Models of q-algebra representations: Matrix elements of the q-oscillator algebra

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This article continues a study of function space models of irreducible representations of q analogs of Lie enveloping algebras, motivated by recurrence relations satisfied by q-hypergeometric functions. Here a q analog of the oscillator algebra (not a quantum algebra) is considered. It is shown that various q analogs of the exponential function can be used to mimic the exponential mapping from a Lie algebra to its Lie group and the corresponding matrix elements of the "group operators" on these representation spaces are computed. This "local" approach applies to more general families of special functions, e.g., with complex arguments and parameters, than does the quantum group approach. It is shown that the matrix elements themselves transform irreducibly under the action of the algebra. q analogs of a formula are found for the product of two hypergeometric functions ${}_1F_1$ and the product of a ${}_1F_1$ and a Bessel function. They are interpreted here as expansions of the matrix elements of a "group operator" (via the exponential mapping) in a tensor product basis (for the tensor product of two irreducible oscillator algebra representations) in terms of the matrix elements in a reduced basis. As a by-product of this analysis an interesting new orthonormal basis was found for a q analog of the Bargmann–Segal Hilbert space of entire functions.

I. INTRODUCTION

This article continues the study of function space models of irreducible representations of q algebras.[1–3] These algebras and models are motivated by recurrence relations satisfied by q-hypergeometric functions[4] and our treatment is an alternative to the theory of quantum groups. Here, we consider the irreducible representations of a q analog of the oscillator algebra (not a quantum algebra). We replace the usual exponential function mapping from the Lie algebra to the Lie group by the q-exponential mappings E_q and e_q. In place of the usual matrix elements on the group (arising from an irreducible representation) which are expressible in terms of Laguerre polynomials and functions, we find seven types of matrix elements expressible in terms of q-hypergeometric series. These q-matrix elements do not satisfy group homomorphism properties, so they do not lead to addition theorems in the usual sense. However, they do satisfy orthogonality relations. Furthermore, in analogy with true group representation theory we can show that each of the seven families of matrix elements determines a two-variable model for irreducible representations of the q-oscillator algebra. In Sec. III we show how this two-variable model leads to orthogonality relations for the matrix elements.

In Sec. IV we find a q analog of a formula for the product of two hypergeometric functions

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${}_1F_1$. This is interpreted here as an expansion of the matrix elements of a "group operator" (via the exponential mapping) in a tensor product basis (for the tensor product of two irreducible oscillator algebra representations) in terms of the matrix elements in a reduced basis. In Sec. V we find a q analog of a formula for the product of a ${}_1F_1$ and a Bessel function. This is interpreted here as an expansion of the matrix elements of the "group operator" in a tensor product basis (for the tensor product of an irreducible oscillator algebra representation and an irreducible representation of the quantum motion group) in terms of the matrix elements in a reduced basis. As a by-product of this analysis we find an interesting new orthonormal basis for a q analog of the Bargmann–Segal Hilbert space of entire functions.

Our approach to the derivation and understanding of q-series identities is based on the study of q algebras as q analogs of Lie algebras.[5,6] We are attempting to find q analogs of the theory relating Lie algebra and local Lie transformation groups.[7,8] A similar approach has been adopted by Floreanini and Vinet.[9–12] This is an alternative to the elegant articles[13–21] which are based primarily on the theory of quantum groups. The main justification of the "local" approach is that it is more general; it applies to more general families of special functions than does the quantum group approach.

The notation used for the q series in this article follows that of Gasper and Rahman.[22]

II. MODELS OF OSCILLATOR ALGEBRA REPRESENTATIONS

In Ref. 1 a q analog of the oscillator algebra was introduced. This is the associative algebra generated by the four elements $H, E_+, E_-, \mathscr{E}$ that obey the commutation relations

$$\begin{aligned}&[H,E_+]=E_+,\ [H,E_-]=-E_-,\\ &[E_+,E_-]=-q^{-H}\mathscr{E},\quad [\mathscr{E},E_\pm]=[\mathscr{E},H]=0.\end{aligned}\tag{2.1}$$

It admits a class of algebraically irreducible representations $\uparrow_{\ell,\lambda}$ where ℓ,λ are complex numbers and $\ell\neq 0$. These are defined on a vector space with basis $\{e_n:n=0,1,2,...,\}$, such that

$$E_+e_n=\ell\sqrt{\frac{q^{-n-1}-1}{1-q}}\,e_{n+1},\quad E_-e_n=\ell\sqrt{\frac{q^{-n}-1}{1-q}}\,e_{n-1},\tag{2.2}$$

$$He_n=(\lambda+n)e_n,\quad \mathscr{E}e_n=\ell^2q^{\lambda-1}e_n.$$

If λ and ℓ are real with $\ell>0$ (as we will assume in this article) then $\uparrow_{\ell,\lambda}$ is defined on the Hilbert space K_0 with orthonormal basis $\{e_n\}$ and on this space we have $E_+=(E_-)^*$, $H^*=H$, and $\mathscr{E}^*=\mathscr{E}$. A second convenient basis for K_0 is $\{f_n:n=0,1,...,\}$ where

$$\begin{aligned}&E_+f_n=\ell q^{-(n+1)/2}f_{n+1},\quad E_-f_n=\ell q^{-n/2}\frac{1-q^n}{1-q}f_{n-1},\\ &Hf_n=(\lambda+n)f_n\quad \mathscr{E}f_n=\ell^2q^{\lambda-1}f_n.\end{aligned}\tag{2.3}$$

Here, $f_n=\sqrt{(q;q)_n/(1-q)^n}e_n$. The elements $\mathscr{C}=qq^{-H}\mathscr{E}+(q-1)E_+E_-$ and $\mathscr{E}$ lie in the center of this algebra, and corresponding to the irreducible representation $\uparrow_{\ell,\lambda}$ we have $\mathscr{C}=\ell^2I$, $\mathscr{E}=\ell^2q^{\lambda-1}I$ where I is the identity operator on K_0.

A convenient one-variable model of $\uparrow_{\ell,\lambda}$ is given by the basis functions $\{f_n(z)=z^n: n=0,1,2,...,\}$ in the complex variable z where the action of the oscillator algebra is

$$E_+=\frac{\ell z}{q^{1/2}}T_z^{-1/2},\quad E_-=\frac{\ell}{(1-q)z}(T_z^{-1/2}-T_z^{1/2}),$$
$$H=\lambda+z\frac{d}{dz},\quad \mathscr{C}=\ell^2q^{\lambda-1}I \tag{2.4}$$

and $T_z^\alpha f(z)=f(q^\alpha z)$.

The inner product on K_0 is

$$(f,g)=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{(1-q)^{-1}}f(re^{i\theta})\overline{g(re^{i\theta})}(q(1-q)r^2;q)_\infty\, d_qr^2\, d\theta,$$

where

$$\int_0^k F(r^2)d_qr^2=k(1-q)\sum_{n=0}^\infty F(kq^n)q^n.$$

The functions

$$e_n=\sqrt{\frac{(1-q)^n}{(q;q)_n}}z^n,\quad n=0,1,...,$$

form an orthonormal basis for the Hilbert space K_0 of all functions

$$f(z)=\sum_{n=0}^\infty c_n z^n$$

such that

$$\sum_{n=0}^\infty \frac{|c_n|^2}{(1-q)^n}<\infty.$$

These functions are analytic in the disk $|z|<(1-q)^{-1/2}$.

A second model of $\uparrow_{\ell,\lambda}$,[1] is determined by the orthonormal basis functions

$$e_n=q^{n(n+1)/4}\sqrt{\frac{(1-q)^n}{(q;q)_n}}z^n,\quad n=0,1,...,$$

and the operators

$$E_+=\ell zI,\quad E_-=-\frac{\ell}{(1-q)z}(1-T_z^{-1}),$$
$$H=\lambda+z\frac{d}{dz},\quad \mathscr{C}=\ell^2q^{\lambda-1}I. \tag{2.5}$$

The inner product is

$$(f,g)=\int\int_{-\infty}^{\infty}f(z)\overline{g(z)}\rho(z,\bar{z})dx\, dy,$$

where $z=x+iy$ and

$$\rho(z,\bar{z}) = \frac{1-q}{(-(1-q)z\bar{z};q)_\infty \pi \ln q^{-1}}.$$

The model Hilbert space $K_0(z)$ consists of all functions

$$f'(z) = \sum_{n=0}^{\infty} c_n z^n$$

such that

$$\sum_{n=0}^{\infty} \frac{|c_n|^2 q^{-n(n+1)/2}}{(1-q)^n} < \infty.$$

This is a space of entire functions; it has the kernel function

$$S(\overline{z'},z) = \sum_{n=0}^{\infty} e'_n(\overline{z'}) e'_n(z) = (-(1-q)q\overline{z'}z;q)_\infty. \tag{2.6}$$

Using the relations (2.3) and the q-exponentials

$$e_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty}, \quad \text{for} \quad |z| < 1,$$
$$E_q(z) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{(q;q)_k} z^k = (-z;q)_\infty \tag{2.7}$$

we can define seven q analogs of the matrix elements of $\uparrow_{\ell,\lambda}$

$$(e+,e-): \quad e_q(\beta E_+) e_q(\alpha E_-) f_n = \sum_{n'} T_{n'n}^{(e+,e-)}(\alpha,\beta) f_{n'},$$

$$(e+,E-): \quad e_q(\beta E_+) E_q(\alpha E_-) f_n = \sum_{n'} T_{n'n}^{(e+,E-)}(\alpha,\beta) f_{n'},$$

$$(e-,E+): \quad e_q(\beta E_-) E_q(\alpha E_+) f_n = \sum_{n'} T_{n'n}^{(e-,E+)}(\alpha,\beta) f_{n'},$$

$$(E+,e-): \quad E_q(\beta E_+) e_q(\alpha E_-) f_n = \sum_{n'} T_{n'n}^{(E+,e-)}(\alpha,\beta) f_{n'}, \tag{2.8}$$

$$(E-,e+): \quad E_q(\beta E_-) e_q(\alpha E_+) f_n = \sum_{n'} T_{n'n}^{(E-,e+)}(\alpha,\beta) f_{n'},$$

$$(E+,E-): \quad E_q(\beta E_+) E_q(\alpha E_-) f_n = \sum_{n'} T_{n'n}^{(E+,E-)}(\alpha,\beta) f_{n'},$$

$$(E-,E+): \quad E_q(\beta E_-) E_q(\alpha E_+) f_n = \sum_{n'} T_{n'n}^{(E-,E+)}(\alpha,\beta) f_{n'}.$$

[The series for the matrix elements $T_{n'n}^{(e-,e+)}(\alpha,\beta)$ does not converge.]

Since $E_+^* = E_-$ the following relationships hold:

$$T_{n'n}^{(e+,e-)}(\alpha,\beta)A_{n'n}=T_{nn'}^{\overline{(e+,e-)}}(\bar{\beta},\bar{\alpha}),$$

$$T_{n'n}^{(e+,E-)}(\alpha,\beta)A_{n'n}=T_{nn'}^{\overline{(E+,e-)}}(\bar{\beta},\bar{\alpha}),\quad T_{n'n}^{(e-,E+)}(\alpha,\beta)A_{n'n}=T_{nn'}^{\overline{(E-,e+)}}(\bar{\beta},\bar{\alpha}), \tag{2.9}$$

$$T_{n'n}^{(E+,E-)}(\alpha,\beta)A_{n'n}=T_{nn'}^{\overline{(E+,E-)}}(\bar{\beta},\bar{\alpha}),\quad T_{n'n}^{(E-,E+)}(\alpha,\beta)A_{n'n}=T_{nn'}^{\overline{(E-,E+)}}(\bar{\beta},\bar{\alpha}).$$

Here

$$A_{n'n}=\frac{(q;q)_{n'}}{(q;q)_n}(1-q)^{n-n'}.$$

Since $e_q(z)E_q(-z)=1$, we have the identities

$$\text{(a)}\quad \sum_h T_{n'h}^{(e+,e-)}(\alpha,\beta)T_{hn}^{(E-,E+)}(-\beta,-\alpha)=\delta_{n'n},$$

$$\text{(b)}\quad \sum_h T_{n'h}^{(E-,e+)}(\alpha,\beta)T_{hn}^{(E+,e-)}(-\beta,-\alpha)=\delta_{n'n}. \tag{2.10}$$

Using the model (2.4) to compute the matrix elements (which are model independent) we obtain the explicit results

$$T_{n'n}^{(e+,e-)}(\alpha,\beta)=\frac{(q^{n'-n+1};q)_\infty(\beta\ell)^{n'-n}}{(q;q)_\infty}q^{(n-n')(n+n'+1)/4}{}_2\phi_1\left(\begin{matrix}q^{-n}, & 0\\ q^{n'-n+1}\end{matrix};q,\frac{-\alpha\beta\ell^2}{1-q}\right)$$

$$=\frac{(q^{n-n'+1};q)_\infty(q^{n'+1};q)_\infty(\alpha\ell)^{n-n'}}{(q;q)_\infty(q^{n+1};q)_\infty(1-q)^{n-n'}}q^{(n'-n)(n'+n+1)/4}{}_2\phi_1$$

$$\times\left(\begin{matrix}q^{-n'}, & 0\\ q^{n-n'+1}\end{matrix};q,\frac{-\alpha\beta\ell^2}{1-q}\right),$$

$$T_{n'n}^{(E+,e-)}(\alpha,\beta)=\frac{(q^{n'+1};q)_\infty(q^{n-n'+1};q)_\infty(\alpha\ell)^{n-n'}}{(q;q)_\infty(q^{n+1};q)_\infty(1-q)^{n-n'}}q^{(n'-n)(n'+n+1)/4}{}_1\phi_1\left(\begin{matrix}q^{-n'}\\ q^{n-n'+1}\end{matrix};q,\frac{\alpha\beta\ell^2}{1-q}\right)$$

$$=\frac{(q^{n'-n+1};q)_\infty(\beta\ell)^{n'-n}}{(q;q)_\infty}q^{(n'-n)(n'-3n-3)/4}{}_1\phi_1\left(\begin{matrix}q^{-n}\\ q^{n'-n+1}\end{matrix};q,\frac{\alpha\beta\ell^2q^{n'-n}}{1-q}\right), \tag{2.11}$$

$$T_{n'n}^{(E+,E-)}(\alpha,\beta)=\frac{(q^{n'-n+1};q)_\infty(\beta\ell)^{n'-n}}{(q;q)_\infty}q^{(n'-n)(n'-3n-3)/4}{}_1\phi_2\left(\begin{matrix}q^{-n}\\ q^{n'-n+1}, & 0\end{matrix};q,\frac{-\alpha\beta\ell^2q^{n'-n}}{1-q}\right)$$

$$=\frac{(q^{n-n'+1};q)_\infty(q^{n'+1};q)_\infty(\alpha\ell)^{n-n'}}{(q;q)_\infty(q^{n+1};q)_\infty(1-q)^{n-n'}}q^{(n-n')(n-3n'-3)/4}{}_1\phi_2$$

$$\times\left(\begin{matrix}q^{-n'}\\ q^{n-n'+1}, & 0\end{matrix};q,\frac{-\alpha\beta\ell^2q^{n-n'}}{1-q}\right),$$

$$T_{n'n}^{(E-,E+)}(\alpha,\beta)=\frac{((-\alpha\beta\ell^2/(1-q)q);q)_\infty(q^{n'-n+1};q)_\infty(\alpha\ell)^{n'-n}}{(q;q)_\infty}$$

$$\times q^{(n'-n)(n'-3n-3)/4}{}_2\phi_1\left(\begin{matrix}q^{-n}, & 0\\ q^{n'-n+1}\end{matrix};q,\frac{-\alpha\beta\ell^2}{(1-q)q}\right)$$

$$=\frac{((-\alpha\beta\ell^2/(1-q)q);q)_\infty(q^{n-n'+1};q)_\infty(q^{n'+1};q)_\infty(\beta\ell)^{n-n'}}{(q;q)_\infty(q^{n+1};q)_\infty(1-q)^{n-n'}}$$

$$\times q^{(n-n')(n-3n'-3)/4}{}_2\phi_1\left(\begin{matrix}q^{-n'}, & 0\\ q^{n-n'+1}\end{matrix};q,\frac{-\alpha\beta\ell^2}{(1-q)q}\right).$$

The matrix elements $T^{(e+,e-)}, T^{(E+,e-)}, T^{(E+,E-)}$ are polynomials in α and β and the matrix elements $T^{(E-,E+)}$ are entire analytic functions of these variables. We will see that the remaining matrix elements can be expressed in terms of these four.

Each of these families of matrix elements determines models of the irreducible representations $\uparrow_{\ell,\lambda}$. This is a consequence of the commutation relations (2.1). To see this we make use of the following formal power series results for linear operators X, Y:

Lemma 1:

$$E_q(\alpha X)Ye_q(-\alpha X)=\sum_{n=0}^{\infty}\frac{\alpha^n}{(q;q)_n}[X,Y]_n,$$

where

$$[X,Y]_0=Y,\quad [X,Y]_{n+1}=X[X,Y]_nq^n-[X,Y]_nX,\quad n=0,1,\dots.$$

Lemma 2:

$$e_q(\beta X)YE_q(-\beta X)=\sum_{n=0}^{\infty}\frac{\beta^n}{(q;q)_n}[X,Y]'_n,$$

where

$$[X,Y]'_0=Y,\quad [X,Y]'_{n+1}=X[X,Y]'_n-q^n[X,Y]'_nX,\quad n=0,1,\dots.$$

Let X and Y be linear operators such that $YX=qXY$. A straightforward formal induction argument using this property[15] (Ref. 22, page 28) yields

Lemma 3:

$$(Y+X)^k=\sum_{\ell=0}^{k}\frac{(q;q)_k}{(q;q)_\ell(q;q)_{k-\ell}}X^\ell Y^{k-\ell},$$

$$e_q(X+Y)=e_q(X)e_q(Y),\quad E_q(X+Y)=E_q(Y)E_q(X).$$

As a consequence of Lemmas 1 and 2 we have

$$\text{(a)}\ E_q(\alpha E_-)E_+e_q(-\alpha E_-)=E_++\frac{\alpha\ell^2}{(1-q)q}q^{-H+\lambda},$$
$$\text{(b)}\ e_q(\beta E_+)E_-E_q(-\beta E_+)=E_--\frac{\beta\ell^2}{(1-q)q}q^{-H+\lambda}. \tag{2.12}$$

Note also the easily verified identities

$$E_q(-\beta E_+)q^{-H}e_q(\beta qE_+)=q^{-H},$$
$$E_q(-\beta qE_-)q^{-H}e_q(\beta E_-)=q^{-H}. \tag{2.13}$$

Iterating Eq. (2.12a) and using Lemma 3 we obtain the operator identity

$$e_q\left(\frac{\alpha\beta\ell^2q^{-H+\lambda-1}}{1-q}\right)e_q(\beta E_+)E_q(\alpha E_-)=E_q(\alpha E_-)e_q(\beta E_+) \tag{2.14}$$

and Eq. (2.12b) yields

$$E_q(\beta E_+)e_q(\alpha E_-)e_q\left(\frac{\alpha\beta\ell^2q^{-H+\lambda-1}}{1-q}\right)=e_q(\alpha E_-)E_q(\beta E_+). \tag{2.15}$$

Note that Eqs. (2.14) and (2.15) imply the relations

$$T_{n'n}^{(E-,e+)}(\beta,\alpha)=e_q\left(\frac{\alpha\beta\ell^2q^{-n'-1}}{1-q}\right)T_{n'n}^{(e+,E-)}(\alpha,\beta),$$
$$T_{n'n}^{(e-,E+)}(\beta,\alpha)=e_q\left(\frac{\alpha\beta\ell^2q^{-n-1}}{1-q}\right)T_{n'n}^{(E+,e-)}(\alpha,\beta). \tag{2.16}$$

Thus the matrix elements $T_{n'n}^{(E-,e+)}$ are well-defined for $|\alpha\beta\ell^2q^{-n'}-1/(1-q)|<1$ and the matrix elements $T_{n'n}^{(e-,E+)}$ are well-defined for $|\alpha\beta\ell^2q^{-n-1}/(1-q)|<1$.

Considering the matrix elements $(e+,e-)$, we see that the operator identities

$$\text{(a)}\ e_q(\beta E_+)e_q(\alpha E_-)E_-=\frac{1}{\alpha}(I-T_\alpha)e_q(\beta E_+)e_q(\alpha E_-),$$
$$\text{(b)}\ e_q(\beta E_+)e_q(\alpha E_-)E_+=\frac{1}{\beta}(I-T_\beta)e_q(\beta E_+)e_q(\alpha E_-)$$
$$+\frac{\alpha\ell^2}{(1-q)q}T_\beta T_\alpha^{-1}q^{-H+\lambda}e_q(\beta E_+)e_q(\alpha E_-), \tag{2.17}$$
$$\text{(c)}\ [H,e_q(\beta E_+)e_q(\alpha E_-)]=(\beta\partial_\beta-\alpha\partial_\alpha)e_q(\beta E_+)e_q(\alpha E_-)$$

imply

$$\text{(a)}\ \ell q^{-n/2}\frac{1-q^n}{1-q}T_{n',n-1}^{(e+,e-)}(\alpha,\beta)=\frac{1}{\alpha}(I-T_\alpha)T_{n'n}^{(e+,e-)}(\alpha,\beta),$$

$$\text{(b)}\ \ell q^{-(n+1)/2}T^{(e+,e-)}_{n',n+1}(\alpha,\beta)=\left(\frac{1}{\beta}(I-T_\beta)+\frac{\alpha\ell^2q^{-n'}}{(1-q)q}T_\beta T_\alpha^{-1}\right)T^{(e+,e-)}_{n'n}(\alpha,\beta), \tag{2.18}$$

$$\text{(c)}\ (n-n')T^{(e+,e-)}_{n'n}(\alpha,\beta)=(\alpha\partial_\alpha-\beta\partial_\beta)T^{(e+,e-)}_{n'n}(\alpha,\beta),$$

where $T_\alpha f(\alpha,\beta)=f(q\alpha,\beta)$. Thus the following set of operators and basis functions defines a realization of the representation $\uparrow_{\ell,-n'}$, $(\mathscr{C}=\ell^2q^{-n'-1}I)$:

$$(e+,e-):\quad E_+=\widetilde{E}^\beta+\frac{\alpha\ell^2}{(1-q)q}q^{-n'}T_\beta T_\alpha^{-1},$$
$$\widetilde{E}^\alpha_-,\quad \widetilde{H},\quad f_{-n'+n}=T^{(e+,e-)}_{n'n}(\alpha,\beta), \tag{2.19}$$

where

$$\widetilde{E}^x=\frac{1}{x}(I-T_x),\quad \hat{E}^x=\frac{q}{x}(T_x^{-1}-I),\quad \widetilde{H}=\alpha\partial_\alpha-\beta\partial_\beta. \tag{2.20}$$

Due to the invariant operator $\mathscr{C}=\ell^2I=q^{-\widetilde{H}+1}\mathscr{C}+(q-1)E_+\widetilde{E}^\alpha_-$, we can write E_+ in different ways. Indeed, eliminating $\hat{E}^\beta$ from E_+ and $(\alpha/q(1-q))T_\alpha^{-1}\mathscr{C}$ we can write the raising operator in the simpler form

$$E'_+=\left(\frac{\alpha\ell^2}{q(1-q)}+\widetilde{E}^\beta\right)T_\alpha^{-1}.$$

For the matrix elements $(e+,E-)$ the operator identities

$$\text{(a)}\ e_q(\beta E_+)E_q(\alpha E_-)E_-=\frac{q}{\alpha}(T_\alpha^{-1}-I)e_q(\beta E_+)E_q(\alpha E_-),$$
$$\text{(b)}\ e_q(\beta E_+)E_q(\alpha E_-)E_+=\frac{1}{\beta}(I-T_\beta)e_q(\beta E_+)E_q(\alpha E_-)$$
$$+\frac{\alpha\ell^2}{(1-q)q}T_\beta q^{-H+\lambda}e_q(\beta E_+)E_q(\alpha E_-) \tag{2.21}$$

and Eq. (2.17c) imply

$$\text{(a)}\ \ell q^{-n/2}\frac{1-q^n}{1-q}T^{(e+,E-)}_{n',n-1}(\alpha,\beta)=\frac{q}{\alpha}(T_\alpha^{-1}-I)T^{(e+,E-)}_{n'n}(\alpha,\beta),$$

$$\text{(b)}\ \ell q^{-(n+1)/2}T^{(e+,E-)}_{n',n+1}(\alpha,\beta)=\left(\frac{1}{\beta}(I-T_\beta)+\frac{\alpha\ell^2q^{-n'}}{(1-q)q}T_\beta\right)T^{(e+,E-)}_{n'n}(\alpha,\beta), \tag{2.22}$$

$$\text{(c)}\ (n-n')T^{(e+,E-)}_{n'n}(\alpha,\beta)=(\alpha\partial_\alpha-\beta\partial_\beta)T^{(e+,E-)}_{n'n}(\alpha,\beta).$$

For the matrix elements $(E+,E-)$ we have

$$\begin{aligned}&\text{(a)}\ \ \ell q^{-n/2}\frac{1-q^n}{1-q}T^{(E+,E-)}_{n',n-1}(\alpha,\beta)=\frac{q}{\alpha}(T_\alpha^{-1}-I)T^{(E+,E-)}_{n'n}(\alpha,\beta),\\ &\text{(b)}\ \ \ell q^{-(n+1)/2}T^{(E+,E-)}_{n',n+1}(\alpha,\beta)=\left(\frac{q}{\beta}(T_\beta^{-1}-I)+\frac{\alpha\ell^2q^{-n'}}{(1-q)q}T_\beta\right)T^{(E+,E-)}_{n'n}(\alpha,\beta),\\ &\text{(c)}\ \ (n-n')T^{(E+,E-)}_{n'n}(\alpha,\beta)=(\alpha\partial_\alpha-\beta\partial_\beta)T^{(E+,E-)}_{n'n}(\alpha,\beta),\end{aligned}\tag{2.23}$$

whereas for the matrix elements $(E+,e-)$ we have

$$\begin{aligned}&\text{(a)}\ \ \ell q^{-n/2}\frac{1-q^n}{1-q}T^{(E+,e-)}_{n',n-1}(\alpha,\beta)=\frac{1}{\alpha}(I-T_\alpha)T^{(E+,e-)}_{n'n}(\alpha,\beta),\\ &\text{(b)}\ \ \ell q^{-(n+1)/2}T^{(E+,e-)}_{n',n+1}(\alpha,\beta)=\left(\frac{q}{\beta}(T_\beta^{-1}-I)+\frac{\alpha\ell^2q^{-n'}}{(1-q)q}T_\beta T_\alpha^{-1}\right)T^{(E+,e-)}_{n'n}(\alpha,\beta),\\ &\text{(c)}\ \ (n-n')T^{(E+,e-)}_{n'n}(\alpha,\beta)=(\alpha\partial_\alpha-\beta\partial_\beta)T^{(E+,e-)}_{n'n}(\alpha,\beta).\end{aligned}\tag{2.24}$$

Due to the invariant operator $\mathscr{C}=\ell^2I$ we can write the raising operator $E_+=\hat{E}^\beta+(\alpha\ell^2q^{-n'}/(1-q)q)T_\beta T_\alpha^{-1}$ in the alternate form

$$E'_+=\left(\frac{\alpha\ell^2}{q(1-q)}+\hat{E}^\beta\right)T_\alpha^{-1}.$$

For the matrix elements $(e-,E+)$ we have

$$\begin{aligned}&\text{(a)}\ \ \ell q^{-n/2}\frac{1-q^n}{1-q}T^{(e-,E+)}_{n',n-1}(\alpha,\beta)=\left(\frac{1}{\beta}(I-T_\beta)-\frac{\alpha\ell^2q^{-n'}}{(1-q)q}T_\beta^{-1}T_\alpha\right)T^{(e-,E+)}_{n'n}(\alpha,\beta),\\ &\text{(b)}\ \ \ell q^{-(n+1)/2}T^{(e-,E+)}_{n',n+1}(\alpha,\beta)=\frac{q}{\alpha}(T_\alpha^{-1}-I)T^{(e-,E+)}_{n'n}(\alpha,\beta),\\ &\text{(c)}\ \ (n-n')T^{(e-,E+)}_{n'n}(\alpha,\beta)=(\beta\partial_\beta-\alpha\partial_\alpha)T^{(e-,E+)}_{n'n}(\alpha,\beta).\end{aligned}\tag{2.25}$$

Due to the invariant operator $\mathscr{C}=\ell^2I$ we can write the lowering operator $E_-=\tilde{E}^\beta-(\alpha\ell^2q^{-n'}/(1-q)q)T_\beta^{-1}T_\alpha$ in the alternate form

$$E'_-=\left(\frac{-\alpha\ell^2}{1-q}+\tilde{E}^\beta\right)T_\alpha.$$

For matrix elements $(E-,e+)$ we have

$$\begin{aligned}&\text{(a)}\ \ \ell q^{-n/2}\frac{1-q^n}{1-q}T^{(E-,e+)}_{n',n-1}(\alpha,\beta)=\left(\frac{q}{\beta}(T_\beta^{-1}-I)-\frac{\alpha\ell^2q^{-n'}}{(1-q)q}T_\beta^{-1}\right)T^{(E-,e+)}_{n'n}(\alpha,\beta),\\ &\text{(b)}\ \ \ell q^{-(n+1)/2}T^{(E-,e+)}_{n',n+1}(\alpha,\beta)=\frac{1}{\alpha}(I-T_\alpha)T^{(E-,e+)}_{n'n}(\alpha,\beta),\end{aligned}\tag{2.26}$$

$$\text{(c)}\ \ (n-n')T^{(E-,e+)}_{n'n}(\alpha,\beta)=(\beta\partial_\beta-\alpha\partial_\alpha)T^{(E-,e+)}_{n'n}(\alpha,\beta).$$

Finally, for the matrix elements $(E-,E+)$ we have

$$\text{(a)}\ \ \ell q^{-n/2}\frac{1-q^n}{1-q}T^{(E-,E+)}_{n',n-1}(\alpha,\beta)=\left(\frac{q}{\beta}(T_\beta^{-1}-I)-\frac{\alpha\ell^2q^{-n'}}{(1-q)q}T_\beta^{-1}T_\alpha\right)T^{(E-,E+)}_{n'n}(\alpha,\beta),$$

$$\text{(b)}\ \ \ell q^{-(n+1)/2}T^{(E-,E+)}_{n',n+1}(\alpha,\beta)=\frac{q}{\alpha}(T_\alpha^{-1}-I)T^{(E-,E+)}_{n'n}(\alpha,\beta), \tag{2.27}$$

$$\text{(c)}\ \ (n-n')T^{(E-,E+)}_{n'n}(\alpha,\beta)=(\beta\partial_\beta-\alpha\partial_\alpha)T^{(E-,E+)}_{n'n}(\alpha,\beta),$$

where the lowering operator $E_-=\tilde{E}^\beta-(\alpha\ell^2q^{-n'}/(1-q)q)T_\beta^{-1}T_\alpha$ can be expressed in the alternate form

$$E'_-=\left(\frac{-\alpha\ell^2}{1-q}+\hat{E}^\beta\right)T_\alpha.$$

These relations are equivalent to q-difference relations satisfied by various q-hypergeometric series. Furthermore, it is easy to verify from the series that the relations hold also for ℓ and n' complex. Thus we have a wide variety of two-variable models of algebraically irreducible representations of the q-oscillator algebra. We note that this approach is closely related to the factorization method of quantum mechanics.[23]

For later use, we also consider a class of algebraically irreducible representations $R(\ell,\delta,\lambda)$ such that the spectrum of H is bounded neither above nor below. Here, ℓ, δ, λ are real numbers and $\ell,\delta>0$. A convenient basis for the representation space K_1 is $\{f_n : n=0,\pm1,\pm2,...,\}$ where

$$E_+f_n=\ell f_{n+1},\quad E_-f_n=\ell\frac{1+\delta q^{-n}}{1-q}f_{n-1}, \tag{2.28}$$

$$Hf_n=(\lambda+n)f_n,\quad \mathscr{E}f_n=-\ell^2q^{\lambda-1}\delta f_n.$$

There is an inner product on K_1 with respect to which the f_n forms an orthogonal basis and $E_+=(E_-)^*$, $H^*=H$, and $\mathscr{E}^*=\mathscr{E}$. We can require that $\|f_n\|^2=(-\delta q^{-n};q)_\infty/(1-q)^n$. The central element $\mathscr{C}=qq^{-H}\mathscr{E}+(q-1)E_+E_-$ corresponding to this irreducible representation is $\mathscr{C}=-\ell^2I$ where I is the identity operator on K_1.

One of the families of matrix elements of $R(\ell,\delta,\lambda)$ is

$$(E+,e-):\ \ E_q(\beta E_+)e_q(\alpha E_-)f_n=\sum_{n'=-\infty}^{\infty}\hat{T}^{(E+,e-)}_{n'n}(\alpha,\beta)f_{n'}.$$

Explicitly

$$\hat{T}^{(E+,e-)}_{n'n}(\alpha,\beta)=\frac{(-q^{n'+1}/\delta;q)_\infty(q^{n-n'+1};q)_\infty(\alpha\ell\delta/q)^{n-n'}}{(q;q)_\infty(-q^{n+1}/\delta;q)_\infty(1-q)^{n-n'}}q^{(n'-n)(n'+n-1)/2}{}_1\phi_1$$
$$\times\left(\begin{matrix}-\delta q^{-n'}\\ q^{n-n'+1}\end{matrix};q,-\frac{\alpha\beta\ell^2}{1-q}\right)$$

$$=\frac{(q^{n'-n+1};q)_\infty(\beta\ell)^{n'-n}}{(q;q)_\infty}q^{(n'-n)(n'-n-1)/2}\,{}_1\phi_1\left(\begin{matrix}-\delta q^{-n}\\ q^{n'-n+1}\end{matrix};q,-\frac{\alpha\beta\ell^2q^{n'-n}}{1-q}\right). \tag{2.29}$$

With respect to the orthonormal basis $\{e_n=f_n/\|f_n\|\}$ the matrix elements of the operator $E_q(\beta E_+)e_q(\alpha E_-)$ are

$$\hat{S}^{(E+,e-)}_{n'n}(\alpha,\beta)=\sqrt{\frac{(-q^{n'+1}/\delta;q)_\infty\delta^{n-n'}}{(-q^{n+1}/\delta;q)_\infty}(1-q)^{n-n'}}\,\frac{(q^{n-n'+1};q)_\infty(\alpha\ell/q)^{n-n'}}{(q;q)_\infty}$$

$$\times q^{(n'-n)(n'+n-3)/4}\,{}_1\phi_1\left(\begin{matrix}-\delta q^{-n'}\\ q^{n-n'+1}\end{matrix};q,-\frac{\alpha\beta\ell^2}{1-q}\right). \tag{2.30}$$

III. ORTHOGONALITY RELATIONS FOR MATRIX ELEMENTS

Identities (2.10) yield orthogonality and biorthogonality relations for q-hypergeometric functions. For example, Eq. (2.10a) can be written in the form

$$\sum_{h=0}^{\infty}\frac{(q;q)_h(zq^{-n-1})^h}{(q;q)_{h-n'}(q;q)_{h-n}}\,{}_2\phi_1\left(\begin{matrix}q^{-n'}, & 0\\ q^{h-n'+1}\end{matrix};q,z\right){}_2\phi_1\left(\begin{matrix}q^{-n} & 0\\ q^{h-n+1}\end{matrix};q,\frac{z}{q}\right)$$

$$=\frac{z^n(q;q)_nq^{-n(n+1)}}{(z/q;q)_\infty}\delta_{n'n},\quad |z/q^{n+1}|<1. \tag{3.1}$$

[By its derivation, identity (2.10a) is valid as a formal power series in the variables α, β. Using the ratio test to determine the domain of convergence corresponding to $\uparrow_{\ell,\lambda}$ we find that the series (3.1) converges for $|z/q^{n+1}|<1$.]

Equation (2.10b) can be written as

$$\sum_{h=0}^{\infty}\frac{(q;q)_h(-zq^{-(n+n'+1)})^hq^{h(h-1)/2}}{(q;q)_{h-n'}(q;q)_{h-n}}\,{}_1\phi_1\left(\begin{matrix}q^{-n'}\\ q^{h-n'+1}\end{matrix};q,zq^{h-n'}\right){}_1\phi_1\left(\begin{matrix}q^{-n}\\ q^{h-n+1}\end{matrix};q,zq^{h-n}\right)$$

$$=(-z)^n(q;q)_n(zq^{-n-1};q)_\infty q^{-3n(n+1)/2}\delta_{n'n} \tag{3.2}$$

convergent for all z.

A nontrivial extension of identity (2.10b) is

$$\sum_{\ell}T^{(E-,e+)}_{n'\ell}(\alpha,\gamma)T^{(E+,e-)}_{\ell n}(\beta,-\alpha)=S_{n'n}(\gamma,\beta),$$

where the matrix elements $S_{n'n}(\gamma,\beta)$ are defined by

$$e_q(\beta E_-)E_q(\gamma E_-)f_n=\sum_{n'}S_{n'n}(\gamma,\beta)f_{n'}.$$

Explicitly

$$S_{n'n}(\gamma,\beta)=\begin{cases}\dfrac{(-\beta\ell)^{n-n'}(q^{-n};q)_{n-n'}}{(1-q)^{n-n'}(q;q)_{n-n'}}\,q^{(n-n')(1+n+n')/4}\left(-\dfrac{\gamma}{\beta};q\right)_{n-n'}, & \text{if } n\geqslant n'\\ 0, \quad \text{if } n<n'.\end{cases}$$

In the special case $\gamma/\beta=-q^{n'-n+k[n-n']}$ where $k[n]\in\{1,2,...,n\}$ for $n>0$ and $k[n]$ is arbitrary for $n\leqslant 0$ we find the result

$$\sum_{h=0}^{\infty}\frac{(q;q)_h(-zq^{-n+k[n-n']})^h q^{h(h-3)/2}}{(q;q)_{h-n'}(q;q)_{h-n}}\,{}_1\phi_1\left(\begin{matrix}q^{-n'}\\ q^{h-n'+1}\end{matrix};q,zq^{h+k[n-n']}\right)\,{}_1\phi_1\left(\begin{matrix}q^{-n}\\ q^{h-n+1}\end{matrix};q,zq^{h-n}\right)$$

$$=(-z)^n(q;q)_n(z;q)_\infty q^{-n(n+1)/2}\delta_{n'n}$$

convergent for all z. (In the case of the Lie algebra of the Euclidean group in the plane, the analogous identities are the Hansen–Lommel identities for q-Bessel functions.[24,13,2] There is a similar extension of (2.10a).

In Ref. 1 orthogonality relations for the matrix elements $T^{(E+,e-)}$ and $T^{(e+,e-)}$ are derived, analogous to the Peter–Weyl-type orthogonality relations for the oscillator group.

IV. A TENSOR PRODUCT IDENTITY

Given the irreducible representations $\uparrow_{\ell_1,\lambda_1}$ and $\uparrow_{\ell_2,\lambda_2}$ on the Hilbert space K_0 we define the tensor product representation $\uparrow_{\ell_1,\lambda_1}\otimes\uparrow_{\ell_2,\lambda_2}$ on the space $K_0\otimes K_0$ by the operators[1]

$$\begin{aligned}F_+&=\Delta(E_+)=E_+\otimes q^{(1/2)H}+q^{-(1/2)H}\otimes E_+,\\ F_-&=\Delta(E_-)=E_-\otimes q^{(1/2)H}+q^{-(1/2)H}\otimes E_-(\kappa_1q^H+\kappa_2),\\ L&=\Delta(H)=H\otimes I+I\otimes H,\\ \mathscr{F}&=\Delta(\mathscr{E})=\mathscr{E}\otimes I+I\otimes\mathscr{E}=(\ell_1^2q^{\lambda_1-1}+\ell_2^2q^{\lambda_2-1})I\otimes I,\end{aligned}\tag{4.1}$$

where

$$\kappa_1=-\frac{\ell_1^2q^{\lambda_1-1}}{\ell_2^2},\quad \kappa_2=\frac{\ell_1^2q^{\lambda_1}+\ell_2^2q^{\lambda_2}}{\ell_2^2q^{\lambda_2}}.\tag{4.2}$$

Then we have

$$\begin{aligned}&[L,F_\pm]=\pm F_\pm,\quad [F_+,F_-]=-\mathscr{F}q^{-L},\\ &[\mathscr{F},F_\pm]=[\mathscr{F},L]=0.\end{aligned}\tag{4.3}$$

We introduce an inner product $\langle\cdot,\cdot\rangle$ on $K_0\otimes K_0$ such that

$$\langle f_h^{(\ell_1,\lambda_1)}\otimes e_j^{(\ell_2,\lambda_2)},f_{h'}^{(\ell_1,\lambda_1)}\otimes e_{j'}^{(\ell_2,\lambda_2)}\rangle=\delta_{hh'}\delta_{jj'}\frac{(-(\kappa_1/\kappa_2)q^{\lambda_2+1};q)_j}{(1-q)^{h+j}}(q;q)_h(q;q)_j\kappa_2^j.\tag{4.4}$$

Then we have

$$\langle F_+p_1,p_2\rangle=\langle p_1,F_-p_2\rangle,\quad \langle Lp_1,p_2\rangle=\langle p_1,Lp_2\rangle$$

for all $p_1,p_2 \in K_0 \otimes K_0$ that are a finite linear combination of the basis vectors $f_h^{(\ell_1,\lambda_1)} \otimes e_j^{(\ell_2,\lambda_2)}$.

In Ref. 1 the representation $\uparrow_{\ell_1,\lambda_1} \otimes \uparrow_{\ell_2,\lambda_2}$ was decomposed into irreducible components, through use of the model (2.4). The result is as follows.

Theorem 1:

$$\uparrow_{\ell_1,\lambda_1} \otimes \uparrow_{\ell_2,\lambda_2} \cong \sum_{s=0}^{\infty} \oplus \uparrow_{\tilde{\ell}_s,\lambda_1+\lambda_2+s}.$$

For $k,s=0,1,2,\ldots$, there is an orthogonal basis $\{f_{s,k}\}$ for $K_0 \otimes K_0$ transforming according to

$$F_+ f_{s,k} = \tilde{\ell}_s q^{-(k+1)/2} f_{s,k+1},$$

$$F_- f_{s,k} = \tilde{\ell}_s q^{-k/2} \frac{1-q^k}{1-q} f_{s,k-1}, \qquad (4.5)$$

$$L f_{s,k} = (\lambda_1+\lambda_2+s+k) f_{s,k},$$

where

$$\tilde{\ell}_s = \sqrt{q^{-s}(\ell_1^2 q^{-\lambda_2} + \ell_2^2 q^{-\lambda_1})}.$$

Furthermore

$$\langle f_{s,k}, f_{s',k'} \rangle = \delta_{ss'} \delta_{kk'} \frac{(q;q)_s (q;q)_k}{(1-q)^{s+k}(-(\kappa_1/\kappa_2)q^{\lambda_2+1};q)_s}.$$

Expanding the orthonormal basis $\{e_k^s\}$ for $K_0 \otimes K_0$

$$e_k^s = \|f_{s,k}\|^{-1} f_{s,k}, \quad s,k=0,1,2,\ldots$$

in terms of the orthonormal basis

$$e_{n_1}^{(\ell_1,\lambda_1)} \otimes e_{n_2}^{(\ell_2,\lambda_2)} \equiv f_h^{(\ell_1,\lambda_1)} \otimes e_j^{(\ell_2,\lambda_2)} / \| f_h^{(\ell_1,\lambda_1)} \otimes e_j^{(\ell_2,\lambda_2)} \|$$

we obtain the Clebsch–Gordon coefficients

$$e_k^s = \sum_{n_1,n_2} \begin{bmatrix} \ell_1,\lambda_1; & \ell_2,\lambda_2; & s \\ n_1; & n_2; & k \end{bmatrix}_q e_{n_1}^{(\ell_1,\lambda_1)} \otimes e_{n_2}^{(\ell_2,\lambda_2)}.. \qquad (4.6)$$

These coefficients vanish unless $n_1+n_2=s+k$. Furthermore, they satisfy the identities

$$\sum_{n_1,n_2} \begin{bmatrix} \ell_1,\lambda_1; & \ell_2,\lambda_2; & s \\ n_1; & n_2; & k \end{bmatrix}_q \begin{bmatrix} \ell_1,\lambda_1; & \ell_2,\lambda_2; & s' \\ n_1; & n_2; & k' \end{bmatrix}_q = \delta_{kk'},$$

$$\sum_{s,k} \begin{bmatrix} \ell_1,\lambda_1; & \ell_2,\lambda_2; & s \\ n_1; & n_2; & k \end{bmatrix}_q \begin{bmatrix} \ell_1,\lambda_1; & \ell_2,\lambda_2; & s \\ n_1'; & n_2'; & k \end{bmatrix}_q = \delta_{n_1 n_1'}, \qquad (4.7)$$

where $n_1+n_2=n_1'+n_2'=s+k=s'+k'$ and we are assuming that $\ell_1,\ell_2>0$ and λ_1,λ_2 are real. Explicitly, we have

$$\begin{bmatrix} \ell_1,\lambda_1; & \ell_2,\lambda_2; & s \\ n_1; & n_2; & k \end{bmatrix}_q = \tilde{\ell}_s^{-k}(q^{(\lambda_2/2)-(s/2)}\ell_1)^k \left(\frac{q^{(\lambda_1/2)+(\lambda_2/2)+s+k}\ell_1}{\kappa_2 \ell_2} \right)^{n_2}$$

$$\times (q^{-s};q)_{n_2} \left[\frac{(-(\kappa_1/\kappa_2)q^{\lambda_2+1};q)_s (q;q)_{n_1} \kappa_2^{n_2}}{(-(\kappa_1/\kappa_2)q^{\lambda_2+1};q)_{n_2} (q;q)_{n_2} (q;q)_s (q;q)_k} \right]^{1/2}$$

$$\times {}_3\phi_2 \left(\begin{matrix} q^{-n_2}, & q^{-k}, & -\frac{\kappa_2}{\kappa_1} q^{-n_2-\lambda_2} \\ q^{1-n_2+s}, & 0 & \end{matrix} ;q;q \right), \tag{4.8}$$

where we have corrected some typographical errors in the corresponding expression (5.11) derived in Ref. 1. The coefficients can be written in the alternate form

$$\begin{bmatrix} \ell_1,\lambda_1; & \ell_2,\lambda_2; & s \\ n_1; & n_2; & k \end{bmatrix}_q = \tilde{\ell}_s^{-k}(q^{(\lambda_2/2)-(s/2)}\ell_1)^k \left(-\frac{q^{(\lambda_1/2)-(\lambda_2/2)-1}\ell_1}{\kappa_1 \ell_2} \right)^{n_2}$$

$$\times \left[\frac{(-(\kappa_1/\kappa_2)q^{\lambda_2+1};q)_{n_2} (-(\kappa_1/\kappa_2)q^{\lambda_2+1};q)_s (q;q)_{n_1} \kappa_2^{n_2}}{(q;q)_{n_2} (q;q)_s (q;q)_k} \right]^{1/2}$$

$$\times {}_2\phi_1 \left(\begin{matrix} q^{-n_2}, & -\frac{\kappa_1}{\kappa_2} q^{1+\lambda_2+s} \\ -\frac{\kappa_1}{\kappa_2} q^{1+\lambda_2} & \end{matrix} ;q,q^{k+1} \right). \tag{4.9}$$

We will compute the matrix elements of the operator $e_q(\beta F_+)E_q(\alpha F_-)$ with respect to both the tensor product basis $e_m^{(\ell_1,\lambda_1)} \otimes e_n^{(\ell_2,\lambda_2)} \equiv e_m \otimes e_n$ and the reduced basis e_k^s.

From Lemma 3 we have the formal identity

$$\begin{aligned} e_q(\beta F_+)E_q(\alpha F_-) &= e_q(\beta q^{-(1/2)H} \otimes E_+) e_q(\beta E_+ \otimes q^{(1/2)H}) E_q(\alpha q^{-(1/2)H} \otimes E_-(\kappa_1 q^H + \kappa_2)) \\ &\quad \times E_q(\alpha E_- \otimes q^{(1/2)H}) \\ &= e_q(\beta q^{-(1/2)H} \otimes E_+) E_q(\alpha q^{-(1/2)H} \otimes E_-(\kappa_1 q^H + \kappa_2)) \\ &\quad \times e_q(\beta E_+ \otimes q^{(1/2)H}) E_q(\alpha E_- \otimes q^{(1/2)H}). \end{aligned}$$

The matrix elements with respect to the tensor product basis are given by

$$\begin{aligned} &\|f_m \otimes f_n\| \cdot \|f_{m'} \otimes f_{n'}\| S_{m'n';mn}^{(e+,E-)}(\alpha,\beta) \\ &= \langle e_q(\beta F_+) E_q(\alpha F_-) f_m \otimes f_n, f_{m'} \otimes f_{n'} \rangle \\ &= \langle e_q(\beta q^{-(1/2)H} \otimes E_+) E_q(\alpha q^{-(1/2)H} \otimes E_-(\kappa_1 q^H + \kappa_2)) \\ &\quad \times e_q(\beta E_+ \otimes q^{1/2H}) E_q(\alpha E_- \otimes q^{(1/2)H}) f_m \otimes f_n, f_{m'} \otimes f_{n'} \rangle \end{aligned}$$

$$=\langle e_q(\beta E_+\otimes q^{(\lambda_2+n)/2})E_q(\alpha E_-\otimes q^{(\lambda_2+n)/2})f_m\otimes f_n,$$

$$E_q(\bar{\alpha}q^{-(\lambda_1+m')/2}\otimes(\kappa_1q^H+\kappa_2)E_+)e_q(\bar{\beta}q^{-(\lambda_1+m')/2}\otimes E_-)f_{m'}\otimes f_{n'}\rangle$$

$$=\langle e_q(\beta q^{(\lambda_2+n)/2}E_+)E_q(\alpha q^{(\lambda_2+n)/2}E_-)f_m,f_{m'}\rangle$$

$$\times\langle e_q(\beta q^{-(\lambda_1+m')/2}E_+)E_q(\alpha q^{-(\lambda_1+m')/2}E_-(\kappa_1q^H+\kappa_2))f_n,f_{n'}\rangle. \qquad (4.10)$$

Thus, the matrix elements factor. Explicitly, we have

$$S^{(e+,E-)}_{m'n',mn}(\alpha,\beta)=\left[\frac{(q;q)_m(q;q)_n(-(\kappa_1/\kappa_2)q^{\lambda_2+1};q)_n(1-q)^{m'+n'-m-n}}{\kappa_2^{n'-n}(q;q)_{m'}(q;q)_{n'}(-(\kappa_1/\kappa_2)q^{\lambda_2+1};q)_{n'}}\right]^{1/2}$$

$$\times\frac{(\alpha\ell_1)^{m-m'}(\alpha\kappa_2\ell_2)^{n-n'}(-(\kappa_1/\kappa_2)q^{\lambda_2+1};q)q^{(m-m')(m-3m'+2n+2\lambda_2-3)/4}}{(q;q)_{n-n'}(q;q)_{m-m'}(-(\kappa_1/\kappa_2)q^{\lambda_2+1};q)_{n'}q^{(n'-n)(n-3n'-2m'-2\lambda_1-3)/4}}$$

$$\times{}_1\phi_1\left(\begin{matrix}q^{-m'}\\ q^{m-m'+1}\end{matrix};q,\frac{\alpha\beta\ell_1^2q^{m-m'+n+\lambda_2}}{1-q}\right)$$

$$\times{}_2\phi_1\left(\begin{matrix}q^{-n'}, & -\frac{\kappa_2}{\kappa_1}q^{-\lambda_2-n'}\\ q^{n-n'+1} & \end{matrix};q,-\frac{\alpha\beta\ell_2^2\kappa_1q^{\lambda_2+n}}{(1-q)q^{\lambda_1+m'}}\right). \qquad (4.11)$$

The matrix elements in the reduced basis are

$$S^{(e+,E-)s}_{k'k}(\alpha,\beta)=\langle e_q(\beta E+)E_q(\alpha E_-)e^s_k,e^s_{k'}\rangle$$

$$=\left[\frac{(q;q)_k(1-q)^{k'-k}}{(q;q)_{k'}}\right]^{1/2}\frac{(\tilde{\ell}_s\alpha)^{k-k'}}{(q;q)_{k-k'}}q^{(k-k')(k-3k'-3)/4}$$

$$\times{}_1\phi_1\left(\begin{matrix}q^{-k'}\\ q^{k-k'+1}\end{matrix};q,\frac{\alpha\beta\tilde{\ell}_s^2q^{k-k'}}{1-q}\right). \qquad (4.12)$$

We note from Eqs. (4.6), (4.10), and (4.12) that the following identity, relating the two classes of matrix elements must hold:

$$S^{(e+,E-)}_{m'n',mn}(\alpha,\beta)=\sum_s\begin{bmatrix}\ell_1,\lambda_1; & \ell_2,\lambda_2; & s\\ m; & n; & m+n-s\end{bmatrix}_q S^{(e+,E-)s}_{m'+n'-s,m+n-s}(\alpha,\beta)$$

$$\times\begin{bmatrix}\ell_1,\lambda_1; & \ell_2,\lambda_2; & s\\ m'; & n'; & m'+n'-s\end{bmatrix}_q.$$

This leads to the identity

$$\frac{d^{n'}(cq;q)_{n'}(dq;q)_\infty}{(cdq;q)_\infty(q;q)_{m'}(w;q)_{n'}}\,{}_1\phi_1\left(\begin{matrix}q^{-m'}\\ dq\end{matrix};q,zd\right)\,{}_2\phi_1\left(\begin{matrix}q^{-n'},w^{-1}q^{1-n'}\\ cq\end{matrix};q,zq^{-1-m'}\right)$$

$$=\sum_{s=0}^{m'+n'}\frac{(w;q)_s w^{m'-s}}{(q;q)_s(q;q)_{m'+n'-s}}\,{}_1\phi_1\left(\begin{matrix}q^{s-m'-n'}\\ cdq\end{matrix};q,\frac{zdq^{-s-n'}}{w}\right)$$

$$\times{}_2\phi_1\left(\begin{matrix}c^{-1}q^{-n'},wq^{s}\\ w\end{matrix};q,cdq^{m'+n'-s+1}\right){}_2\phi_1\left(\begin{matrix}q^{-n'},wq^{s}\\ w\end{matrix};q,q^{m'+n'-s+1}\right). \qquad (4.13)$$

The result (4.13) is established, initially, only for $c=q^t$, $d=q^p$ where p, t are non-negative integers. However, a standard analytic continuation argument extends it to all complex values of z, w, c, d such that $|cdq|<1$.

V. A SECOND TENSOR PRODUCT IDENTITY

The q-oscillator algebra, modulo the ideal generated by $\mathscr{E}$, is isomorphic to the enveloping algebra of the Lie algebra $m(2)$ of the Euclidean motion group in the plane.[2,8,11,13] Thus the irreducible representations of $m(2)$ induce irreducible representations of the q-oscillator algebra. We focus our attention on the induced representation $(\omega),\omega>0$. The spectrum of the operator H corresponding to (ω) is the set $Z=\{0,\pm 1,\pm 2,...,\}$ and the complex representation space K_2 has basis vectors p_m, $m\in Z$, such that

$$E_\pm p_m=\omega p_{m\pm 1},\quad Hp_m=mp_m,\quad \mathscr{E}p_m=0. \qquad (5.1)$$

There is an inner product on K_2 such that $\langle p_m,p_{m'}\rangle=\delta_{mm'},m,m'\in Z$. On the dense subspace $\mathscr{K}$ of all finite linear combinations of the basis vectors we have

$$\langle E_+f,f'\rangle=\langle f,E_-f'\rangle,\quad \langle Hf,f'\rangle=\langle f,Hf'\rangle \qquad (5.2)$$

for all $f,f'\in\mathscr{K}$, so $H=H^*$ and $E_+^*=E_-$.

A simple realization of (ω) is given by the operators

$$H=u\frac{d}{du},\quad E_+=\omega u,\quad E_-=\frac{\omega}{u},\quad \mathscr{E}=0 \qquad (5.3)$$

acting on the space of all linear combinations of the functions u^m, u is a complex variable, $m\in Z$, with basis vectors $p_m(u)=u^m$.

Consider the following q analog of matrix elements of (ω):[2]

$$E_q(\beta E_+)e_q(\alpha E_-)p_n=\sum_{n'=-\infty}^{\infty}T_{n'n}^{(E,e)}(\alpha,\beta)p_{n'},\quad |\omega\beta|<1.$$

Explicitly we have

$$T_{n'n}^{(E,e)}(\alpha,\beta)=\frac{(q^{n'-n+1};q)_\infty(\beta\omega)^{n'-n}}{(q;q)_\infty}q^{(n'-n)(n'-n-1)/2}{}_1\phi_1\left(\begin{matrix}0\\ q^{n'-n+1}\end{matrix};q,-\alpha\beta\omega^2q^{n'-n}\right)$$

$$=\frac{(q^{n-n'+1};q)_\infty(\alpha\omega)^{n-n'}}{(q;q)_\infty}\,{}_1\phi_1\left(\begin{matrix}0\\ q^{n-n'+1}\end{matrix};q,-\alpha\beta\omega^2\right). \qquad (5.4)$$

If $\alpha\beta\neq 0$ we can express these elements in terms of the Hahn–Exton q-Bessel function[17]

$$J_\nu(z;q)=\frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}z^\nu{}_1\phi_1\left(\begin{matrix}0\\ q^{\nu+1}\end{matrix};q,qz^2\right). \qquad (5.5)$$

Indeed, setting $\alpha=ire^{i\psi}$, $\beta=ire^{-i\psi}$, we see that in terms of the new complex coordinates $[r, e^{i\psi}]$ we have

$$T^{(e,E)}_{n'n}[r, e^{i\psi}]=e^{i((\pi/2)+\psi)(n-n')}q^{(n-n')/2}J_{n-n'}(r\omega q^{-1/2};q). \tag{5.6}$$

[Note that $J_{-n}(z;q)=(-1)^n q^{n/2}J_n(zq^{n/2};q)$, for integer n.]

We define the tensor product representation $(\omega)\otimes\uparrow_{\ell,\lambda}$, acting on the space $K_2\otimes K_0$, by

$$\begin{aligned}
F_+&=\Delta(E_+)=E_+\otimes q^{(1/2)H}+q^{-(1/2)H}\otimes E_+,\\
F_-&=\Delta(E_-)=E_-\otimes q^{(1/2)H}+q^{-(1/2)H}\otimes E_-,\\
L&=\Delta(H)=H\otimes I+I\otimes H,\\
\mathscr{F}&=\Delta(\mathscr{E})=I\otimes\mathscr{E}.
\end{aligned} \tag{5.7}$$

By construction

$$\begin{aligned}
&[L,F_\pm]=\pm F_\pm,\quad [F_+,F_-]=-\mathscr{F}q^{-L},\\
&[\mathscr{F},F_\pm]=[\mathscr{F},L]=0.
\end{aligned} \tag{5.8}$$

Note also that with respect to the induced inner product we have $F_+=F_-^*$, $L=L^*$, $\mathscr{F}=\mathscr{F}^*$ on the dense subspace of all finite linear combinations of the basis vectors $p_m\otimes f_n$.

We will decompose $(\omega)\otimes\uparrow_{\ell,\lambda}$ into a direct sum of irreducible representations of the q-oscillator algebra. To carry out the decomposition explicitly it is very useful to employ a function space realization of the tensor product representation. Using the models (2.5) and (5.3) we find

$$F_+=q^{\lambda/2}\omega uT_z^{1/2}+\ell_z T_u^{-1/2},\quad F_-=q^{\lambda/2}\frac{\omega}{u}T_z^{1/2}-\frac{\ell}{(1-q)z}T_u^{-1/2}(I-T_z^{-1}),$$

$$L=u\partial_u+z\partial_z+\lambda,\quad \mathscr{F}=\ell^2q^{\lambda-1}.$$

For the decomposition we first determine a basis for $K_2\otimes K_0$ that consists of simultaneous eigenvectors of the commuting operators L and

$$\mathscr{C}=qq^{-L}\mathscr{F}+(q-1)F_+F_-. \tag{5.9}$$

Introducing new variables u and $r=z/u$ in place of u, z we see that the functions $\{\sqrt{(1-q)^n/(q;q)_n}u^\xi q^{n(n+1)/4}r^n=u^\xi e_n(r):n=0,1,2,...;\xi=0,\pm 1,\pm 2,...\}$ form an orthonormal basis for $K_2\otimes K_0$. Clearly, the possible eigenvalues of L on the space are $\{\xi+\lambda:\xi=0,\pm 1,\pm 2,...\}$ and the eigenspace $\mathscr{S}_\xi$ corresponding to eigenvalue $\xi+\lambda$ consists of the functions j_ξ such that

$$j_\xi(u,z)=u^\xi h(r)=u^\xi\sum_{n=0}^\infty c_n e_n(r),\quad \sum_{n=0}^\infty |c_n|^2<\infty.$$

Furthermore $K_2\otimes K_0\equiv\Sigma_{\xi=-\infty}^\infty\oplus\mathscr{S}_\xi$. The restriction of $\mathscr{C}$ to $\mathscr{S}_\xi$ takes the form $\mathscr{C}j_\xi(u,z)=u^\xi\mathscr{C}_\xi h(r)$ where $j_\xi=u^\xi h(r)$ and

$$\mathscr{C}_\xi h(r)=\left[\ell^2 q^{-\xi}+(q-1)(q^{\lambda}\omega^2 T_r+q^{(\lambda+1-\xi)/2}\omega\ell r T_r)-\left(q^{(\lambda-1-\xi)/2}\frac{\omega\ell}{r}+q^{-\xi}\ell^2\right)\right.$$
$$\left.\times(T_r-I)\right]h(r). \tag{5.10}$$

The symmetric operator $\mathscr{C}_\xi:K_0(r)\to K_0(r)$ is bounded and its closure is self-adjoint; further it belongs to the Hilbert–Schmidt class.[25] To see this we use Eq. (5.10) to determine the action of $\mathscr{C}_\xi$ on the orthonormal basis $\{e_n(r)\}$

$$\mathscr{C}_\xi e_n=-\sqrt{(1-q)(1-q^{n+1})}q^{(\lambda-\xi+n)/2}\omega\ell e_{n+1}+(q^{-\xi}\ell^2$$
$$-(1-q)q^\lambda\omega^2)q^n e_n-\sqrt{(1-q)(1-q^n)}q^{(\lambda-\xi+n-1)/2}\omega\ell e_{n-1}. \tag{5.11}$$

We see from this result that

$$\sum_{n=0}^{\infty}\langle\mathscr{C}_\xi^2 e_n,e_n\rangle=\frac{q^{-2\xi}\ell^4+(1-q)^2q^{2\lambda}\omega^4}{1-q^2}<\infty, \tag{5.12}$$

which implies that the closure of the domain of the operator defined by Eq. (5.11) is $\mathscr{S}_\xi$ and that this bounded self-adjoint operator $\mathscr{C}_\xi$ is Hilbert–Schmidt.

The eigenvalue equation $\mathscr{C}_\xi h(r)=ch(r)$ can, from Eq. (5.10), be expressed in the form

$$T_r h(r)=\frac{(1+q^{(-\lambda+1+\xi)/2}cr/\omega\ell)}{(1+(q-1)q^{(\lambda+1+\xi)/2}r\omega/\ell)(1+q^{(-\lambda+1-\xi)/2}r\ell/\omega)}h(r),$$

with solution

$$h_\xi^c(r)=\frac{((1-q)q^{(\lambda+1+\xi)/2}r\omega/\ell;q)_\infty(-q^{(-\lambda+1-\xi)/2}r\ell/\omega;q)_\infty}{(-q^{(-\lambda+1+\xi)/2}cr/\omega\ell;q)_\infty} \tag{5.13}$$

unique up to a multiplicative constant. Note that the functions $f_\xi^c(u,z)=u^\xi h_\xi^c(r)$ satisfy the relations

$$F_+f_\xi^c=q^{\lambda/2}\omega f_{\xi+1}^c,\quad F_-f_\xi^c=q^{-\xi-\lambda/2}\frac{\ell^2}{(1-q)\omega}\left(1-q^\xi\frac{c}{\ell^2}\right)f_{\xi-1}^c,$$
$$Lf_\xi^c=(\lambda+\xi)f_\xi^c,\quad \mathscr{F}f_\xi^c=q^{\lambda-1}\ell^2 f_\xi^c. \tag{5.14}$$

It remains to determine for which values of c the functions $h_\xi^c(r)$ belong to the Hilbert space $K_0(r)$. Since the elements of $K_0(r)$ are entire functions, there are only three possibilities

case 1: $c=q^{-\xi_0}\ell^2$, $\xi_0\in Z$, $\xi=\xi_0,\xi_0+1,\xi_0+2,\ldots$,

case 2: $c=-(1-q)q^{\lambda+s}\omega^2$, $\xi\in Z$, $s=0,1,2,\ldots$,

case 3: $c=0$.

In case 1 we have basis eigenvectors $(m=\xi-\xi_0)$

$$f_m^{(\xi_0)}(u,z)=((1-q)q^{(\lambda+\xi_0+m+1)/2}r\omega/\ell;q)_\infty(-q^{(-\lambda-\xi_0-m+1)/2}r\ell/\omega;q)_m u^{\xi_0+m}, \tag{5.15}$$

$$c=q^{-\xi_0}\ell^2, \quad m=0,1,2,\ldots .$$

A direct computation for $m=0$ gives $\|f_0^{(\xi_0)}\|^2=(-(1-q)q^{\lambda+\xi_0}\omega^2/\ell^2;q)_\infty$. Then from the recurrence relations (5.14) and the fact that $F_+=F_-^*$ we find

$$\|f_m^{(\xi_0)}\|^2=\left(\frac{\ell^2 q^{-(\lambda+\xi_0+1)}}{(1-q)\omega^2}\right)^m q^{-m(m-1)/2}(q;q)_m(-(1-q)q^{\lambda+\xi_0}\omega^2/\ell^2;q)_\infty .$$

In case 2 the eigenvectors are $(n=\xi)$

$$f_n^{[s]}(u,z)=(-q^{(-\lambda-n+1)/2}r\ell/\omega;q)_\infty((1-q)q^{(\lambda+n+1)/2}r\omega/\ell;q)_s u^n,$$
$$c=-(1-q)q^{\lambda+s}\omega^2, \quad n=0,\pm 1,\pm 2,\ldots, \tag{5.16}$$

and the normalization is

$$\|f_n^{[s]}\|^2=(q;q)_s q^{-s}\left(-(1-q)q^{\lambda+n+1}\frac{\omega^2}{\ell^2};q\right)_s\left(-q^{-\lambda-n}\frac{\ell^2}{(1-q)\omega^2};q\right)_\infty .$$

Case 3 does not occur because h_ξ^0 does not belong to the Hilbert space $K_0(r)$. To see this, we expand h_ξ^0 in terms of the orthonormal basis: $h_\xi^0=\Sigma_{m=0}^\infty a_m e_m$. From Eq. (5.11) and the defining equation $\mathscr{C}_\xi h_\xi^0=0$ we obtain the recurrence relation

$$(q^{-\xi}\ell^2-(1-q)q^\lambda\omega^2)q^m a_m-\sqrt{(1-q)(1-q^{m+1})}q^{(\lambda-\xi+m)/2}\omega\ell a_{m+1}$$
$$-\sqrt{(1-q)(1-q^m)}q^{(\lambda-\xi+m-1)/2}\omega\ell a_{m-1}=0 \tag{5.17}$$

for $m\geqslant 1$ and

$$(q^{-\xi}\ell^2-(1-q)q^\lambda\omega^2)a_0-(1-q)q^{(\lambda-\xi)/2}\omega\ell a_1=0.$$

We require that $a_0\neq 0$ is real, so that all a_m are real. Setting $\rho_m=a_{m+1}/a_m$ we see from Eq. (5.17) that

$$\rho_{m+1}\rho_m=\frac{\gamma q^{(m+1)/2}\rho_m}{(1-q^{m+2})^{1/2}}-\left(\frac{1-q^{m+1}}{1-q^{m+2}}\right)^{1/2}q^{-1/2}=A_m\rho_m-B_m, \tag{5.18}$$

where γ is real and does not depend on m. Since $0<q<1$ it is clear that we can choose an ϵ with $0<\epsilon<1$ and an integer m_0 such that $|B_m|>1+\epsilon$ and $|A_m|<\epsilon$ for all $m>m_0$. Since $h_\xi^0(r)$ is not a polynomial we can find an $m'>m_0$ such that $a_{m'}\neq 0$. Now either $|\rho_{m'}|>1$ or $|\rho_{m'}|\leqslant 1$. If $|\rho_{m'}|>1$ then $a_{m'+1}^2>a_{m'}^2$. If $|\rho_{m'}|\leqslant 1$ then from the fact that $\rho_{m'+1}\rho_{m'}=A_{m'}\rho_{m'}-B_{m'}$ we have $|\rho_{m'+1}\rho_{m'}|\geqslant|B_{m'}|-|A_{m'}\rho_{m'}|>(1+\epsilon)-\epsilon$ so $|\rho_{m'+1}\rho_{m'}|>1$ which implies that $a_{m'+2}^2>a_{m'}^2$. Proceeding in this manner we can construct a sequence of integers $p_1<p_2<p_3<\cdots$ such that $a_{p_1}\neq 0$ and $a_{p_{k+1}}^2>a_{p_k}^2$ for $k=1,2,\ldots$. Thus $\Sigma_{n=0}^\infty a_n^2$ diverges and case 3 cannot occur.

We have computed the following eigenvalues of $\mathscr{C}_\xi$, each with multiplicity one:

$$q^{-\xi+m}\ell^2, \quad -(1-q)q^{\lambda+m}\omega^2, \quad m=0,1,2,\ldots .$$

Summing the squares of these eigenvalues we find

$$\sum_{m=0}^{\infty}(q^{-2\xi+2m}\ell^4+(1-q)^2q^{2\lambda+2m})=\frac{q^{-2\xi}\ell^4+(1-q)^2q^{2\lambda}\omega^4}{1-q^2}$$

in agreement with Eq. (5.12). Thus we have obtained the spectral resolution of the operator $\mathscr{C}$.

Theorem 2:

$$(\omega)\otimes\uparrow_{\ell,\lambda}\cong\oplus\sum_{\xi_0\in Z}\uparrow_{q^{-\xi_0/2}\ell,\lambda+\xi_0}\oplus\sum_{s=0}^{\infty}R(\sqrt{1-q}q^{(\lambda+s)/2}\omega,q^{-\lambda-s}\ell^2/(1-q)\omega^2,\lambda).$$

Expanding the orthonormal basis $\{e_m^{(\xi_0)}, e_n^{[s]}\}$ for $K_2\otimes K_0$

$$e_m^{(\xi_0)}=\|f_m^{(\xi_0)}\|^{-1}f_m^{(\xi_0)},\quad \xi_0\in Z,\quad m=0,1,2,...,$$

$$e_n^{[s]}=\|f_n^{[s]}\|^{-1}f_n^{[s]},\quad s,\pm n=0,1,2,...,$$

in terms of the orthonormal basis $p_j\otimes e_h$, $\pm j,h=0,1,2,...$ we obtain the Clebsch–Gordan coefficients

$$e_m^{(\xi_0)}=\sum_{j,h}\begin{pmatrix}\omega; & \ell,\lambda; & \xi_0\\ j; & h; & m\end{pmatrix}_q^1 p_j\otimes e_h,$$

$$e_n^{[s]}=\sum_{j,h}\begin{pmatrix}\omega; & \ell,\lambda; & s\\ j; & h; & n\end{pmatrix}_q^2 p_j\otimes e_h. \tag{5.19}$$

These coefficients of the first and second kind vanish unless $j+h=\xi_0+m$, $j+h=n$, respectively. Furthermore, they satisfy the identities

$$\sum_{j,h}\begin{pmatrix}\omega; & \ell,\lambda; & \xi_0\\ j; & h; & m\end{pmatrix}_q^1\begin{pmatrix}\omega; & \ell,\lambda; & \xi_0'\\ j; & h; & m'\end{pmatrix}_q^1=\delta_{mm'}\delta_{\xi_0\xi_0'}, \tag{5.20a}$$

where $j+h=\xi_0+m=\xi_0'$

$$\sum_{j,h}\begin{pmatrix}\omega; & \ell,\lambda; & s\\ j; & h; & n\end{pmatrix}_q^2\begin{pmatrix}\omega; & \ell,\lambda; & s'\\ j; & h; & n\end{pmatrix}_q^2=\delta_{ss'}, \tag{5.20b}$$

$$\sum_{j,h}\begin{pmatrix}\omega; & \ell,\lambda; & \xi_0\\ j; & h; & m\end{pmatrix}_q^1\begin{pmatrix}\omega; & \ell,\lambda; & s\\ j; & h; & n\end{pmatrix}_q^2=0, \tag{5.20c}$$

where $j+h=\xi_0+m=n$, and

$$\sum_{m=0}^{\infty}\begin{pmatrix}\omega; & \ell,\lambda; & j+h-m\\ j; & h; & m\end{pmatrix}_q^1\begin{pmatrix}\omega; & \ell,\lambda; & j'+h'-m\\ j'; & h'; & m\end{pmatrix}_q^1$$

$$+\sum_{s=0}^{\infty}\begin{pmatrix}\omega; & \ell,\lambda; & s\\ j; & h; & j+h\end{pmatrix}_q^2\begin{pmatrix}\omega; & \ell,\lambda; & s\\ j'; & h'; & j'+h'\end{pmatrix}_q^2=\delta_{jj'}, \tag{5.20d}$$

where $j+h=j'+h'$. Explicitly, we have

$$\begin{pmatrix} \omega; & \ell,\lambda; & \xi_0 \\ \xi_0+m-h; & h; & m \end{pmatrix}_q^1 = \sqrt{\frac{(q;q)_m(1-q)^{m-h}}{(q;q)_h(-(1-q)\omega^2 q^{\lambda+\xi_0}/\ell^2;q)_\infty}} \left(\frac{\omega q^{(\lambda+\xi_0+m)/2}}{\ell}\right)^{m-h}$$

$$\times \frac{(q^{m-h+1};q)_\infty}{(q;q)_\infty} q^{(h-m)(h+m-1)/4}$$

$$\times {}_1\phi_1\left(\begin{matrix} q^{-h} \\ q^{m-h+1} \end{matrix};q,\frac{-(1-q)\omega^2 q^{\lambda+\xi_0+m+1}}{\ell^2}\right) \quad (5.21)$$

and

$$\begin{pmatrix} \omega; & \ell,\lambda; & s \\ n-h; & h; & n \end{pmatrix}_q^2 = \sqrt{\frac{(q;q)_s(1-q)^{-h}q^s}{(q;q)_h(-(1-q)\omega^2 q^{\lambda+n+1}/\ell^2;q)_s(-\ell^2 q^{-\lambda-n}/\omega^2(1-q);q)_\infty}}$$

$$\times \left(\frac{-(1-q)\omega q^{(\lambda+n)/2}}{\ell}\right)^h \frac{(q^{s-h+1};q)_\infty}{(q;q)_\infty} q^{h(h-1)/4}$$

$$\times {}_1\phi_1\left(\begin{matrix} q^{-h} \\ q^{s-h+1} \end{matrix};q,\frac{-\ell^2 q^{-\lambda-n+1}}{(1-q)\omega^2}\right). \quad (5.22)$$

The completeness relation (5.20d) leads to the special function identity

$$\sum_{m=0}^{\infty} \frac{(q;q)_m q^m}{(q;q)_{m-h}(q;q)_{m-h'}} \left[\frac{{}_1\phi_1\left(\begin{matrix} q^{-h} \\ q^{m-h+1} \end{matrix};q,-zq\right) {}_1\phi_1\left(\begin{matrix} q^{-h'} \\ q^{m-h'+1} \end{matrix};q,-zq\right)}{z^{(h+h')/2}(-q/z;q)_m(-z;q)_\infty} \right.$$

$$\left. + \frac{(-1)^{h+h'} z^{(h+h')/2} {}_1\phi_1\left(\begin{matrix} q^{-h} \\ q^{m-h+1} \end{matrix};q,-q/z\right) {}_1\phi_1\left(\begin{matrix} q^{-h'} \\ q^{m-h'+1} \end{matrix};q,-q/z\right)}{(-zq;q)_m(-1/z;q)_\infty} \right]$$

$$= \delta_{hh'} q^{-h(h-1)/2}(q;q)_h, \quad z>0, \quad h,h'=0,1,\ldots .$$

From Lemma 3 we have the identity

$$E_q(\beta F_+)e_q(\alpha F_-) = E_q(\beta q^{-(1/2)H}\otimes E_+)e_q(\alpha q^{-(1/2)H}\otimes E_-)E_q(\beta E_+\otimes q^{(1/2)H})$$

$$\times e_q(\alpha E_-\otimes q^{(1/2)H})$$

so the matrix elements with respect to the tensor product basis are given by

$$S^{(E+,e-)}_{j'h';jh}(\alpha,\beta) = \langle E_q(\beta F_+)e_q(\alpha F_-)p_j\otimes e_h, p_{j'}\otimes e_{h'}\rangle$$

$$= \langle E_q(\beta E_+\otimes q^{(\lambda+h)/2})e_q(\alpha E_-\otimes q^{(\lambda+h)/2})p_j\otimes e_h, e_q(\bar{\alpha} q^{-j'/2}\otimes E_+)$$

$$\times E_q(\bar{\beta} q^{-j'/2}\otimes E_-)p_{j'}\otimes e_{h'}\rangle$$

$$= \langle E_q(\beta q^{(\lambda+h)/2}E_+)e_q(\alpha q^{(\lambda+h)/2}E_-)p_j, p_{j'}\rangle$$

$$\times\langle E_q(\beta q^{-j'/2}E_+)e_q(\alpha q^{-j'/2}E_-)e_h,e_{h'}\rangle$$

$$=T^{(E,e)}_{j'j}(\alpha q^{(\lambda+h)/2},\beta q^{(\lambda+h)/2})T^{(E+,e-)}_{h'h}(\alpha q^{-j'/2},\beta q^{-j'/2})\sqrt{\frac{(q;q)_{h'}(1-q)^{h-h'}}{(q;q)_h}}. \tag{5.23}$$

Thus the following identity, relating the matrix elements in the tensor product and reduced $(S^{(E+,e-)\ell,\lambda}_{j',j}(\alpha,\beta),\hat{S}^{(E+,e-)\ell,\delta,\lambda}_{j',j}(\alpha,\beta))$ bases must hold:

$$S^{(E+,e-)}_{j'h',jh}(\alpha,\beta)=\sum_{\xi_0=-\infty}^{\infty}\begin{pmatrix}\omega; & \ell,\lambda; & \xi_0\\ j; & h; & j+h-\xi_0\end{pmatrix}^1_q S^{(E+,e-)q^{-\xi_0/2}\ell,\lambda+\xi_0}_{j'+h'-\xi_0,j+h-\xi_0}(\alpha,\beta)$$

$$\times\begin{pmatrix}\omega; & \ell,\lambda; & \xi_0\\ j'; & h'; & j'+h'-\xi_0\end{pmatrix}^1_q+\sum_{s=0}^{\infty}\begin{pmatrix}\omega; & \ell,\lambda; & s\\ j; & h; & j+h\end{pmatrix}^2_q$$

$$\times\hat{S}^{(E+,e-)\sqrt{1-q}q^{(\lambda+s)/2}\omega,q^{-\lambda-s}\ell^2/(1-q)\omega^2,\lambda}_{j'+h',j+h}(\alpha,\beta)\begin{pmatrix}\omega; & \ell,\lambda; & s\\ j'; & h'; & j'+h'\end{pmatrix}^2_q.$$

This leads to the identity

$$\frac{(q;q)_h}{(q;q)_{j-j'}(q;q)_{h-h'}}\,{}_1\phi_1\begin{pmatrix}0\\ q^{j-j'+1};q,-yq^h\end{pmatrix}\,{}_1\phi_1\begin{pmatrix}q^{-h'}\\ q^{h-h'+1};q,zq^{-j'}\end{pmatrix}$$

$$=\sum_{\xi=-\infty}^{\min(j+h,j'+h')}\frac{(q;q)_{j+h-\xi}(y/z)^{j'-\xi}q^{[(j+h-j'-h')(j+h-j'-h'-1)-\xi(\xi+1)-\xi(j+h)]/2}}{(q;q)_{j+h-j'-h'}(q;q)_{j'-\xi}(q;q)_{j-\xi}(-yq^{\xi}/z;q)_\infty}$$

$$\times{}_1\phi_1\begin{pmatrix}q^{-h}\\ q^{j-\xi+1};q,-yq^{j+h+1}/z\end{pmatrix}\,{}_1\phi_1\begin{pmatrix}q^{-h'}\\ q^{j'-\xi+1};q,-yq^{j'+h'+1}/z\end{pmatrix}$$

$$\times q^{h(j'-j)+j'(j'+1)/2}\,{}_1\phi_1\begin{pmatrix}q^{\xi-j'-h'}\\ q^{j+h-j'-h'+1};q,zq^{-\xi+j+h-j'-h'}\end{pmatrix}$$

$$+\sum_{s=0}^{\infty}\frac{(q;q)_s(z/y)^{s-h}(1-q)^{j+h-j'-h'}q^{-s(j'+h')+h(h+j')+h'(h'-1)/2}}{(q;q)_{j+h-j'-h'}(q;q)_{s-h}(q;q)_{s-h'}(-zq^{-j'-h'-s}/y;q)_\infty}$$

$$\times\,{}_1\phi_1\begin{pmatrix}q^{-h}\\ q^{s-h+1};q,-zq^{-j-h+1}/y\end{pmatrix}\,{}_1\phi_1\begin{pmatrix}q^{-h'}\\ q^{s-h'+1};q,-zq^{-j'-h'+1}/y\end{pmatrix}(-1)^{h+h'}$$

$$\times{}_1\phi_1\begin{pmatrix}-zq^{-s-j'-h'}/y\\ q^{j+h-j'-h'+1};q,-yq^s\end{pmatrix},$$

where $y/z>0$, $h,h',\pm j,\pm j'=0,1,\ldots$.

VI. A BARGMANN–SEGAL HILBERT SPACE BASIS

It follows from the proof of Theorem 2 that for any real constant $\mu>0$ the functions $\{h_m^\mu(z),j_m^\mu(z):m=0,1,2,...,\}$ form an orthonormal basis for the Hilbert space $K_0(z)$, Eqs. (2.5), (2.6). Here

$$h_m^\mu(z)=\frac{(1-q)^{m/2}\mu^m((1-q)q^{1/2}z\mu;q)_\infty(-q^{1/2}z/\mu;q)_m}{q^{m(m-1)/4}\sqrt{(q;q)_m(-(1-q)\mu^2q^{-m};q)_\infty}}, \tag{6.1}$$

$$j_m^\mu(z)=\frac{q^{m/2}(-q^{1/2}z/\mu;q)_\infty((1-q)q^{1/2}z\mu;q)_m}{\sqrt{(q;q)_m(-(1-q)\mu^2;q)_m(-\mu^{-2}/(1-q);q)_\infty}}. \tag{6.2}$$

Using relations (5.11) and the operators (2.2) associated with the representation $\uparrow_{1,0}$ on $K_0(z)$ we can characterize $\{h_m^\mu,j_m^\mu\}$ as the orthonormal basis of eigenfunctions of the self-adjoint operator

$$-(1-q)q^{1/2}\mu^{-1}(E_+q^H+q^HE_-)+(\mu^{-2}-(1-q))q^H. \tag{6.3}$$

Indeed h_m^μ corresponds to eigenvalue $q^m\mu^{-2}$ and j_m^μ to eigenvalue $-(1-q)q^m$.

To get a better understanding of this orthogonality we make use of the kernel function $S(\bar{z'},z) = (-(1-q)q\bar{z'}z;q)_\infty$, Eq. (2.6), for $K_0(z)$. This function has the property that $f(b)=(f,S(\bar{b},\cdot))$ for any $f\in K_0(z)$ and $b\in\mathbb{C}$. It follows immediately that $f\in K_0(z)$ is orthogonal to the basis function h_0^μ if and only if $f(-q^{-1/2}\mu)=0$. Similarly, $f\in K_0(z)$ is orthogonal to the basis function j_0^μ if and only if $f((1-q)^{-1}q^{-1/2}/\mu)=0$. To extend this observation we make use of the following version of Heine's q-analog of Gauss' summation formula (Ref. 22, p. 11).

Lemma 4: Let α,β be complex numbers with $\beta\neq 0$. Then

$$(\alpha z;q)_k=\sum_{\ell=0}^{k}\frac{(q;q)_k}{(q;q)_\ell(q;q)_{k-\ell}}\left(\frac{\alpha q^\ell}{\beta}\right)^\ell(\alpha q^{\ell+1}/\beta;q)_{k-\ell}(\beta q^{-\ell}z;q)_\ell,\quad k=0,1,2,....$$

Setting $\alpha=-q^{1/2}/\mu$, $k=m$, and $\beta=(1-q)q^{1/2}\mu$ in Lemma 4 we can express h_m^μ as a linear combination of the functions $S(-q^{-1/2}q^{-k}\mu,z)$, $k=0,1,...,m$. Similarly, setting $\alpha=(1-q)q^{1/2}\mu$, $k=m$, and $\beta=-q^{1/2}/\mu$ in Lemma 4 we can express j_m^μ as a linear combination of the functions $S((1-q)^{-1}q^{-1/2}q^{-k}/\mu,z)$, $k=0,1,...,m$. In each case the expansion coefficients are all nonzero. These expansions yield an independent proof that the elements of the set $B=\{h_m^\mu,j_m^\mu:\ m=0,1,...,\}$ are mutually orthogonal. Moreover, we see that $f\in K_0(z)$ is orthogonal to all elements of B if and only if $f(z)$ vanishes for all $z\in M$ where $M=\{-q^{-1/2}q^{-k}\mu,(1-q)^{-1}q^{-1/2}q^{-k}/\mu,\ k=0,1,...,\}$. However, we know from the proof of Theorem 2 that B is a basis for $K_0(z)$. Hence, $f\in K_0(z)$ vanishes for all $z\in M$ if and only if $f\equiv 0$.

Following Bargmann[26] we say that a *characteristic set* D for the Hilbert space $K_0(z)$ is a subset of $\mathbb{C}$ such that if $f\in K_0(z)$ and $f(z)=0$ for all $z\in D$ then $f\equiv 0$.

Theorem 3: *The set*

$$\{-q^{-1/2}q^{-k}\mu,(1-q)^{-1}q^{-1/2}q^{-k}/\mu,\quad k=0,1,...\}$$

is a characteristic set for the Hilbert space $K_0(z)$.

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