Completely integrable relativistic Hamiltonian systems and separation of variables in Hermitian hyperbolic spaces

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The Hamilton–Jacobi and Laplace–Beltrami equations on the Hermitian hyperbolic space HH(2) are shown to allow the separation of variables in precisely 12 classes of coordinate systems. The isometry group of this two-complex-dimensional Riemannian space, SU(2,1), has four mutually nonconjugate maximal abelian subgroups. These subgroups are used to construct the separable coordinates explicitly. All of these subgroups are two-dimensional, and this leads to the fact that in each separable coordinate system two of the four variables are ignorable ones. The symmetry reduction of the free HH(2) Hamiltonian by a maximal abelian subgroup of SU(2,1) reduces this Hamiltonian to one defined on an O(2,1) hyperboloid and involving a nontrivial singular potential. Separation of variables on HH(2) and more generally on HH(n) thus provides a new method of generating nontrivial completely integrable relativistic Hamiltonian systems.

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I. INTRODUCTION

The purpose of this article is to discuss the separation of variables in the four (real) -dimensional Hermitian hyperbolic space HH(2) for the following two equations:

(i) The Hamilton–Jacobi equation (HJ)

$$\sum_{i,j} g^{ij} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^j} = E; \tag{1.1}$$

(ii) the Laplace–Beltrami equation (LB)

$$\Delta \psi = \sum_{i,j} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \sqrt{g}\, g^{ij} \frac{\partial}{\partial x^j} = \lambda \psi . \tag{1.2}$$

In a previous paper[1] (further to be referred to as I) we have considered the separation of variables in complex projective spaces $\mathbb{C}P(n)$. The isometry group of $\mathbb{C}P(n)$ is the compact group $SU(n+1)$, and its Cartan subgroup was used to generate n ignorable variables and to reduce the problem of variable separation on $\mathbb{C}P(n)$ to the separation of variables on the real sphere S^n. We refer to this paper for a discussion of the motivation and for some historical background.

Here let us just mention the relation between separation of variables in the HJ equation and complete integrability of the corresponding Hamiltonian system. Indeed, separability for the HJ equation is defined to mean that a solution S of (1.1) exists satisfying

$$S = \sum_i S_i(x^i, \lambda_1, \ldots, \lambda_n), \quad \det \frac{\partial^2 S}{\partial x^i \partial \lambda_j} \neq 0, \tag{1.3}$$

where λ_i are n constants: the separation constants. We associate n second-order operators in involution with each separable coordinate system in an n-dimensional space (one of them is the Hamiltonian); the constants λ_i are the eigenvalues of these operators. The existence of these operators assures that the system is integrable.

For studies of the separation of variables in Hamilton–Jacobi equations on Riemannian and pseudo-Riemannian manifolds, see also Refs. 2–5.

The additive separation of variables (1.3) in the HJ equation corresponds to multiplicative separation in the LB equation (1.2):

$$\psi = \prod_i \psi_i(x^i, \lambda_1, \ldots, \lambda_n). \tag{1.4}$$

Indeed, for Einstein spaces every coordinate system that separates the HJ equation will also separate the LB equation[2–4] (the converse is always true). Separation of variables in LB equations makes it possible to use powerful methods of group theory to study broad classes of special functions.[5–9]

II. THE SPACE HH(n) AND ITS ISOTROPY GROUP SU(n,1)

We introduce the Hermitian hyperbolic (or complex hyperbolic) space HH(n) following Kobayashi and Nomizu[10] and Helgason.[11] Let $(e_0, e_1, \ldots, e_n)$ be a standard basis in $\mathbb{C}^{n+1}$ and consider the Hermitian form

$$F(x,y) = -\bar{x}_0 y_0 + \sum_{k=1}^{n} \bar{x}_k y_k, \tag{2.1}$$

where the overbar denotes complex conjugation. This form is invariant under the action of the group $U(n,1)$:

$$g \in U(n,1), \quad F(gx, gy) = F(x,y), \quad x, y \in \mathbb{C}^{n+1}, \tag{2.2}$$

which acts transitively on the real hypersurface M in $\mathbb{C}^{n+1}$ defined by

$$F(y,y) = -1. \tag{2.3}$$

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The group $U(1) = \{e^{i\theta}\}$ acts freely on this manifold by $y \to e^{i\theta} y$. The space of orbits with suitable complex manifold structure and Kaehler metric is identified as HH(n). The corresponding natural projection $\pi: M \to \mathrm{HH}(n)$ defines a principal bundle with U(1) as structure group. The U(n,1) action commutes with that of U(1), and it hence projects to an action on the base HH(n). The isotropy subgroup of U(n,1) at the point $p_0 = \pi(e_0)$ is $U(1) \times U(n)$, and we obtain the diffeomorphism

$$U(n,1)/[U(n) \times U(1)] \sim \mathrm{HH}(n). \tag{2.4}$$

The group SU(n,1) acts almost effectively on this space.

In addition to the homogeneous coordinates $\{y_0, y_1, \ldots, y_n\}$, let us introduce affine coordinates on HH(n):

$$\pi(y_0, y_1, \ldots, y_n) = (z_1, \ldots, z_n), \quad z_k = y_k / y_0, \quad k = 1, \ldots, n. \tag{2.5}$$

The space HH(n) can then be identified with an open unit ball in $\mathbb{C}^n$

$$z \in \mathbb{C}^n, \quad \sum_{k=1}^{n} \bar{z}_k z_k < 1. \tag{2.6}$$

The real part of the Hermitian form (2.1) determines in a natural manner a metric on HH(n), which is the noncompact version of the well-known Fubini-Study metric[10]:

$$ds^2 = -\frac{4}{c} \times \frac{\left(1 - \sum \bar{z}_k z_k\right)\left(\sum d\bar{z}_k dz_k\right) + \left(\sum \bar{z}_k dz_k\right)\left(\sum z_k d\bar{z}_k\right)}{\left(1 - \sum \bar{z}_k z_k\right)^2}, \tag{2.7}$$

where $c < 0$ is the (constant) holomorphic sectional curvature.

We now limit ourselves to the case under consideration, namely $n = 2$.

The Hamiltonian associated with the metric (2.7) for $n = 2$ ($c = -4$) is

$$H = 4(1 - |z_1|^2 - |z_2|^2)[(|z_1|^2 - 1) p_1 \bar{p}_1 + (|z_2|^2 - 1) \times p_2 \bar{p}_2 + z_1 \bar{z}_2 p_1 \bar{p}_2 + \bar{z}_1 z_2 \bar{p}_1 p_2]. \tag{2.8}$$

The Lie algebra u(2,1) in the representation acting on the homogeneous coordinates (y_0, y_1, y_2) is realized by 3×3 complex matrices X satisfying

$$X^+ J + JX = 0, \quad J = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \tag{2.9}$$

(the superscript + denotes Hermitian conjugation).

Two convenient bases are given by the matrices X_i, or alternatively Y_i, $i = 0, 1, \ldots, 8$:

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = Y_7, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} = Y_8,$$

$$X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} = \frac{Y_1 + Y_6}{2},$$

$$X_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = Y_5, \quad X_5 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = Y_6 - Y_4,$$

$$X_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = Y_7 - Y_2,$$

$$X_7 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} = Y_8 - Y_3,$$

$$X_8 = \frac{i}{\sqrt{3}} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \frac{Y_1 - 3Y_6}{2\sqrt{3}},$$

$$X_0 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = Y_0 \tag{2.10}$$

[the Y_i basis is particularly appropriate for considering solvable subalgebras of su(2,1)].

With these conventions the second order Casimir operator of su(2,1) can be written as

$$\begin{aligned} C_2 &= X_1^2 + X_2^2 + X_3^2 - X_4^2 - X_5^2 - X_6^2 - X_7^2 + X_8^2 \\ &= \tfrac{1}{3} Y_1^2 - Y_2^2 - Y_3^2 - Y_4^2 - Y_5^2 \\ &\quad + \{Y_2, Y_7\} + \{Y_3, Y_8\} + \{Y_4, Y_6\}, \end{aligned} \tag{2.11}$$

where $\{\ ,\ \}$ denotes the anticommutator.

A Killing vector L on the cotangent bundle with local coordinates $(z_i, \bar{z}_i, p_i, \bar{p}_i, i = 1,2)$ is a linear polynomial in p_i, $\bar{p}_i$:

$$L = \sum_i c_i(z_1, z_2, \bar{z}_1, \bar{z}_2) p_i + \text{c.c.} \tag{2.12}$$

(where c.c. indicates the complex conjugate quantity), such that

$$[H, L]_P = 0, \tag{2.13}$$

i.e., the Poisson bracket of H with L is zero. The Killing vectors for HH(2) provide a realization of the algebra su(2,1). Using the basis X_i ($i = 1, \ldots, 8$) of (2.10) for the infinitesimal operators, we calculate the corresponding Killing vectors in affine and homogeneous coordinates to be, respectively,

$$\begin{aligned} X_1 &= -z_2 p_{z_1} + z_1 p_{z_2} + \text{c.c.} = -y_2 p_{y_1} + y_1 p_{y_2} + \text{c.c.}, \\ X_2 &= -i(z_2 p_{z_1} + z_1 p_{z_2}) + \text{c.c.} = i(y_2 p_{y_1} + y_1 p_{y_2}) + \text{c.c.}, \\ X_3 &= i(-z_1 p_{z_1} + z_2 p_{z_2}) + \text{c.c.} = i(y_1 p_{y_1} - y_2 p_{y_2}) + \text{c.c.}, \\ X_4 &= (z_1^2 - 1) p_{z_1} + z_1 z_2 p_{z_2} + \text{c.c.} = y_1 p_{y_0} + y_0 p_{y_1} + \text{c.c.}, \\ X_5 &= i[(z_1^2 + 1) p_{z_1} + z_1 z_2 p_{z_2}] + \text{c.c.} \\ &= i(-y_1 p_{y_0} + y_0 p_{y_1}) + \text{c.c.}, \\ X_6 &= z_1 z_2 p_{z_1} + (z_2^2 - 1) p_{z_2} + \text{c.c.} = y_2 p_{y_0} + y_0 p_{y_2} + \text{c.c.}, \end{aligned}$$

$$X_7 = i[z_1 z_2 p_{z_1} + (z_2^2+1) p_{z_2}] + \text{c.c.}$$
$$= i(-y_2 p_{y_0} + y_0 p_{y_2}) + \text{c.c.},$$
$$X_8 = i\sqrt{3}(z_1 p_{z_1} + z_2 p_{z_2}) + \text{c.c.} \qquad (2.14)$$
$$= (i/\sqrt{3})(2y_0 p_{y_0} - y_1 p_{y_1} - y_2 p_{y_2}) + \text{c.c.}$$

Throughout we shall make use of the moment map; whenever convenient we use the operators $\partial/\partial z_i$ or $\partial/\partial y_\mu$ instead of the functions P_{z_i} or P_{y_μ} and commutator brackets instead of Poisson brackets.

III. SUBGROUPS OF SU(2,1) AND COMPLETE SETS OF COMMUTING SECOND-ORDER OPERATORS

According to the operator approach to the separation of variables,[6–9] each separable system on HH(2) will be characterized by four second-order operators $\{H,T_1,T_2,T_3\}$ that are in involution with respect to the appropriate Lie bracket (one of them being the Hamiltonian H, or correspondingly the Laplace operator Δ). The first task is to classify the triplets of operators $\{T_1,T_2,T_3\}$ into equivalence classes under the action of the group SU(2,1), leaving H invariant.

The task in the present case of HH(2) is greatly simplified by two circumstances:

(1) It has recently been shown[12] that for HH(2) all second-order Killing tensors, i.e., operators

$$T = \sum_{i,k=1}^{2} \{c_{ik}(z_1,\bar{z}_1,z_2,\bar{z}_2)\, p_i\, p_k + d_{ik}(z_1,\bar{z}_1,z_2,\bar{z}_2)\, p_i\, \bar{p}_k + \text{c.c.}\}, \qquad (3.1)$$

satisfying

$$[T,H] = 0 \qquad (3.2)$$

lie in the enveloping algebra of su(2,1). Each of the operators T_i can hence be written in the form

$$T_i = \sum_{a,b=1}^{8} A^i_{ab} X_a X_b, \quad A^i_{ab} = A^i_{ba} \in \mathbb{R}. \qquad (3.3)$$

(2) We have shown in I, Theorem 4, that every separable coordinate system in $\mathbb{C}$P(2) and HH(2) has precisely two ignorable variables. We recall that an ignorable variable in a certain coordinate system is a variable that does not figure in the metric tensor g_{ik} expressed in this system.[4] An ignorable variable ϕ is obtained by setting a Killing vector, say L_1, equal to the momentum p_ϕ canonically conjugate to ϕ. The square of this Killing vector is then a second-order Killing tensor

$$T_1 = L_1^2 = p_\phi^2. \qquad (3.4)$$

This can be done[13] if the corresponding Killing tensor T_1 is the square of a Killing vector, i.e., in our case the square of an element of su(2,1). Since two variables must be ignorable in each separable coordinate system, it follows that two of the operators T_i, say T_1 and T_2, must be squares of elements of su(2,1):

$$T_1 = L_1^2 = \Big(\sum_{\alpha=1}^{8} a_\alpha X_\alpha\Big)^2,$$
$$T_2 = L_2^2 = \Big(\sum_{\alpha=1}^{8} b_\alpha X_\alpha\Big)^2. \qquad (3.5)$$

Since T_1 and T_2 commute, the operators L_1 and L_2 must generate an abelian subalgebra of su(2,1). All subalgebras of su[2,1) are known,[14] and work is in progress on the classification of the maximal abelian subalgebras (MASA's) of all classical Lie algebras.[15,16] In particular, su(2,1) has four different MASA's [each representing a conjugacy class of MASA's under the action of SU(2,1)]. Each of them is two-dimensional.

The procedure of finding all triplets of operators $\{T_1,T_2,T_3\}$ related to separable coordinates on HH(2) thus reduces to the following:

(i) Take T_1 and T_2 as in (3.5), where L_1 and L_2 run through the four different MASA's of su(2,1).

(ii) For each MASA L_1,L_2, find the most general operator $Q = T_3 \in S^2(\text{su}(2,1))$ [second-order symmetric tensor in the enveloping algebra of su(2,1)] commuting with L_1 and L_2. The operator T_3 has the form (3.3).

(iii) Simplify each T_3 by linear combinations, with L_1^2, L_2^2, L_1L_2 and C_2 (2.11) and classify the operators T_3 into conjugacy classes under the action of the normalizer of $\{L_1,L_2\}$ in SU(2,1) (the normalizer is the group of transformations leaving the algebra $\{L_1,L_2\}$ invariant).

A particularly important and simple class of coordinates are called "subgroup type coordinates,"[5,7,8,13] and they occur when T_3 is the Casimir operator of a subgroup of SU(2,1).

In Fig. 1 we show all subalgebras of su(2,1) that are relevant for our purposes (for a complete classification see Ref. 14). The basis elements $\{X_\alpha\}$ and $\{Y_\alpha\}$ are defined in (2.10), we use the two bases interchangeably. The lowest row in Fig. 1 is occupied by the four MASA's: $\{X_3,X_8\}$ and $\{Y_1,Y_6-Y_4\}$ are the compact and noncompact Cartan subalgebras, respectively, $\{Y_1,Y_4\}$ contains a nilpotent element

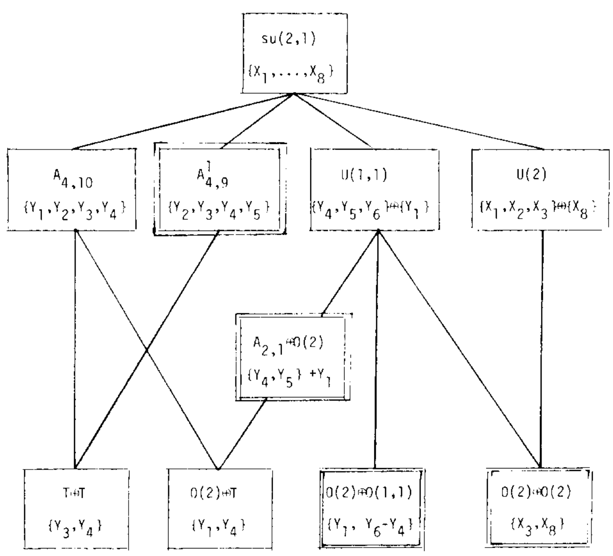

FIG. 1. Maximal abelian subalgebras of su(2,1) and some subalgebras containing them. The basis elements X_i and Y_i are defined in (2.10). The four MASA's constitute the lowest row, and double boxes indicate their normalizers; $A_{4,10}$, $A^1_{4,9}$, and $A_{2,1}$ are solvable algebras, and T denotes a translation type subalgebra.

Y_4 (Y_4 is represented by a nilpotent matrix in any finite-dimensional representation). All elements of $\{Y_2, Y_4\}$ are nilpotent, i.e., this is a maximal abelian nilpotent subalgebra (MANS).[15,16] The letter T in the boxes denotes the presence of such nilpotent elements ("translations" on a light cone). The double boxes indicate normalizers of the MASA's. By definition, Cartan subalgebras are self-normalizing. A classification of all real Lie algebras of dimension $d \leqslant 5$ exists[17]; the notation $A_{4,10}$, $A^{1}_{4,9}$, $A_{2,1}$ refers to that article. The algebras $A_{4,10}$, U(1,1), and u(2) are the only subalgebras of su(2,1) [up to conjugacy under SU(2,1)] containing at least one MASA and having a second-order Casimir operator.

These algebras and their Casimir operators play an important role below; so let us discuss them in more detail.

(1) The su(2) subalgebra of u(2) is $\{X_1, X_2, X_3\}$ and its Casimir operator is

$$I(\text{su}(2)) = X_1^2 + X_2^2 + X_3^2. \tag{3.6}$$

(2) Two mutually conjugate su(1,1) subalgebras and their Casimir operators are

$$\{X_4, X_5, \tfrac{1}{2}(X_3 - \sqrt{3}\, X_8)\} \sim \{Y_4, Y_5, Y_6\},$$
$$I_1(\text{su}(1,1)) = X_4^2 + X_5^2 - \tfrac{1}{4}(X_3 - \sqrt{3}X_8)^2 \tag{3.7}$$

and

$$\{X_6, X_7, \tfrac{1}{2}(X_3 + \sqrt{3}X_8)\},$$
$$I_2(\text{su}(1,1)) = X_6^2 + X_7^2 - \tfrac{1}{4}(X_3 + \sqrt{3}X_8)^2. \tag{3.8}$$

(3) The solvable algebra $A_{4,10}$:

$$\{Y_1, Y_2, Y_3, Y_4\} \sim \{X_3 + (1/\sqrt{3})X_8,\ -X_5 + \tfrac{1}{2}(X_3 - \sqrt{3}X_8), X_1 - X_6, X_2 - X_7\}.$$

Its invariant is[14]

$$I_{4,10} = 4Y_1Y_4 + 3(Y_2^2 + Y_3^2). \tag{3.9}$$

Notice that one realization of $A_{4,10}$ is related to the one-dimensional harmonic oscillator. If we put

$$Y_1 \equiv \frac{3}{2}\left(\frac{\partial^2}{\partial x^2} + x^2\right), \quad Y_2 = x, \quad Y_3 = \frac{\partial}{\partial x}, \quad Y_4 = \tfrac{1}{2},$$

then the commutation relations for Y_i are satisfied, and we have $I_{4,10} = 4Y_1$.

Let us now return to the classification of triplets of operators outlined above.

A. The compact Cartan subalgebra

We have

$$T_1 = X_3^2, \quad T_2 = X_8^2,$$

and $[T_1, T_3] = 0$, $[T_2, T_3] = 0$ implies

$$Q_{\text{I}} = T_3 = aI(\text{su}(2)) + bI_1(\text{su}(1,1)) + cI_2(\text{su}(1,1)). \tag{3.10}$$

The Cartan subalgebras are self-normalizing; hence the only freedom left is to subtract some multiple of C_2. The following possibilities occur:

(1) $b = c \neq -a$: $Q_1 = I(\text{su}(2))$,
(2) $c = -a \neq b$: $Q_2 = I_1(\text{su}(1,1))$,
(3) $b \neq c \neq -a \neq b$: $Q_{3,4} = I_1(\text{su}(1,1)) + \mu I_2(\text{su}(1,1))$,
Q_3: $0 < \mu < 1$, Q_4: $-1 \leqslant \mu < 0$,

(the case $|\mu| > 1$ can be rotated into one of the cases with $|\mu| \leqslant 1$).

B. The noncompact Cartan subalgebra

$$T_1 = \tfrac{9}{4}[X_3 + (1/\sqrt{3})X_8]^2, \quad T_2 = X_5^2,$$
$$Q_{\text{II}} = T_3 = aI_1(\text{su}(1,1)) + b(X_1X_6 + X_6X_1 + X_2X_7 + X_7X_2). \tag{3.11}$$

Two possibilities should be distinguished:

Q_5: $b = 0$, $a = 1$,

Q_6: $b = 1$, $a \geqslant 0$

(the relative sign of b and a can be changed by a rotation through the angle π, hence the restriction $a \geqslant 0$ in Q_6).

C. The MASA $\{Y_1, Y_4\}$

$$T_1 = Y_1^2 = \tfrac{9}{4}[X_3 + (1/\sqrt{3})X_8]^2,$$
$$T_2 = Y_4^2 = (-X_5 + \tfrac{1}{2}X_3 - \tfrac{1}{2}\sqrt{3}\,X_8)^2,$$
$$Q_{\text{III}} = T_3 = aI_{4,10} + bI_1(\text{su}(1,1)). \tag{3.12}$$

Four possibilities occur:

Q_7: $a = 0$, $b = 1$,

Q_8: $a = 1$, $b = 0$,

Q_9: $a = b = 1$,

Q_{10}: $a = -b = 1$.

Indeed, if $ab \neq 0$, we make use of the external part of the normalizer of $\{Y_1, Y_4\}$, namely the operator Y_5 to scale a with respect to b: For $ab > 0$ we can scale so that we get $a = b$; for $ab < 0$ so that we get $a = -b$.

D. The maximal abelian nilpotent subalgebra

$$T_1 = Y_3^2 = (X_2 - X_7)^2,$$
$$T_2 = (Y_4)^2 = (-X_5 + \tfrac{1}{2}X_3 - \tfrac{1}{2}\sqrt{3}\,X_8)^2,$$
$$Q_{\text{IV}} = T_3 = aI_{4,10} + b[Y_1Y_3 + Y_3Y_1 - 3(Y_2Y_5 + Y_5Y_2) - 6(Y_4Y_8 + Y_8Y_4)]. \tag{3.13}$$

Two cases should be distinguished:

Q_{11}: $a = 1$, $b = 0$,

Q_{12}: $a = 0$, $b = 1$.

Indeed, if $a \neq 0$, we set $a = 1$ and use the external part of the normalizer of $\{Y_3, Y_4\}$ to transform $b \to 0$ [this is achieved by a transformation of the type $Q' = \exp(\alpha Y_2)Q\exp(-\alpha Y_2)$].

We have thus obtained 12 orbits of operators $\{T_1, T_2, T_3\}$. Among them six are of the subgroup type, i.e, such that Q is the Casimir operator of some subgroup of SU(2,1). These are the sets involving Q_1, Q_2, Q_5, Q_7, Q_8, and Q_{11}.

In the following section we shall establish a one-to-one correspondence between the above-classified triplets of operators in involution and 12 types of separable coordinates on HH(2).

IV. SEPARABLE COORDINATES ON HH (2)

A. Introduction of ignorable coordinates and reduction to separation on an O(2,1) hyperboloid

Our purpose now is to find all separable coordinates in HH(2), i.e., to transform from the affine coordinates $\{z_1,z_2,\bar{z}_1,\bar{z}_2\}$ to four real variables $\{A,B,x,y\}$ such that x and y are ignorable and that Eqs. (1.1) and (1.2) separate in the new variables. This transformation can be performed in two different manners, starting with the affine coordinates z_i ($i=1,2$) or the homogeneous coordinates y_μ ($\mu=0,1,2$), respectively. In each case the procedure is repeated four times, separately for each MASA of su(2,1).

Using affine coordinates, we proceed as follows:

(1) Choose a basis $\{L_1,L_2\}$ for the considered MASA, express L_1 and L_2 in terms of z_i as in (2.14) and put

$$L_1 = P_x, \quad L_2 = P_y. \tag{4.1}$$

Solve equations (4.1): This provides the explicit dependence of z_1 and z_2 on the ignorable variables. The dependence on the essential variables A, B is as yet unknown and is contained in the integration "constants" of (4.1).

(2) To obtain the dependence on A, B make use of a procedure outlined in Ref. 4, for arbitrary four-dimensional Riemannian spaces. Since HH(2) is a positive-definite metric space and since each separable system must involve precisely two ignorable variables, only case "C" of Ref. 4 occurs. Hence a pseudogroup P of coordinate transformations (described in I and Ref. 4) must exist, transforming the Fubini–Study metric (2.7) into a form in which the metric tensor satisfies:

$$\begin{aligned} &g^{AA} = g^{BB} = \frac{1}{k_1(A)+k_2(B)}, \quad g^{AB}=0,\\ &g^{xx} = \frac{e_1(A)+e_2(B)}{k_1(A)+k_2(B)}, \quad g^{yy} = \frac{f_1(A)+f_2(B)}{k_1(A)+k_2(B)},\\ &g^{xy} = \frac{h_1(A)+h_2(B)}{k_1(A)+k_2(B)},\\ &g^{Ax}=g^{Ay}=g^{Bx}=g^{By}=0, \end{aligned} \tag{4.2}$$

where k_i, e_i, f_i, and h_i are functions of the indicated variables satisfying

$$\frac{\partial^2}{\partial A\partial B}\ln\left[\frac{(k_1+k_2)^2}{(e_1+e_2)(f_1+f_2)-(h_1+h_2)^2}\right]=0 \tag{4.3}$$

[i.e., $R_{AB}=0$, where R_{ij} is the Ricci tensor]. Solve Eqs. (4.2) and (4.3) to obtain the dependence of $\{z_1,z_2\}$ on A and B.

Following this procedure, we find that the MASA $\{X_3,X_8\}$ leads to four different types of coordinates, $\{Y_1,Y_6-Y_4\}$ to two types, $\{Y_1,Y_4\}$ to four types, and finally $\{Y_3,Y_4\}$ to two. The computations are quite long and involved, but the results are relatively simple and coincide with those obtained using a different, more geometrical and group-theoretical method, described below.

The second procedure is an adaptation of the general method of the reduction of phase space in classical mechanics by ignorable variables.[18] The procedure is related to that used by Marsden and Weinstein[19] and Kazhdan, Kostant, and Sternberg[20] to obtain completely integrable Hamiltonian systems. In I we applied this procedure to reduce by the maximal torus, i.e., the Cartan subgroup of SU($n+1$). We thus reduced the problem of separating variables on $\mathbb{C}$P(n) to that of separating on the sphere S_{n+1}. The free Hamiltonian on $\mathbb{C}$P(n) was reduced to a singular Hamiltonian on S_{n+1} with a specific inverse square type potential. We shall see that the situation is very similar for HH(2) and that the reduction can be performed by any of the maximal abelian subgroups (not just the maximal torus).

Instead of MASA's of su(2,1), we shall use MASA's of u(2,1), i.e., to the basis L_1,L_2 of each MASA we add a further operator

$$X_0 = y_0 p_{y_0} + y_1 p_{y_1} + y_2 p_{y_2} + \text{c.c.} \tag{4.4}$$

When acting on functions $f(y_0,y_1,y_2)$ that project properly onto HH(2), i.e., homogeneous functions satisfying

$$f(y_0,y_1,y_2) = f(y_1/y_0, y_2/y_0) \tag{4.5}$$

we have

$$\left(y_0\frac{\partial}{\partial y_0} + y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}\right)f=0 \tag{4.6}$$

and for the corresponding constant of motion on HH(2) we have

$$X_0 = y_0 p_{y_0} + y_1 p_{y_1} + y_2 p_{y_2} = 0. \tag{4.7}$$

The procedure is:

(1) Choose a basis $\{L_1,L_2\}$ for the considered MASA, express L_1, L_2, and X_0 in terms of y as in (2.14) and (4.4) and put

$$L_1 = p_x, \quad L_2 = p_y, \quad X_0 = p_\rho. \tag{4.8}$$

Solve equations (4.8) to obtain the explicit dependence of y on the ignorable variables x, y, and ρ [upon projection from $\mathbb{C}$(3) to HH(2) ρ will cancel out].

The variables y_μ depend on three more real variables, say s_0, s_1, and s_2, which are contained in the integration constants of Eqs. (4.8). These must be introduced in such a manner that s_μ, x, y, and ρ parametrize all of $\mathbb{C}$(3), that x and y project into ignorable variables on HH(2), and that the variables s_μ are compatible with the projection, i.e.,

$$\begin{aligned} |y|^2 &\equiv |y_0^2| - |y_1|^2 - |y_2|^2 = s_0^2 - s_1^2 - s_2^2\\ &\equiv s^2 = \text{const.} \end{aligned} \tag{4.9}$$

In order to obtain the space HH(2), we put $s^2=1$; other homogeneous spaces with SU(2,1) actions are obtained by putting $s^2=-1$ or $s^2=0$.

(2) Express the su(2,1) infinitesimal operators X_i ($i=1,\ldots,8$) the Hamiltonian H and the Killing tensor $T_3=Q$ in terms of the variables $\{x,y,s_0,s_1,s_2\}$ (setting $p_\rho=0$, or correspondingly dropping a term containing $\partial/\partial\rho$). The essential variables s_μ are constrained by the condition (4.9). The corresponding momenta p_{s_μ} figure in the infinitesimal operators X_i only via the expressions

$$\begin{aligned} &I_{12} = s_1 p_{s_2} - s_2 p_{s_1}, \quad I_{01} = s_0 p_{s_1} + s_1 p_{s_0},\\ &I_{02} = s_0 p_{s_2} + s_2 p_{s_0}. \end{aligned} \tag{4.10}$$

The quantities $I_{\mu\nu}$ ($\mu,\nu=0,1,2$) generate an o(2,1) algebra under the corresponding Lie bracket. This o(2,1) is in general not a subalgebra of su(2,1); however, if we restrict ourselves to the manifold (4.9) by setting the ignorable variables equal to zero, then we obtain

$$X_1 = I_{12}, \quad X_4 = I_{01}, \quad X_6 = I_{02}, \tag{4.11}$$

i.e., the O(2,1) group acting on the variables s coincides with the real O(2,1) subgroup of SU(2,1). In the new variables the Hamiltonian H and the Killing tensor Q are expressed as

$$H = I_{12}^2 - I_{01}^2 - I_{02}^2 + f_1(s_\mu) p_x^2 + f_2(s_\mu) p_y^2 + f_3(s_\mu) p_x p_y, \tag{4.12}$$

$$Q = \sum_{\substack{\mu\nu \\ \mu'\nu'}} A_{\mu\nu,\mu'\nu'} I_{\mu\nu} I_{\mu'\nu'} + h_1(s_\mu) p_x^2 + h_2(s_\mu) p_y^2 + h_3(s_\mu) p_x p_y, \tag{4.13}$$

where f_i and h_i are functions of the essential variables s_μ and $A_{\mu\nu,\mu'\nu'} = A_{\mu'\nu',\mu\nu}$ is a symmetric constant matrix. The problem of separating variables for the free Hamiltonian on HH(2) has thus been reduced to that of separating variables in the Hamiltonian (4.12). This is an O(2,1) Hamiltonian, which is, however, not a free one: It includes a "potential" term depending on the O(2,1) variables s_μ. We recall that the momenta p_x and p_y corresponding to the ignorable variables should be set equal to constants

$$p_x = c_1, \quad p_y = c_2. \tag{4.14}$$

Notice that we have

$$I_{12}^2 - I_{01}^2 - I_{02}^2 = (p_{s_0}^2 - p_{s_1}^2 - p_{s_2}^2), \tag{4.15}$$

where we have used the fact that

$$\sum_{\mu=0}^{2} s_\mu p_{s_\mu} = 0. \tag{4.16}$$

(3) Introduce separable coordinates on the hyperboloid (4.9), compatible with the form of the operator Q and the potential in (4.12).

Let us now implement the first two steps of this procedure for each of the four MASA's of su(2,1).

1. The compact Cartan subalgebra $\{X_3, X_8\}$

We first introduce the ignorable variables $(\rho, \alpha_1, \alpha_2)$, putting

$$\tfrac{1}{2}[X_3 - (1/\sqrt{3})X_8] = p_{\alpha_1}, \quad -\tfrac{1}{2}[X_3 + (1/\sqrt{3})X_8] = p_{\alpha_2}, \quad X_0 = p_\rho. \tag{4.17}$$

Using (2.14), we obtain a system of equations that is easily solved to express the homogeneous coordinates as

$$y_0 = s_0 e^{i(3\rho - \alpha_1 - \alpha_2)/3}, \quad y_1 = s_1 e^{i(3\rho + 2\alpha_1 - \alpha_2)/3}, \quad y_2 = s_2 e^{i(3\rho - \alpha_1 + 2\alpha_2)/3}. \tag{4.18}$$

The infinitesimal operators are expressed in these coordinates in the Appendix. Putting $\alpha_1 = \alpha_2 = 0$, we obtain (4.11); X_2, X_3, X_5, X_7, X_8 then involve only the essential variables and the momenta conjugate to the ignorable ones. Expressions (4.12) and (4.13) for the Hamiltonian H and Killing tensor $Q_{\rm I}$ (3.10) reduce to

$$H = -I_{12}^2 + I_{01}^2 + I_{02}^2 + \left[\frac{1}{s_1^2} p_{\alpha_1}^2 + \frac{1}{s_2^2} p_{\alpha_2}^2 - \frac{1}{s_0^2}(p_{\alpha_1} + p_{\alpha_2})^2\right], \tag{4.19}$$

$$\begin{aligned} Q_{\rm I} = {} & a\left[I_{12}^2 + \left(1 + \frac{s_2^2}{s_1^2}\right)p_{\alpha_1}^2 + \left(1 + \frac{s_1^2}{s_2^2}\right)p_{\alpha_2}^2\right] \\ & + b\left[I_{01}^2 + \left(-1 + \frac{s_0^2}{s_1^2}\right)p_{\alpha_1}^2 + \left(-1 + \frac{s_1^2}{s_0^2}\right)(p_{\alpha_1} + p_{\alpha_2})^2\right] \\ & + c\left[I_{02}^2 + \left(-1 + \frac{s_0^2}{s_2^2}\right)p_{\alpha_2}^2 + \left(-1 + \frac{s_2^2}{s_0^2}\right) \times (p_{\alpha_1} + p_{\alpha_2})^2\right]. \end{aligned} \tag{4.20}$$

Setting $p_{\alpha_i} = 0$ we obtain a free O(2,1) Hamiltonian and a Killing tensor of a specific type: it involves the squares $I_{\mu\nu}^2$ only. Separation of variables on an O(2,1) hyperboloid H_2 is discussed below.[5,7,21] Nine distinct separable coordinate systems exist on H_2 but only four of them have Killing tensors of the type $Q_{\rm I}$. Precisely these four occur in our HH(2) problem.

Setting $p_{\alpha_i} = c_i \neq 0$, we reduce (4.19) to an O(2,1) Hamiltonian with an inverse square type singular potential, and $Q_{\rm I}$ reduces to the corresponding integral of motion. We have thus generated a nontrivial relativistic completely integrable Hamiltonian system. Similar systems with singular inverse square potentials have been studied in a nonrelativistic context.[22–24]

2. The noncompact Cartan subalgebra $\{X_3 + (1/\sqrt{3})X_8, X_5\}$

Introduce the ignorable variables (ρ, α, u) by putting

$$-\tfrac{1}{2}[X_3 + (1/\sqrt{3})X_8] = p_\alpha, \quad X_5 = p_u, \quad X_0 = p_\rho. \tag{4.21}$$

Expressing X_i in terms of the homogeneous coordinates y_μ, we obtain a system of partial differential equations that can be solved to yield

$$\begin{aligned} y_0 &= e^{i(3\rho - \alpha)/3}(is_0 \operatorname{ch} u + s_1 \operatorname{sh} u), \quad 0 \leqslant \rho < 2\pi, \ 0 \leqslant \alpha < 2\pi, \\ y_1 &= e^{i(3\rho - \alpha)/3}(is_1 \operatorname{ch} u - s_0 \operatorname{sh} u), \quad 0 \leqslant u < \infty, \\ y_2 &= e^{i(3\rho + 2\alpha)/3} is_2. \end{aligned} \tag{4.22}$$

The infinitesimal operators are given in the Appendix. Putting $\alpha = u = 0$, we again obtain (4.11). The Hamiltonian and Killing tensor Q [(3.11)] in this case are

$$H = -I_{12}^2 + I_{01}^2 + I_{02}^2 + \left\{-\frac{s_0^2 - s_1^2}{(s_0^2 + s_1^2)^2} p_u^2 + \left[\frac{s_0^2 - s_1^2}{(s_0^2 + s_1^2)^2} - \frac{1}{s_2^2}\right] p_\alpha^2 + \frac{4 s_0 s_1}{(s_0^2 + s_1^2)^2} p_u p_\alpha\right\}, \tag{4.23}$$

$$\begin{aligned} Q_{\rm II} = {} & a\left[I_{01}^2 + \frac{(s_0^2 - s_1^2)^2}{(s_0^2 + s_1^2)^2}(p_u^2 - p_\alpha^2) - 4\frac{s_0 s_1 (s_0^2 - s_1^2)}{(s_0^2 + s_1^2)^2} p_u p_\alpha\right] \\ & + b\left[\{I_{12}, I_{02}\} + 2\frac{s_0 s_1 s_2^2}{(s_0^2 + s_1^2)^2} p_u^2 + 2\frac{s_0 s_1 [(s_0^2 + s_1^2)^2 - s_2^4]}{(s_0^2 + s_1^2)^2 s_2^2} p_\alpha^2 + 2\frac{(s_0^2 + s_1^2) + s_2^2(s_0^2 - s_1^2)}{(s_0^2 + s_1^2)^2} p_u p_\alpha\right]. \end{aligned} \tag{4.24}$$

Setting $p_u = p_\alpha = 0$, we again obtain a free 0(2,1) Hamiltonian and a specific O(2,1) Killing tensor (leading to only two of the nine separable systems on H_2). For $p_u = c_1$ and $p_\alpha = c_2$, we obtain a new nontrivial completely integrable Hamiltonian system with a singular potential.

3. The orthogonally decomposable MASA $\{Y_1, Y_4\}$

To introduce the ignorable variables (ρ,α,t), we put

$$-\tfrac{1}{3}Y_1 = p_\alpha, \quad -Y_4 = p_t, \quad X_0 = p_\rho \tag{4.25}$$

and obtain

$$\begin{aligned} y_0 &= e^{i(3\rho-\alpha)/3}[s_0 + i(s_0 - s_1)t], \quad -\infty < t < \infty, \\ y_1 &= e^{i(3\rho-\alpha)/3}[s_1 + i(s_0 - s_1)t], \\ &\quad 0 \leqslant \rho < 2\pi, \ 0 \leqslant \alpha < 2\pi, \\ y_2 &= e^{i(3\rho+2\alpha)/3}s_2 . \end{aligned} \tag{4.26}$$

The infinitesimal operators are given in the Appendix. The Hamiltonian and Killing tensor (3.12) are

$$\begin{aligned} H = & -I_{12}^2 + I_{01}^2 + I_{02}^2 \\ & + \left[\frac{1}{s_2^2}p_\alpha^2 + \frac{s_0+s_1}{(s_0-s_1)^3}p_t^2 + \frac{2}{(s_0-s_1)^2}p_\alpha p_t\right], \end{aligned} \tag{4.27}$$

$$\begin{aligned} Q_{\rm III} = 3a\bigg[& (I_{02} - I_{12})^2 + \frac{(s_0-s_1)^2}{s_2^2}p_\alpha^2 \\ & + \frac{s_2^2}{(s_0-s_1)^2}p_t^2 + 2p_\alpha p_t\bigg] \\ & + b\left[I_{01}^2 + \frac{(s_0+s_1)^2}{(s_0-s_1)^2}p_t^2 + 2\frac{s_0+s_1}{s_0-s_1}p_t p_\alpha\right]. \end{aligned} \tag{4.28}$$

For $\alpha = t = 0$ we again have pure O(2,1) quantities. The specific form of $Q_{\rm III}$ leads to four of the nine separable 0(2,1) systems. For $p_\alpha = c_1$ and $p_t = c_2$, we obtain yet another O(2,1) Hamiltonian with a new nontrivial singular interaction.

4. The maximal abelian nilpotent subalgebra $\{Y_3, Y_4\}$

To introduce the ignorable variables (ρ,t,u), we put

$$Y_3 = p_t, \quad Y_4 = -p_u, \quad X_0 = p_\rho \tag{4.29}$$

and obtain

$$\begin{aligned} y_0 &= e^{i\rho}[(s_0 - s_1)(u - \tfrac{1}{2}it^2) + s_2 t - is_0], \quad -\infty < u < \infty, \\ y_1 &= e^{i\rho}[(s_0 - s_1)(u - \tfrac{1}{2}it^2) + s_2 t - is_1], \quad -\infty < t < \infty, \\ y_2 &= e^{i\rho}[-is_2 - (s_0 - s_1)t], \quad 0 \leqslant \rho < 2\pi. \end{aligned} \tag{4.30}$$

The infinitesimal operators are in the Appendix; the Hamiltonian and Killing tensor (3.13) are

$$\begin{aligned} H = & -I_{12}^2 + I_{01}^2 + I_{02}^2 + \bigg[\frac{1}{(s_0-s_1)^2}p_t^2 \\ & - \frac{4s_2}{(s_0-s_1)^3}p_u p_t + \frac{3s_2^2 - s_0^2 - s_1^2}{(s_0-s_1)^4}p_u^2\bigg], \end{aligned} \tag{4.31}$$

$$\begin{aligned} Q_{\rm IV} = 3a\bigg[& (I_{12} - I_{02})^2 + \left(p_t - \frac{2s_2}{s_0-s_1}p_u\right)^2 \\ & + 3b\left[\{I_{02} - I_{12}, I_{01}\} + \frac{2s_2}{s_0-s_1}p_t^2\right. \\ & + 4s_2\frac{s_0^2 - s_1^2 + s_2^2}{(s_0-s_1)^3}p_u^2 \\ & + 2\frac{2s_1^2 - 3s_2^2 - 2s_0 s_1}{(s_0-s_1)^2}p_u p_t\bigg]. \end{aligned} \tag{4.32}$$

For $p_t = p_u = 0$ the operator $Q_{\rm IV}$ reduces to an O(2,1) operator related to variable separation in two of the nine separable systems on H_2. For $p_t = c_1$ and $p_u = c_2$ we again obtain a nontrivial interaction term in (4.31).

B. Separation of variables on an O(2,1) hyperboloid

Let us now consider the separation of variables in the free Hamilton–Jacobi equation or free Laplace–Beltrami equation on the O(2,1) homogeneous space

$$s^2 = s_0^2 - s_1^2 - s_2^2 = K^2 \quad (K^2 = \pm 1 \text{ or } 0). \tag{4.33}$$

Nine separable coordinate systems have been shown to exist[21] and to be in one-to-one correspondence with orbits of second-order operators in the enveloping algebra of O(2,1).[5,7] Since the results are not readily available and were not presented in a convenient form for our purposes, we summarize them here.

Let $I_{\mu\nu}$ be the O(2,1) operators (4.10), satisfying

$$[I_{01}, I_{02}] = -I_{12}, \quad [I_{12}, I_{01}] = I_{02}, \quad [I_{12}, I_{02}] = -I_{01}. \tag{4.34}$$

A general second-order operator in the O(2,1) enveloping algebra can be written as

$$R = (I_{12} I_{01} I_{02}) X \begin{pmatrix} I_{12} \\ I_{01} \\ I_{02} \end{pmatrix}, \quad X = X^T \in \mathbb{R}^{3\times 3}. \tag{4.35}$$

Under an O(2,1) transformation, R is transformed into R' given by (4.35) with X replaced by X':

$$X' = G^T X G, \quad GJG^T = J, \tag{4.36}$$

where J is a nonsingular 3×3 real symmetric matrix with signature $(-\ +\ +)$. We rewrite (4.36) as

$$\tilde{X}' = G^{-1}\tilde{X}G, \quad \tilde{X} = JX, \quad \tilde{X}^T J = J\tilde{X}. \tag{4.37}$$

Thus, X is symmetric under the involution that defines o(2,1). Such symmetric matrices have recently been classified for all classical Lie algebras.[25] For O(2,1) the results are quite simple, namely any pair of matrices $(\tilde{X}, J)$ satisfying (4.37) is SL(3,$\mathbb{R}$) conjugate to one of the following:

(I) $\tilde{X}_{\rm I}$ orthogonally decomposable with three real eigenvalues:

$$\tilde{X}_{\rm I} = \begin{pmatrix} -c & & \\ & a & \\ & & b \end{pmatrix}, \quad J = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad a,b,c \in \mathbb{R}. \tag{4.38}$$

(II) $\tilde{X}_{\rm II}$ orthogonally decomposable with one real eigenvalue and one pair of complex conjugate eigenvalues:

$$\widetilde{X}_{\rm II}=\begin{pmatrix} a & b & 0\\ -b & a & 0\\ 0 & 0 & c\end{pmatrix},\quad J=\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix},\quad a,b,c\in\mathbb{R},\ b>0\,. \tag{4.39}$$

(III) $\widetilde{X}_{\rm III}$ orthogonally decomposable with two real eigenvalues:

$$\widetilde{X}_{\rm III}=\begin{pmatrix} a & 0 & 0\\ 1 & a & 0\\ 0 & 0 & b\end{pmatrix},\quad J=\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix},\quad a,b\in\mathbb{R}\,. \tag{4.40}$$

(IV) $\widetilde{X}_{\rm IV}$ indecomposable (one real eigenvalue):

$$\widetilde{X}_{\rm IV}=\begin{pmatrix} a & 0 & 0\\ 1 & a & 0\\ 0 & 1 & a\end{pmatrix},\quad J=\begin{pmatrix} & & 1\\ & 1 & \\ 1 & & \end{pmatrix},\quad a\in\mathbb{R}\,. \tag{4.41}$$

Returning to a basis in which J is as (4.38) and simplifying by linear combinations with the O(2,1) Casimir operator

$$\Delta = I_{01}^2 + I_{02}^2 - I_{12}^2, \tag{4.42}$$

we obtain four classes of quadratic operators R:

$$\begin{aligned} R_{\rm I} &= \lambda_3 I_{12}^2 + \lambda_1 I_{01}^2 + \lambda_2 I_{02}^2, \quad \lambda_i\in\mathbb{R},\\ R_{\rm II} &= \lambda I_{01}^2 + \{I_{02},I_{12}\},\\ R_{\rm III} &= \lambda I_{01}^2 + \mu(I_{02}-I_{12})^2, \quad \mu\neq 0,\ \lambda,\mu\in\mathbb{R},\\ R_{\rm IV} &= \{I_{02}-I_{12},I_{01}\} \end{aligned} \tag{4.43}$$

(the brackets { , } denote an anticommutator). The operator $R_{\rm I}$ can be further simplified by combinations with Δ; in $R_{\rm II}$ we can assume $\lambda\geqslant 0$; in $R_{\rm III}$ we can scale μ with respect to λ by means of the O(2,1) transformation $\exp\alpha\, I_{01}$ and hence only distinguish three cases: $\lambda=0, \mu=1$; $\lambda=\mu=1$; $\lambda=-\mu=1$.

Finally we obtain nine classes of operators R_a ($a=1,...,9$) and the corresponding coordinate systems for which the O(2,1) Hamilton-Jacobi and Laplace–Beltrami equations separate. The separable coordinates, Hamiltonians H and integrals of motion R_a, for the two-sheeted hyperboloid, i.e., $K^2=1$ are as follows.

1. Spherical: R_I with $\lambda_1=\lambda_2\neq\lambda_3$

$$\begin{aligned} &s_0=\cosh A,\quad s_1=\sinh A\cos B,\quad s_2=\sinh A\sin B,\\ &0\leqslant A<\infty,\ 0\leqslant B<2\pi, \\ &H=p_A^2+\frac{1}{\sinh^2 A}p_B^2,\quad R_1=I_{12}^2=p_B^2. \end{aligned} \tag{4.44}$$

2. Hyperbolic: R_I with $-\lambda_3=\lambda_2\neq\lambda_1$

$$\begin{aligned} &s_0=\cosh A\cosh B,\quad s_1=\cosh A\sinh B,\\ &s_2=\sinh A,\quad A,B\in\mathbb{R},\\ &H=p_A^2+\frac{1}{\cosh^2 A}p_B^2,\quad R_2=I_{01}^2=p_B^2. \end{aligned} \tag{4.45}$$

3. Elliptic I: R_I with $-\lambda_3\neq\lambda_1\neq\lambda_2\neq-\lambda_3$, $(\lambda_1+\lambda_3)/(\lambda_2+\lambda_3)>0$

$$\begin{aligned} &s_0^2=\nu\rho/a,\quad s_1^2=(\nu-1)(\rho-1)/(a-1),\\ &s_2^2=(\nu-a)(a-\rho)/(a-1)a,\quad 1\leqslant\rho\leqslant a\leqslant\nu<\infty,\ 1<a \end{aligned} \tag{4.46}$$

$$\begin{aligned} H&=[4/(\nu-\rho)]\,[\nu(\nu-1)(\nu-a)p_\nu^2\\ &\quad+\rho(\rho-1)(a-\rho)p_\rho^2]\,,\\ R_3&=aI_{01}^2+I_{02}^2\\ &=[4\nu\rho/(\nu-\rho)]\,[(\nu-1)(\nu-a)p_\nu^2\\ &\quad+(\rho-1)(a-\rho)p_\rho^2]\,. \end{aligned}$$

4. Elliptic II: R_I with $-\lambda_3\neq\lambda_1\neq\lambda_2\neq-\lambda_3$, $(\lambda_1+\lambda_3)/(\lambda_2+\lambda_3)<0$

$$\begin{aligned} &s_0^2=(\nu-1)(1-\rho)/(a-1),\quad s_1^2=-\nu\rho/a,\\ &s_2^2=(\nu-a)(a-\rho)/(a-1)a,\\ &\rho\leqslant 0,\ 1<a\leqslant\nu,\ 0<a-1\leqslant 1, \end{aligned} \tag{4.47}$$

$$\begin{aligned} H&=[4/(\nu-\rho)]\,[\nu(\nu-1)(\nu-a)p_\nu^2\\ &\quad+\rho(\rho-1)(a-\rho)p_\rho^2]\,,\\ R_4&=(a-1)I_{01}^2-I_{02}^2\\ &=-[4(1-\rho)(\nu-1)/(\nu-\rho)]\ [\nu(\nu-a)p_\nu^2\\ &\quad+(a-\rho)\rho p_\rho^2]. \end{aligned}$$

5. Complex elliptic: R_{II}

$$\begin{aligned} &\tfrac{1}{2}(s_0+is_1)^2=(\nu-a)(\rho-a)/a(a-a^*),\quad s_2^2=-\nu\rho/|a|^2,\\ &\nu<0<\rho,\ a=\alpha+i\beta,\ \beta>0,\ \alpha,\beta\in\mathbb{R}, \end{aligned} \tag{4.48}$$

$$\begin{aligned} H&=[4/(\rho-\nu)]\ [\rho(\rho-a)(\rho-a^*)p_\rho^2\\ &\quad-\nu(\nu-a)(\nu-a^*)p_\nu^2]\,,\\ R_5&=aI_{01}^2-\beta\{I_{12},I_{02}\}\\ &=[4\rho\nu/(\rho-\nu)]\ [(\rho-a)(\rho-a^*)p_\rho^2\\ &\quad-(\nu-a)(\nu-a^*)p_\nu^2]\,. \end{aligned}$$

6. Horospheric: R_{III} with $\lambda=0$

$$\begin{aligned} &s_0=\cosh A+\tfrac{1}{2}r^2e^{-A},\quad s_1=\sinh A+\tfrac{1}{2}r^2e^{-A},\\ &s_2=re^{-A},\quad -\infty<A<\infty,\ -\infty<r<\infty,\\ &H=p_A^2+e^{2A}p_r^2,\quad R_6=(I_{02}-I_{12})^2=p_r^2. \end{aligned} \tag{4.49}$$

7. Elliptic parabolic: R_{III} with $\lambda\mu>0$

$$\begin{aligned} &s_0^2=\tfrac{1}{4}(\nu+\rho)^2/\nu\rho,\quad s_1^2=\tfrac{1}{4}(\nu+\rho-2\nu\rho)^2/\nu\rho,\\ &s_2^2=(1-\nu)(\rho-1),\quad 0<\nu<1<\rho\,, \end{aligned} \tag{4.50}$$

$$\begin{aligned} H&=[4/(\rho-\nu)]\ [\rho^2(\rho-1)p_\rho^2+\nu^2(1-\nu)p_\nu^2]\,,\\ R_7&=I_{01}^2+(I_{02}-I_{12})^2\\ &=[4\rho\nu/(\rho-\nu)]\ [\rho(\rho-1)p_\rho^2+\nu(1-\nu)p_\nu^2]\,. \end{aligned}$$

8. Hyperbolic parabolic: R_{III} with $\lambda\mu<0$

$$s_0^2=(\nu+\rho-2\nu\rho)^2/(-4\nu\rho),\quad s_1^2=(\nu+\rho)^2/(-4\nu\rho),$$
$$s_2^2=(1-\nu)(\rho-1),\quad \nu<0<1<\rho, \tag{4.51}$$
$$H=[4/(\rho-\nu)]\,[\rho^2(\rho-1)p_\rho^2+\nu^2(1-\nu)p_\nu^2],$$
$$R_8=I_{01}^2-(I_{02}-I_{12})^2$$
$$=[4\rho\nu/(\rho-\nu)]\,[\rho(\rho-1)p_\rho^2+\nu(1-\nu)p_\nu^2].$$

9. Semicircular parabolic: R_{IV}

$$s_0^2=1/[-16(\rho\nu)^3]\,[(\rho-\nu)^2+\rho^2\nu^2]^2,$$
$$s_1^2=1/[-16(\rho\nu)^3]\,[(\rho-\nu)^2-\rho^2\nu^2]^2,$$
$$s_2^2=(\rho+\nu)^2/(-4\rho\nu),\quad \nu<0<\rho, \tag{4.52}$$
$$H=[4/(\rho-\nu)](\rho^3p_\rho^2-\nu^3p_\nu^2),\quad R_9=\{I_{02}-I_{12},I_{01}\}$$
$$=2[\nu\rho/(\nu-\rho)](\rho^2p_\rho^2-\nu^2p_\nu^2).$$

Three of these coordinate systems are of the "subgroup type," namely spherical, hyperbolic, and horospheric, corresponding to the group reductions

$$O(2,1)\supset O(2),\quad O(2,1)\supset O(1,1),\text{ and } O(2,1)\supset T,$$

respectively (T being the group of translations generated by $I_{02}-I_{12}$).

All coordinate systems are written so as to parametrize the upper sheet of a one-sheeted hyperboloid. It is not difficult to modify the coordinates so as to parametrize the one sheeted hyperboloid ($s^2=-1$).

C. Separable coordinates on HH(2) and the Hamiltonian systems

In Sec. III we have classified triplets of operators $\{T_1,T_2,T_3\}$ into 12 orbits under SU(2,1). In Sec. IVA we have introduced ignorable variables on HH(2). Each different MASA of SU(2,1) leads to specific coordinates in which the Hamiltonian H and integral of motion $Q=T_3$ reduce to an O(2,1) form corresponding to an O(2,1) Hamiltonian system with a nontrivial interaction. In Sec. IVB we reviewed separation on the O(2,1) hyperboloid $s^2=1$. Combining all these results together, we obtain the following theorem.

Theorem 1: (1) There exist precisely 12 systems of coordinates on HH (2) in which the Hamiltonian–Jacobi and Laplace–Beltrami equations separate.

(2) Each separable system has two ignorable and two nonignorable variables. The nonignorable variables are introduced so as to separate variables on the O(2,1) hyperboloid $s^2=s_0^2-s_1^2-s_2^2=1$.

(3) The separable coordinate systems in HH (2) are in one-to-one correspondence with orbits of triplets of second-order operators $\{T_1,T_2,T_3\}$ in the enveloping algebra of su(2,1). The operators T_i are in involution, two of them, $T_1=L_1^2$ and $T_2=L_2^2$, are squares of the generators L_1,L_2 of a MASA of su(2,1), the third $Q=T_3$ is a general operator of the form (3.3). The operator Q takes one of the forms $Q_1,\ldots,Q_{12}$ listed in Sec. III.

(4) The compact Cartan subalgebra $\{X_3,X_8\}$ for which Q has the form Q_1 of (4.20) leads to four types of coordinate systems, namely, (4.18) with (s_0,s_1,s_2), expressed in spherical (Q_1), hyperbolic (Q_2), elliptic I (Q_3), or elliptic II (Q_4) coordinates on the O(2,1) hyperboloid H_2.

(5) The noncompact Cartan subalgebra $\{X_3+(1/\sqrt{3})X_8,X_5\}$ for which Q has the form Q_{II} of (4.24) leads to two types of coordinate systems, namely, (4.22) with (s_0,s_1,s_2) expressed in hyperbolic (Q_5) or complex elliptic (Q_6) coordinates on H_2.

(6) The decomposable non-Cartan subalgebra $\{Y_1,Y_4\}$ for which Q has the form Q_{III} of (4.28) leads to four separable coordinate systems, namely, (4.26) with (s_0,s_1,s_2) expressed in hyperbolic (Q_7), horospheric (Q_8), elliptic parabolic (Q_9), or elliptic hyperbolic (Q_{10}) coordinates on H_2.

(7) The MANS $\{Y_3,Y_4\}$ for which Q has the form Q_{IV} of (4.32) leads to two separable systems, namely, (4.30) with (s_0,s_1,s_2) expressed in horospheric (Q_{11}) or semicircular parabolic coordinates (Q_{12}) on H_2.

Finally, let us list the 12 separable coordinate systems and in the process show that the "potentials" in the O(2,1) Hamiltonians are indeed compatible with separation in each of the 12 cases. We shall use the affine coordinates (2.5).

1. The compact Cartan subalgebra $\{X_3,X_8\}$

$$\tfrac{1}{2}[X_3-(1/\sqrt{3})X_8]=p_{\alpha_1}=c_1,$$
$$-\tfrac{1}{2}[X_3+(1/\sqrt{3})X_8]=p_{\alpha_2}=c_2.$$

a. Spherical coordinates:

$$z_1=\tanh A\cos Be^{i\alpha_1},\quad z_2=\tanh A\sin Be^{i\alpha_2}, \tag{4.53}$$
$$Q_1=p_B^2+(1/\cos^2B)p_{\alpha_1}^2+(1/\sin^2B)p_{\alpha_2}^2=c_3,$$
$$H=p_A^2+(1/\sinh^2A)Q_1-(1/\cosh^2A)(p_{\alpha_1}+p_{\alpha_2})^2=E.$$

b. Hyperbolic coordinates:

$$z_1=\tanh Be^{i\alpha_1},\quad z_2=(\tanh A/\cosh B)e^{i\alpha_2}, \tag{4.54}$$
$$Q_2=p_B^2+(1/\sinh^2B)p_{\alpha_1}^2-(1/\cosh^2B)(p_{\alpha_1}+p_{\alpha_2})^2=c_3,$$
$$H=p_A^2+(1/\cosh^2A)Q_2+(1/\sinh^2A)p_\alpha^2=E.$$

c. Elliptic I coordinates:

$$z_1^2 = [a(\nu-1)(\rho-1)/(a-1)\nu\rho]\, e^{2i\alpha_1},$$
$$z_2^2 = [(\nu-a)(a-\rho)/(a-1)\nu\rho]\, e^{2i\alpha_2}, \tag{4.55}$$
$$Q_3 = [1/(\nu-\rho)]\{4\rho\nu(\nu-1)(\nu-a)p_\nu^2 + 4\nu\rho(\rho-1)(a-\rho)p_\rho^2$$
$$+ [(a-\rho)\nu/(\rho-1) + (\nu-a)\rho/(\nu-1)]\, p_{\alpha_1}^2 + a[(\rho-1)\nu/(a-\rho) + (\nu-1)\rho/(\nu-a)]\, p_{\alpha_2}^2$$
$$- a(\nu/\rho - \rho/\nu)(p_{\alpha_1} + p_{\alpha_2})^2\} = c_3,$$
$$H = [1/(\nu-\rho)]\, [4\nu(\nu-1)(\nu-a)p_\nu^2 + 4\rho(\rho-1)(a-\rho)p_\rho^2$$
$$+ (a-1)[1/(\rho-1) - 1/(\nu-1)]\, p_{\alpha_1}^2 + a(a-1)[1/(a-\rho) + 1/(\nu-a)]\, p_{\alpha_2}^2$$
$$- a(1/\rho - 1/\nu)(p_{\alpha_1} + p_{\alpha_2})^2] = E.$$

d. Elliptic II coordinates:

$$z_1^2 = -(a-1)\nu\rho/a(\nu-1)(1-\rho), \quad z_2^2 = (\nu-a)(a-\rho)/a(\nu-1)(1-\rho), \tag{4.56}$$
$$\rho\leqslant 0, \quad 1<a\leqslant\nu, \quad 0<a-1\leqslant 1,$$
$$H = [1/(\nu-\rho)]\, \{4\nu(\nu-1)(\nu-a)p_\nu^2 + 4\rho(\rho-1)(a-\rho)p_\rho^2 + a(-1/\rho + 1/\nu)p_{\alpha_1}^2$$
$$+ a(a-1)[1/(a-\rho) + 1/(\nu-a)]\, p_{\alpha_2}^2 - (a-1)[1/(1-\rho) + 1/(\nu-1)](p_{\alpha_1} + p_{\alpha_2})^2 = E,$$
$$Q_4 = [4(1-\rho)(\nu-1)/(\nu-\rho)]\, [\nu(\nu-a)p_\nu^2 + \rho(a-\rho)p_\rho^2] + [(\nu\rho - a\rho - a\nu + a)/\nu\rho]\, p_{\alpha_1}^2$$
$$+ (a-1)(\rho\nu - a)p_{\alpha_2}^2 + [(a-1)(2-\nu-\rho)/(\nu-1)(1-\rho)]\, (p_{\alpha_1} + p_{\alpha_2})^2 = c_3.$$

2. The noncompact Cartan subalgebra $\{X_3 + (1/\sqrt{3})X_8, X_5\}$

$$-\tfrac{1}{2}[X_3 + (1/\sqrt{3})X_8] = p_\alpha = c_1, \quad X_5 = p_u = c_2.$$

e. Hyperbolic coordinates:

$$z_1 = \frac{i\sinh B\cosh u - \cosh B\sinh u}{i\cosh B\cosh u + \sinh B\sinh u}, \quad z_2 = ie^{i\alpha}\frac{\tanh A}{i\cosh B\cosh u + \sinh B\sinh u}, \tag{4.57}$$
$$Q_5 = p_B^2 + (1/\cosh^2 2B)(p_u^2 - p_\alpha^2) - [2\sinh 2B/\cosh^2 2B]\, p_u p_\alpha,$$
$$H = p_A^2 + (1/\cosh^2 A)\, Q_5 + (1/\sinh^2 A)\, p_\alpha^2.$$

f. Complex elliptic coordinates: The coordinates are

$$z_1 = \frac{is_1\cosh u - s_0\sinh u}{is_0\cosh u + s_1\sinh u}, \quad z_2 = ie^{i\alpha}\frac{s_2}{is_0\cosh u + s_1\sinh u} \tag{4.58}$$

with s_0, s_1, and s_2 as in (4.48)

$$Q_6 = [1/(\rho-\nu)]\, \Big\{4\rho\nu(\rho-a)(\rho-a^*)p_\rho^2 - 4\rho\nu(\nu-a)(\nu-a^*)p_\nu^2$$
$$+ \tfrac{1}{4}|a-a^*|^2\rho\nu\, [1/(\rho-a)(\rho-a^*) - 1/(\nu-a)(\nu-a^*)](-p_u^2 + p_\alpha^2)$$
$$+ (|a|^2/\rho\nu)(\rho^2 - \nu^2)p_\alpha^2$$
$$+ \tfrac{1}{2}i(a-a^*)\{[(a^*+a)\nu\rho - 2|a|^2\rho]/(\nu-a)(\nu-a^*) - [(a^*+a)\nu\rho - 2|a|^2\nu]/(\rho-a)(\rho-a^*)\}\, p_u p_\alpha\Big\},$$
$$H = [1/(\rho-\nu)]\, \Big\{4\rho(\rho-a)(\rho-a^*)p_\rho^2 - 4\nu(\nu-a)(\nu-a^*)p_\nu^2$$
$$+ \tfrac{1}{4}|a-a^*|^2\, [\rho/(\rho-a)(\rho-a^*) - \nu/(\nu-a)(\nu-a^*)]\, (-p_u^2 + p_\alpha^2)$$
$$+ (|a|^2/\rho\nu)(\rho-\nu)p_\alpha^2$$
$$+ \tfrac{1}{2}i(a-a^*)\{[(a^*+a)\nu + 2]|a|^2/(\nu-a)(\nu-a^*) - [(a^*+a)\rho - 2|a|^2]/(\rho-a)(\rho-a^*)\}\, p_u p_\alpha\Big\}.$$

3. The orthogonally decomposable MASA $\{Y_1, Y_4\}$

$$-\tfrac{1}{3}Y_1 = -\tfrac{1}{2}[X_3 + (1/\sqrt{3})X_8] = p_\alpha = c_1,$$
$$-Y_2 = X_5 - \tfrac{1}{2}(X_3 - \sqrt{3}X_8) = p_t = c_2.$$

g. Hyperbolic coordinates [*a = 0 in (4.28)*]:

$$z_1 = (\sinh B + ite^{-B})/(\cosh B + ite^{-B}),$$
$$z_2 = \tanh A e^{i\alpha}/(\cosh B + ite^{-B}), \tag{4.59}$$
$$Q_7 = p_B^2 + (e^{2B}p_t + p_\alpha)^2 - p_\alpha^2 = c_3,$$
$$H = p_A^2 + (1/\cosh^2 A)\, Q_7 + (1/\sinh^2 A)p_\alpha^2 = E.$$

h. Horospheric coordinates [$b=0$ *in* (4.28)]:

$$z_1=(-1+e^{2A}+B^2+2it)/(1+e^{2A}+B^2+2it),$$
$$z_2=Be^{i\alpha}/(1+e^{2A}+B^2+2it), \quad (4.60)$$
$$Q_8=p_B^2+[(1/B)p_\alpha+Bp_t]^2,$$
$$H=p_A^2+e^{2A}Q_8+e^{4A}p_t^2.$$

i. Elliptic parabolic coordinates [$3a=b$ *in* (4.28)]:

$$z_1=(\nu+\rho-2\nu\rho+2i\nu\rho t)/(\nu+\rho+2i\nu\rho t),$$
$$z_2^2=4\nu\rho(1-\nu)(\rho-1)e^{2i\alpha}/(\nu+\rho+2i\nu\rho t)^2, \quad (4.61)$$
$$Q_9=[1/(\rho-\nu)]\{4\rho\nu[\rho(\rho-1)p_\rho^2+\nu(1-\nu)p_\nu^2]+[\rho(1-\nu)/\nu^2+\nu(\rho-1)/\rho^2]p_t^2+\nu\rho[1/(1-\nu)+1/(\rho-1)]p_\alpha^2+2(\rho/\nu-\nu/\rho)p_tp_\alpha\}=c_3,$$
$$H=[1/(\rho-\nu)]\{4\rho^2(\rho-1)p_\rho^2+4\nu^2(1-\nu)p_\nu^2+[1/(1-\nu)+1/(\rho-1)]p_\alpha^2+[(1-\nu)/\nu^2+(\rho-1)/\rho^2]p_t^2+2(1/\nu-1/\rho)p_\alpha p_t\}=E.$$

j. Hyperbolic parabolic coordinates [$3a=-b$ *in* (4.28)]:

$$z_1=(\nu+\rho-2i\nu\rho t)/(\nu+\rho-2\nu\rho-2i\nu\rho t),\quad z_2^2=[-4\nu\rho(1-\nu)(\rho-1)e^{2i\alpha}]/(\nu+\rho-2\nu\rho-2i\nu\rho t)^2, \quad (4.62)$$
$$Q_{10}=[1/(\rho-\nu)]\{4\rho\nu[\rho(\rho-1)p_\rho^2+\nu(1-\nu)p_\nu^2]+[\rho(1-\nu)/\nu^2+\nu(\rho-1)/\rho^2]p_t^2+\nu\rho[1/(1-\nu)+1/(\rho-1)]p_\alpha^2-2(\rho/\nu-\nu/\rho)p_\alpha p_t\},$$
$$H=[1/(\rho-\nu)]\{4\rho^2(\rho-1)p_\rho^2+4\nu^2(1-\nu)p_\nu^2+[1/(1-\nu)+1/(\rho-1)]p_\alpha^2+[(1-\nu)/\nu^2+(\rho-1)/\rho^2]p_t^2-2(1/\nu-1/\rho)p_\alpha p_t\}.$$

4. The maximal abelian nilpotent subalgebra $\{Y_3,Y_4\}$

$$Y_3=X_2-X_7=p_t=c_1,\quad -Y_4=X_5-\tfrac{1}{2}X_3+\tfrac{1}{2}\sqrt{3}X_8=p_u=c_2.$$

k. Horospheric coordinates [$b=0$ *in* (4.32)]:

$$z_1=[2(u+Bt)-i(e^{2A}+B^2+t^2-1)]/[2(u+Bt)-i(e^{2A}+B^2+t^2+1)],$$
$$z_2=-2(t+iB)/[2(u+Bt)-i(e^{2A}+B^2+t^2+1)], \quad (4.63)$$
$$Q_{11}=p_B^2+(p_t-2Bp_u)^2=c_3,$$
$$H=p_A^2+e^{2A}Q_{11}+e^{4A}p_u^2=E.$$

l. Semicircular parabolic coordinates [$a=0$ *in* (4.32)]:

$$z_1=\frac{2\rho^2\nu^2u-2\rho\nu(\rho+\nu)t-i[(\rho-\nu)^2+\rho^2\nu^2(t^2-1)]}{2\rho^2\nu^2u-2\rho\nu(\rho+\nu)t-i[(\rho-\nu)^2+\rho^2\nu^2(t^2+1)]}, \quad (4.64)$$
$$z_2=\frac{2\rho\nu t-2i(\rho+\nu)}{2\rho^2\nu^2u-2\rho\nu(\rho+\nu)t-i[(\rho-\nu)^2+\rho^2\nu^2(t^2+1)]},$$
$$Q_{12}=[2/(\rho-\nu)]\{\nu\rho^3p_\rho^2-\nu^3\rho p_\nu^2+(\nu/\rho-\rho/\nu)p_t^2+4(\nu/\rho^3-\rho/\nu^3)p_u^2+(4\nu/\rho^2-4\rho/\nu^2+\rho-\nu)p_up_t\},$$
$$H=[4/(\rho-\nu)]\{\rho^3p_\rho^2-\nu^3p_\nu^2+(1/\rho-1/\nu)p_t^2+4(1/\rho^3-1/\nu^3)p_u^2+4(1/\rho^2-1/\nu^2)p_up_t\}.$$

To summarize: The nonsubgroup type coordinates on H_2, namely elliptic I and II, complex elliptic, elliptic parabolic, hyperbolic parabolic, and semicircular parabolic each occur precisely once. The subgroup type coordinates on H_2 occur as follows: spherical coordinates once [since the compact subalgebra u(2) contains only one MASA], hyperbolic coordinates three times [u(1,1) contains three MASA's] and horospheric coordinates twice [$A_{4,10}$ contains two MASA's (see Fig. 1)].

V. CONCLUSION

The results of this article should be viewed in the context of three different but related research programs. One is a systematic study of the group theoretical, algebraic, and geometrical aspects of the separation of variables in linear and nonlinear partial differential equations. From this point of view we should stress that the hermitian hyperbolic space HH(2) is a noncompact manifold of nonconstant curvature (it does, however, have constant holomorphic sectional curvature). The fact that it has a large isometry group, namely SU(2,1), made it possible to apply essentially the same techniques as for spaces with constant curvature. We have shown that all 12 separable coordinate systems on HH(2) have their origin in the properties of the algebra su(2,1), its subalgebras, and its enveloping algebra.

The second context is that of the classification of subgroups of Lie groups, in particular, maximal Abelian subgroups of classical Lie groups, and its application to the study of differential equations. Indeed, the classification of all MASA's of su(2,1) into four conjugacy classes was the basis of our calculations providing the explicit forms of the 12 separable coordinate systems. In passing, we comment that other applications of this classification are being pursued. In addition to the separation of variables, these include the derivation of superposition principles for certain systems of nonlinear differential equations[26–28] and the symmetry reduction of certain nonlinear partial differential equations to ordinary ones.[29]

Finally, the reduction of the problem of separating variables for the free Hamiltonian on HH(2) to that of a Hamiltonian with a nontrivial interaction, defined on a lower-dimensional manifold, namely the O(2,1) hyperboloid H_2, is an example of a more general method of introducing interactions, in particular completely integrable Hamiltonian systems, by symmetry reduction on group manifolds or homogeneous spaces.

All three above aspects are being actively pursued. In particular, we are currently generalizing the results of this paper to the space HH(n) making use of the MASA's of SU(n,1). The completely integrable Hamiltonian systems obtained in this article are being investigated (explicit solutions, properties of trajectories, special functions occurring as solutions of the Laplace–Beltrami equations, etc.). The related problem of separating variables in Hamiltonians on HH(2) with specific potentials that reduce by symmetry to more general completely integrable relativistic Hamiltonian systems than the ones treated in this article is also under consideration.

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APPENDIX: THE su(2,1) INFINITESIMAL OPERATORS IN TERMS OF IGNORABLE VARIABLES AND O(2,1) VARIABLES

1. Compact Cartan subalgebra variables

$$X_1 = \cos(\alpha_1-\alpha_2)I_{12} + \sin(\alpha_1-\alpha_2)[(s_2/s_1)p_{\alpha_1} + (s_1/s_2)p_{\alpha_2}],$$

$$X_2 = -\sin(\alpha_1-\alpha_2)I_{12} + \cos(\alpha_1-\alpha_2)[(s_2/s_1)p_{\alpha_1} + (s_1/s_2)p_{\alpha_2}],$$

$$X_3 = p_{\alpha_1} - p_{\alpha_2},$$
$$X_4 = \cos\alpha_1 I_{01} - \sin\alpha_1[(s_1/s_0+s_0/s_1)p_{\alpha_1} + (s_1/s_0)p_{\alpha_2}],$$
$$X_5 = \sin\alpha_1 I_{01} + \cos\alpha_1[(s_1/s_0+s_0/s_1)p_{\alpha_1} + (s_1/s_0)p_{\alpha_2}],$$
$$X_6 = \cos\alpha_2 I_{02} - \sin\alpha_2[(s_2/s_0)p_{\alpha_1} + (s_2/s_0+s_0/s_2)p_{\alpha_2}],$$
$$X_7 = \sin\alpha_2 I_{02} + \cos\alpha_2[(s_2/s_0)p_{\alpha_1} + (s_2/s_0+s_0/s_2)p_{\alpha_2}],$$
$$X_8 = -\sqrt{3}(p_{\alpha_1}+p_{\alpha_2}).$$

2. Noncompact Cartan subalgebra variables

$$X_1 = \cosh u\,\cos\alpha I_{12} + \sinh u\,\sin\alpha I_{02} - [s_2/(s_0^2+s_1^2)](s_0\cosh u\,\sin\alpha - s_1\sinh u\,\cos\alpha)p_u - [1/s_2(s_0^2+s_1^2)][-s_0(s_0^2+s_1^2+s_2^2)\sinh u\,\cos\alpha + s_1(s_0^2+s_1^2-s_2^2)\cosh u\,\sin\alpha]p_\alpha,$$

$$X_2 = \cosh u\,\sin\alpha I_{12} + \sinh u\,\cos\alpha I_{02} + [s_2/(s_0^2+s_1^2)](s_0\cosh u + \cos\alpha + s_1\sinh u\,\sin\alpha)p_u + [1/s_2(s_0^2+s_1^2)][s_0(s_0^2+s_1^2+s_2^2)\sinh u\,\sin\alpha + s_1(s_0^2+s_1^2-s_2^2)\cosh u\,\cos\alpha]p_\alpha,$$

$$X_3 = \tfrac{1}{2}(-\sinh 2u I_{01} + [2s_0s_1/(s_0^2+s_1^2)]\cosh 2u p_u + \{[(s_0^2-s_1^2)/(s_0^2+s_1^2)]\cosh 2u - 3\}p_\alpha),$$

$$X_4 = \cosh 2u I_{01} - [2s_0s_1/(s_0^2+s_1^2)]\sinh 2u p_u - [(s_0^2-s_1^2)/(s_0^2+s_1^2)]\sinh 2u p_\alpha,$$

$$X_5 = p_u,$$

$$X_6 = -\sinh u\,\sin\alpha I_{12} + \cosh u\,\cos I_{02} - [s_2/(s_0^2+s_1^2)](s_0\sinh u\,\cos\alpha + s_1\cosh u\,\sin\alpha)p_u - [1/s_2(s_0^2+s_1^2)][s_0(s_0^2+s_1^2+s_2^2)\cosh u\,\sin\alpha + s_1(s_0^2+s_1^2-s_2^2)\sinh u\,\cos\alpha]p_\alpha,$$

$$X_7 = \sinh u\,\cos\alpha I_{12} + \cosh u\,\sin\alpha I_{02} - [s_2/(s_0^2+s_1^2)](s_0\sinh u\,\sin\alpha - s_1\cosh u\,\cos\alpha)p_u - [1/s_2(s_0^2+s_1^2)][-s_0(s_0^2+s_1^2+s_2^2)\cosh u\,\cos\alpha + s_1(s_0^2+s_1^2-s_2^2)\sinh u\,\sin\alpha]p_\alpha,$$

$$X_8 = \tfrac{1}{2}\sqrt{3}(\sinh 2u I_{01} - [2s_0s_1/(s_0^2+s_1^2)]\cosh 2u p_u - \{[(s_0^2-s_1^2)/(s_0^2+s_1^2)]\cosh 2u + 1\}p_\alpha).$$

3. Variables corresponding to orthogonally decomposable non-Cartan subalgebra

$$Y_1 = -3p_\alpha,$$

$$Y_2 = -\cos\alpha(I_{02}-I_{12}) + \sin\alpha\{[(s_0-s_1)/s_2]p_\alpha - [s_2/(s_0-s_1)]p_t\},$$

$$Y_3 = -\sin\alpha(I_{02}-I_{12}) - \cos\alpha\{[(s_0-s_1)/s_2]p_\alpha - [s_2/(s_0-s_1)]p_t\},$$

$$Y_4 = -p_t, \quad Y_5 = I_{01} + 2tp_t,$$

$$Y_6 = -2tI_{01} + [(s_0+s_1)/(s_0-s_1)]p_\alpha + \{[2s_0s_1 - 2t^2(s_0-s_1)^2]/(s_0-s_1)^2\}p_t,$$

$$Y_7 = \cos\alpha I_{12} + t\sin\alpha(I_{02}-I_{12}) + \left\{\{[-s_1(s_0-s_1)-s_2^2]\sin\alpha + (s_0-s_1)^2 t\cos\alpha\}/s_2(s_0-s_1)\right\}p_\alpha - \{s_2[s_0\sin\alpha + t(s_0-s_1)\cos\alpha]/(s_0-s_1)^2\}p_t,$$

$$Y_8 = \sin\alpha I_{12} - t\cos\alpha(I_{02}-I_{12}) + (\{[s_1(s_0-s_1)+s_2^2]\cos\alpha + (s_0-s_1)^2 t\sin\alpha\}/s_2(s_0-s_1))p_\alpha + \{s_2[s_0\cos\alpha - t(s_0-s_1)\sin\alpha]/(s_0-s_1)^2\}p_t.$$

4. Variables corresponding to the maximal abelian nilpotent subalgebra

$$Y_1 = 3\{t(I_{12}-I_{02}) + [s_2/(s_0-s_1)]p_t + (t^2 - s_2^2/(s_0-s_1)^2)p_u\},$$

$$Y_2 = I_{12} - I_{02} + 2tp_u, \quad Y_3 = p_t,$$
$$Y_4 = -p_u, \quad Y_5 = I_{01} + tp_t + 2up_u,$$
$$Y_6 = -2uI_{01} + t^3(I_{02} - I_{12}) + t(I_{02} + I_{12}) - \left[\frac{3s_2t^2}{s_0 - s_1} + 2tu + \frac{(s_0 + s_1)s_2}{(s_0 - s_1)^2}\right]p_t + \left[-\tfrac{1}{2}t^4 + \frac{3t^2s_2^2}{(s_0 - s_1)^2} - 2u^2 + \frac{s_2^2(s_0 + s_1)}{(s_0 - s_1)^3} + \frac{2s_0s_1}{(s_0 - s_1)^2}\right]p_u,$$
$$Y_7 = I_{12} + \frac{3t^2}{2}(I_{02} - I_{12}) - \left(u + \frac{3s_2t}{s_0 - s_1}\right)p_t - t\left[t^2 - 1 - \frac{3s_2^2}{(s_0 - s_1)^2}\right]p_u,$$
$$Y_8 = tI_{01} + u(I_{12} - I_{02}) + \left[\frac{t^2}{2} + \frac{s_1^2 - s_2^2 - s_0s_1}{(s_0 - s_1)^2}\right]p_t + \left[2tu + \frac{s_2(s_0^2 - s_1^2 + s_2^2)}{(s_0 - s_1)^3}\right]p_u.$$

[1]C. P. Boyer, E. G. Kalnins, and P. Winternitz, "Separation of Variables for the Hamilton Jacobi Equation on Complex Projective Spaces," Preprint CRMA-1064, Montréal, 1981.
[2]S. Benenti and M. Francoviglia, "The Theory of Separability of the Hamilton-Jacobi Equation and Its Application to General Relativity," in *General Relativity and Gravitation*, edited by A. Held (Plenum, New York, 1980), Vol. 1.
[3]E. G. Kalnins and W. Miller, Jr., SIAM J. Math. Anal. **12**, 617 (1981).
[4]C. P. Boyer, E. G. Kalnins, and W. Miller, Jr., Comm. Math. Phys. **59**, 285 (1978).
[5]C. P. Boyer, E. G. Kalnins, and W. Miller, Jr., J. Math. Phys. **19**, 200 (1978).
[6]P. Winternitz and I. Friš, Yad. Fiz. **1**, 889 (1965) [Sov. J. Nucl. Phys. **1**, 636 (1965)].
[7]P. Winternitz, I. Lukač, and Ya. A. Smorodinskiı, Yad. Fiz. **7**, 192 (1968) [Sov. J. Nucl. Phys. **7**, 139 (1968)].
[8]W. Miller, Jr., *Symmetry and Separation of Variables* (Addison-Wesley, Reading, Mass., 1977).
[9]J. Patera and P. Winternitz, J. Math. Phys. **14**, 1130 (1973); J. Chem. Phys. **65**, 2725 (1976).
[10]S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, (Interscience, New York, 1969), Vol. II.
[11]S. Helgason, *Diffferential Geometry, Lie Groups and Symmetric Spaces* (Academic, New York, 1978).
[12]P. Delong, Ph. D. thesis, University of Minnesota, 1982.
[13]W. Miller, Jr., J. Patera, and P. Winternitz, J. Math. Phys. **22**, 251 (1981).
[14]J. Patera, P. Winternitz, and H. Zassenhaus, J. Math. Phys. **15**, 1378 (1974).
[15]J. Patera, P. Winternitz, and H. Zassenhaus, Math. Rep. Acad. Sci. (Can.) **2**, 231,237 (1980).
[16]J. Patera, P. Winternitz, and H. Zassenhaus, J. Math. Phys. **24**, 1973 (1983).
[17]J. Patera, R. T. Sharp, P. Winternitz, and H. Zassenhaus, J. Math. Phys. **17**, 986 (1976).
[18]R. Abraham and J. E. Marsden, *Foundations of Mechanics* (Benjamin, Reading, Mass., 1978).
[19]J. Marsden and A. Weinstein, Rep. Math. Phys. **5**, 121 (1974).
[20]D. Kazhdan, B. Kostant, and S. Sternberg, Comm. Pure Appl. Math. **31**, 481 (1978).
[21]M. P. Olevsky, Mat. Sb. **27**(69), 379 (1950).
[22]P. Winternitz, I. Friš, M. Uhlı̌r, and Ya. A. Smorodinskiĭ, Yad. Fiz. **4**, 625 (1966) [Sov. J. Nucl. Phys. **4**, 444 (1967)].
[23]P. Kramer, M. Moshinsky, and T. Seligman, "Complex Extensions of Canonical Transformation and Quantum Mechanics," in *Group Theory and Applications*, edited by E. Loebl (Academic, New York, 1975), Vol. 3.
[24]M. Moshinsky, J. Patera, and P. Winternitz, J. Math. Phys. **16**, 82 (1975).
[25]D. Ž. Djokokić, J. Patera, P. Winternitz, and H. Zassenhaus, J. Math. Phys. **24**, 1363 (1983).
[26]R. L. Anderson, Lett. Math. Phys. **4**, 1 (1980).
[27]R. L. Anderson, J. Harnad, and P. Winternitz, Lett. Math. Phys. **5**, 143 (1981); Physica D **4**, 164 (1982); J. Math. Phys. **23**, 1062 (1982).
[28]P. Winternitz, Physica A **114**, 105 (1982).
[29]A. M. Grundland, J. Harnad, and P. Winternitz, Kinam **4**, 333 (1982).