Symmetries of the Hamilton–Jacobi equation

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We present a detailed discussion of the infinitesimal symmetries of the Hamilton–Jacobi equation (an arbitrary first order partial differential equation). Our presentation elucidates the role played by the characteristic system in determining the symmetries. We then specialize to the case of a free particle in one space and one time dimension, and study the local Lie group of point transformations locally isomorphic to O(3,2). We show that the separation of variables of the corresponding Hamilton–Jacobi equation in the form of a sum is related to orbits in the Schrödinger subalgebra of o(3,2). The remaining orbits of o(3,2) yield symmetry related solutions which separate in more complicated product forms. Finally some connections with the primordial equation of hydrodynamics (without force terms) are made.

INTRODUCTION

One of the most important techniques in finding explicit solutions of partial differential equations is that of Lie group theory. This is said while keeping in mind the recent developments which illustrate the intimate connection of the time honored method of separation of variables with the theory of Lie groups.[1–5] Up to now most of this development has treated only second order linear partial differential equations, although the first and perhaps best understood example of separation of variables occurred for the nonlinear Hamilton–Jacobi equation.[6–9] Indeed there is a close connection between the separation of variables for second order linear partial differential equations of hyperbolic–elliptic type and the corresponding quadratic Hamilton–Jacobi equation which describe the characteristic surfaces of the former. This connection is usually described in the dual formulation in terms of a covariant Riemannian metric[10] $ds^2=g_{ij}dx^i dx^j$. However, even for parabolic equations like the time dependent Schrödinger and heat equations we will see that the connection with a Hamilton–Jacobi equation of first degree in the temporal derivative remains, in the sense that they both admit the same type of separable coordinates. This is no doubt related to the fact that such coordinates are projectively related to quadratic surfaces in a higher dimensional pseudo-Riemannian space. However, we will show shortly how the elliptic Hamilton–Jacobi equation (sums of quadratics) is related by a simple point transformation to the parabolic Hamilton–Jacobi equation (first order derivative in time). It is also emphasized that the separation of the parabolic type presents a unified picture of four types[2,3] of potentials V, the free particle ($V=0$), the linear potential ($V=ax$), and the attractive and repulsive harmonic oscillators ($V=\pm\omega x^2$).

Now generally any first order partial differential equation can be cast by the process of embedding in a space of one higher dimension, into the Hamilton–Jacobi form

$$S_t + H(x^i, p_i, t)=0,\qquad p_i=S_{x^i} \tag{0.1}$$

(subindices with respect to variables denote differentiation). The importance of this equation in geometrical optics, the calculus of variations, and obtaining explicit solutions of Hamilton's equations of classical mechanics is well known. (For the classical treatment see Chap. 2 of Ref. 11; for modern treatments see Chap. 13 of Ref. 12 and Chap. 4 of Ref. 13.) There is also a close connection with the theory of canonical transformations which we mention briefly here since the treatment in the sequel is complementary to this in the sense that it relates to contact transformations. Indeed consider a manifold (Hamiltonian manifold) with local coordinates (x^i, p_i, t) which has a closed 2-form ω and a function H such that

$$\omega = dp_i\wedge dx^i - dH\wedge dt. \tag{0.2}$$

Now each submanifold such that $\omega=0$ implies the existence of a function $S(x^i,t)$ which is a solution of the Hamilton–Jacobi equation (0.1) (for more details see, e.g., Chap. 13 of Ref. 12). On the other hand, if we consider $-H$ as a coordinate, then the transformations which leave ω invariant form the pseudogroup of canonical transformations over a $(2n+2)$-dimensional manifold. Then restricting H to be a function will give a subpseudogroup which depends upon H, of course.

We now consider the special case of a free particle in a Riemannian (or pseudo-Riemannian) n-space with contravariant metric g^{ij}. Then (0.1) becomes

$$S_t + g^{ij}S_{x^i}S_{x^j}=0\ . \tag{0.3}$$

If we introduce a change of variables $T=t+S$, $z=t-S$, an easy calculation shows that (0.3) is equivalent to

$$T_z^2 + g^{ij}T_{x^i}T_{x^j}=1 \tag{0.4}$$

as long as both sets (x^i, t) and (x^i, z) can be treated as independent variables. Two comments are in order: First, the local symmetry group of point transformations of (0.3) and (0.4) are isomorphic. It was shown in Ref. 14 that when g^{ij} is the flat Euclidean metric, the local symmetry group of point transformations of (0.3) is a factor group of order 2 of $O(n+2,2)$. Second, the above change of coordinates involving the dependent variable shows that (0.3) is equivalent to a Riemannian (pseudo-Riemannian) metric.

In this paper we study in detail the symmetries and separable coordinates of the equation

$$S_t + S_x^2 = q + p^2 = 0. \qquad (*)$$

This equation can be obtained from (0.3) by partial separation, at least in the case when g^{ij} admits a Killing vector. Thus from the point of view of separable coordinates we only study here subgroup coordinates. In fact the more general point transformation symmetries of (*) will yield coordinates not associated with the usual separation of variables. From this point of view the similarity solutions or complete integrals we obtain are more general than ordinary R-separation; however, we do not study here the usual quadratic orthogonal separation involving quadratic forms. Those, of course, do not appear in (*), but they will appear in the analog of (0.4), i.e.,

$$T_z^2 + T_x^2 = 1. \qquad (**)$$

We plan, to treat these in a subsequent work. Recently[15] it was shown that in a Riemannian or pseudo-Riemannian metric space there are two types of separation, those coming from local symmetry groups and those coming from the usual orthogonal separation, and that the latter are described by contravariant quadratic symmetric forms (Killing tensors).

The outline of the paper is as follows: In Sec. 1 we compute the Lie algebra of vector fields depending on both coordinates and momenta which are infinitesimal symmetries for an arbitrary first-order partial-differential equation. This computation elucidates the role played by the characteristic system in determining the symmetries. We discuss some of the underlying structure of this infinite-dimensional Lie algebra. Then we specialize to the subalgebra of point transformation symmetries of (*). These generate a finite-dimensional local Lie group-conformal transformations in R^3, locally isomorphic to O(3, 2). We then classify the orbits in the Lie algebra O(3, 2) under conjugacy with respect to the group. In Sec. 2 we obtain all R-separable coordinates systems for (*). In Sec. 3 we present a similarity solution[16] for each of the orbit representatives found in Sec. 1 and discuss the connection with the separation of variables of Sec. 2. Some remarks concerning the general solution and characteristic vector fields are also made.

Finally, in Sec. 4 we present a discussion of symmetries which derives from the fact that the x derivative of (*) yields the primordial equation of hydrodynamics without force terms[11,16]

$$p_t + 2pp_x = 0 \qquad (***)$$

(the connection holds for n spatial dimensions). This allows one to relate a subalgebra of symmetries of (***) to a subalgebra of symmetries of (*). Moreover, even symmetries of (***) which are not symmetries of (*) can be used to determine complete integrals of the latter, or vice-versa.

1. THE INFINITESIMAL SYMMETRIES OF THE HAMILTON-JACOBI EQUATION

Consider an n dimensional manifold M with local coordinates[17] x^i and an arbitrary first order differential equation on M

$$G(x^i, u_{x^i}, u) = 0. \qquad (1.1a)$$

We wish to determine the infinitesimal symmetries of such an equation which depend on all the variables present. To do this we consider the cotangent bundle $T^*(M)$ over M with local coordinates (x^i, p_i), and construct the product manifold $T^*(M)\times R$. Now $T^*(M)$ has a canonical 1-form $p_i\,dx^i$ which provides the contact 1-form

$$\alpha = du - p_i\,dx^i$$

on $T^*(M)\times R$. Solutions of (1.1a) will be surfaces in $T^*(M)\times R$

$$G(x^i, p_i, u) = 0,$$

which also annul the 1-form α. Now, following Cartan,[18] we construct the closed ideal (closed refers to exterior differentiation) I defined by

$$G(x^i, p_i, u), \qquad (1.1b)$$

$$\alpha = du - p_i\,dx^i, \qquad (1.1c)$$

$$dG = G_{x^i}\,dx^i + G_{p_i}\,dp_i + G_u\,du, \qquad (1.1d)$$

$$d\alpha = dx^i \wedge dp_i. \qquad (1.1e)$$

The surfaces in $T^*(M)\times R$ which annul I will be the solutions of the differential equation (1.1a). Stated more precisely we look for immersed submanifolds whose pullback annuls I.

Now the symmetries of the differential equation (1.1a) will be those local C^2 diffeomorphisms on $T^*(M)\times R$ whose pullback maps I into I. Stated infinitesimally this reads[12,19]

$$\underset{X}{\pounds} G = \xi G, \qquad (1.2a)$$

$$\underset{X}{\pounds}\alpha = \lambda\alpha + \eta dG + (A_i\,dx^i + B^i\,dp_i)G, \qquad (1.2b)$$

where $\underset{X}{\pounds}$ denotes the Lie derivative with respect to the vector field X, and $\xi, \lambda, \eta, A_i, B^i$ are functions on $T^*(R^n)\times R$, where ξ, A_i, B^i must be nonsingular in a neighborhood of $G=0$ but are other wise arbitrary. We have replaced M by the Euclidean manifold R^n. It should be mentioned here that the commutivity of the exterior derivative and the Lie derivative guarantee that dG and $d\alpha$ are back in I when an infinitesimal transformation is applied, and so Eqs. (1.2) suffice to define the symmetry condition for all of I. Notice that the Lie algebra $\mathcal{G}$ of symmetries is more general than just contact transformations since it is not necessary that the contact 1-form α be preserved. The contact transformations which are symmetries of (1.1a) form a Lie subalgebra $\mathcal{G}_C \subset \mathcal{G}$ given by the special case $\eta = A_i = B^i = 0$. To determine the Lie algebra $\mathcal{G}$, we use the expressions[12] valid for a 0-form f and any form ω

$$\underset{X}{\mathcal{L}} f = X \lrcorner\, df,$$
$$\underset{X}{\mathcal{L}} \omega = d(X \lrcorner\, \omega) + X \lrcorner\, d\omega, \tag{1.3}$$

where $\lrcorner$ denotes the natural inner product between vector fields and exterior differential forms. Applying (1.3) to (1.2a) and (1.2b) and defining the function on $T^*(R^n)\times R$, $F = X \lrcorner\, \alpha$, we equate coefficients of the independent 1-forms in (1.2b) to obtain

$$X^{x^i} = -F_{p_i} + \eta G_{p_i} + B^i G, \tag{1.4a}$$

$$X^{p_i} = F_{x^i} + p_i F_u - \eta(G_{x^i} + p_i G_u) - A_i G, \tag{1.4b}$$

$$X^u = F + p_i X^{x^i} = F - p_i F_{p_i} + \eta p_i G_{p_i} + B^i p_i G, \tag{1.4c}$$

and from (1.2a) we find

$$X \lrcorner\, dG = G_{x^i} X^{x^i} + G_{p_i} X^{p_i} + G_u X^u = \xi G, \tag{1.4d}$$

where the superscripts on the vector field X denote its component, i.e.,

$$X = X^{x^i}\partial_{x^i} + X^{p_i}\partial_{p_i} + X^u \partial_u. \tag{1.4e}$$

Now, inserting (1.4a)—(1.4c) into (1.4d), we obtain a linear first-order partial-differential equation for the function F which immediately yields the system

$$\frac{dx^i}{d\tau} = G_{p_i}, \quad \frac{du}{d\tau} = p_i G_{p_i}, \quad \frac{dp_i}{d\tau} = -(G_{x^i} + p_i G_u), \tag{1.5a}$$

$$\frac{dF}{d\tau} = (\xi - G_{x^i}B^i + G_{p_i}A_i - p_i B^i G_u)G + G_u F. \tag{1.5b}$$

We recognize that Eqs. (1.5a) describe nothing more than the *characteristic system*[11,12] of Eq. (1.1a). Thus the function F has two parts; one determined by Eqs. (1.5b), plus an arbitrary function which depends only on the characteristic curves of (1.1a).

Now the characteristic vector fields $\mathcal{J}$ in $\mathcal{G}$ are those which satisfy $X \lrcorner\, \omega \in I$ for all ω in I. By using the identity

$$\underset{X}{\mathcal{L}}(Y \lrcorner\, \omega) = [X, Y] \lrcorner\, \omega + Y \lrcorner\, \underset{X}{\mathcal{L}}\omega, \tag{1.6}$$

it is easy to show[12] that $\mathcal{J}$ is in fact an ideal in $\mathcal{G}$ for any ideal of forms I.

However, in some sense the terms in Eqs. (1.4) proportional to G are trivial, e.g., A_i and B^i, since if we restrict the vector fields to the surface in $T^*(R^n)\times R$ defined by (1.1a), these parts vanish. Indeed we can consider all vector fields in (1.4) which satisfy

$$Y \lrcorner\, \alpha = EG, \quad Y \lrcorner\, d\alpha = \beta G, \tag{1.7}$$

where E and β are arbitrary 0- and 1-forms, respectively, on $T^*(R^n)\times R$ which are nonsingular near $G=0$. Clearly all such vector fields are characteristic. Moreover, by using (1.3), (1.4d), and (1.6b), it is not difficult to show that they form an ideal $\tilde{\mathcal{J}}$ in $\mathcal{G}$. Thus it is often convenient to consider the factor algebra $\mathcal{G}/\tilde{\mathcal{J}}$. We can always choose A_i and B^i such that the term multiplying G in (1.5b) vanishes in which case we have

$$\frac{dF}{d\tau} = G_u F. \tag{1.8}$$

In general when $G_u = 0$, (1.1a) takes the standard Hamilton—Jacobi form (0.1) and the symmetries are determined by an arbitrary function of the characteristic strips. In this case the first two of Eqs. (1.5a) are just Hamilton's equation of classical mechanics. For example, for the free particle in Euclidean space, the function F takes the form

$$F = F(x^i - 2p_i t, S - 2p^2 t - qt, p_i, q). \tag{1.9a}$$

The point transformation symmetries are locally isomorphic to $O(n+2,2)$ as shown in Ref. 14. Now $\mathcal{G}/\tilde{\mathcal{J}}$ admits a Lie algebra semidirect sum

$$\mathcal{G}/\tilde{\mathcal{J}} = \mathcal{G}_c/\tilde{\mathcal{J}} \,\supset\!\!\!\!+\, \mathcal{J}/\tilde{\mathcal{J}} \tag{1.9b}$$

where $\mathcal{G}_c/\tilde{\mathcal{J}}$ is generated by the contact symmetries given by the function F which satisfies the characteristic system (1.5a) and (1.8), and $\mathcal{J}/\tilde{\mathcal{J}}$ describes the characteristics given by the function η.

For the remainder of this section we will discuss only point transformation symmetries $\mathcal{G}_p \subset \mathcal{G}_c/\tilde{\mathcal{J}}$ for (*). To find them from (1.4), we set $A^i = B_i = \eta = 0$ and impose the condition

$$X^{x^i}_{p_j} = 0,$$

i.e., the transformations on the base space are independent of p_j. Doing this explicitly for the case when $G=0$ is given by (*) and using (1.9c) will determine the point transformation symmetries of (*). From this analysis one can find that the vector fields span the finite dimensional Lie algebra o(3, 2). [In n space and 1 time dimensions, $o(n+2,2)$.] However, to understand better the appearance of the Lie algebra o(3, 2) of the conformal group, we introduced in Ref. 14 the graph $W(t, x, S) = 0$ of solutions of (*). Then upon computing the derivatives $W_x + W_S S_x = W_t + W_S S_t = 0$ and introducing the Minkowski variables

$$x^0 = 2^{-1/2}(t + 2S), \quad x^2 = 2^{-1/2}(t - 2S), \quad x^1 = x, \tag{1.10a}$$

we find that W satisfies

$$(W_{x^0})^2 - (W_{x^1})^2 - (W_{x^2})^2 = 0. \tag{1.10b}$$

Thus the point transformation symmetries of (*) are precisely the conformal transformations of the cone (1.10b).

This global approach[20] has distinct advantages over the infinitesimal method: (i) Without much work we have reduced the problem to known results; (ii) the geometry elucidates the meaning of the symmetries; (iii) we obtain certain symmetries which are not connected to the identity component of the group and thus are not obtainable through infinitesimal methods. However, it should also be mentioned that in general it is not always so easy to find such a nice geometrical situation in which case infinitesimal methods provide the most straightforward approach.

The symmetries of a cone in a pseudo-Euclidean space of three dimensions with signature $(+,-,-)$ form the conformal group $C^{1,2}$ which is a certain factor group of the pseudo-orthogonal group O(3, 2). More precisely we can consider the group O(3, 2) as a group of transformations in a five-dimensional pseudo-Euclidean space with signature $(+,-,-,-,+)$ which leaves the quadratic form $\eta_0^2-\eta_1^2-\eta_2^2-\eta_3^2+\eta_4^2$ invariant. We now consider the 5-cone $\eta_a\eta^a=0$ and define homogeneous coordinates

$$x^\mu=\eta^\mu/(\eta^3+\eta^4), \tag{1.11}$$

where $\mu=0,1,2$. The linear action of $O(3,2)$ on the 5-cone given by

$$\eta'_a=\Lambda_a{}^b\eta_b,$$

with $\Lambda_a{}^b\in O(3,2)$, then induces through (1.11) a nonlinear action on the Minkowski space $M=\{x^\mu\}$, which we will give shortly. However, it is seen that the action of $O(3,2)$ on M is not effective. Indeed, there are two members of $O(3,2)$ which act as the identity transformation on M, namely the subgroup $Z_2=\{\Lambda\in O(3,2):\ \Lambda\eta=\pm\eta\}$. Hence, the conformal group $C^{1,2}\sim O(3,2)/Z_2$.

Now the group $O(3,2)$ consists of four components, where the component connected to the identity is $SO_0(3,2)\equiv\{\Lambda\in O(3,2):\ \det\Lambda=1,\ \Lambda_0{}^0\Lambda_4{}^4-\Lambda_0{}^4\Lambda_4{}^0>1\}$. The other three components are obtained by reversing the signs of $\det\Lambda$ and $\Lambda_0{}^0\Lambda_4{}^4-\Lambda_0{}^4\Lambda_4{}^0$. Notice that $SO_0(3,2)\subset C^{1,2}$. The whole $O(3,2)$ can be obtained by extending $SO_0(3,2)$ by two discrete operations P-parity and T-covariant time reversal given by

$$P=\{x^0\to x^0, x^1\to -x^1, x^2\to x^2\},$$

$$T=\{x^0\to -x^0, x^1\to x^1, x^2\to x\},$$

respectively. In terms of (t,x,S), we have

$$P=\{t\to t, x\to -x, S\to S\}, \tag{1.12a}$$

$$T=\{t\to -2S, x\to x, S\to -\tfrac{1}{2}t\}. \tag{1.12b}$$

We will also be interested in a discrete symmetry R obtained by combining P with a certain member of $SO_0(2,1)\subset SO_0(3,2)$, namely

$$R=\{t\to S, x\to x, S\to t\}. \tag{1.12c}$$

Finally we mention the well-known inversion symmetry

$$I=\{(t,x,S)\to (t,x,S)/(4tS-x^2)\}. \tag{1.12d}$$

It is emphasized that the symmetries (1.12b)–(1.12d) are nontrivial symmetries of the Hamilton–Jacobi equation (*). Indeed, (1.12b) and (1.12c) imply that, given a solution $S(x,t)$ of (*), we can use the implicit function theorem and solve for $t=t(x,S)$, which again satisfies

$$t_x^2+t_S=0,$$

i.e., is another solution of (*).

We now give the group transformations of $SO_0(3,2)$ in terms of the original Hamilton–Jacobi variables (t,x,S):

(1) *$O(2,1)$ transformations*:

$$x'=\Lambda^1{}_1x+\frac{\Lambda^1{}_0+\Lambda^1{}_2}{\sqrt{2}}\,t+\sqrt{2}(\Lambda^1{}_0-\Lambda^1{}_2)S,$$

$$t'=\frac{(\Lambda^0{}_1+\Lambda^2{}_1)}{\sqrt{2}}x+\frac{(\Lambda^0{}_0+\Lambda^0{}_2+\Lambda^2{}_0+\Lambda^2{}_2)}{2}t+(\Lambda^0{}_0+\Lambda^2{}_0-\Lambda^0{}_2-\Lambda^2{}_2)S,$$

$$S'=\frac{(\Lambda^0{}_1-\Lambda^2{}_1)}{2\sqrt{2}}x+\frac{(\Lambda^0{}_0+\Lambda^0{}_2-\Lambda^2{}_0-\Lambda^2{}_2)}{4}t+\frac{(\Lambda^0{}_0-\Lambda^0{}_2-\Lambda^2{}_0+\Lambda^2{}_2)}{2}S, \tag{1.13a}$$

where $\Lambda^i{}_j\in O(2,1)$, $i,j=0,1,2$.

(2) *Translations*:

$$x'=x+a,\quad t'=t+\tau,\quad S'+S+\sigma, \tag{1.13b}$$

with $a,\ \tau,\sigma\in R$.

(3) *Dilatations*:

$$x'=\rho x,\quad t'=\rho t,\quad S'=\rho S, \tag{1.13c}$$

with $\rho>0$.

(4) *Special conformal transformations*:

$$\begin{aligned}x'&=\sigma^{-1}(x,t,S)[x+C_1(x^2-4tS)],\\ t'&=\sigma^{-1}(x,t,S)[t+C_+(x^2-4tS)],\\ S'&=\sigma^{-1}(x,t,S)[S+C_-(x^2-4tS)],\end{aligned} \tag{1.13d}$$

where

$$\sigma(x,t,S)=1-2C_+t-4C_-S+2C_1x+(C_+C_--C_1^2)(4tS-x^2)$$

and $C_1, C_\pm\in R$. It is mentioned that the special conformal transformations can be generated by a translation, an inversion, and another translation.

Now the group action (1.13) is really only a local group since the points where $\sigma(x,t,S)$ vanishes map finite points to infinity. Nevertheless, a global Lie group can be defined if we consider the "cone" compactification of R^3, making the manifold homeomorphic with the sphere S^3. Although this is necessary for a global Lie group, for our purposes it is more convenient to work with the local coordinates (x,t,S), keeping in mind that under finite group transformations singularities can occur. Hence, what we are really dealing with is a finite pseudogroup. Although the study of such singularities is of interest, we will not consider them further here. We only mention that Sard's theorem[13] guarantees that they form a set of measure zero.

In what follows we will be interested in two different formulations of the Lie algebra $o(3,2)$. The first is the covariant formulation with a basis given by M_{ab} with $a,b=0,\ldots,4$, which satisfy the Lie brackets

$$[M_{ab},M_{cd}]=g_{ad}M_{bc}+g_{bc}M_{ad}-g_{ac}M_{bd}-g_{bd}M_{ac}. \tag{1.14}$$

On the η-space realization used previously the M_{ab} can be realized as $\eta_a\partial_b-\eta_b\partial_a$. However, on R^3 it is more convenient to consider the realization[14] [these are the point transformations of 1.4 for (*) projected onto R^3]

$$\begin{aligned}&X_1=\partial_t,\quad X_2=t\partial_t+\tfrac{1}{2}x\partial_x,\quad X_3=t^2\partial_t+tx\partial_x+\tfrac{1}{4}x^2\partial_S,\\ &X_4=\partial_x,\quad X_5=t\partial_x+\tfrac{1}{2}x\partial_S,\quad X_6=\partial_S,\\ &X_7=\tfrac{1}{2}x\partial_x+S\partial_S,\\ &X_8=\tfrac{1}{2}xt\partial_t+(tS+\tfrac{1}{4}x^2)\partial_x+\tfrac{1}{2}xS\partial_S,\\ &X_9=\tfrac{1}{4}x^2\partial_t+Sx\partial_x+S^2\partial_S,\quad X_{10}=\tfrac{1}{2}x\partial_t+S\partial_x.\end{aligned} \tag{1.15}$$

It is not difficult to see that the generators $X_1,\ldots,X_7$ form a subalgebra of $o(3,2)$. In fact this subalgebra is maximal and generates the subgroup of $so_0(3,2)$ which leaves a lightlike two-plane invariant. It has the structure $gl(2,R) \ni w$, i.e., the general linear algebra with the Heisenberg–Weyl subalgebra as an ideal. However, we will be more interested in the subalgebra formed by the generators $X_1, \ldots, X_6$ whose structure is $s_1 \sim sl(2,R) \ni w$. This algebra generates a group known as the Schrödinger group $\mathcal{S}_1$ since it is the group which leaves invariant the Schrodinger equation for a free particle in one space and one time dimension.[14,21,22] The existence of the Schrödinger group $\mathcal{S}_1$ as a subgroup of $O(3,2)$, or more generally[14,23] $\mathcal{S}_n \subset O(n+2,2)$, emphasizes the close connection between the Schrödinger and heat equations on one hand and the Hamilton–Jacobi equation on the other. The subgroup $\mathcal{S}_1$ will play an important role in what follows. It is also seen that the discrete symmetry R given by (1.8c) provides us with another Schrödinger subgroup $\mathcal{S}_1$, conjugate to $\mathcal{S}_1$, through the mappings $X_1 \leftrightarrow X_6$, $X_2 \leftrightarrow X_7$, $X_3 \leftrightarrow X_9$, $X_4 \leftrightarrow X_4$, $X_5 \leftrightarrow X_{10}$, $X_8 \leftrightarrow X_8$.

Rather than write down the commutation relations explicitly for the generators (1.15), it is more convenient to express them in terms of generators M_{ab} satisfying (1.14), viz.,

$$\begin{aligned}
M_{10} &= -(1/\sqrt{2})(X_5 + 2X_{10}),\\
M_{20} &= X_2 - X_7,\\
M_{30} &= (1/2\sqrt{2})(X_1 + \tfrac{1}{2}X_6 - 2X_3 - 4X_9),\\
M_{21} &= -(1/\sqrt{2})(X_5 - 2X_{10}),\\
M_{31} &= \tfrac{1}{2}(X_4 + 4X_8),\\
M_{32} &= (1/2\sqrt{2})(X_1 - \tfrac{1}{2}X_6 + 2X_3 - 4X_9),\\
M_{40} &= (1/2\sqrt{2})(X_1 + \tfrac{1}{2}X_6 + 2X_3 + 4X_9),\\
M_{41} &= \tfrac{1}{2}(X_4 - 4X_8),\\
M_{42} &= (1/2\sqrt{2})(X_1 - \tfrac{1}{2}X_6 - 2X_3 + 4X_9),\\
M_{43} &= X_2 + X_7.
\end{aligned} \tag{1.16}$$

Now we are interested in the orbit structure of $O(3,2)$ under the adjoint action of the conformal group $C^{1,2}$. In fact this problem has been solved in several places[4,24,25]; however, in none of these are the results in a form particularly suited for our needs. As will be seen in the next section, for the purpose of separation of variables the subgroup $\mathcal{S}_1$ plays a distinguished role. Therefore, we want to pick orbit representatives which are members of the Lie algebra s_1 of $\mathcal{S}_1$ if possible. The procedure we use to do this is to notice that every member of $O(3,2)$ stabilizes a timelike, spacelike, or lightlike vector. Of course, specific elements may stabilize more than one type of vector. We then study each case separately by looking at the adjoint action of the stability subgroup and picking orbit representatives in s_1 when possible. When this is done, we must then check for conjugacy under the full $C^{1,2}$ group, again picking member of s_1 when possible. In this way we obtain a complete set of orbit representatives emphasizing which are conjugate to members of s_1 and which are not.

We begin by classifying the orbits of s_1. Now in the case of the linear Schrödinger equation treated in Refs. 2 and 3, the orbits of the factor algebra of s_1 by the central element X_6 were considered. The reason for this is that for all linear equations it is convenient to think in terms of diagonalizing operators and from this point of view X_6 is irrelevant. However, in the case of nonlinear equations one cannot always diagonalize operators in this sense. Instead, we can construct relative invariants,[26] i.e., if the infinitesimal generator X is a symmetry of the differential equation (1.1a), we can construct the graph $f(x^i, u_{x^i}, u) = 0$ of a solution u which satisfies $X \lrcorner\, df = Xf = 0$. In the special case when (1.1a) is a linear equation, this is equivalent to diagonalization of operators in the factor algebra as long as we consider general orbit representatives which include the central operators. The case at hand should illustrate the point. Thus we are interested in classifying orbits in s_1 under three particular groups: (i) the Galilei group G_1 extended by dilatations, $D \otimes G_1$; (ii) the Schrödinger group $\mathcal{S}_1$; (iii) the full conformal group $C^{1,2}$. The first group $D \otimes G_1$ is of particular interest since this is the geometrical group closely associated with the separation of variables. That is, two coordinate system which differ by dilatations of (x,t), or by Galilei transformations, essentially look the same. In Refs. 2, 3 there are some inconsistencies concerning this point. Conjugacy under $\mathcal{S}_1$ and $C^{1,2}$ are of interest for obvious reasons.

The orbits of s_1 under $D \otimes G_1$ are

$$\begin{aligned}
&X_1 \pm X_6,\quad X_2 + aX_6,\quad X_3 \pm X_6,\quad X_1 + X_3 + aX_6,\\
&X_1 - X_3 + aX_6,\quad X_1 \pm X_5,\quad X_3 \pm X_4,\\
&X_1,\quad X_3,\quad X_4,\quad X_5,\quad X_6,
\end{aligned} \tag{1.17}$$

where $-\infty < a < \infty$. We will discuss the connection of these orbits with the separation of variables of (*) in the next section.

Under $\mathcal{S}_1$ we gain the type of equivalences discussed in Refs. 2, 3, viz.,

$$\begin{aligned}
&X_1 \pm X_6,\quad X_2 + aX_6,\quad X_1 + X_3 + aX_6,\\
&X_1 + X_5,\quad X_1,\quad X_4,\quad X_6,
\end{aligned} \tag{1.18}$$

where again $-\infty < a < \infty$. In both the above cases $a = 0$ is a degenerate orbit.

Under the full conformal group $C^{1,2}$ the orbits of s_1 become

$$\begin{aligned}
&X_1 \pm X_6,\quad X_2 + X_6,\quad X_1 + X_3 \pm X_6,\quad X_1 + X_5,\\
&X_1 + X_3,\quad X_2,\quad X_1.
\end{aligned} \tag{1.19}$$

Thus under $C^{1,2}$ we can dilate a to ± 1 using X_7, and interestingly enough we find that X_4 is on the same orbit as $X_1 - X_6$ through a rotation generated by $X_5 - X_{10}$. Again the last three entries in (1.15) correspond to degenerate orbits.

Now we wish to clasify the orbit structure of $o(3,2)$ under the conformal group. As mentioned previously we first classify the one-parameter subalgebras of the stability subgroups and then later take into account conjugacy under the full $C^{1,2}$.

A. Timelike

We take the vector $(1,0,0,0,0)$ for which the stability subgroup is $O(3,1)$ generated by the rotations $\{M_{21}, M_{31},$

$M_{32}\}$ and the boosts $\{M_{41}, M_{42}, M_{43}\}$. The one-parameter subalgebras are well known,[27,28] and, using (1.16), we have the orbits

$$M_{21} \sim (X_5 - 2X_{10}), \quad M_{43} = X_2 + X_7,$$
$$M_{32} + M_{42} \sim (X_1 - \tfrac{1}{2}X_6), \tag{1.20}$$
$$M_{21} + aM_{43} \sim (X_5 - 2X_{10}) + a(X_2 + X_7),$$

where here $0 < a < \infty$. Our conjugacy is under O(3, 1) and not just the connected component $SO_0(3, 1)$.

B. Spacelike

We choose the vector (0, 1, 0, 0, 0) for which the stability subgroup is O(2, 2) generated by $\{M_{20}, M_{30}, M_{40}, M_{32}, M_{42}, M_{43}\}$. Here it is convenient to employ the well-known Lie algebra isomorphism $o(2,2) \sim o(2,1) \oplus o(2,1)$, where $\oplus$ is a Lie algebra direct sum. Explicitly, we construct

$$J_3^{\pm} = \tfrac{1}{2}(M_{40} \pm M_{32}),$$
$$K_1^{\pm} = \tfrac{1}{2}(M_{43} \pm M_{20}), \tag{1.21a}$$
$$K_2^{\pm} = \tfrac{1}{2}(M_{30} \pm M_{42}),$$

which can be seen to generate a commuting pair of O(2, 1) algebras which satisfy

$$[J_3, K_2] = K_2, \quad [J_3, K_2] = -K_1, \quad [K_1, K_2] = -J_3. \tag{1.21b}$$

To find the orbits of this $o(2,1)^+ \oplus o(2,1)^-$ under O(2, 2), we first notice that the $o(2,1)^-$ is conjugate to $o(2,1)^+$ by a discrete transformation in O(2, 2) [explicitly in terms of our model this is the transformation R given by (1.8c) combined with certain dilatations in $SO_0(2,2)$]. Thus we have the usual one-parameter subalgebras of $o(2,1)^+$. Then we must find the nontrivial extensions of these orbits by the orbits of $o(2,1)^-$. This is done by the method of the Goursat twist as discussed for example in Ref. 28. Finally one checks for conjugacy of the extensions under O(2, 2). Accordingly, we find the orbits

$$J_3^+ + aJ_3^- \sim X_1 + X_3 + a(X_6 + X_9), \quad -1 < a < 1, \quad a \neq 0,$$
$$J_3^+ + aK_1^- \sim X_1 + X_3 + aX_7, \quad 0 < a < \infty,$$
$$J_3^+ \pm (J_3^- + K_2^-) \sim X_1 + X_3 \pm X_6,$$
$$K_1^+ + (J_3^- + K_2^-) \sim X_2 + X_6, \tag{1.22}$$
$$J_3^+ + K_2^+ \pm (J_3^- + K_2^-) \sim X_1 \pm X_6,$$
$$K_1^+ + aK_1^- \sim X_2 + aX_7, \quad -1 \leq a \leq 1, \quad a \neq 0,$$
$$J_3^+ \sim X_1 + X_3, \quad K_1^+ \sim X_2, \quad J_3^+ + K_2^+ \sim X_1.$$

In arriving at (1.22) we have taken full advantage of the dilatations in O(2, 2) generated by X_2 and X_7 to remove some of the annoying constants which multiply the various X's in the expression in (1.16).

C. Lightlike

We choose the vector (0, 0, 0, 1, 1) for which the stability subgroup $D \otimes E(2,1)$ is generated by the o(2, 1) subalgebra $\{M_{21}, M_{10}, M_{20}\}$, the translations $\{M_{31} + M_{41}, M_{32} + M_{42}, M_{30} + M_{40}\}$, and the dilatation M_{43}. Again we use a modified Goursat twist[28] method to find the nontrivial extensions of the o(2, 1) subalgebra (modified since the ideal is solvable rather than Abelian). In order to simplify the notation, we introduce $J_3 = M_{21}$, $K_1 = M_{20}$, $K_2 = M_{10}$ which satisfy (1.17b), for the Abelian subalgebra $P_1 = M_{31} + M_{41}$, $P_2 = M_{32} + M_{42}$, $P_3 = M_{30} + M_{40}$, which transform as the designated components of an O(2, 1) vector, and $D = M_{43}$ which commutes with o(2, 1) and satisfies $[D, P_i] = -P_i$. We thus obtain the following orbit representatives:

$$J_3 + aD \sim X_5 - X_{10} + a(X_2 + X_7), \quad 0 < a < \infty,$$
$$J_3 + P_3 \sim X_1 + X_5 - X_{10},$$
$$K_1 + bD \sim X_2 + aX_7, \quad -1 \leq a \leq 1,$$
$$K_1 + P_2 \sim X_2 - X_7, \tag{1.23}$$
$$J_3 + K_2 + D \sim X_2 + X_5 + X_7,$$
$$J_3 + K_2 + P_2 \sim X_1 + X_5,$$
$$P_2 \sim X_1 - X_6, \quad P_3 \sim X_1 + X_6,$$
$$P_2 + P_3 \sim X_1, \quad D \sim X_2 + X_7.$$

Again we have made use of the dilatations in $D \otimes E(2,1)$ to simplify the operators in terms of the X's.

Now in order to obtain all orbits of o(3, 2), we only have to check the above results for conjugacy under the full $C^{1,2}$. Since we have already done this for the s_1 subalgebra, we can restrict our attention to the remaining cases. Indeed for the timelike case we can use dilatations to adjust some of the constants appearing in (1.16), and we see that all of the orbits (1.20) also appear as orbits in the other two cases. In fact, there are no further simplifications due to conjugacy other than identifying those orbits which appear in both cases. We have collected our results in Table I, indicating in which of the three cases the various orbits appear as well as which are members of the Schrödinger subalgebra s_1 as well as its maximal proper extension $gl(2, R) \otimes w$ in o(3, 2).

2. SEPARATION OF VARIABLES

For the purpose of separating variables in (*) it is more convenient to use the equivalent homogeneous equation

$$W_x^2 + WW_t = 0 \tag{2.1}$$

obtained from (*) by the substitution $S = \ln W$. We are in general interested in R-separability, that is, we look for a transformation of coordinates

$$x = F(v_1, v_2), \quad t = G(v_1, v_2), \tag{2.2}$$

$v_1, v_2 \in R$, where F and G are once differentiable real functions, such that the solution of (2.1) takes the form

$$W = \exp[Q(v_1, v_2)]A(v_1)B(v_2), \tag{2.3}$$

where Q can not be written as the sum of functions of the single variables unless it vanishes. It is clear that a solution of (2.1) of the form (2.3) implies a solution of (*) of the form

$$S = Q(v_1, v_2) + \ln A(v_1) + \ln B(v_2).$$

We proceed by considering the cases $Q = 0$ and $Q \neq 0$ separately. First, it is convenient to introduce a notion of equivalence. Two coordinates will be said to be equivalent if they can be related by a member of the group

TABLE I. Orbits in o(3, 2) classified under $C^{1,2}$. t, s, l denote respectively timelike, spacelike, and lightlike.

Orbit		Representative	Type	Remarks
$gl(2,R) \oplus w$	s_1	$X_1+\epsilon X_6$	$t(\epsilon=-1), s, l$	$\epsilon=\pm 1, 0$
		X_2+X_6	s	
		X_2	s, l	
		X_1+X_5	l	
		$X_1+X_3+\epsilon X_6$	s	$\epsilon=\pm 1, 0$
		X_2+aX_7	$t(a=1), s, l$	$-1\le a\le 1$
		$X_1+X_3+aX_7$	s	$0<a<\infty$
		$X_2+X_5+X_7$	l	
		$X_1+X_5-X_{10}$	l	0
		$X_5-X_{10}+a(X_2+X_7)$	t, l	$-0\le a<\infty$
		$X_1+X_3+a(X_6+X_9)$	s	$-1\le a\le 1,\ a\ne 0$

$D\otimes G_1$ discussed previously. We also consider any two systems to be equivalent if they differ by a constant multiple, i.e., $(v_1,v_2)\sim(v_1',v_2')$ if $(v_1',v_2')=\alpha(v_1,v_2)$, α constant.

A. Pure separability, $Q=0$

Rewriting (2.1) in terms of the coordinates v_1 and v_2, we obtain

$$a_{11}W_1^2+a_{12}W_1W_2+a_{22}W_2^2+a_1WW_1+a_2WW_2=0, \qquad (2.4)$$

where $a_{11}=(G_2/D)^2$, $a_{12}=-2G_1G_2/D^2$, $a_{22}=(G_1/D)$, $a_1=-F_2/D$, $a_2=F_1/D$, $D=F_1G_2-F_2G_1$, and the subscripts on W, G, F indicate differentiation with the respective variable. The conditions for separability can be further subdivided into two cases:

(i) $a_{12}\ne 0$: This is only possible if W is an exponential in one variable, say v_2, and the coefficients depend only on the remaining variable v_1. Upon redefining the variable v_1 this gives rise to coordinates of the form $t=v_2+h(v_1)$, $x=v_1$, where h is an arbitrary function of v_1. These coordinates describe nonorthogonal coordinate axes and always give rise to exponential solutions. We will not consider these any further in this article.

(ii) $a_{12}=0$: Without loss of generality we can take $G_1=0$ and hence $t=v_2$. By multiplying (2.4) by F_1^2 we can take the coefficients as $a_{11}=2$, $a_1=-F_1F_2$, and $a_2=F_1^2$. The conditions for separability are then

$$F_1^2=f(v_1)g(v_2), \quad F_1F_2=h(v_1), \qquad (2.5)$$

with f, g, and h arbitrary functions of their respective variables. By redefining the variable v_1, the conditions (2.5) imply

$$F=v_1p(v_2)+q(v_2), \quad pp_2=\alpha, \quad pq_2=\beta,$$

where α and β are constants. Without loss of generality we can put $q=0$, and we find two cases:

(1) $\alpha=0$, $x=v_1$, $t=v_2$,

(2) $\alpha\ne 0$, $x=v_1v_2^{1/2}$, $t=v_2$.

B. R-separability, $Q\ne 0$

We now wish to classify all coordinate systems for which (2.1) admits solutions of the form (2.3) for nontrivial real function Q. The appearance of the Q will give rise to a factor a_0W^2 added to Eq. (2.4). We now only consider the case $a_{12}=0$ and we obtain, preceeding as before, the nonzero coefficients

$$a_{11}=1, \quad a_1=2Q_1-F_1F_2, \quad a_2=F_1^2,$$
$$a_0=Q_1^2+F_1(F_1Q_2-F_2Q_1).$$

The condition for separability then gives

$$F_1^2=f(v_1)g(v_2), \quad 2Q_1-F_1F_2=h(v_1),$$
$$Q_1^2+F_1(F_1Q_2-F_2Q_1)=f(v_1)g(v_2)+p(v_1), \qquad (2.6)$$

where again f, g, h, p, q are arbitrary functions of their denoted variables. By suitably redefining the variable v_1, we have from the first of Eqs. (2.6)

$$F=v_1u(v_2)+W(v_2)$$

and from the second

$$Q=\tfrac{1}{4}v_1^2uu_2+\tfrac{1}{2}v_1uW_2.$$

Then from the third equation in (2.6), a straightforward computation yields

$$u^3u_{22}=A,$$
$$u^3W_{22}=B, \qquad (2.7)$$

where A and B are constants. Now we can integrate the first of these equations to give $u=(av_2^2+b)^{1/2}$. We consider the following cases:

(1) $a=0$: We can take $u=1$. Then by using equivalence under space translations, Galilei transformations, and dilatations, the coordinates can be brought to the form

$$x=v_1\pm v_2^2, \quad v_2=t \quad \text{with} \quad Q=\pm v_1v_2.$$

(2) $b=0$: We may take $u=v_2$ and similarly bring the coordinates to one of the forms

$$x=v_1v_2\pm 1/v_2, \quad t=v_2 \quad \text{with } Q=\tfrac{1}{4}v_1^2v_2v_1/2v_2,$$
$$x=v_1v_2, \qquad t=v_2, \quad Q=\tfrac{1}{4}v_1^2v_2.$$

(3) $a/b>0$, $a,b\ne 0$: Using dilatation, we can take $u=(v_2^2+1)^{1/2}$. Again using Galilei and space translation, we find

$$x=v_1(v_2^2+1)^{1/2}, \quad t=v_2, \quad Q=\tfrac{1}{4}v_1^2v_2.$$

(4) a, $b\ne 0$, $a/b<0$: Similarly we find

$$x=v_1|1-v_2^2|^{1/2}, \quad t=v_2, \quad Q=(\epsilon/4)v_1^2v_2,$$

where $\epsilon=\operatorname{sgn}(1-v_2^2)$. Thus we have shown that up to equivalence under the group $D\otimes G_1$, there are precisely seven coordinate systems such that (2.1) and hence the Hamilton–Jacobi equation (*) is separable. Moreover, these coordinates coincide with the separable coordinate system[2] for the Schrödinger equation $U_{xx}+iU_t=0$ and the heat equation $U_{xx}+U_t=0$. The list of separable coordinates is presented in Table II, where equivalences under the full Schrödinger group is also noted. It is also mentioned here that the separation of variables for (*) also implies the equivalence of the four types of potentials; i.e., free particle, linear potential, and attractive and repulsive harmonic oscillator. Indeed it is not difficult to give explicitly the transformations which map the time dependent Hamilton–Jacobi equation with a linear potential, attractive, or repulsive harmonic oscillator potential onto (*). Thus it follows also that their local symmetry groups of point transformations are all isomorphic to O(3, 2). A closer connection will be seen explicitly in the next section.

3. SIMILARITY SOLUTIONS

In this section we give a systematic treatment of similarity solutions of (*) by giving the solution which corresponds to each of the orbit representatives in Table I. We can then say that any similarity solution obtainable from point transformations must be related to one of our representative solutions by at most a transformation in $C^{1,2}$. Moreover, we will show how the orbits of the subalgebra s_1 relate to the method of separation of variables presented in Sec. 2, or more specifically that to each system of separable coordinates (ξ, τ), there corresponds an orbit representative of s_1 such that the similarity variable is ξ and the similarity solution is the solution obtained by the separation of variables of (*). In this way we will obtain complete integrals of (*). Any arbitrary parameter which has been transformed away by our orbit analysis can, of course, always be reinstated. As is well known,[11] then, the general solution can always be obtained by forming the envelope of any complete integral. It seems likely that all known explicit complete integrals of (*) can be obtained by group theoretical methods.

More generally let $f(x^i, u)=0$ be the graph of a solution u of (1.1a) and $X \in \mathcal{G}_p$; then f is called a *relative invariant* with respect to X if

$$\underset{X}{\pounds} f = X \lrcorner\, df = Xf = 0. \tag{3.1}$$

For every such f which satisfies (3.1), we can solve implicitly for u which when combined with the original differential equation (1.1a) reduces (1.1) to a differential equation with one less variable. Any solution u obtained in this way is called a *similarity solution*.[16] It is clear in general that in order to specify a unique solution for an equation in n independent variables, we must demand that f be a relative invariant for $n-1$ members X_α of g, $\alpha = 1, \ldots, n-1$. The X_α's need not commute, but owing to (3.1) they must form a subalgebra of g. Thus, the problem of finding complete similarity solutions relates to the problem of classifying all subalgebras of a given Lie algebra.[28] The preceeding discussion of similarity solutions has a simple geometric interpretation. We restrict ourselves here to R^3. Indeed (3.1) says that for any vector field X we construct surfaces in R^3 such that X lies in its tangent plane at each point. The tangent planes to all integral surfaces at a point intersect along X, i.e., X defines the characteristics of (3.1). If in addition X is a symmetry of a differential equation as given by (1.1a), $\underset{X}{\pounds}$ describes the infinitesimal dragging of the tangent plane to an integral surface of the equation (1.1a) in such a way that the tangent plane lines up with the tangent plane of another solution. For a general first order equation the possible tangent planes form a one-parameter family which envelops the Monge cone at a given point. Now, choosing a tangent plane defined by a generator of the Monge cone and X, we are guaranteed that, by moving along the curve generated by X, there will be a generator of the Monge cone which lies in the tangent plane at each point. In this way we describe an integral surface which satisfies both (1.1a) and (3.1). There are two qualifications to be made: First X cannot be collinear to the generator of the Monge cone; second X must not imply a relationship between the independent variables for (1.1a).

Now in the practical computation of relative invariants one uses the characteristic equations of a given vector field, viz.,

$$X = \xi^i(\mathbf{x}, u)\partial_{x^i} + \eta(\mathbf{x}, u)\partial_u, \tag{3.2}$$

then $u(\mathbf{x})$ can be obtained by solving

$$\frac{dx^1}{\xi^1(\mathbf{x}, u)} = \cdots = \frac{dx^n}{\xi^n(\mathbf{x}, u)} = \frac{du}{\eta(\mathbf{x}, u)}. \tag{3.3}$$

In our case any $X \in o(3, 2)$ takes the form

$$X = a(x, t)\partial_t + b(x, t, S)\partial_x + c(x, S)\partial_S, \tag{3.4}$$

where the coefficients can be read off from (1.11). The characteristic equations for (3.4) are then

$$\frac{dt}{a(x, t)} = \frac{dx}{b(x, t, S)} = \frac{dS}{c(x, S)}. \tag{3.5}$$

Solving any two of the Eqs. (3.5) when combined with (*) will then give the similarity solution corresponding to the vector field (3.4).

We now proceed to discuss the similarity solutions for the subgroup $\mathcal{S}_1$ and their relation to the separation of variables of the previous section. For the s_1 subalgebra we see from (1.11) that both b and c are independent

TABLE II. Separable coordinates (*) classified under $D \otimes G_1$. Subgroupings indicate equivalence under $\mathcal{S}_1$.

Coordinates	Multiplier	Operator	Remarks	
$x=v_1,\quad t=v_2$	$Q=0$	$X_1+\epsilon X_6$	$\epsilon=\pm 1,\ 0$	
$x=v_1v_2,\quad t=v_2$	$Q=\frac{1}{4}v_1^2v_2$	$X_3+\epsilon X_6$	"	
$x=v_1+\epsilon v_2^2,\quad t=v_2$	$Q=\epsilon v_1v_2$	$X_1+\epsilon X_5$	"	
$x=v_1v_2+\epsilon/v_2,\quad t=v_2$	$Q=v_1^2v_2/4-\epsilon v_1/2v_2$	$X_3+\epsilon X_4$	"	
$x=v_1v_2^{1/2},\quad t=v_2$	$Q=0$	$2X_2+aX_6$		$-\infty<a<\infty$
$x=v_1\lvert 1-v_2^2\rvert^{1/2},\quad t=v_2$	$Q=\frac{1}{4}\epsilon v_1^2v_2$	$X_1-X_3+aX_6$	$\epsilon=\mathrm{sgn}(1-v_2^2)$	"
$x=v_1\lvert 1+v_2^2\rvert^{1/2},\quad t=v_2$	$Q=v_1^2v_2/4$	$X_1+X_3+aX_6$		"

of S (transformations which act linearly on $w=e^S$), and a is independent of x. Thus we can integrate the first two of Eqs. (3.5) to give the similarity variable $\xi=\xi(x,t)$. Then expressing x as a function of ξ and t, we have

$$dS=\frac{C(x(\tau,t))}{a(x)}dt. \tag{3.6}$$

Integrating along the characteristic ξ, we obtain S as

$$S=\int_{\xi=\text{const}}\frac{C(x(\xi,t))}{a(t)}dt+F(\xi). \tag{3.7}$$

Substituting (3.7) back into (*) yields a first order ordinary differential equation for F which can then be integrated to give the explicit similarity solution.

As mentioned previously, it is the geometric subgroup $D\otimes\mathcal{G}_1$, which is relevant for the separation of variables; therefore, we consider the orbit representatives given by (1.13) for the similarity solutions. We will see that for each orbit in (1.13) the similarity variable ξ will correspond precisely to the variable v_1 for one of the separable coordinate systems listed in Table II, although there are degenerate cases. The separation constant corresponds to the parameter a in (1.13), i.e., to the one-parameter extensions by the central element X_6. In some cases the separation constant can be transformed to ± 1 or 0 by a member of $D\otimes\mathcal{G}_1$ which alters only slightly the functional form of the solution. We also group together those orbits (1.14) and separable systems which are inequivalent under the Schrödinger group $\mathcal{S}_1$. As in Refs. 2, 3, these systems are denoted by the appelations, harmonic oscillator, repulsive harmonic oscillator, free particle, and linear potential, since they reduce (*) to the time-independent Hamilton–Jacobi equation with the corresponding type of potential. Within this grouping we label by 1 and 2 coordinates which are equivalent under $\mathcal{S}_1$ but inequivalent the subgroup $D\otimes\mathcal{G}_1$ since they appear differently from a geometric point of view. We will give the details for the first case only.

A. Harmonic oscillator

The separable coordinates are

$$\xi=v_1=x/(1+t^2)^{1/2},\quad \tau=v_2=t. \tag{3.8}$$

Substituting these into (*) and using the ansatz

$$S=\tfrac{1}{4}\xi^2\tau+F(\xi)+G(\tau), \tag{3.9}$$

we obtain

$$F_\xi^2+\tfrac{1}{4}\xi^2+(1+\tau^2)G_\tau=0. \tag{3.10}$$

Separation implies

$$(1+\tau^2)G_\tau=a, \tag{3.11a}$$

which reduces (3.9) to the time-independent Hamilton–Jacobi equation with a harmonic oscillator potential

$$F_\xi^2+\tfrac{1}{4}\xi^2+a=0. \tag{3.11b}$$

Integrating Eqs. (3.11) and placing into (3.9), we find

$$S=\tfrac{1}{4}\xi^2\tau+a\tan^{-1}\tau-a\sin^{-1}(\xi/2\sqrt{-a})+\tfrac{1}{2}\sqrt{-a}\,\xi(1+\xi^2/4a)^{1/2}. \tag{3.12}$$

From the point of view of similarity solutions it is easy to see that the coordinates (3.8) correspond to the orbit $X_1+X_3+aX_6$ of (1.13) for which Eq. (3.5) is

$$\frac{dt}{1+t^2}=\frac{dx}{tx}=\frac{dS}{\frac{1}{4}x^2+a}. \tag{3.13}$$

The first two of these equations give precisely the variable ξ of (3.8), while the first and third [or what amounts to (3.7)] gives, integrating along the characteristic ξ, (3.9) with $G=a\tan^{-1}\tau$. Then substituting (3.9) back into (*) gives (3.11b) and hence the similarity solution (3.12). We point out that the case $a=0$ is degenerate.

B. Repulsive harmonic oscillator

(1) The separable coordinates are

$$\xi=v_1=x/|t|^{1/2},\quad \tau=v_2=t, \tag{3.14}$$

which correspond to the orbit $2X_2+aX_6$ whose subsidiary conditions are

$$\frac{dt}{2t}=\frac{dx}{x}=\frac{dS}{a}. \tag{3.15}$$

Integrating (3.15) gives

$$S=\tfrac{1}{2}a\ln\tau+F(\xi), \tag{3.16a}$$

which upon substituting into (*) gives

$$F_\xi^{\,2}-\tfrac{1}{2}\xi F_\xi+a=0, \tag{3.16b}$$

yielding the solutions

$$\begin{aligned} S&=\tfrac{1}{2}a\ln\tau+\tfrac{1}{8}\xi^2-a\cosh^{-1}\left(\frac{\xi}{\sqrt{8a}}\right)+\left(\frac{a}{8}\right)^{1/2}\xi\left(\frac{\xi^2}{8a}-1\right)^{1/2},\quad \tau>0\\ S&=\tfrac{1}{2}a\ln\tau-\tfrac{1}{8}\xi^2-a\sinh^{-1}\left(\frac{\xi}{\sqrt{8a}}\right)+\left(\frac{a}{8}\right)^{1/2}\xi\left(\frac{\xi^2}{8a}+1\right)^{1/2},\quad \tau<0. \end{aligned} \tag{3.17}$$

Again the case $a=0$ is degenerate.

(2) The separable coordinates are

$$\xi=v_1=x/|t^2-1|^{1/2},\quad \tau=v_2=t, \tag{3.18}$$

corresponding to the orbit $X_1-X_3+aX_6$ in (1.13) whose equations are

$$\frac{dt}{t^2-1}=\frac{dx}{tx}=\frac{dS}{\frac{1}{4}x^2-a}. \tag{3.19}$$

Integrating, we obtain

$$\begin{aligned} S&=\tfrac{1}{4}\xi^2\tau+a\coth^{-1}\tau+F(\xi),\qquad \tau^2>1,\\ S&=\tfrac{1}{4}\xi^2\tau+a\tanh^{-1}\tau+F(\xi),\qquad \tau^2<1, \end{aligned} \tag{3.20a}$$

where $F(\xi)$ satisfies

$$F_\xi^{\,2}-\tfrac{1}{4}\xi^2-\operatorname{sgn}(\tau^2-1)a=0, \tag{3.20b}$$

leading to the solutions

$$S=\tfrac{1}{4}\xi^2\tau+a\coth^{-1}\tau+a\sinh^{-1}\left(\frac{\xi}{2\sqrt{a}}\right)+\tfrac{1}{2}\sqrt{a}\,\xi\left(\frac{\xi^2}{4a}+1\right)^{1/2},\quad \tau^2>1,$$

$$S=-\tfrac{1}{4}\xi^2\tau+a\tanh^{-1}\tau-a\cosh^{-1}\left(\frac{\xi}{2\sqrt{a}}\right)$$
$$+\tfrac{1}{2}\sqrt{a}\,\xi\left(\frac{\xi^2}{4a}-1\right)^{1/2},\quad \tau^2<1. \qquad (3.21)$$

As mentioned previously cases (1) and (2) are related by a transformation in S_1. The transformation which takes (3.21) into (3.17) is given by

$$t'=\frac{1+t}{1-t},\quad x'=\frac{2^{1/2}x}{(1-t)},\quad S'=S+\frac{x^2}{4(1-t)}. \qquad (3.22)$$

It is also mentioned that Eq. (3.16b) can be cast into the form of a repulsive harmonic oscillator by replacing F by $F\pm\xi^2/8$. Again in both cases (1) and (2), $a=0$ is degenerate.

C. Free particle

(1) The separable coordinates are

$$\xi=v_1=x/t,\quad \tau=v_2=t, \qquad (3.23a)$$

corresponding to the orbit $X_3+\epsilon X_6$ in (1.13).

The subsidiary conditions (3.5) are

$$\frac{dt}{t^2}=\frac{dx}{tx}=\frac{dS}{\frac{1}{4}x^2+\epsilon}, \qquad (3.23b)$$

giving rise to

$$S=\tfrac{1}{4}\xi^2\tau-\epsilon/\tau+F(\xi), \qquad (3.24a)$$

where

$$F_\xi^{\,2}+\epsilon=0. \qquad (3.24b)$$

Thus we have the solution

$$S=\tfrac{1}{4}\xi^2\tau-\epsilon/\tau\pm\sqrt{-\epsilon}\,\xi. \qquad (3.25)$$

(2) The coordinates are simply the usual Cartesian ones $\xi=x$, $\tau=t$, corresponding to the orbit representative $X_1+\epsilon X_6$ whose equations are

$$\frac{dt}{1}=\frac{dx}{0}=\frac{dS}{\epsilon}, \qquad (3.26)$$

giving rise to

$$S=\epsilon t+F(x) \qquad (3.27a)$$

with

$$F_x^{\,2}+\epsilon=0. \qquad (3.27b)$$

Hence, the similarity solution is simply

$$S=\epsilon t\pm\sqrt{-\epsilon}\,x+c. \qquad (3.28)$$

Here we allow $\epsilon=0$ as well as $\epsilon=\pm1$ so as to include the degenerate orbits X_3 and X_1.

D. Linear potential

(1) The separable coordinates are

$$\xi=v_1=\frac{x}{t}+\frac{\epsilon}{2t^2},\quad \tau=v_2=t, \qquad (3.29a)$$

corresponding to the orbit $X_3+\epsilon X_4$ with the subsidiary conditions

$$\frac{dt}{t^2}=\frac{dx}{tx+\epsilon}=\frac{dS}{\frac{1}{4}x^2}, \qquad (3.29b)$$

which gives rise to

$$S=\frac{\xi^2\tau}{4}+\frac{\epsilon\xi}{4\tau}-\frac{\epsilon^2}{48\tau^3}+F(\xi) \qquad (3.30a)$$

with

$$F_\xi^{\,2}-\tfrac{1}{2}\epsilon\xi=0. \qquad (3.30b)$$

Integrating (3.41b), we find the solution

$$S=\frac{\xi^2\tau}{4}+\frac{\epsilon\xi}{4\tau}-\frac{\epsilon^2}{48\tau^3}+\tfrac{1}{3}\sqrt{2\epsilon}\,\xi^{3/2}. \qquad (3.31)$$

(2) The separable coordinates are

$$\xi=v_1=x-\tfrac{1}{2}\epsilon t^2,\quad \tau=v_2=t, \qquad (3.32)$$

corresponding to $X_1+\epsilon X_5$ with the equations

$$\frac{dt}{1}=\frac{dx}{\epsilon t}=\frac{dS}{\frac{1}{2}\epsilon x}. \qquad (3.33)$$

Integrating, we find

$$S=\frac{\epsilon\xi\tau}{2}+\frac{\epsilon^2\tau^3}{12}+F(\xi), \qquad (3.34a)$$

with

$$F_\xi^{\,2}+\tfrac{1}{2}\epsilon\xi=0, \qquad (3.34b)$$

giving rise to the solution

$$S=\tfrac{1}{2}\epsilon\xi\tau+\tfrac{1}{12}\epsilon^2\tau^3+\tfrac{1}{3}\sqrt{-2\epsilon}\,\xi^{3/2}. \qquad (3.35)$$

Again we allow $\epsilon=0$ as well as ±1 in order to include the degenerate cases. The group transformation which takes (3.35) to (3.31) and (3.28) to (3.25) is

$$t'=-1/t,\quad x'=x/t,\quad S'=S-x^2/4t. \qquad (3.36)$$

It can be seen that this is the square of the transformation (3.22).

The remaining orbits in (1.13) and (1.14) are degenerate in the sense that they give rise to special cases. X_4 gives the usual cartesian separation and the special solution $S=$const, where as X_5 which is equivalent to X_4 under S_1, gives the degenerate solution $a=0$ in (3.17). A relative invariant of X_6 violates the condition that x and t be independent (in involution). However, we should notice that it does not violate the independence of x and z in (**) and thus gives rise to a nontrivial solution. It is interesting that under the full conformal group these cases are equivalent to those already discussed. In fact under $C^{1,2}$ we have only the four types given by the potentials and their degenerate cases as noted in (1.15). We can always set the separation constant equal to ±1 or 0.

As mentioned previously the subalgebra of o(3,2) generated by $X_1,\ldots,X_7$ is maximal and contains s_1. Moreover, its structure is $gl(2,R)\otimes w_1$, but now X_6 is not in the center. However, we notice from (1.11) that for this subalgebra the coefficient b given by (3.4) still has no S dependence; hence, we should obtain a similarity variable $\xi(x,t)$ upon integrating the first two of Eqs. (3.4). Indeed this suggests that there may be some type of separation of variables not considered in Sec. 2 which lead to these solutions. We will now show that this is indeed the case. We will only consider orbits inequivalent under the full conformal group $C^{1,2}$; however, we expect that again classifying the subalgebra $\mathrm{gl}(2,R)\oplus w$, under its subgroup $D\otimes G_1$, will lead to a more geometric picture compatible with the separation of variables. From Table I we pick out the following orbit representatives of $\mathrm{gl}(2,R)\otimes w_1$ which are not in s_1:

(1) $X_2+aX_7,\quad -1\le a\le 1,\quad a\ne 0$: The subsidiary equations are

$$\frac{dt}{t}=\frac{dx}{\frac{1}{2}(a+1)x}=\frac{dS}{aS}. \tag{3.37}$$

The similarity variable is

$$\xi=t^{-(a+1)/2}x, \tag{3.38a}$$

giving the form

$$S=\xi^a F(\xi). \tag{3.38b}$$

Plugging (3.48b) back into (*), we find that $F(\xi)$ satisfies

$$F_\xi{}^2-\tfrac{1}{2}(a+1)\xi F_\xi+aF=0. \tag{3.38c}$$

Thus we see that we have the separation of (*) in the form of a product instead of a sum. If we look into the separation process in some detail, we will see that the conditions for separation involve a coupling between the coordinate functions (2.2) and the separable solution in the variable $v_2=t$. For this reason this type of separation is much more complicated and usually not considered for equations of the kind of (*). However, here we are led to these naturally by considering similarity solutions. Now Eq. (3.38c) is a special case of Chrystal's equation[29] whose solution is given implicitly by

$$F=\left[\frac{(a+1)^2}{4}-u^2\right]\frac{\xi^2}{4a},$$
$$\xi[u\mp(a+1)/2]^{(a+1)/2}[u\mp(a-1)/2]^{(1-a)/2}=C \tag{3.39a}$$

with $a\ne\pm1$ and C a constant. For the degenerate cases $a=\pm1$ we have the regular solutions

$$F=\tfrac{1}{2}(\xi+C)^2,\quad a=-1, \tag{3.39b}$$

$$F=-\tfrac{1}{4}C^2\mp\tfrac{1}{2}C\xi,\quad a=1, \tag{3.39c}$$

and, for $a=1$, the singular solution[29]

$$F=\tfrac{1}{4}\xi^2+C,\quad a=1. \tag{3.39d}$$

(2) $X_1+X_3+aX_7,\quad 0<a<\infty$: The Pfaffian equations are

$$\frac{dt}{1+t^2}=\frac{dx}{tx+\frac{1}{2}ax}=\frac{dS}{\frac{1}{4}x^2+aS}, \tag{3.40}$$

which upon integrating the first two of these equations gives the similarity variable

$$\xi=\frac{x}{(1+t^2)^{1/2}}\left(\frac{1+it}{1-it}\right)^{ia/4}, \tag{3.41a}$$

while the first and third gives

$$S=\frac{\xi^2\tau}{4}\left(\frac{1+i\tau}{1-i\tau}\right)^{-ia/2}+\left(\frac{1+i\tau}{1-i\tau}\right)^{-ia/2}F(\xi), \tag{3.41b}$$

where F satisfies

$$F_\xi{}^2-\tfrac{1}{2}a\xi F_\xi+aF+\tfrac{1}{4}\xi^2=0. \tag{3.42c}$$

This equation has the form of the general Chrystal's equation.[29] Its general solution is given by

$$\xi^2(u-2i\mp\tfrac{1}{2}a)^{1\pm a/2i}(u+2i\mp\tfrac{1}{2}a)^{-1\pm a/2i}=C, \tag{3.43a}$$

where C is an arbitrary constant and

$$F=(\xi^2/4a)(\tfrac{1}{4}a^2-1-u^2). \tag{3.43b}$$

We mention that (3.42c) has a singular solution which we ignor since it occurs when a is pure imaginary.

(3) $X_2+X_5+X_7$: The subsidiary equations are

$$\frac{dt}{t}=\frac{dx}{t+x}=\frac{dS}{S+\frac{1}{2}x}, \tag{3.44}$$

which yields the similarity variable

$$\xi=x/t-\ln t \tag{3.45a}$$

and the form

$$S=\tfrac{1}{2}\xi\tau\ln\tau+\tfrac{1}{4}\tau\ln^2\tau+\tau F(\xi), \tag{3.45b}$$

where F satisfies

$$F_\xi{}^2-(\xi+1)F_\xi+\tfrac{1}{2}\xi+F=0. \tag{3.45c}$$

The general solution of this equation is given by

$$(\pm u-1)\exp(\pm u-1)=C^{e^\xi}, \tag{3.46a}$$

where C is an arbitrary constant and

$$F=\tfrac{1}{4}(1+\xi^2-u^2). \tag{3.46b}$$

Thus it is seen that the remaining two cases [(2) and (3) above] separate in the product form with an additional multiplier term $Q(\xi,\tau)$ present.

There now remains from Table I only three cases of orbit representatives of $o(3,2)$ which are not in $\mathrm{gl}(2,R)\otimes w_1$. Of these the first two to be considered are in fact closer related to (**).

(4) $X_5-X_{10}+X_1$: The Pfaffian subsidiary equations are

$$\frac{dt}{1-\frac{1}{2}x}=\frac{dx}{t-S}=\frac{dS}{\frac{1}{2}x} \tag{3.47a}$$

or alternatively in terms of $z=t-S$, $T=t+S$, we have

$$\frac{dz}{1-x}=\frac{dx}{z}=\frac{dT}{1} \tag{3.47b}$$

from which we find the similarity variable

$$\xi^2=z^2+(x-1)^2 \tag{3.48a}$$

and the solution

$$T=\sin^{-1}[(x-1)/\xi]+F(\xi), \tag{3.48b}$$

where F satisfies

$$\xi^2F_\xi^2+1-\xi^2=0, \tag{3.48c}$$

giving rise to the general solution

$$T=\sin^{-1}[(x-1)/\xi]+\xi^2-1-\tan^{-1}\xi^2-1+C. \tag{3.49}$$

Clearly this case is related to the separation of (**) in polar coordinates.

(5) $X_5-X_{10}+a(X_2+X_7)$: The subsidiary equations are

$$\frac{dt}{at-\frac{1}{2}x}=\frac{dx}{t-S+ax}=\frac{dS}{\frac{1}{2}x+aS} \tag{3.50a}$$

or in terms of z and T

$$\frac{dz}{az-x}=\frac{dx}{z+ax}=\frac{dT}{aT}. \tag{3.50b}$$

From the first two equations the similarity variable is

$$\xi=(x^2+z^2)^{1/2}\left(\frac{z-ix}{z+ix}\right)^{a/2i} \tag{3.51a}$$

while the second two equations give the form

$$T=\left(\frac{z+ix}{z-ix}\right)^{a/2i}F(\xi), \tag{3.51b}$$

where $F(\xi)$ satisfies

$$(a^2+1)\xi^2F_\xi{}^2-2a^2\xi F_\xi F+a^2F^2-\xi^2=0. \tag{3.51c}$$

The general solution of this equation is given implicitly by

$$\frac{a}{a^2+1}\operatorname{sgn}\xi=\left(\frac{aF+i(a^2+1)\xi^2-a^2F^2}{aF-i(a^2+1)\xi^2-a^2F^2}\right)^{\pm a/2i}\times\left|\frac{F}{F\pm(a^2+1)\xi^2-a^2F^2}\right|. \tag{3.52a}$$

The case $a=0$ is degenerate and leads to

$$T=\pm x^2+z^2+C, \tag{3.52b}$$

which in terms of S gives a certain translation in S and t of the fundamental solution $x^2/4t$.

(6) $X_1+X_3+a(X_6+X_9)$: The Pfaffian equations are

$$\frac{dt}{1+t^2+\frac{1}{4}ax^2}=\frac{dx}{x(t+aS)}=\frac{dS}{a(1+S^2)+\frac{1}{4}x^2}. \tag{3.53}$$

We have not been able to find a simple way to integrate these equations explicitly. This ends the list of similarity solutions for (*). We mention also that it would be interesting to see if there is any relation (perhaps of a projective nature) between the solutions presented here and the semisubgroup separation of variables for the graph equation (1.10b) and hence the wave equation in 3-space.[4]

Before ending this section we briefly comment on one other solution generated by a symmetry, namely the general solution generated by the characteristcis. However, since (*) is not quasilinear, this solution cannot be written as a similarity solution. The characteristics for any first order equation are determined from the Eqs. (1.5a) or equivalently from the characteristic vector fields (1.7). The relative invariant[30] obtained from the vector fields in $\mathcal{G}/\mathcal{J}$ given by

$$Y=\eta(x,t,S,p,q)(2p\partial_x+\partial_t+p^2\partial_S) \tag{3.54a}$$

is determined by the equations

$$\frac{dx}{2p}=\frac{dt}{1}=\frac{dS}{p^2}=\frac{dp}{0}=\frac{dq}{0}, \tag{3.54b}$$

giving rise to the general solution of (*) in terms of the characteristic strips[11]

$$S=p^2t+F(x-2pt,p), \tag{3.54c}$$

where F is an arbitrary function of its arguments. Indeed the above analysis can be made much simpler if we use the characteristic $\xi=x-2pt$ as an underlying variable. We can consider (*) to be generated by the ideal

$$r=p^2+q=0,\quad dr=0, \tag{3.55a}$$

$$dx\wedge dp+dt\wedge dq=0. \tag{3.55b}$$

Then clearly $dx=d\xi+2p\,dt+2t\,dp$, so (3.55b) becomes

$$d\xi\wedge dp+dt\wedge dr=0, \tag{3.55c}$$

which implies the existence of a function $V(\xi,t)$ with $p=V_\xi$ and $r=V_t$. Then (3.55a) implies that it is independent of t, and thus the general solution is given by

$$p=V_\xi(\xi), \tag{3.55d}$$

which is equivalent to (3.54c) as long as $d\xi\wedge dt\neq 0$. In fact, it can easily be seen that $V(\xi)$ is equal to F in (3.54c), modulo an additive constant. In the next section we will see that (3.55d) is closely related to prolongations of (*).

4. PROLONGATIONS

The concept of prolongation was first introduced by Cartan[18,31] in his study of what has since been called infinite pseudogroups. His idea was to obtain and classify certain pseudogroups (infinite groups in Cartan's language) by taking successively higher derivatives of Lie's differential equations for finite Lie groups. Indeed a classification of certain types of pseudogroups has by now been rigorously established, using essentially this idea.[13,32] However, here we wish only to apply the first prolongation of (*), that is we take the derivative with respect to x of (*) and notice that it gives precisely (***). The question that is raised is then what is the connection between the symmetries of (*) and (***)? We do not intend to give here a full analysis of this question but only to point out some interesting relationships.

Since (***) is a first order quasilinear partial differential equation, the analysis performed in the beginning of Sec. 1 applies. We are only interested in the point transformation symmetries of (***) since only these can be projected to symmetries of $R^2\times R^1$ with local coordinates (x,t,p). Then, using (1.4) and (1.5), we find the pseudogroup of point transformations of (***) to be generated by the vector fields (projections onto $R^2\times R^1$)

$$X=2pF^1(x,t,p)\partial_x-2pF^0(x-2pt,p)\partial_p+\left[F^1(x,t,p)+\frac{x+2pt}{2p}F^0(x-2pt,p)+g(x-2pt,p)\right]\partial_t, \tag{4.1}$$

where F^0, F^1, g are arbitrary function of their arguments. It is easy to see that the ideal I of characteristic vector fields of (***) is generated by $F^1(x,t,p)$.

We now look for those members of the symmetry algebra $\mathcal{G}_*$ of * given by (1.4) and (1.5b) which can be related to a subalgebra of (4.1) whose vector fields when prolonged[33] to act on the variables S and q can be identified with a subalgebra of $\mathcal{G}_*$. This prolongation can be accomplished through the use of (1.4c) and (1.4d) and give precisely those vector fields in $\mathcal{G}_*$ for which X^x, X^t, and X^p are independent of S and q. A straightforward computation gives constraints on the vector fields (4.1) which imply the existence of a function $H(x-2pt,p)$

such that

$$F^0 = \frac{1}{2p} H_x = \frac{1}{2p'} H_t,$$
$$g = \frac{1}{2p}\left(H_{p'} + \frac{\xi}{2p'} H_t + c\xi\right), \tag{4.2}$$

where c is a constant and we use the change of variables $p' = p$, $\xi = x - 2pt$. The prolongation to the S and q components of the vector fields now proceeds via (1.4c) and (1.4d) respectively. These prolonged vector fields can be written

$$\begin{aligned}
X^x &= 2p\tilde{F}^1(x,t,p) + cx,\\
X^t &= \tilde{F}^1 - (1/2p)H_p + ct,\\
X^p &= -H_x,\\
X^q &= -2pH_x + E^0(x,t,S,p,q)(p^2+q)\\
X^S &= p^2\tilde{F}^1(x,t,p) + \tfrac{1}{2}pH_p - H + cS\\
&\quad + E^1(x,t,S,p,q)(p^2+q),
\end{aligned} \tag{4.3}$$

where $\tilde{F}^1$ is an arbitrary function of its arguments and E^1 and E^0 are arbitrary except for being nonsingular, at $p^2 + q = 0$. Again as in Sec. 1 it is convenient to factor these terms out and use (4.3) modulo E^0 and E^1. We now consider some explicit examples.

The first example to be considered is the characteristic collineation given by the arbitrary functions F^1 in (4.1). For (***) this gives rise to the general solution

$$p = f(x - 2pt). \tag{4.4}$$

Now the prolonged vector fields given by $\tilde{F}^1$ in (4.3) will generate the general solution of (*) given by (3.54c) or (3.55d). In fact we can easily identify f in (4.4) with $V_\xi = V_x$ in (3.55d).

As another example we consider those point transformation symmetries of (***) which can be prolonged to point transformation symmetries of (*) or vice-versa. These can be found by simply demanding the condition that the x, t, and S components of the vector fields in (4.3) be independent of p and q. Through a straightforward calculation we arrive at a finite-dimensional subalgebra spanned by the vector fields[34]

$$\begin{aligned}
&Y_1 = \partial_t, \quad Y_4 = \partial_x,\\
&Y_2 = t\partial_t + \tfrac{1}{2}x\partial_x - \tfrac{1}{2}p\partial_p, \quad Y_5 = t\partial_x + \tfrac{1}{2}\partial_p,\\
&Y_3 = t^2\partial_t + tx\partial_x + (\tfrac{1}{2}x - tp)\partial_p,\\
&Y_7 = \tfrac{1}{2}x\partial_x + \tfrac{1}{2}p\partial_p.
\end{aligned} \tag{4.5}$$

We have used a notation suggested by (1.15); the prolongation of (4.5) by adding the q and S components via (4.3) gives precisely the corresponding Y's in (1.15). Conversely, we can obtain the above vector fields from the corresponding ones in (1.15) [the subalgebra $\mathrm{gl}(2,R) \otimes w$] by lifting the latter to $T^*(R^2) \times R^1$ and projecting onto a surface with S and q constant. We notice that Y_6 is missing from (4.5) since X_6 projects to the identity for constant S, i.e., $Y_6 = 0$. The structure of the generators (4.5) is $\mathrm{gl}(2,R) \oplus a_2$, where a_2 is a two-dimensional Abelian ideal generated by Y_4 and Y_5. Hence the prolongation process *does not* conserve Lie brackets. However, X_6 generates a one-dimensional ideal of $\mathrm{gl}(2,R) \otimes w$, and thus there is a Lie algebra isomorphism between the factor algebra $[\mathrm{gl}(2,R) \otimes w]/\{X_6\}$ in (1.15) and $\mathrm{gl}(2,R) \otimes a_2$ given by (4.5). Of course, the subalgebra $s_1 \sim \mathrm{sl}(2,R) \otimes w$ of $\mathcal{G}_*$ obtained by removing X_7 is a central extension of the subalgebra $\mathrm{sl}(2,R) \otimes a_2$ of $\mathcal{G}_{***}$ obtained by removing Y_7 from (4.5).

Now there is an interesting connection between the similarity solutions of (4.5) and the corresponding ones for (1.15) given in Sec. 3. Indeed the orbit representatives of $\mathrm{gl}(2,R) \otimes a_2$ under the adjoint action of the group are

$$\begin{aligned}
&Y_2 + aY_7 \quad (a \geq 0), \quad Y_1 + Y_3 + aY_7 \quad (-\infty < a > \infty)\\
&Y_1 + Y_5, \quad Y_2 + Y_5 + Y_7, \quad Y_1 + Y_7, \quad Y_1, \quad Y_7.
\end{aligned} \tag{4.6}$$

Comparing (4.6) with the orbit representatives of $\mathrm{gl}(2,R) \otimes w$ in Table I and considering the factor algebra $[\mathrm{gl}(2,R) \otimes w]/\{X_6\}$, we see that the only difference is the appearance of $Y_1 + Y_7$ and Y_7 and the ranges of a in (4.6). This is so since X_7 and $X_1 + X_7$ are conformally equivalent to X_2 and $X_2 + X_6$ respectively. Similarly, the differences in the ranges of a are explained by conformal equivalence. Now the connection of the corresponding similarity solutions of (*) and (***) is this: Take the x derivative of one of the similarity solutions in $\mathrm{gl}(2,R) \otimes w$ obtained in Sec. 3 and put $p = S_x$; then this solution is precisely the similarity solution obtained from the corresponding orbit representative in $\mathrm{gl}(2,R) \otimes a_2$ for (***). It should be added that the multiple of X_6 for a similarity solution of (*) becomes an integration constant for the corresponding similarity solution of (***). A simple example should illustrate the point. Consider the similarity solution for $2X_2 + X_6$ given by (3.17). Considering only $t > 0$, we find

$$p = S_x = \frac{1}{2t^{1/2}} \quad \frac{\xi}{2} + \frac{\xi^2}{4} - 4a \quad , \quad \xi = \frac{x}{t^{1/2}}, \tag{4.7a}$$

which is the similarity solution of (***) obtained from $2Y_2$ with proper identification of the integration constant. Indeed

$$\frac{dt}{2t} = \frac{dx}{x} = \frac{-dp}{p} \tag{4.7b}$$

gives the similarity variable $\xi = x/t^{1/2}$ and

$$p = t^{-1/2} f(\xi). \tag{4.7c}$$

If we call $f = F_\xi$ and substitute (4.7c) into (***), we get

$$2F_\xi F_{\xi\xi} - \tfrac{1}{2}\xi F_{\xi\xi} - \tfrac{1}{2}F_\xi = 0, \tag{4.7d}$$

which is precisely the x derivative of (3.16b).

More generally, we can consider the entire subalgebra $\mathcal{H} \subset \mathcal{G}_{***}$ determined by (4.2). Now looking at (4.3) we see as before that we must not only factor out E^0 and E^1 but also the constant part in the function H, i.e., the generator X_6 in (1.15). That this can be done follows readily from the form of the generators in (4.3), namely, that the only S dependence of the vector fields in (4.3), mod (E^0, E^1), is of the type $S\partial_S$. Then the prolongation process defines an isomorphism of $\mathcal{H}$ onto the subalgebra of $\mathcal{G}_*$ given by (4.3) modulo the above equivalences. We mention that one can find nontrivial similarity solutions for (***) which upon integration give solutions of (*) and that the prolonged vector field corresponding to such solutions are vector fields on $T^*(R^2)\times R$ which are not the lifts of vector fields on R^3.

Finally we make a few comments on the members of $\mathcal{G}_*$ and $\mathcal{G}_{***}$ which are not related by prolongations. For example, looking at (1.15), we seen that all the members of o(3, 2) which cannot be prolonged to members of $\mathcal{G}_{***}$ are those vector fields whose components involve the variable S. Nevertheless, they yield similarity solutions of (*) for which we can determine, in principal, S and hence $p=S$ at nonsingular points, and are guaranteed that p will satisfy (***). Conversely, from those members of $\mathcal{G}_{***}$ that cannot be prolonged to symmetries of (*), we can also determine a p through the similarity methods which upon integration with respect to x provides a solution of (*). The problem is from the group theoretical standpoint that the prolongation process discussed above no longer gives a symmetry. However, they can be interpreted as generalized symmetries since they are a symmetry of one equation and give rise to solutions of both. In this connection it would be interesting to study further the symmetries of the complete prolonged ideal of differential forms which contains both (*) and (***) and possibly any further prolongations in the spirit of Ref. 35.

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[1]For a review see W. Miller, Jr., in *Proceedings of the Advanced Seminar on Special Functions* (Academic, New York, 1975). There is also a forthcoming book; W. Miller, Jr., *Symmetry and Separation of Variables for Linear Differential Equations* (Addison-Wesley, Reading, Mass., to appear). Also for a selective choice see Refs. 2–5.

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