Killing–Yano tensors and variable separation in Kerr geometry

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A complete analysis of the free-field massless spin-s equations ($s = 0, \frac{1}{2}, 1$) in Kerr geometry is given. It is shown that in each case the separation constants occurring in the solutions obtained from a potential function can be characterized in an invariant way. This invariant characterization is given in terms of the Killing–Yano tensor admitted by Kerr geometry.

I. INTRODUCTION AND MATHEMATICAL PRELIMINARIES

A complete understanding of the characterization of solutions of spin-s free-field equations in Kerr geometry has yet to be achieved. Interest in these equations originated with the investigations of Teukolsky,[1] who showed that in the Newman–Penrose[2] formalism separable solutions were possible for certain Maxwell and Weyl scalars in Kerr geometry.[3] (Kerr geometry is the space-time geometry of the gravitational background due to a rotating black hole.)

Chandrasekhar[4] has shown that these results can be extended to the Dirac equation. These results have been further extended[5,6] and shown to hold for more general classes of space-time. In the original work of Carter[7] it was established that the Hamilton–Jacobi and Schrödinger equations admitted a solution for the Kerr geometry via standard separation of variables techniques. Because of this property, Kerr space-time admits a quadratic constant of the motion in addition to the already known two Killing vector fields. However, the key property at the heart of the solution of the equations for spin-s $(0,\frac{1}{2},1)$ is the existence of a Killing–Yano tensor.[8] The role played by such a tensor for the solutions of the Dirac equation has been explained in Refs. 9 and 10. In this paper we indicate how this characterization works for massless particles with spins 0, $\frac{1}{2}$, and 1 and massive particles with spins-0, $\frac{1}{2}$. In so doing we clarify the role of the Killing–Yano tensor. The results for spin-1 are new and the treatment of spins-0, $\frac{1}{2}$, while not new, is presented in a unified way.

Once this work is extended we expect to better understand the methods by which a theory of "variable separation" can be constructed for general spin-s equations. Earlier work by the authors,[11] although not incorrect, did not succeed in giving an intrinsic characterization of the separation parameters appearing in the solution of Maxwell's equations. What was in fact achieved in Ref. 11 was a characterization of a particular choice of gauge. The contents of the present paper are arranged as follows. In Sec. I we outline the conventions and notations used, together with the relevant definitions and properties of Killing–Yano tensors. In Secs. II and III we deal with the zero-mass equations of spin-0, $\frac{1}{2}$, and 1, respectively.

In this paper we consistently use the spinor notation of Penrose and Rindler.[12] In addition, we employ the null tetrad formalism as described by Chandrasekhar.[4] Specifically, we restrict ourselves to the Kinnersley null tetrad of vectors with the components

$$\begin{aligned} l^i &= (1/\Delta)(r^2+a^2,\Delta,0,a),\\ n^i &= (1/2\rho^2)(r^2+a^2,-\Delta,0,a),\\ m^i &= (1/\sqrt{2}\bar{\rho})(ia\sin\theta,0,1,i\csc\theta),\\ \overline{m}^i &= (1/\sqrt{2}\bar{\rho}^*)(-ia\sin\theta,0,1,-i\csc\theta), \end{aligned} \tag{1.1}$$

where

$$\Delta = r^2+a^2-2Mr,\ \rho^2 = r^2+a^2\cos^2\theta,$$
$$\bar{\rho} = (r+ia\cos\theta).$$

The Kerr solution of the Einstein equations has the line element

$$\begin{aligned} ds^2 &= \left(1-\frac{2Mr}{\rho^2}\right)dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2\,d\theta^2\\ &\quad + \frac{4aMr\sin^2\theta}{\rho^2}\,dt\,d\phi\\ &\quad - \left((r^2+a^2)+\frac{2a^2Mr\sin^2\theta}{\rho^2}\right)\sin^2\theta\,d\phi^2. \end{aligned} \tag{1.2}$$

A Killing–Yano tensor $K_{AA'BB'}$ is a (skew symmetric) tensor satisfying

$$\nabla_{(CC'}K_{AA')BB'} = 0,\ K_{AA'BB'} + K_{BB'AA'} = 0. \tag{1.3}$$

The Killing–Yano tensor can also be equivalently represented in terms of the pair of symmetric Killing spinors $K_{AB'}$, $\overline{K}_{A'B'}$ via

$$K_{AA'BB'} = \tfrac{1}{2}(\epsilon_{A'B'}K_{AB} + \epsilon_{AB}\overline{K}_{A'B'}). \tag{1.4}$$

Conditions (1.3) are then equivalent to

$$\begin{aligned} &\nabla_{(AA'}K_{BC)} = 0,\ \nabla_{A(A'}\overline{K}_{B'C')} = 0,\\ &\nabla_{BA'}K_A{}^B + \nabla_{AB'}\overline{K}_{A'}{}^{B'} = 0. \end{aligned} \tag{1.5}$$

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We have the following result: In Kerr space-time the equations for a Killing–Yano tensor have only one solution. The nonzero components of this tensor in the null tetrad formalism using the Kinnersley tetrad are

$$K_{00'11'} = ia\cos\theta,\ K_{01'10'} = r. \tag{1.6}$$

The Klein–Gordon equation for a spin-0 free-field is

$$\Box\phi = (\nabla_{AA'}\nabla^{AA'})\phi = m^2\phi. \tag{1.7}$$

In Newman–Penrose notation (1.7) has the form

$$\begin{aligned}\Box\phi = [&(D-\rho-\rho^*)\Delta + (\Delta-\gamma-\gamma^*+\mu+\mu^*)D\\ &-(\delta^*-\alpha+\beta^*-\tau^*+\pi)\delta\\ &-(\delta+\beta-\alpha^*-\tau+\pi^*)\delta^*]\phi = m^2\phi.\end{aligned} \tag{1.8}$$

In terms of the coordinates used to describe the line element (1.2) this equation reads

$$\begin{aligned}\Box\phi = (-1/2\rho^2)\{&\Delta(\mathscr{D}_1\mathscr{D}_0^+ + \mathscr{D}_1^+\mathscr{D}_0)\\ &+(\mathscr{L}_1\mathscr{L}_0^+ + \mathscr{L}_1^+\mathscr{L}_0)\}\phi = m^2\phi.\end{aligned} \tag{1.9}$$

Equation (1.9) admits a separable solution

$$\phi = R_0(r)S_0(\theta)e^{im\phi+i\sigma t}, \tag{1.10}$$

where the separation equations are

$$\begin{aligned}&[\Delta(\mathscr{D}_1\mathscr{D}_0^+ + \mathscr{D}_1^+\mathscr{D}_0) + 2m^2r^2 + \lambda]R_0 = 0,\\ &[\mathscr{L}_1\mathscr{L}_0^+ + \mathscr{L}_0^+\mathscr{L}_0 + 2m^2a^2\cos^2\theta - \lambda]S_0 = 0.\end{aligned} \tag{1.11}$$

The directional derivatives in expression (1.11) are defined by

$$\begin{aligned}\mathscr{D}_n &= \partial_r + iK/\Delta + 2n(r-M)/\Delta,\\ \mathscr{D}_n^+ &= \partial_r - iK/\Delta + 2n(r-M)/\Delta,\\ \mathscr{L}_n &= \partial_\theta + Q + n\cot\theta,\\ \mathscr{L}_n^+ &= \partial_\theta - Q + n\cot\theta,\end{aligned} \tag{1.12}$$

where $K = (r^2+a^2)\sigma + am$ and $Q = a\sigma\sin\theta + m\csc\theta$.

From the theory of separation of variables for the Klein–Gordon equation it follows that there exists a second-order symmetry operator U such that

$$U\phi = \lambda\phi \tag{1.13}$$

for a separable solution ϕ. (We say that U is a symmetry operator if it commutes with $\Box$: $[\Box, U] = 0$.)

In terms of the Killing–Yano tensor,

$$\begin{aligned}U &= (K^{AA'BB'}\nabla_{BB'})(K_{AA'}{}^{CC'}\nabla_{CC'}) - K^{AA'BB'}M_{BB'}\nabla_{AA'}\\ &= (1/2\rho^2)[a^2\cos^2\theta[\Delta(\mathscr{D}_1\mathscr{D}_0^+ + \mathscr{D}_1^+\mathscr{D}_0)]\\ &\quad - r^2[\mathscr{L}_1\mathscr{L}_0^+ + \mathscr{L}_1^+\mathscr{L}_0]],\end{aligned} \tag{1.14}$$

where

$$M_{AB'} = \tfrac{1}{3}\nabla^{BA'}K_{BB'AA'}.$$

We also note here that the symmetric tensor

$$\mathscr{K}^{AA'CC'} = K^{AA'}{}_{BB'}K^{BB'CC'} \tag{1.15}$$

is a second-order Killing tensor satisfying the Killing equation

$$\nabla_{(AA'}\mathscr{K}_{BB'CC')} = 0. \tag{1.16}$$

This fact is crucial in the separability of the corresponding Hamilton–Jacobi and Schrödinger equations.

II. THE DIRAC EQUATION

In spinor notation the Dirac equation has the form

$$\nabla_{AX'}\chi^{X'} = (im_e\sqrt{2})\phi_A,\ \nabla_{AX'}\phi^A = -(im_e/\sqrt{2})\chi_{X'}. \tag{2.1}$$

Equations (2.1), when written in Newman–Penrose notation, are

$$\begin{aligned}&(D-\rho^*)\chi_{1'} - (\delta+\pi^*-\alpha^*)\chi_{0'} = (im_e/\sqrt{2})\phi_0,\\ &(\delta^*+\beta^*-\tau^*)\chi_{1'} - (\Delta+\mu^*-\gamma^*)\chi_{0'} = (im_e/\sqrt{2})\phi_1,\\ &(D-\rho)\phi_1 - (\delta^*+\pi-\alpha)\phi_0 = -(im_e/\sqrt{2})\chi_{0'},\\ &(\delta+\beta-\tau)\phi_1 - (\Delta+\mu-\gamma)\phi_0 = -(im_e/\sqrt{2})\chi_{1'}.\end{aligned} \tag{2.2}$$

Chandrasekhar[4] found solutions of the form

$$\begin{aligned}\phi_1 &= (1/\bar\rho^*)R_{-1/2}S_{-1/2}e^{i\sigma t+im\phi},\\ \phi_0 &= -R_{1/2}S_{1/2}e^{i\sigma t+im\phi},\\ \chi_{1'} &= -(1/\bar\rho)R_{-1/2}S_{1/2}e^{i\sigma t+im\phi},\\ \chi_{0'} &= -R_{1/2}S_{-1/2}e^{i\sigma t+im\phi}.\end{aligned} \tag{2.3}$$

The second-order separation equations are

$$\begin{aligned}&\{\Delta\mathscr{D}_{1/2}^+\mathscr{D}_0 - [im_e/(\lambda+im_er)]\Delta\mathscr{D}_0\\ &\quad -(\lambda^2+m_e^2r^2)\}R_{-1/2} = 0,\\ &\{\Delta\mathscr{D}_{1/2}\mathscr{D}_0^+ + [im_e/(\lambda-im_er)]\Delta\mathscr{D}_0^+\\ &\quad -(\lambda^2+m_e^2r^2)\}\Delta^{1/2}R_{1/2} = 0,\\ &\{\mathscr{L}_{1/2}\mathscr{L}_{1/2}^+ + [am_e\sin\theta/(\lambda+am_e\cos\theta)]\mathscr{L}_{1/2}^+\\ &\quad +(\lambda^2-a^2m_e^2\cos^2\theta)\}S_{-1/2} = 0,\\ &\{\mathscr{L}_{1/2}^+\mathscr{L}_{1/2} - [am_e\sin\theta/(\lambda-am_e\cos\theta)]\mathscr{L}_{1/2}\\ &\quad +(\lambda^2-a^2m_e^2\cos^2\theta)\}S_{1/2} = 0.\end{aligned} \tag{2.4}$$

The separated solutions satisfy the eigenvalue equations

$$\begin{aligned}L_{AA'}\chi^{A'} &= (K_{AA'}{}^{BB'}\nabla_{BB'} - M_{AA'})\chi^{A'} = (\lambda/\sqrt{2})\phi_A,\\ N_{AA'}\phi^A &= (K_{AA'}{}^{BB'}\nabla_{BB'} + M_{AA'})\phi^A = (\lambda/\sqrt{2})\chi_{A'}.\end{aligned} \tag{2.5}$$

From Eqs. (2.5) follows the conditions

$$\begin{aligned}&[\nabla_{AA'}L^{AX'} + N_{AA'}\nabla^{AX'}]\chi_{X'} = 0,\\ &[\nabla^{CA'}N_{AA'} + L^{CA'}\nabla_{AA'}]\phi^A = 0.\end{aligned} \tag{2.6}$$

From (2.5) we can construct the operator

$$\Lambda = \begin{bmatrix} 0 & L_A^{A'}\\ N^A_{A'} & 0\end{bmatrix} \tag{2.7}$$

acting on the Dirac spinors

$$\begin{bmatrix}\phi_A\\ \chi_{A'}\end{bmatrix}.$$

The operator (2.7) anticommutes with the Dirac Hamiltonian

$$H = \begin{bmatrix}(im_e/\sqrt{2})\epsilon_B{}^A & -\nabla_B{}^{A'}\\ \nabla^A{}_{B'} & -(im_e/\sqrt{2})\epsilon_{B'}{}^{A'}\end{bmatrix}. \tag{2.8}$$

The proof of relations (2.6) is instructive; we now prove the first of these relations. Consider the operator

$$Q_{A'C'} = N_{AA'}\nabla^A{}_{C'} - \nabla^A{}_{A'}L_{AC'} \tag{2.9}$$

using

$$\begin{aligned}\nabla_{AA'}K_{BC} &= \epsilon_{AB}M_{CA'} + \epsilon_{AC}M_{BA'},\\ \nabla_{AA'}\bar{K}_{B'C'} &= -\epsilon_{A'B'}M_{AC'} - \epsilon_{A'C'}M_{AB'}.\end{aligned} \tag{2.10}$$

We find that

$$\begin{aligned}Q_{A'C'} = \tfrac{1}{2}[&K_A{}^B\nabla_{BA'}\nabla^A{}_{C'} + \bar{K}_{A'}{}^{B'}\nabla_{AB'}\nabla^A{}_{C'}\\ &- K_A{}^B\nabla^A{}_{A'}\nabla_{BC'} - \bar{K}_{C'}{}^{B'}\nabla^A{}_{A'}\nabla_{AB'}\\ &+ (\epsilon_{A'C'}M^{AB'} + \epsilon_{A'}{}^{B'}M^A{}_{C'})\nabla_{AB'}\\ &+ 3M^B{}_{A'}\nabla_{BC'}]\\ &+ M_{AA'}\nabla^A{}_{C'} + M_{AC'}\nabla^A{}_{A'} + (\nabla^A{}_{A'}M_{AC'}).\end{aligned} \tag{2.11}$$

Noting that

$$K_A{}^B\nabla_{BA'}\nabla^A{}_{C'} = K_A{}^B\nabla^A{}_{A'}\nabla_{BC'} \tag{2.12}$$

since K_{AB} is symmetric,

$$\begin{aligned}&\bar{K}_{A'}{}^{B'}\nabla_{AB'}\nabla^A{}_{C'} - \bar{K}_{C'}{}^{B'}\nabla^A{}_{A'}\nabla_{AB'}\\ &= \tfrac{1}{2}\bar{K}_{A'}{}^{B'}(\nabla_{AB'}\nabla^A{}_{C'} + \nabla^A{}_{C'}\nabla_{AB'} + [\nabla_{AB'},\nabla^A{}_{C'}])\\ &\quad - \tfrac{1}{2}\bar{K}_{C'}{}^{B'}(\nabla^A{}_{A'}\nabla_{AB'} + \nabla_{AB'}\nabla^A{}_{A'} + [\nabla^A{}_{A'},\nabla_{AB'}])\\ &= \tfrac{1}{2}\epsilon_{A'C'}\bar{K}_{D'}{}^{B'}(\nabla_{AB'}\nabla^{AD'} + \nabla^{AD'}\nabla_{AB'})\\ &\quad + \tfrac{1}{2}\bar{K}_{A'}{}^{B'}[\nabla_{AB'},\nabla^A{}_{C'}] + \tfrac{1}{2}\bar{K}_{C'}{}^{B'}[\nabla_{AB'},\nabla^A{}_{A'}]\\ &= \tfrac{1}{2}\bar{K}_{A'}{}^{B'}[\nabla_{AB'},\nabla^A{}_{C'}] + \tfrac{1}{2}\bar{K}_{C'}{}^{B'}[\nabla_{AB'},\nabla^A{}_{A'}],\end{aligned} \tag{2.13}$$

$$\epsilon_{A'C'}M^{AB'}\nabla_{AB'} = M^A{}_{C'}\nabla_{AA'} - M^A{}_{A'}\nabla_{AC'}, \tag{2.14}$$

we can write

$$\begin{aligned}Q_{A'C'} &= \tfrac{1}{4}\bar{K}_{A'}{}^{B'}[\nabla_{AB'},\nabla^A{}_{C'}]\\ &\quad + \tfrac{1}{4}\bar{K}_{C'}{}^{B'}[\nabla_{AB'},\nabla^A{}_{A'}] + (\nabla^A{}_{A'}M_{AC'}).\end{aligned} \tag{2.15}$$

Now consider

$$\begin{aligned}\nabla_{AA'}M_{BB'} &= \tfrac{1}{3}\nabla_{AA'}\nabla_{CB'}K^C{}_B\\ &= \tfrac{1}{3}(\nabla_{CB'}\nabla_{AA'} + [\nabla_{AA'},\nabla_{CB'}])K^C{}_B\\ &= \tfrac{1}{3}(\nabla_{AB'}M_{BA'} + \epsilon_{AB}\nabla_{CB'}M^C{}_{A'}\\ &\quad + \epsilon_{A'B'}\Psi_{ABCD}K^{CD}),\end{aligned} \tag{2.16}$$

from which the following results can be obtained:

$$\begin{aligned}&\nabla_{(A(A'}M_{B)B')} = 0,\quad \nabla_{AA'}M^{AA'} = 0,\\ &\nabla_{(AA'}M_{B)}{}^{A'} = \tfrac{1}{2}\Psi_{ABCD}K^{CD} = W_{AB},\quad \text{defining } W_{AB}.\end{aligned} \tag{2.17}$$

Note that we can also write $\nabla_{AA'}M_{BB'} = -\tfrac{1}{3}\nabla_{AA'}\nabla_{BC'}\bar{K}^{C'}{}_{B'}$ and proceed in a similar manner as before to obtain the additional result

$$\begin{aligned}\nabla_{A(A'}M^A{}_{B')} &= -\tfrac{1}{2}\bar{\Psi}_{A'B'C'D'}\bar{K}^{C'D'}\\ &= -\bar{W}_{A'B'},\quad \text{defining } \bar{W}_{A'B'}.\end{aligned} \tag{2.18}$$

Now since (by reducing to symmetric spinors) we can write for any $T_{ABA'B'}$,

$$\begin{aligned}T_{ABA'B'} &= T_{(AB)(A'B')} + \tfrac{1}{2}\epsilon_{A'B'}T_{(AB)K'}{}^{K'}\\ &\quad + \tfrac{1}{2}\epsilon_{AB}T_K{}^K{}_{(A'B')} + \tfrac{1}{4}\epsilon_{AB}\epsilon_{A'B'}T_K{}^K{}_{K'}{}^{K'}\end{aligned} \tag{2.19}$$

it follows that

$$\nabla_{AA'}M_{BB'} = \tfrac{1}{2}\epsilon_{A'B'}W_{AB} - \tfrac{1}{2}\epsilon_{AB}\bar{W}_{A'B'}. \tag{2.20}$$

We also note in passing that $\nabla_{AA'}M_{BB'}$ is a skew-symmetric tensor, i. e., M_a satisfies

$$\nabla_{(b}M_{a)} = 0, \tag{2.21}$$

i.e., M_a is a Killing vector.

Returning to the operator $Q_{A'C'}$, we can now write

$$\begin{aligned}Q_{A'C'} &= \tfrac{1}{4}\bar{K}_{A'}{}^{B'}[\nabla_{AB'},\nabla^A{}_{C'}]\\ &\quad + \tfrac{1}{4}\bar{K}_{C'}{}^{B'}[\nabla_{AB'},\nabla^A{}_{A'}] + \bar{W}_{A'C'},\end{aligned} \tag{2.22}$$

from which its action on a spinor $\phi^{C'}$ is as follows:

$$\begin{aligned}Q_{A'C'}\phi^{C'} &= \tfrac{1}{4}\bar{K}_{A'}{}^{B'}\epsilon_A{}^A\bar{\Psi}_{B'C'}{}^{C'}{}_{M'}\phi^{M'}\\ &\quad + \tfrac{1}{4}\bar{K}_{C'}{}^{B'}\epsilon_A{}^A\bar{\Psi}_{B'A'}{}^{C'}{}_{M'}\phi^{M'}\\ &\quad + \tfrac{1}{2}\bar{\Psi}_{A'C'L'M'}\bar{K}^{L'M'}\phi^{C'} = 0.\end{aligned} \tag{2.23}$$

Thus

$$N_{AA'}\nabla^A{}_{C'}\phi^{C'} = \nabla^A{}_{A'}L_{AC'}\phi^{C'} \tag{2.24}$$

If we consider only the neutrino equation $\nabla^{AA'}\phi_A = 0$ for the case in which $m_e = 0$ (massless spin-$\tfrac{1}{2}$), then the separation constant λ^2 stems from the eigenvalue equation

$$L_B{}^{A'}N^A{}_{A'}\phi_A = (\lambda^2/2)\phi_B. \tag{2.25}$$

III. THE MAXWELL EQUATIONS

For the case of the Maxwell equations corresponding to mass-zero spin-1 the characterization of separation parameters in terms of the components of the Killing–Yano tensor can also be achieved. Maxwell's equations are commonly formulated in terms of the skew-symmetric energy momentum tensor $F_{AA'BB'}$, which satisfies

$$\begin{aligned}&\nabla_{AA'}F_{BB'CC'} + \nabla_{CC'}F_{AA'BB'} + \nabla_{BB'}F_{CC'AA'} = 0,\\ &\nabla^{AA'}F_{AA'BB'} = 0,\quad F_{AA'BB'} + F_{BB'AA'} = 0.\end{aligned} \tag{3.1}$$

As with the case of Killing–Yano tensor, $F_{AA'BB'}$ can be realized via the symmetric spinors ϕ_{AB}, $\phi_{A'B'}$ according to

$$F_{AA'BB'} = \epsilon_{AB}\bar{\phi}_{A'B'} + \epsilon_{A'B'}\phi_{AB}. \tag{3.2}$$

In terms of these symmetric spinors, Maxwell's equations have the form

$$\nabla^A{}_{A'}\phi_{AB} = 0, \tag{3.3a}$$

$$\nabla_A{}^{C'}\bar{\phi}_{C'B'} = 0. \tag{3.3b}$$

In Ref. 4 Chandrasekhar has obtained explicit solutions for these equations: viz.

$$\begin{aligned}&(D - 2\rho)\phi_{01} - (\delta^* + \pi - 2\alpha)\phi_{00} = 0,\\ &(D - \rho)\phi_{11} - (\delta^* + 2\pi)\phi_{01} = 0,\\ &(\delta - 2\tau)\phi_{01} - (\Delta + \mu - 2\gamma)\phi_{00} = 0,\\ &(\delta - \tau + 2\beta)\phi_{11} - (\Delta + 2\mu)\phi_{01} = 0.\end{aligned} \tag{3.4}$$

From the crucial observation that

$$\bar{\rho}(\delta - 2\tau)(D - 2\rho) = (D - 2\rho)\bar{\rho}(\delta - 2\tau), \tag{3.5}$$

Teukolsky[1] deduced that if $\phi_{00} = \psi_{00}e^{i\sigma t + im\phi}$, then the function ψ_{00} satisfies

$$[\Delta\mathscr{D}_1\mathscr{D}_1^+ + \mathscr{L}_0^+\mathscr{L}_1 - 2i\sigma\bar{\rho}]\psi_{00} = 0. \tag{3.6}$$

This function admits a separable solution $\psi_{00} = R_1S_1$, where the separation equations are

$$\begin{aligned}&(\Delta\mathscr{D}_1\mathscr{D}_1^+ - 2i\sigma r - \lambda)R_1 = 0,\\ &(\mathscr{L}_0^+\mathscr{L}_1 + 2a\sigma\cos\theta + \lambda)S_1 = 0.\end{aligned} \tag{3.7}$$

If Eqs. (3.4) are analyzed further and we write

$$\phi_{11} = (2(\bar\rho^*)^2)^{-1}\psi_{11}e^{i\sigma t + im\phi},$$

we find that the function ψ_{11} satisfies

$$[\Delta\mathscr{D}_0^+\mathscr{D}_0 + \mathscr{L}_0\mathscr{L}_1^+ + 2i\sigma\bar\rho]\psi_{11} = 0, \quad (3.8)$$

which admits a separable solution $\psi_{11} = R_{-1}S_{-1}$ with the separation equations

$$[\Delta\mathscr{D}_0^+\mathscr{D}_0 + 2i\sigma r + \lambda]R_{-1} = 0,$$
$$[\mathscr{L}_0\mathscr{L}_1^+ - 2a\sigma\cos\theta - \lambda]S_{-1} = 0. \quad (3.9)$$

Equations (3.7) and (3.9) were first derived by Teukolsky.[1] The functions $R_{\pm 1}$, $S_{\pm 1}$ are called Teukolsky functions by Chandrasekhar. If instead of $R_{\pm 1}$ we choose the function $P_{-1} = R_{-1}$, $P_{+1} = \Delta R_{+1}$, then the functions exhibit interesting properties, which are summarized in the Appendix. Chandrasekhar proceeded further and showed that ϕ_{01} can be written in the form

$$\phi_{01} = (1/\sqrt{2}\bar\rho^*\mathscr{C})[\mathscr{D}_0\mathscr{L}_1 - (1/\bar\rho^*)$$
$$\times(\mathscr{L}_1 + ia\sin\theta\mathscr{D}_0)]P_{-1}S_{+1}, \quad (3.10)$$

where $\mathscr{C}$ is as in (A1).

We now seek the invariant characterization of the parameters λ and $\mathscr{C}$. To determine this we draw on the results of Cohen and Kegeles,[12] who showed how to obtain solutions of (3.3) via the use of a Debye potential $\bar P^{X'Y'}$ and a gauge degree of freedom $G_B^{\ W'}$. If these functions satisfy

$$\nabla^{A(M'}\nabla_{AW'}\bar P^{N')W'} = 2\nabla^{A(M'}G_A^{\ N')}, \quad (3.11)$$

then

$$\phi_{AB} = \nabla_{(AW'}\nabla_{B)X'}\bar P^{W'X'} - 2\nabla_{(AW'}G_{B)}^{\ \ W'} \quad (3.12)$$

is a solution of (3.3). More specifically, if one chooses $G_A^{\ W'} = -U_{AA'}\bar P^{A'W'}$, where

$$U_{00'} = \rho^*_1,\ U_{10'} = \tau^*_1,\ U_{01'} = -\pi^*,\ U_{11'} = -\mu^*,$$

then $\bar P^{A'W'}$ satisfies the decoupled equation

$$[(1-p/2)(\Delta + \mu - \gamma + (1-2p)\gamma^* - p\mu^*)(D + (1-p)\rho^*)$$
$$- (1-p/2)(\delta + \beta - \tau + (1-2p)\alpha^* - p\pi^*)(\delta^* + 2(1-p)\beta^* + (1-p)\tau^*)$$
$$- (p/2)(\delta^* + \pi - \alpha + (3-2p)\beta^* + (2-p)\tau^*)(\delta + 2(1-p)\alpha^* - (p-1)\pi^*)$$
$$+ (p/2)(D - \rho + (2-p)\rho^*)(\Delta + 2(1-p)\gamma^* - (p-1)\mu^*)]\bar P^{A'W'} = 0, \quad (3.13)$$

where p is the number of "ones" appearing in the indices of $\bar P^{A'W'}$.

The choice of $G_A^{\ W'}$ made above is particularly interesting since it yields three equivalent representations for the same function, viz.

(i) $p = 0$, $\bar P^{0'0'} = P_{-1}S_1e^{i\sigma t + im\phi}$:

$$\phi_{00} = \mathscr{D}_0\mathscr{D}_0\bar P^{0'0'},$$
$$\phi_{01} = (1/\sqrt{2}\bar\rho^*)[\mathscr{D}_0\mathscr{L}_1 - (1/\bar\rho^*)$$
$$\times(\mathscr{L}_1 + ia\sin\theta\mathscr{D}_0)]\bar P^{0'0'}, \quad (3.14)$$
$$\phi_{11} = [1/2(\bar\rho^*)^2]\mathscr{L}_0\mathscr{L}_1\bar P^{0'0'}.$$

(ii) $p = 2$, $\bar P^{1'1'} = (\bar\rho)^2\Delta^{-1}P_{+1}S_{-1}e^{i\sigma t + im\phi}$:

$$\phi_{00} = \mathscr{L}_0^+\mathscr{L}_1^+\bar P^{1'1'},$$
$$\phi_{01} = -(\Delta/\sqrt{2}\bar\rho^*)[\mathscr{D}_1^+\mathscr{L}_1^+ - (1/\bar\rho^*)$$
$$\times(\mathscr{L}_1^+ + ia\sin\theta\mathscr{D}_1^+)]\bar P^{1'1'}, \quad (3.15)$$
$$\phi_{11} = [\Delta^2/2(\bar\rho^*)^2]\mathscr{D}_1^+\mathscr{D}_1^+\bar P^{1'1'}.$$

From the identities given in the Appendix it is straightforward to establish that (3.14) and (3.15) are representations of the same functions ϕ_{AB}.

(iii) $p = 1$; in this case $\bar P^{0'1'}$ satisfies

$$\left[\Delta\left(\mathscr{D}_1^+ - \frac{1}{\bar\rho}\right)\left(\mathscr{D}_0 + \frac{1}{\bar\rho}\right) + \left(\mathscr{L}_1^+ + \frac{ia\sin\theta}{\rho}\right)\right.$$
$$\left.\times\left(\mathscr{L}_0 - \frac{ia\sin\theta}{\bar\rho}\right)\right]\frac{1}{\bar\rho}\bar P^{0'1'} = 0.$$

An examination of this equation shows that $\bar P^{0'1'}$ satisfies the same equation as $\bar\rho\bar\phi_{0'1'}$. Hence a solution may be taken to be

$$\bar P^{0'1'} = (1/\sqrt{2}\mathscr{C})[\mathscr{D}_0\mathscr{L}_1^+ - (1/\rho)(\mathscr{L}_1^+ - ia\sin\theta\mathscr{D}_0)]$$
$$\times P_{-1}S_{-1}e^{i\sigma t + im\phi}.$$

With this choice the components of ϕ_{AB} can be written as

$$\phi_{00} = (\sqrt{2}/\bar\rho)\mathscr{D}_0\mathscr{L}_0^+\bar P^{0'1'},$$
$$\phi_{01} = \frac{-1}{4\rho^2}\left[\Delta\left(\mathscr{D}_1 - \frac{2}{\bar\rho^*}\right)\mathscr{D}_0^+ + \Delta\left(\mathscr{D}_1^+ - \frac{2}{\bar\rho^*}\right)\mathscr{D}_0\right.$$
$$+ \left(\mathscr{L}_1^+ - \frac{ia\sin\theta}{\bar\rho}\right)\mathscr{L}_0\Big]$$
$$\left.+ \left(\mathscr{L}_1^+ \frac{ia\sin\theta}{\bar\rho}\right)\mathscr{L}_0^+\right]\bar P^{0'1'}, \quad (3.16)$$
$$\phi_{11} = (\Delta/\sqrt{2}\rho^2\bar\rho^*)\mathscr{D}_0^+\mathscr{L}_0\bar P^{0'1'}.$$

Again, using the identities in the Appendix it can be verified that expression (3.16) for ϕ_{AB} is identical to those given when $p = 0$ or 2.

These representations (and the corresponding ones for $\phi_{A'B'}$) are invaluable for the proof of our principal result.

Theorem: If the functions ϕ_{AB} are the solutions of $\nabla^A_{\ A'}\phi_{AB} = 0$ as represented by, say, (3.14), then the parameters λ and $\mathscr{C}$ are intrinsically defined via the relations

$$C^{AB}{}_{A'B'}\phi_{AB}$$
$$= (K^A{}_{(A'}{}^{CC'}K^B{}_{B')}{}^{DD'}\nabla_{CC'}\nabla_{DD'}$$
$$+ 4M^A{}_{(A'}K^B{}_{B')}{}^{DD'}\nabla_{DD'} + 2M^A{}_{(A'}M^B{}_{B')})\phi_{AB}$$
$$= \tfrac{1}{2}\mathscr{C}\phi_{A'B'}, \quad (3.17)$$

$$\Lambda_{(A}{}^{B}\phi_{C)B} = (K_A{}^{A'EE'}\nabla_{EE'} - M_A{}^{A'}) \times (K^B{}_{A'}{}^{DD'}\nabla_{DD'} + 2M^B{}_{A'})\phi_{BC} = (\lambda/2)\phi_{AC}. \quad (3.18)$$

The proof of relations (3.17) and (3.18) is (in principle) straightforward. The use of the algebraic computing language MACSYMA has been particularly useful in this respect. For relations (3.17) the given result can simply be verified using identities (A4) of the Appendix.

Relation (3.18) is somewhat more difficult. For the cases when $A = C = 0$ or 1 the result is relatively straightforward to establish. However, the result when $A = 0$, $C = 1$ requires extensive computation. In particular, the verification of the identity

$$\frac{a^2\cos^2\theta\Delta}{4\rho^2\bar\rho^*}(\mathscr{D}_1\mathscr{D}_0^+ + \mathscr{D}_1^+\mathscr{D}_0)\bar\rho^*\phi_{10} - \frac{r^2}{4\rho^2\bar\rho^*}(\mathscr{L}_1\mathscr{L}_0^+\mathscr{L}_1^+\mathscr{L}_0)\bar\rho^*\phi_{10} + \frac{Ma^2\cos^2\phi}{4(\bar\rho^*)^3}\phi_{10} - \frac{iar\cos\theta(\bar\rho^*+5\bar\rho)}{4\rho^2\bar\rho^*}\phi_{10} - \frac{a^2}{(\bar\rho^*)^2}\phi_{10} - \frac{ia\sin\theta}{2(\bar\rho^*)^3}\phi_{00} + \frac{ia\sin\theta}{2(\bar\rho^*)}\phi_{11} = \frac{\lambda}{2}\phi_{01} \quad (3.19)$$

is nontrivial.

It can also be verified that the following holds.

(i) If ϕ_{AB} is a solution of (3.3a), then so is $\phi'_{AB} = \Lambda_{(A}{}^{C}\phi_{B)C}$.

In fact,

$$\nabla^{CC'}\Lambda_{(C}{}^{A}\phi_{B)A} = [\Lambda_B{}^A\epsilon_{A'}{}^{C'} - (K_B{}^{C'EE'}\nabla_{EE'} - M_B{}^{C'})M^A{}_{A'} + M_B{}^{C'}(K^A{}_{A'}{}^{DD'}\nabla_{DD'} + 2M^A{}_{A'}) - 2M_B{}^{C'}M^A{}_{A'} + \tfrac{1}{2}\nabla_B{}^{C'}K^A{}_{A'}{}^{DD'}M_{DD'}]\nabla^{CA'}\phi_{AC}. \quad (3.20)$$

(ii) If $\phi_{A'B'}$ is a solution of (3.3b), then $\phi'_{AB} = \mathrm{C}^{A'B'}{}_{AB}\phi_{A'B'}$ is a solution of (3.3a). Moreover,

$$\phi'_{C'D'} = \mathbb{C}_{C'D'}{}^{AB}\mathrm{C}^{A'B'}{}_{AB}\phi_{A'B'} = \mathbb{C}_{C'D'}{}^{A'B'}\phi_{A'B'}$$

is a solution of (3.3b) if $\phi_{A'B'}$ is a solution as well.

(iii) If ϕ_{AB} is a solution of (3.3a) then

$$\Lambda_C{}^F\mathbb{C}_{FD}{}^{KL}\phi_{KL} + \Lambda_D{}^F\mathbb{C}_{FC}{}^{KL}\phi_{KL} = \mathbb{C}_{CD}{}^{AB}[\Lambda_A{}^E\phi_{EB} + \Lambda_B{}^E\phi_{EA}].$$

The operator $\mathrm{C}^{A'B'}_{AB}$ is essentially the operator introduced by Torres del Castillo.[13]

We take the opportunity here to give a more complete discussion of the vector potential $A_{BB'}$ which gives rise to the corresponding $F_{AA'BB'}$:

$$F_{CC'BB'} = \nabla_{CC'}A_{BB'} - \nabla_{BB'}A_{CC'}. \quad (3.21)$$

As is well known, the choice of vector potential is not unique. A derivative of a gauge function can always be added according to $A_{CC'} \to A_{CC'} + \nabla_{CC'}\phi$. As in Ref. 11, we choose the gauge in which the components $A_{CC'}$ are divergenceless; then these functions satisfy

$$\Box A_{CC'} = (\nabla^{BB'}\nabla_{BB'})A_{CC'} = 0, \quad \nabla^{BB'}A_{BB'} = 0. \quad (3.22)$$

There are two independent solutions for the above equation which correspond to the same $F_{AA'BB'}$. These solutions are the analogs of electric and magnetic multipoles,[14]

$$A_{00'} = [P_{+1}(\mathscr{L}_1S_{+1} - \mathscr{L}_1^+S_{-1})\Delta^{-1}]e^{i\sigma t + im\phi},$$
$$A_{11'} = [P_{-1}(\mathscr{L}_1^+S_{-1} - \mathscr{L}_1S_{+1})(2\rho^2)^{-1}]e^{i\sigma t + im\phi},$$
$$A_{01'} = -(\mathscr{D}_0^+P_{+1} + \mathscr{D}_0P_{-1})S_{+1}(\sqrt{2}\bar\rho)^{-1}e^{i\sigma t + im\phi},$$
$$A_{10'} = (\mathscr{D}_0^+P_{+1} + \mathscr{D}_0P_{-1})S_{-1}(\sqrt{2}\bar\rho^*)^{-1}e^{i\sigma t + im\phi}; \quad (3.23a)$$

$$A_{00'} = [P_{+1}(ia\cos\theta\mathscr{L}_1^+ + ia\sin\theta)S_{-1}\Delta^{-1}]e^{i\sigma t + im\phi},$$
$$A_{11'} = [P_{-1}(ia\cos\theta\mathscr{L}_1 + ia\sin\theta)S_{+1}(2\rho^2)^{-1}]e^{i\sigma t + im\phi},$$
$$A_{01'} = -(r\mathscr{D}_0 - 1)P_{-1}S_{+1}e^{i\sigma t + im\phi},$$
$$A_{10'} = -(r\mathscr{D}_0^+ - 1)P_{+1}S_{-1}e^{i\sigma t + im\phi}. \quad (3.23b)$$

Indeed, the (3.23a) corresponds to electric multipoles and (3.23b) corresponds to magnetic multipoles. In establishing (3.23a) use was made of the identity

$$\mathscr{E}[(\mathscr{L}_1 - ia\sin\theta\mathscr{D}_0^+)P_{+1}S_{+1} + (\mathscr{L}_1^+ - ia\sin\theta\mathscr{D}_0)P_{-1}S_{+1}] = i\sigma\bar\rho[\bar\rho^*\mathscr{D}_0\mathscr{L}_1 - (\mathscr{L}_1 + ia\,\theta\mathscr{D}_0)]P_{-1}S_+1.$$

It should be noted that the method of Cohen and Kegeles[12] also gives expressions for the vector potential. More specifically, the vector

$$A_{CC'} = (\nabla_{CE'}\bar{P}^{E'}{}_{C'} - 2G_{CC'}) + \text{complex conjugate}$$

is such that $A_{CC'}$ is a solution of the Maxwell equations. However, the choice of functions $\bar{P}^{X'Y'}$ and $G_A^{\ Y'}$ given previously does not lead to solutions in the divergence-free gauge.

IV. CONCLUSION

In this paper we have explicitly shown how the separation parameters that occur for spin-$s = 0, \frac{1}{2}, 1$ equations can be intrinsically characterized in terms of covariant operators whose coefficients can be written in terms of the Killing–Yano tensor and its covariant derivatives. In Minkowski space we subsequently show that these characterizations and their natural generalizations hold true for any s. There are well-known difficulties with the generalizations of equations of type (3.3).[15] In this respect it is our intention to examine the nature of the intrinsic operator characterization of the functions $A_{CC'}$ and their generalizations for higher spin. All these results provide a nontrivial example of solutions to spin-s equations. Ideally, a suitable theory of such solutions to this type of equation would enable us to derive the existence of such solutions from intrinsic properties only.

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APPENDIX: SUMMARY OF THE CHANDRASEKHAR[4] RESULTS

Chandrasekhar,[4] in his treatment of electromagnetic waves in Kerr geometry, has thoroughly developed the properties of the Teukolsky[1] functions. We summarize his results in the following theorems.

Theorem A1: For a suitable choice of the relative normalization of the functions $P_{\pm 1}$ it is possible to arrange that

$$\Delta \mathscr{D}_0 \mathscr{D}_0 P_{-1} = \mathscr{C} P_{+1}, \quad \Delta \mathscr{D}_0^+ \mathscr{D}_0^+ P_{+1} = \mathscr{C} P_{-1}, \tag{A1}$$

where

$$\mathscr{C}^2 = \lambda^2 - 4(a^2\sigma^2 + am\sigma).$$

Theorem A2: If the functions $S_{\pm 1}$ are normalized to unity,[4] then it is possible to arrange that

$$\mathscr{L}_0 \mathscr{L}_1 S_{+1} = \mathscr{C} S_{-1}, \quad \mathscr{L}_0^+ \mathscr{L}_1^+ S_{-1} = \mathscr{C} S_{+1}, \tag{A2}$$

with $\mathscr{C}$ as in Theorem A1.

Corollary: The derivatives of the functions $P_{\pm 1}$ and $S_{\pm 1}$ can again be expressed as combinations of the same functions:

$$\begin{aligned} &\mathscr{D}_0^+ P_{+1} = (-i/2K)[(\lambda + 2i\sigma r)P_{+1} - \mathscr{C} P_{-1}], \quad \mathscr{D}_0 P_{-1} = (i/2K)[(\lambda - 2i\sigma r)P_{-1} - \mathscr{C} P_{+1}], \\ &\mathscr{L}_1^+ S_{-1} = (-1/2Q)[(\lambda - 2a\sigma\cos\theta)S_{-1} + \mathscr{C} S_{+1}], \quad \mathscr{L}_1 S_{+1} = (1/2Q)[(\lambda + 2a\sigma\cos\theta)S_{+1} + \mathscr{C} S_{-1}]. \end{aligned} \tag{A3}$$

In addition to identities (A3) the following relations are instrumental in the establishment of (3.17):

$$\begin{aligned} &\mathscr{D}_0^+ \mathscr{L}_0^+ = (\mathscr{C}/\Delta)\bar{\psi}_2 + \mathscr{L}_0^+ (1/\bar{\rho}^*)[-\psi_1 + ia\sin\theta(1/\Delta)\psi_2], \\ &\mathscr{D}_0 \mathscr{L}_0 \psi_1 = \mathscr{C}\bar{\psi}_0 - \mathscr{L}_0(1/\bar{\rho}^*)(\psi_1 + ia\sin\theta\psi_0), \\ &\mathscr{L}_1^+ \mathscr{L}_0 \psi_1 = \mathscr{C}\bar{\psi}_1 + (1/\bar{\rho})(\mathscr{L}_1^+ - ia\sin\theta\mathscr{D}_0)\psi_2 - \mathscr{L}_1^+(1/\bar{\rho}^*)(\psi_2 + ia\sin\theta\psi_1), \\ &\Delta\mathscr{D}_1\mathscr{D}_0^+\psi_1 = -\mathscr{C}\bar{\psi}_1 - (1/\bar{\rho})(\mathscr{L}_1^+ - ia\sin\theta\mathscr{D}_0)\psi_2 - \mathscr{D}_0(1/\bar{\rho}^*)(\Delta\psi_1 - ia\sin\theta\psi_2), \\ &\Delta\mathscr{D}_1^+\mathscr{D}_0\psi_1 = -\mathscr{C}\bar{\psi}_1 + (1/\bar{\rho})(\mathscr{L}_1 - ia\sin\theta\mathscr{D}_0^+)\Delta\psi_0 - \mathscr{D}_0^+(\Delta/\bar{\rho}^*)(\psi_1 + ia\sin\theta\psi_0), \\ &\mathscr{L}_1\mathscr{L}_0^+\psi_1 = \mathscr{C}\bar{\psi}_1 - (1/\bar{\rho})(\mathscr{L}_1 - ia\sin\theta\mathscr{D}_0^+)\Delta\psi_0 + \mathscr{L}_1(1/\bar{\rho}^*)(\Delta\psi_0 - ia\sin\theta\psi_1), \\ &\Delta\mathscr{D}_1^+\mathscr{L}_1\psi_0 = -\mathscr{C}\bar{\psi}_1 + (1/\bar{\rho})(\mathscr{L}_1 - ia\sin\theta\mathscr{D}_0^+)\Delta\psi_0, \\ &\mathscr{D}_0\mathscr{L}_1^+\psi_2 = \mathscr{C}\bar{\psi}_1 + (1/\bar{\rho})(\mathscr{L}_1^+ - ia\sin\theta\mathscr{D}_0)\psi_2, \end{aligned} \tag{A4}$$

where

$$\psi_0 = 2\phi_{00}, \quad \psi = \sqrt{2}\bar{\rho}^*\phi_{01}, \quad \psi_2 = 2(\bar{\rho}^*)^2\phi_{11}.$$

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