \qquad (1.1)$$

The coordinate systems related to the attractive harmonic oscillator are written as Oc, Or, and Oe [corresponding to cartesian, radial (polar), and elliptic coordinates respectively], and are presented in Table I of 6. The associated pairs of operators are $(K_{-2} - K_2, P_1^2 + B_1^2)$, $(K_{-2} - K_2, M^2)$, and $(K_{-2} - K_2, M^2 - P_2^2 - B_2^2)$ respectively, as listed in Table II of 6.

The spectral resolutions of these pairs as given in 6 with $L_3 = K_{-2} - K_2$ are

Oc: $(iL_3, P_1^2+B_1^2)$ with eigenvalues (λ,μ) and basis functions

$$\mathrm{Oc}_{n_1n_2}(\mathbf{x},t) = (2^{n_1+n_2}\pi n_1!\, n_2!)^{-1/2}\exp[i\pi(n_1+n_2+1)]^{1/2} \exp[-\tfrac{1}{4}(x_1^2+x_2^2)(1-it)]\left(\frac{t+i}{t-i}\right)^{(n_1+n_2)/2} \times (t-i)^{-1}H_{n_1}(x_1/[2(1+t^2)]^{1/2})H_{n_2}(x_2H_n[2/(1+t^2)]^{1/2}), \tag{1.2}$$

where $\lambda=n_1+n_2+1$, $\mu=-n_1-\frac{1}{2}$, and $H_n(x)$ are Hermite polynomials.

Or: (iL_3, M^2) with eigenvalues (λ,μ) and basis functions

$$\mathrm{Or}^+_{n,m}(\mathbf{x},t)=K\left(\frac{m!}{\pi^3 2^m(n+m)!}\right)^{1/2}\frac{(-1)^{m+n}}{2^{2m}}\frac{(t+i)^{n+m/2}}{(t-i)^{n+m/2+1}} \times\exp\left(\frac{r^2(it-1)}{4(1+t^2)}\right)L_n^m\left(\frac{r^2}{2(1+t^2)}\right)\cos m\theta \tag{1.3}$$

$$\mathrm{Or}^-_{n,m}(\mathbf{x},t)=\tan m\theta\,\mathrm{Or}^+_{n,m}(\mathbf{x},t) \quad m=1,2,\cdots,$$

where $L_n^m(r)$ are Laguerre polynomials, $K=\sqrt{2}$ for $m=0$ and 1 otherwise, $x=r\cos\theta$, $y=r\sin\theta$, $\lambda=2n+m+1$, and $\mu=-m^2$.

Oe: $(iL_3,\ M^2-P_2^2-B_2^2)$ with eigenvalues (λ,μ) and basis functions

$$\mathrm{Oe}^+(\mathbf{x},t)=\frac{\lambda_p^{m+}}{\pi}\exp[(i/4)t(\sinh^2 v_1+\cos^2 v_2)] \times(t-i)^{p/2+1}(t+1)^{-p/2}\,\mathrm{hc}_p^m(iv_1,\tfrac{1}{2})\,\mathrm{hc}_p^m(v_2,\tfrac{1}{2}), \tag{1.4}$$

where $x_1=(1+t^2)^{1/2}\cosh v_1\cos v_2$, $x_2=(1+x^2)^{1/2}\times\sinh v_1\sin v_2$, $\lambda=p+1$, $\mu^+=\frac{1}{2}\lambda+a_p^m(\frac{1}{2})$, and the functions $\mathrm{hc}_p^m(v,\xi)$ are periodic solutions of the Whittaker—Hill[5] equation and are related to the even-parity Ince polynomials through

$$\mathrm{hc}_p^m(v,\xi)=\exp[-\xi\cos(2v)/4]C_p^m(v,\xi). \tag{1.5}$$

The numbers $a_p^m(\xi)$ denote the characteristic values for the even Ince polynomials. The functions $\mathrm{Oe}^-(\mathbf{x},t)$ are obtained by replacing the even Ince polynomials by odd-parity Ince polynomials with corresponding characteristic values denoted by $b_p^m(\xi)$. These functions are thus denoted

$$\mathrm{hs}_p^m(v,\xi)=\exp[-\xi\cos(2v)/4]C_p^m(v,\xi). \tag{1.6}$$

We mention here that for our purposes it is *not* convenient to normalize the Ince polynomials as done by Arscott.[5] A full discussion of our normalization is given in Sec. 3, where it is also seen that much information about Ince polynomials follows from the representation theory of $SU(2)$. Once our normalization is fixed the constants $\lambda_p^{m\pm}$ can be determined.

To conclude this section, we give the unitary mapping which describes the time evolution of the solutions of (*), viz.,

$$\exp(t\mathcal{K}_{-2})f(\mathbf{x})=\mathrm{l.i.m.}\frac{1}{4it}\int_{-\infty}^{\infty}\int dy_1dy_2 \times\exp[-(\mathbf{x}-\mathbf{y})^2/4it]f(\mathbf{y}), \tag{1.7}$$

where $\mathcal{K}_{-2}=i(\partial_{x_1x_1}+\partial_{x_2x_2})$. We note that $\exp(-t\mathcal{K}_{-2})$ applied to any of the basis functions (1.2), (1.3), or (1.4) will give the same functions evaluated at $t=0$. In the next section we compose (1.7) with Bargmann's transform to relate the basis states in Bargmann Hilbert space to the solutions (1.2), (1.3), and (1.4) of the free particle Schrödinger equation (*).

2. BARGMANN'S REALIZATION

Bargmann's transformation[2] (we consider only the case of two spatial dimensions) is a unitary mapping of $L_2(R_2)$ onto the Hilbert space $\mathcal{J}(\mathbf{z})$ of functions f of two complex variables $\mathbf{z}=(z_1,z_2)$ completed with the norm $\|f\|$ induced by the inner product

$$(g,f)=\int_{R_4}d\mu(\mathbf{z})\bar{g}(\mathbf{z})f(\mathbf{z}) \tag{2.1}$$

with $d\mu(\mathbf{z})=\pi^{-2}\exp(-\bar{\mathbf{z}}\cdot\mathbf{z})d^2\mathrm{Re}z\,d^2\mathrm{Im}z$, and $\|f\|=(f,f)^{1/2}$. The mapping is given by

$$f(\mathbf{z})=(\mathbf{A}\psi)(\mathbf{z})\equiv\mathrm{l.i.m.}\int_{R_2}d^2x A(\mathbf{z},\mathbf{x})\psi(\mathbf{x}), \tag{2.2}$$

where $\psi(\mathbf{x})\in L_2(R_2)$ and

$$A(\mathbf{z},\mathbf{x})=\pi^{-1/2}\exp\{-\tfrac{1}{2}(\mathbf{z}^2+\mathbf{x}^2)+\sqrt{2}\mathbf{z}\cdot\mathbf{x}\}. \tag{2.3}$$

The inverse mapping $\mathbf{A}^{-1}$ is given by

$$\psi(\mathbf{x})=(\mathbf{A}^{-1}f)(\mathbf{x})=\mathrm{l.i.m.}\int_{R_4}d\mu(\mathbf{z})\bar{A}(\mathbf{z},\mathbf{x})g(\mathbf{z}) \tag{2.4}$$

for any $a\in\mathcal{J}_2$.

The composition of the two unitary maps $\exp(t\mathcal{K}_{-2})$ and $\mathbf{A}^{-1}$ will then map entire functions $f\in\mathcal{J}_2$ onto $L_2(R_2)$ functions which are solutions of (*). This mapping is given by

$$[\exp(t\mathcal{K}_{-2})\mathbf{A}^{-1}g](\mathbf{x},t)=\int_{R_4}d\mu(\mathbf{z})\overline{K_t(\mathbf{z},\mathbf{x})}g(\mathbf{z}), \tag{2.5}$$

where

$$\overline{K_t(\mathbf{z},\mathbf{x})}=\frac{1}{(1+2it)\sqrt{\pi}}\exp\left(\frac{-\frac{1}{2}(1-2it)\bar{\mathbf{z}}^2-\frac{1}{2}\mathbf{x}^2+\sqrt{2}\bar{\mathbf{z}}\cdot\mathbf{x}}{1+2it}\right). \tag{2.6}$$

Notice that when $t\to 0$, we recover Bargmann's mapping (2.4) as we must. The inverse map $\mathbf{A}\exp(-t\mathcal{K}_{-2})$ with the kernel $K_t(\mathbf{z},\mathbf{x})$ is then obtained by complex conjugation of (2.6), viz.,

$$[A\exp(-t\mathcal{K}_{-2})\psi](\mathbf{z})=\int_{R_2}d^2x\,K_t(\mathbf{z},\mathbf{x})\psi(\mathbf{x},t). \tag{2.7}$$

Thus we have established the one-to-one correspondence between Bargmann's Hilbert space of entire functions $\mathcal{J}_2$ with the $L_2(R_2)$ solution[6] of the free particle Schrödinger equation (*). One can also use (2.7) to construct the Lie algebra $\mathcal{G}$ in the Bargmann realization; however, it is easier to evaluate the generators (1.1) at $t=0$ and make the replacement $\partial_t\to-\mathcal{K}_{-2}$ as done in 6 and then pass to Bargmann's realization by replacing the annilihation operator $\frac{1}{2}x_i-\partial_{x_i}$ by its analytic representation z_i. In this way the generators of $\mathcal{G}$ take the form

$$\begin{aligned}\mathcal{L}_3&=-i(z_1\partial_{z_1}+z_2\partial_{z_2}+1),\\ \mathcal{L}_2&=\tfrac{1}{2}(\partial_{z_1z_1}+\partial_{z_2z_2}+z_1^2+z_2^2),\\ \mathcal{D}&=\tfrac{1}{2}(\partial_{z_1z_1}+\partial_{z_2z_2}-z_1^2-z_2^2),\quad \mathcal{B}_i=\tfrac{1}{2}i(z_i+\partial_{z_i}),\\ \mathcal{P}_i&=-\tfrac{1}{2}(z_i-\partial_{z_i}),\quad \mathcal{M}=(z_1\partial_{z_2}-z_2\partial_{z_1}),\quad \mathcal{E}=i,\end{aligned} \tag{2.8}$$

where the script letters correspond to the block letters in (1.1) and we have used, instead of $\mathcal{K}_{-2}$ and $\mathcal{K}_2$, the combinations $\mathcal{L}_3=\mathcal{K}_{-2}-\mathcal{K}_2$ and $\mathcal{L}_2=\mathcal{K}_{-2}+\mathcal{K}_2$, which take a simpler form in the (z_1,z_2) formalism. Indeed the harmonic oscillator Hamiltonian $i\mathcal{L}_3$ now appears as a dilatation operator making its spectral resolution in $\mathcal{J}_2$ very simple. As well we can give the integrated group action of (2.8) as done in 6. However, as we use only (2.5) and (2.7), we omit this.

Now the second order operator $\mathcal{P}_1^2+\mathcal{B}_1^2$ for the oc system takes the form

$$\mathcal{P}_1^2+\mathcal{B}_1^2=-(z_1\partial_{z_1}+1)$$

and, hence, the normalized eigenfunctions of the pair $(i\mathcal{L}_3, \mathcal{P}_1^2+\mathcal{B}_1^2)$ with eigenvalues (λ,μ) yield Bargmann's well-known result

$$gc_{\lambda,\mu}(\mathbf{z})=z_1^{n_1}z_2^{n_2}/\sqrt{n_1!n_2!}\,, \tag{2.9}$$

where $\mu=-n_1-\frac{1}{2}$, $\lambda=n_1+n_2+1$. These functions form an ON basis in $\mathcal{J}_2$ and map onto the Oc functions (1.2) via the unitary map (2.5).

In order to treat the systems or and oe it is expedient to introduce the complex polar coordinates[7]

$$\begin{aligned} z_1&=\rho\cos\zeta, \quad 0\leqslant \mathrm{Re}\rho<\infty, \ -\infty<\mathrm{Im}\rho<\infty \\ z_2&=\rho\sin\zeta, \quad -\pi<\mathrm{Re}\zeta<\pi, \ -\infty<\mathrm{Im}\zeta<\infty. \end{aligned} \tag{2.10}$$

In these coordinates the operators $i\mathcal{L}_3$ and $\mathcal{M}$ take the simple form

$$i\mathcal{L}_3=\rho\partial\rho+1, \quad \mathcal{M}=\partial_\zeta,$$

and hence the spectral resolution of the pair $(i\mathcal{L}_3/\mathcal{M}^2)$ with eigenvalues (λ,μ) yields the eigen functions

$$\mathrm{gr}^+_{n,m}(\mathbf{z})=K[2^{2n+m-1}n!\,\Gamma(n+m+1)]^{-1/2}\rho^{2n+m}\cos m\zeta, \tag{2.11a}$$

$$\mathrm{gr}^-_{n,m}(\mathbf{z})=\tan m\zeta\,\mathrm{gr}^+_{n,m}(\mathbf{z}), \tag{2.11b}$$

where K, n, m are as in (1.3). These basis functions form an ON basis in $\mathcal{J}_2$ which map onto the Or functions (1.3) by (2.5).

For the elliptic system oe we consider the spectral resolution of the pair $(i\mathcal{L}_3, \mathcal{M}^2-\mathcal{P}_2^2-\mathcal{B}_2^2)$ with eigenvalues (λ,μ). It is easy to see that the second of these operators gives the differential equation for Ince functions in the complex variable ζ, which we discuss in more detail in the next section. Suffice it now to write down the eigenfunctions (S_p^m is an odd-parity Ince polynomial)

$$\mathrm{ge}^+_{p,m}(\mathbf{z})=2^{-p/2}\rho^p\,\mathrm{C}_p^m(\zeta), \tag{2.12a}$$

$$\mathrm{ge}^-_{p,m}(\mathbf{z})=2^{-p/2}\rho^p\,\mathrm{S}_p^m(\zeta), \tag{2.12b}$$

where the notation follows from (1.4) and (1.5). The functions (2.12) form an ON basis in $\mathcal{J}_2$ which map onto the functions (1.4) through the unitary map (2.5).

3. INCE POLYNOMIALS AND $SU(2)$

As is well known[7] the degeneracy group for the harmonic oscillator in two spatial dimensions is $SU(2)$. Although $SU(2)$ is not a subgroup of G, a representation of its Lie algebra appears as a subalgebra of the 20-dimensional vector space of second-order elements in the enveloping algebra of $\mathcal{G}$. Rather than give immediately the representations of the Lie algebra $SU(2)$ in terms of these operators, we prefer to develop the abstract formalism along the lines presented by Patera and Winternitz[4] for Lamé polynomials, establishing the connection with the preceding section at the end.

A. The algebraic approach

Denote by $\mathcal{U}$ the universal enveloping algebra of the Lie algebra $\mathcal{L}$ [here $\mathcal{L}=SU(2)$], $\mathcal{C}$ the center of $\mathcal{U}$, $\mathcal{U}_2$ the symmetric second-order elements of $\mathcal{U}$, and define $\mathcal{U}^{(2)}=\mathcal{L}+\mathcal{U}_2$. Let $J_i, i=1,2,3$, be the standard basis for $SU(2)$. Then a general element of $\mathcal{U}^{(2)}$ can be written as $a_{ij}(J_iJ_j+J_jJ_i)+a_iJ_i$, $a_{ij},a_i\in R$. Note that for $SU(2)$, $\mathcal{C}\subset\mathcal{U}_2$. It suffices to consider only elements of the factor algebra $\mathcal{U}^{(2)}/\mathcal{C}$. Now an arbitrary element of $\mathcal{U}^{(2)}/\mathcal{C}$ can be brought to the form $J_3^2+rJ_1^2+a_iJ_i$ through an inner automorphism of $SU(2)$. The symmetric second order elements $\mathcal{U}_2/\mathcal{C}$ have been studied by Patera and Winternitz,[4] and they have shown the one-to-one correspondence between the two $SU(2)$ orbits and separation of variables on the sphere S^2. In any case a general element of $\mathcal{U}^{(2)}/\mathcal{C}$ describes an eigenvalue problem with four free parameters giving rise to special functions which have as limiting cases both Lamé polynomials and polynomials arising from the element $J_3^2+aJ_2$, which we will show to be Ince polynomials.

The Lie algebra $SU(2)$ with the basis of Hermitian generators J_i takes the form

$$[J_i,J_j]=i\epsilon_{ijk}J_k. \tag{3.1}$$

The canonical basis for the representation space is defined by

$$\begin{aligned} J_\pm\psi_{jr}&=[(j\mp r)(j\pm r+1)]^{1/2}\psi_{jr\pm1} \\ J_3\psi_{jr}&=r\psi_{jr} \end{aligned} \tag{3.2}$$

with $J_\pm=J_1\pm iJ_2$, where we employ Vilenkin's[8] phase convention

$$\exp(i\pi J_1)\psi_{jr}=\exp(i\pi j)\psi_{j-r}.$$

We are interested in the eigenvalue problem defined by the operator

$$E=J_3^2+aJ_2, \tag{3.3}$$

with eigenvalue taken for later convenience to be $\frac{1}{4}\eta$, viz.,

$$E\psi_{j\eta}=\tfrac{1}{4}\eta\psi_{j\eta}. \tag{3.4}$$

First we consider some symmetries of E in the group of automorphisms of $SU(2)$. Now any such symmetry must map $J_2\to J_2$ and $J_3\to\pm J_3$. It is not difficult to see that any transformation R of this type necessarily takes one of two possible forms:

(i) $R^+=\alpha I$, $\alpha\in\mathbb{C}$, $I=$identity in $SU(2)$,

(ii) $R^-=\beta\exp(-i\pi J_2)$, $\beta\in\mathbb{C}$.

From the existence of R^- and Schur's lemma it is clear that the functions $\psi_{j\eta}$ do not completely specify a basis for an irreducible representation of $SU(2)$. We can define a complete basis by further specifying the eigenvalues of R^-. Furthermore, since $(R^-)^2$ is a multiple of the identity, we can take these eigenvalues to be ±1 which then determines β to be $\exp(i\pi j)$. We hereafter drop the minus supercript on R and write

$$R\psi^\pm_{j\eta}=\pm\psi^\pm_{j\eta}, \tag{3.5}$$

where $R=\exp(i\pi j)\exp(-i\pi J_2)$. The hermiticity of E and R then guarantees the orthogonality conditions

$$(\psi_{j\eta'}^{\epsilon'}, \psi_{j\eta}^{\epsilon})=\delta_{\eta\eta'}\delta_{\epsilon\epsilon'}, \tag{3.6}$$

where we have properly normalized $\psi_{j\eta}^{\epsilon}$.

The determination of $\psi_{j\eta}^{\pm}$ is then tantamount to the determination of the overlap functions $(\psi_{jr}, \psi_{j\eta}^{\pm})$. From (3.5) we find

$$(\psi_{jr}, \psi_{j\eta}^{\pm})=\pm\exp(-i\pi r)(\psi_{j-r}, \psi_{j\eta}^{\pm}) \tag{3.7}$$

and from (3.2), (3.3), and (3.4) we obtain the three-term recursion formula

$$-(a/2i)[(j-r)(j+r+1)]^{1/2}(\psi_{jr+1}, \psi_{j\eta}^{\pm}) + (a/2i)[(j+r)(j-r+1)]^{1/2}(\psi_{jr-1}, \psi_{j\eta}^{\pm})=(r^2-\tfrac{1}{4}\eta)(\psi_{jr}, \psi_{j\eta}^{\pm}). \tag{3.8}$$

It is now convenient to introduce new coefficients $A_r^{\pm}$ as

$$A_r^{\pm}=\frac{\exp[i\pi(j-r)/2](\psi_{jr}, \psi_{j\eta}^{\pm})}{\sqrt{(j-r)!(j+r)!}}, \quad 0<m\le j,$$
$$A_0^{\pm}=\frac{\exp(i\pi j/2)}{2(j!)}(\psi_{j0}, \psi_{j\eta}^{\pm}), \tag{3.9}$$

while from (3.7) $A_r^{\pm}$ can be defined for negative r as

$$A_{-r}^{\pm}=\pm A_r^{\pm}.$$

We see immediately that $A_0^-=0$. Upon substituting into (3.8) our recursion formula takes precisely the form given by Arscott[5] for Ince polynomials with j integer, viz.,

$$\xi(j+r+2)A_{r+2}^{\pm}+(4r^2+4-\eta)A_{r+1}^{\pm}+\xi(j-r)A_r^{\pm}=0, \quad r>0, \tag{3.10a}$$

$$\xi(j+2)A_2^{\pm}+(4-\eta)A_1^{\pm}+2\xi jA_0^{\pm}=0, \tag{3.10b}$$

$$\xi(j+1)A_1^{\pm}-\eta A_0^{\pm}=0, \tag{3.10c}$$

$$(4j^2-\eta)A_j^{\pm}+\xi A_{j-1}^{\pm}=0, \tag{3.10d}$$

where $\xi=-2a$ and we have identified A_r^+ and A_r^- with Arscott's trigonometric coefficients A_r and B_r, respectively, up to normalization. Notice also that our r takes on both integer and half-integer values. Now for j half-integer we merely delete Eqs. (3.10b, c). Moreover, Arscott's parameter p is identified with our $2j$ ($p=2j$). Thus even p corresponds to integer IR's (irreducible representations) of $SU(2)$ and odd p to half-integer IR's.

Following Arscott, we denote the characteristic values η by $a_j^m(\xi)$ and $b_j^m(\xi)$ for $\psi_{j\eta}^{\pm}$ respectively. Now the dimension of an IR is $(2j+1)$, and from (3.10) we conclude that for integer j there are $j+1$ even parity characteristic values $a_j^m(\xi)$ and j odd parity characteristic values $b_j^m(\xi)$, whereas for half-integer j there are $(j+\frac{1}{2})$ of each type.

From the structure of the operator E in (3.3), there is a further interesting symmetry property noticed by Arscott. Putting $a=-2\xi$ and writing the ξ dependence explicitly, i.e., $E(\xi)=J_3^2-2\xi J_2$, we notice that

$$\exp(i\pi\,\mathrm{Ad}J_1)E(\xi)=E(-\xi) \tag{3.11}$$

and a similar relation is obtained by replacing J_1 by J_3. It follows from (3.11) that if $a_j^m(\xi)$ or $b_j^m(\xi)$ are characteristic values for $E(\xi)$, then $a_j^m(-\xi)$ and $b_j^m(-\xi)$ are also characteristic values for $E(\xi)$. Furthermore, a short computation demonstrates that

$$\exp(i\pi\,\mathrm{Ad}J_1)R=R \quad \text{for integer } j,$$
$$\exp(i\pi\,\mathrm{Ad}J_1)R=-R \quad \text{for half-integer } j. \tag{3.12}$$

Hence, it follows that for half-integer j the set $\{b_j^m(\xi)\}$ is given by the set $\{a_j^m(-\xi)\}$, whereas for j integer $a_j^m(-\xi)\in\{a_j^m(\xi)\}$ and $b_j^m(-\xi)\in\{b_j^m(\xi)\}$.

The expansion of the $\psi_{j\eta}^{\pm}$ basis in terms of the canonical basis is readily obtained:

$$\psi_{j\eta}^{\pm}=\sum_r\sqrt{(j-r)!(j+r)!}\exp[-i\pi(j-r)/2] \times A_r^{\pm}(\eta)(\psi_{jr}\pm\exp(-i\pi r)\psi_{j-r}), \tag{3.13}$$

where the sum over r runs $r=0,\ldots,j$ for j integer and $r=\frac{1}{2}, \frac{3}{2}, \ldots, j$ for j half-integer. From the orthonormalization condition (3.6), we find

$$4(j!)\overline{A_0^{\epsilon'}(\eta')}A_0^{\epsilon}(\eta)+2\sum_{r=1}^{j}(j-r)!(j+r)!\overline{A_r^{\epsilon'}(\eta')}A_r^{\epsilon}(\eta) =\delta_{\eta'\eta}\delta_{\epsilon'\epsilon} \quad (j=\text{integer}) \tag{3.14a}$$

$$2\sum_{r=1/2}^{j}(j-r)!(j+r)!\overline{A_r^{\epsilon'}(\eta')}A_r^{\epsilon}(\eta) =\delta_{\eta'\eta}\delta_{\epsilon'\epsilon} \quad (j=\tfrac{1}{2}\text{integer}). \tag{3.14b}$$

Notice that our normalization for $A_r^{\epsilon}(\eta)$ is different from that of Arscott. The inverse expansion is easily obtained from (3.13):

$$\psi_{jr}=\exp[i\pi(j-r)/2]\sqrt{(j-r)(j+r)!}\sum_{\eta,\epsilon}\overline{A_r^{\epsilon}(\eta)}\psi_{j\eta}^{\epsilon}, \quad r\ne 0,$$
$$\psi_{j0}=2(j!)\exp(i\pi j/2)\sum_{\eta}\overline{A_0^{+}(\eta)}\psi_{j\eta}^{+}. \tag{3.15}$$

From the orthonormality of the ψ_{jr}'s we find

$$\sum_{\eta,\epsilon}A_r^{\epsilon}(\eta) =(j-r)!(j+r)!\delta_{rr'} \quad (r \text{ and } r' \text{ not both } 0), \tag{3.16a}$$

$$\sum_{\eta}\overline{A_0^{+}(\eta)}A_0^{+}(\eta)=\tfrac{1}{2}(j!)^2 \tag{3.16b}$$

B. One variable model

A well-known[8] model of $SU(2)$ on the space of polynomials of degree $2j$ in one complex variable is given by the realization

$$J'_+=\frac{d}{dz}, \quad J'_-=2jz-z^2\frac{d}{dz}, \quad J'_3=j-z\frac{d}{dz}. \tag{3.17}$$

The canonical basis states are then realized as

$$\psi'_{jr}(z)=z^{j-r}/\sqrt{(j-r)!(j+r)!}. \tag{3.18}$$

In this realization the operator E [Eq. (3.3)] takes the form

$$E=z\frac{d^2}{dz^2}+\left(\frac{a}{2i}(z^2+1)-(2j-1)z\right)\frac{d}{dz}+j(j+iaz). \tag{3.19}$$

However, for our purposes it is more convenient to consider another one variable model of $SU(2)$ obtained from (3.17) by a similarity transformation. Set $z=\exp(i\pi/2)\exp(2i\zeta)$ and consider the operators $J_i=z^{-j}\times J'_i z^j$. In the new variable ζ the generators J_3, $J_{\pm}$ take the form

$$J_3=\frac{i}{2}\frac{d}{d\zeta}, \quad J_{\pm}=-\exp(2i\zeta)\left(\frac{1}{2i}\frac{d}{d\zeta}\pm j\right) \tag{3.20}$$

and the canonical basis states are

$$\psi_{j,r}(\zeta)=\exp(i\pi(j-r)/2)\exp(-2ir\zeta)/\sqrt{(j-r)!\,(j+r)!}\,. \quad (3.21)$$

It is easy to check that the operators (3.20) satisfy the relations (3.2) on the states (3.21). Furthermore, the operator E takes the form

$$E=\frac{1}{4}\frac{d^2}{d\zeta^2}-\frac{a}{2}\sin 2\zeta\frac{d}{d\zeta}+ja\cos 2\zeta \quad (3.22)$$

and the eigenvalue equation (3.4) becomes

$$\psi''+\xi\sin 2\zeta\,\psi'+(\eta-2j\xi\cos 2\zeta)\psi=0, \quad (3.23)$$

which is precisely what Arscott[4] calls Ince's equation with $2j$ identified with Arscott's p.

We construct a realization for the scalar product (3.6) which covers the complex ζ plane once and for which (3.21) forms an orthonormal basis for each integer or half-integer j, viz.

$$(f,g)_j=\frac{\Gamma(2j+2)}{2^{2j+1}\pi}\int_{-\infty}^{\infty}d\zeta_2(\cosh 2\zeta_2)^{-2j-2}\int_{-\pi}^{\pi}d\zeta_1\overline{f(\zeta)}\,g(\zeta), \quad (3.24)$$

where $\zeta=\zeta_1+i\zeta_2$, ζ_1, $\zeta_2\in R$. Writing the expansion formula (3.13) explicitly with the state (3.21), we obtain

$$\psi_{jn}^{+}(\zeta)=2\sum_r A_r^{+}(\eta)\cos 2r\zeta=2\,\mathrm{C}_{2j}^{m}(\zeta,\xi) \quad (3.25a)$$

$$\psi_{jn}^{-}(\zeta)=-2i\sum_r A_r^{-}(\eta)\sin 2r\zeta=-2i\,\mathrm{S}_{2j}^{m}(\zeta,\xi). \quad (3.25b)$$

It is readily verified by substitution that the solutions (3.24) satisfy the differential equation (3.23) with the recursion formulas (3.10).

It is now a simple task to make the connection of our model in this section with the previous section. It can be seen that the spectral resolution of the operator $\mathcal{M}^2-\mathcal{P}_2^2-\mathcal{B}_2^2$ of Sec. 2 gives exactly the differential equation (3.23) with the identification $p=\lambda-1=2j$, $\frac{1}{2}-\mu+\frac{1}{2}p=\eta$, and $\xi=-\frac{1}{2}$.

Now the Lie algebra model (3.17) has been integrated to the group $SU(2)$ by Vilenkin,[8] and it is a simple task to express his representation in terms of our functions $\psi(\zeta)$. In so doing we can express the cross-basis matrix elements of $\exp(-i\theta J_1)$ in terms of a finite sum of Jacobi polynomials.

4. OVERLAP FUNCTIONS

In this section we calculate the overlap functions between the bases oc, or, and oe, respectively. However, since these functions are invariant under the unitary transformations of G as well as Bargmann's transformation **A**, they also apply to the bases Oc, Or, and Oe in Sec. 1. Thus we obtain expansion formulas for each one of these functions in terms of the others. Those expansions involving the Oe basis are probably new.

The overlap function for oc—or systems has been calculated for the case of three-dimensions.[9] In the two-dimensional case here we find

$$(\mathrm{gc}_{n_1,n_2},\mathrm{gr}_{nm}^{\pm})=K\delta_{n_1+n_2,2n+m}\,i^{n_1}\left(\frac{n_2!}{n_1!}2^{2n+m+1}n!\,(n+m)!\right)^{1/2}$$
$$\times\binom{1}{-i}\left(\frac{i^m\,{}_2F_1(-n_1,1-(n_1+n_2+m)/2;(n_2-n_1-m)/2;-1)}{\Gamma((n_1+n_2-m)/2)\Gamma((n_2-n_1+m)/2)}\right.$$
$$\left.\pm\frac{i^{-m}\,{}_2F_1(-n_1,1-(n_1+n_2-m)/2;(n_2-n_1+m)/2;-1)}{\Gamma((n_1+n_2-m)/2)\Gamma((n_2-n_1+m)/2)}\right). \quad (4.1)$$

These coefficients allow us to expand[10] the Hermite functions (1.2) in terms of the Laguerre functions (1.3) and vice-versa.

The overlap functions for the system or$^{\pm}$—oe$^{\pm}$ are even easier to calculate, viz.,

$$(\mathrm{gr}_{n,m'}^{\pm},\mathrm{ge}_{p,m}^{\pm})=K\delta_{p,2n+m'}2^{n+(m'-s-1)/2}[\Gamma(n+m'+1)n!]^{1/2}$$
$$\times A_{m'/2}^{\pm}(\eta_m), \quad (4.2)$$

whereas the overlap between different parity states vanishes. These coefficients allow us to expand the functions (1.3) in terms of the functions (1.4) and vice-versa. The composition of (4.1) and (4.2) gives us the overlap functions as an infinite series

$$(\mathrm{gc}_{n_1n_2},\mathrm{ge}_{p,m}^{\pm})$$
$$=\sum_n(\mathrm{gc}_{n_1n_2},\mathrm{gr}_{n,p-2n}^{\pm})(\mathrm{gr}_{n,p-2n}^{\pm},\mathrm{ge}_{nm}^{\pm}). \quad (4.3)$$

Furthermore, we can combine the above results with those of 6 to obtain further overlap functions. However, we present here only those which can be readily obtained in close form and were not given in 6, viz., for the free particle radial coordinates and harmonic oscillator elliptic coordinates:

$$(\mathrm{fr}_{\gamma,m'}^{\pm},\mathrm{oe}_{p,m}^{\pm})=2^{1-m'/2}K^2\sqrt{n!\,m'!}\,A_{m'/2}^{\pm}(\eta_m)\gamma^{m'+1/2}$$
$$\times\exp(-\gamma^2/4)L_{(p-m')/2}^{m'}(\tfrac{1}{2}\gamma^2). \quad (4.4)$$

These functions allow us to expand the Bessel functions given by Eq. (4.24) in 6 in terms of the Ince polynomials (1.4) and conversely to write the functions (1.4) as an integral and sum of Bessel functions.

Similarly, for the repulsive oscillator, radial coordinates, and the harmonic oscillator, elliptic coordinates,

$$(\mathrm{rr}_{\lambda m'}^{+},\mathrm{oe}_{pm}^{+})=K^2A_{m'/2}(\eta_m)\left(\frac{2}{[(p-m')/2]!}\right)^{(m'-1)/2-i\lambda}$$
$$\times\frac{\Gamma((m'+1-i\lambda)/2)}{\sqrt{m'!}}\,{}_2F_1(-(p-m')/2,(m'+1-i\lambda)/2,$$
$$\times m'+1;2), \quad (4.5a)$$

whereas for the negative parity solutions we have

$$(\mathrm{rr}_{\lambda,m'}^{-},\mathrm{oe}_{pm}^{-})=-i(\mathrm{rr}_{\lambda m'}^{+},\mathrm{oe}_{pm}^{+}). \quad (4.5b)$$

Accordingly these coefficients allow us to expand the Whittaker functions, Eq. (4.38) of 6, in terms of the Ince polynomials (1.4) and conversely to express the Ince polynomials as an integral and sum over Whittaker functions.

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