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Coulomb-oscillator duality in spaces of constant curvature

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In this paper we construct generalizations to spheres of the well-known Levi-Civita, Kustaanheimo–Steifel, and Hurwitz regularizing transformations in Euclidean spaces of dimensions two, three, and five. The corresponding classical and quantum mechanical analogs of the Kepler–Coulomb problem on these spheres are discussed. © *2000 American Institute of Physics.* [S0022-2488(00)05205-1]

I. INTRODUCTION

It is well known that the problem of a body moving under the influence of a central force field with potential $V(r)=-\mu/r$ has a singularity at the origin. We refer to this as the *Kepler problem.* This problem is usually posed in three dimensions, but since the motion is always constrained to a plane perpendicular to the constant angular momentum vector we can reduce it to two dimensions with Newtonian equations of motion and energy integral

$$\frac{d^2}{dt^2}\mathbf{r}=-\frac{\mu}{r^3}\mathbf{r}, \quad \frac{1}{2}\left(\frac{dr}{dt}\right)^2-\frac{\mu}{r}+\frac{1}{2r^2}=h, \tag{1}$$

where $r^2=\mathbf{r}\cdot\mathbf{r}$, $r^2(d\theta/dt)=c$ and $\mathbf{r}=(x,y)=(r\cos\theta,\ r\sin\theta)$. As is well known,[1,2] in two dimensions the Levi-Civita transformation effectively removes the singularity and rewrites this problem in terms of the classical harmonic oscillator. In this process the original problem has been regularized. To achieve the regularization, instead of t we use the variable s defined by

$$s=\int\frac{dt}{r}, \quad \frac{d}{dt}=\frac{1}{r}\frac{d}{ds}. \tag{2}$$

With $x'=dx/ds$, etc., the original equations (1) are

$$\mathbf{r}''-\frac{r'}{r}\mathbf{r}'+\frac{\mu}{r}\mathbf{r}=\mathbf{0}, \quad \frac{1}{2r^2}\mathbf{r}'\cdot\mathbf{r}'-\frac{\mu}{r}=h. \tag{3}$$

Instead of using the variables (x,y) it is convenient to make the transformation[1]

$$\begin{vmatrix}x\\y\end{vmatrix}=\begin{vmatrix}u_1 & -u_2\\ u_2 & u_1\end{vmatrix}\begin{vmatrix}u_1\\u_2\end{vmatrix} \quad \text{or} \quad \mathbf{r}=L(\mathbf{u})\mathbf{u}. \tag{4}$$

From the explicit form of these relations it follows that $\mathbf{r}'=2L(\mathbf{u})\mathbf{u}'$. The equations of motion are equivalent to

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$$\mathbf{u}''+\frac{\mu/2-\mathbf{u}'\cdot\mathbf{u}'}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}=\mathbf{u},\quad \frac{\mu}{2}=\mathbf{u}'\cdot\mathbf{u}'-\frac{h}{2}\mathbf{u}\cdot\mathbf{u}. \tag{5}$$

Consequently we have the regularized equation of motion

$$\mathbf{u}''-\frac{h}{2}\mathbf{u}=\mathbf{0}.$$

This is essentially the equation for the harmonic oscillator if $h<0$. The solution $u_1=\alpha\cos(\omega s)$, $u_2=\beta\sin(\omega s)$, $\omega^2=-h/2$ is equivalent to elliptical motion.

The relationship between the harmonic oscillator and the corresponding Kepler problem can also be easily seen from the point of view of Hamilton–Jacobi theory. Indeed the Hamiltonian can be written in the two equivalent forms

$$H=\frac{1}{2}(p_x^2+p_y^2)+\frac{\mu}{\sqrt{x^2+y^2}}=\frac{1}{8(u_1^2+u_2^2)}[p_{u_1}^2+p_{u_2}^2+8\mu]. \tag{6}$$

If we now write down the corresponding Hamilton–Jacobi equation via the substitutions

$$p_{u_1}\to\partial_{u_1}S=S_{u_1},\quad p_{u_2}\to\partial_{u_2}S=S_{u_2},$$

we obtain

$$S_{u_1}^2+S_{u_2}^2+8\mu-8E(u_1^2+u_2^2)=0. \tag{7}$$

This is just the Hamilton–Jacobi equation for a mechanical system with Hamiltonian

$$H'=p_{u_1}^2+p_{u_2}^2-8E(u_1^2+u_2^2)$$

and energy -8μ. (This is the pseudo-Coulomb problem, see Refs. 3, and 4. Reference 3 also obtains (7) as an application of Stäckel transform theory.)

This transformation also achieves a regularization of the corresponding quantum mechanical problem, which we call the *quantum Coulomb* problem. Indeed, the Schrödinger equation in the presence of the potential $V(r)=-\mu/r$ in two dimensions has the form

$$-\frac{1}{2}(\partial_x^2+\partial_y^2)\Psi-\frac{\mu}{\sqrt{x^2+y^2}}\Psi=E\Psi. \tag{8}$$

In the coordinates (u_1,u_2), (8) becomes[5]

$$(\partial_{u_1}^2+\partial_{u_2}^2)\Phi+\{8\mu+8E(u_1^2+u_2^2)\}\Phi=0. \tag{9}$$

Here, (9) has all the appearances of the Schrödinger equation in an oscillator potential $V(u_1,u_2)=-4E(u_1^2+u_2^2)$ and energy $\mathcal{E}=4\mu$. Note that for scattering state $E>0$ we have the repulsive oscillator potential and for $E=0$ the free motion. For $E<0$ we get the attractive oscillator potential and the corresponding bound state energy spectrum can be easily computed from this reformulation of the Coulomb problem, although the weight function for the inner product is no longer the same.[2,5,3,6] (Indeed, the Virial Theorem states that for the Coulomb problem the change in weight function does not alter the bound state spectrum.[6]) The wave functions have the form $\Phi=\varphi_1(u_1)\varphi_2(u_2)$ where the functions φ_λ satisfy

$$(\partial_{u_\lambda}^2+\kappa_\lambda+8Eu_\lambda^2)\varphi_\lambda=0,\quad \lambda=1,2,\quad \kappa_1+\kappa_2=8\mu.$$

The bound state eigenvalues are quantized according to

$$\kappa_\lambda = 2\sqrt{-2E}(2n_\lambda + 1), \quad \lambda = 1,2, \tag{10}$$

where n_1, n_2 are integers. Taking into account[7] $\Phi(-u_1,-u_2) = (-1)^{n_1+n_2}\Phi(u_1,u_2)$ and using that $\Psi(x)$ is even in variable u: $\Psi[x(u)] = \Psi[x(-u)]$ [because two points $(-u_1,-u_2)$ and (u_1,u_2) in u-space map to the same point in the plane (x,y)], we find from (10) the energy spectrum of the two-dimensional Coulomb system[8,9]

$$E_N = -\frac{\mu^2}{2\left(N+\frac{1}{2}\right)^2}, \quad N = \frac{n_1+n_2}{2} = 0,1,2,\ldots .$$

It is well known that the regularizing transformations (4) that we have discussed for the Kepler and Coulomb problems in two-dimensional Euclidean spaces are also possible in the case of three (Kustaanheimo–Stiefel transformation for mapping $R_4 \to R_3$)[2,10–12] and five (Hurwitz transformation for mapping $R_8 \to R_5$)[13–19] dimensions. The only difference in these cases is that additional constraints are required. These transformations have been employed to solve many problems in classical and quantum mechanics (see Ref. 14 and references therein).

As in flat space, the study of the Kepler–Coulomb system in constant curvature spaces has a long history. It was first introduced in quantum mechanics by Schrödinger,[20] who used the factorization method to solve the Schrödinger equation and to find the energy spectrum for the harmonic potential as an analog of the Kepler–Coulomb potential on the three-dimensional sphere. Later, two- and three-dimensional Coulomb and oscillator systems were investigated by many authors in Refs. 21–31.

However, in spite of these achievements the question of finding all transformations that both generalize the Levi-Civita, Kustaanheimo–Steifel (KS), and Hurwitz transformations for spaces with constant curvature and preserve the Kepler–Coulomb and oscillator duality has been open until now. The answer to this question is a main aim of our paper.

The paper is organized as follows. In Sec. II we present the transformations that generalize the flat space Levi-Civita transformation and correspond to the map $S_{2C} \to S_2$ from complex into real two-dimensional spheres. We also show that this transformation establishes the correspondence between Kepler–Coulomb and oscillator systems in classical and quantum mechanics. In Sec. III, in analogy with Sec. II, we construct the Kustaanheimo–Steifel and Hurwitz transformation and show Kepler–Coulomb and oscillator duality for mappings $S_{4C} \to S_3$ and $S_{8C} \to S_5$, respectively. Section IV is devoted to a summary and discussion of our findings. In the Appendix we give some formulas determining the connections between Laplace–Beltrami operators and the volume elements in different spaces.

II. THE TRANSFORMATION ON THE TWO-SPHERE

The potential, which is the analog of the Coulomb potential in quantum mechanics and the gravitational potential for the Kepler problem, is taken to be[20,21]

$$V = -\frac{\mu}{R}\frac{s_3}{\sqrt{s_1^2+s_2^2}}, \tag{11}$$

where (s_1,s_2,s_3) are the Cartesian coordinates in the ambient Euclidean space and R is the radius of the sphere

$$\mathbf{s}\cdot\mathbf{s} \equiv s_1^2+s_2^2+s_3^2 = R^2.$$

(Note that $V=-(\mu/R)\cot\alpha$ where α is the arclength distance from $\mathbf{s}$ to the north pole of the sphere. Furthermore, the leading term in the Laurent series expansion in α about the north pole is $-\mu/R\alpha$.)

This problem is easily transformed into a much simpler one via the transform

$$s_1=i\sqrt{u_1^2+u_2^2+u_3^2}\cdot\frac{u_1^2-u_2^2}{2u_3},$$

$$s_2=i\sqrt{u_1^2+u_2^2+u_3^2}\cdot\frac{u_1u_2}{u_3}, \tag{12}$$

$$s_3=\sqrt{u_1^2+u_2^2+u_3^2}\cdot\left(u_3+\frac{u_1^2+u_2^2}{2u_3}\right),$$

or in matrix form

$$\begin{vmatrix} s_1 \\ s_2 \\ s_3 \end{vmatrix} = \frac{\sqrt{u_1^2+u_2^2+u_3^2}}{2u_3} \begin{vmatrix} iu_1 & -iu_2 & 0 \\ iu_2 & iu_1 & 0 \\ u_1 & u_2 & 2u_3 \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix}. \tag{13}$$

The advantage of this transform is the Euler identity[5]

$$s_1^2+s_2^2+s_3^2=(u_1^2+u_2^2+u_3^2)^2, \tag{14}$$

from which we see that the point $\mathbf{u}=(u_1,u_2,u_3)$ lies on the complex ''sphere'' S_{2C}: $u_1^2+u_2^2+u_3^2=D^2$ with the real radius D if $\mathbf{s}=(s_1,s_2,s_3)$ lies on the real sphere S_2 with radius R, and $R=D^2$.

In the general case the two-dimensional complex sphere S_{2C} may be parametrized by four real variables (the constraint $u_1^2+u_2^2+u_3^2=D^2$ includes two equations for real and imaginary parts). The requirement of reality of the Cartesian variables s_i leads to two more equations and the formula (12) corresponds to the mapping from a two-dimensional submanifold (or surface) in the complex sphere S_{2C} (four-dimensional real space) to the sphere S_2. To verify we introduce ordinary spherical coordinates on S_2:

$$s_1=R\sin\chi\cos\varphi,\quad s_2=R\sin\chi\sin\varphi,\quad s_3=R\cos\chi. \tag{15}$$

From transformation (12) we have

$$\frac{s_3}{R}=\frac{1}{2}\left(\frac{u_3}{D}+\frac{D}{u_3}\right). \tag{16}$$

Putting $s_3=R\cos\chi$ in formula (16) we get $u_3=De^{i\chi}$ and then the corresponding points on the complex sphere S_{2C} are

$$u_1=D\sqrt{1-e^{2i\chi}}\cos\frac{\varphi}{2},\quad u_2=D\sqrt{1-e^{2i\chi}}\sin\frac{\varphi}{2},\quad u_3=De^{i\chi}, \tag{17}$$

where $0\leqslant\chi\leqslant\pi$, $0\leqslant\varphi\leqslant4\pi$. Note that the transformation (12) is not one to one; two points $(-u_1,-u_2,u_3)$ and (u_1,u_2,u_3) on the sphere in u-space correspond to one point on the sphere in s-space. Thus, when the variables (u_1,u_2,u_3) cover the sphere in u-space, the variables s_i cover the sphere in s-space twice.

Let us now introduce nonhomogeneous coordinates according to[32]

$$\overline{s_i}=R\frac{s_i}{s_3},\quad \overline{u_i}=D\frac{u_i}{u_3},\quad D^2=R,\quad i=1,2. \tag{18}$$

Then formula (12) transforms to

$$\overline{s_1}=\frac{i(\overline{u_1}^2-\overline{u_2}^2)}{2\left(1+\dfrac{\overline{u_1}^2+\overline{u_2}^2}{2D^2}\right)},\quad \overline{s_2}=\frac{i\overline{u_1}\cdot\overline{u_2}}{\left(1+\dfrac{\overline{u_1}^2+\overline{u_2}^2}{2D^2}\right)}. \tag{19}$$

In the contraction limit $D\to\infty$ we obtain

$$\overline{s_1}=i\frac{\overline{u_1}^2-\overline{u_2}^2}{2},\quad \overline{s_2}=i\overline{u_1}\cdot\overline{u_2}, \tag{20}$$

which coincides with the flat space Levi-Civita transformation (4) up to the additional mapping $\overline{u_i}\to e^{-i(\pi/4)}\sqrt{2}\tilde{u}_i$.

The relationship between the infinitesimal distances is

$$d\mathbf{s}\cdot d\mathbf{s}=(u_1^2+u_2^2+u_3^2)\left[\frac{(\mathbf{u}\cdot d\mathbf{u})^2}{u_3^2}-\left(\frac{u_1^2+u_2^2}{u_3^2}\right)d\mathbf{u}\cdot d\mathbf{u}\right]+3(\mathbf{u}\cdot d\mathbf{u})^2. \tag{21}$$

Thus, when restricted to the sphere, the infinitesimal distances are related by

$$\frac{d\mathbf{s}\cdot d\mathbf{s}}{R}=-\left(\frac{u_1^2+u_2^2}{u_3^2}\right)d\mathbf{u}\cdot d\mathbf{u}, \tag{22}$$

and we see that as in flat space the transformation (12) is conformal.

A. Classical motion

Just as in the case of Euclidean space, the classical equations of motion under the influence of a Coulomb potential can be simplified. The classical equations are

$$\ddot{\mathbf{s}}=-(\dot{\mathbf{s}}\cdot\dot{\mathbf{s}})\mathbf{s}-\boldsymbol{\nabla}V, \tag{23}$$

where the first term on the right-hand side is the centripetal force term, corresponding to the constraint of the motion to the sphere, and the potential satisfies

$$\mathbf{s}\cdot\boldsymbol{\nabla}V=0. \tag{24}$$

Here, $\dot{\mathbf{s}}=(d/dt)\mathbf{s}$. [In studying (23) and (24) we initially regard the coordinates $\mathbf{s}$ as unconstrained and then restrict our attention to solutions on the sphere.] In the case of potential (11) these equations become

$$\frac{d^2}{dt^2}s_j=-s_j(\dot{\mathbf{s}}\cdot\dot{\mathbf{s}})-\frac{\mu}{R}\frac{s_js_3}{(s_1^2+s_2^2)^{3/2}},\quad j=1,2,$$

$$\frac{d^2}{dt^2}s_3=-s_3(\dot{\mathbf{s}}\cdot\dot{\mathbf{s}})+\frac{\mu}{R}\frac{1}{(s_1^2+s_2^2)^{1/2}},$$

subject to the constraints

$$\mathbf{s}\cdot\mathbf{s}=R^2 \tag{25}$$

and its differential consequences

$$\mathbf{s}\cdot\dot{\mathbf{s}}=0, \quad \mathbf{s}\cdot\ddot{\mathbf{s}}+\dot{\mathbf{s}}\cdot\dot{\mathbf{s}}=0.$$

From the equations of motion we immediately deduce the energy integral

$$\tfrac{1}{2}\dot{\mathbf{s}}\cdot\dot{\mathbf{s}}+V=E. \tag{26}$$

We choose a new variable τ such that

$$\frac{d\tau}{dt}=\frac{1}{D^2}\cdot\frac{u_3^2}{u_1^2+u_2^2}.$$

In terms of the variables τ and u_i, the equations of motion can now be written in the form

$$(u_1')^2+(u_2')^2+(u_3')^2-2D^2\left(E+\frac{i\mu}{D^2}\right)+\frac{2D^4}{u_3^2}\left(E-\frac{i\mu}{D^2}\right)=0, \tag{27}$$

$$u_1''+2\left(E+\frac{i\mu}{D^2}\right)u_1=0, \quad u_2''+2\left(E+\frac{i\mu}{D^2}\right)u_2=0, \tag{28}$$

$$u_3''+2\left(E+\frac{i\mu}{D^2}\right)u_3-\frac{2D^4}{u_3^3}\left(E-\frac{i\mu}{D^2}\right)=0, \tag{29}$$

subject to the constraint $\mathbf{u}\cdot\mathbf{u}=D^2$ and its differential consequences $\mathbf{u}\cdot\mathbf{u}'=0$, $\mathbf{u}\cdot\mathbf{u}''+\mathbf{u}'\cdot\mathbf{u}'=0$, where $u_i'=du_i/d\tau$. These equations are equivalent to the equations of motion we would obtain by choosing the Hamiltonian

$$H=\frac{1}{2}(p_{u_1}^2+p_{u_2}^2+p_{u_3}^2)-\left(E+\frac{i\mu}{D^2}\right)(u_1^2+u_2^2+u_3^2)+\frac{D^4}{u_3^2}\left(E-\frac{i\mu}{D^2}\right), \tag{30}$$

regarding the variables u_i as independent and using the variable τ as time. In fact, to solve the classical mechanical problem from the point of view of the Hamilton–Jacobi equation, we use the relation

$$\begin{aligned}&\frac{1}{2}(p_{s_1}^2+p_{s_2}^2+p_{s_3}^2)-\frac{\mu}{R}\frac{s_3}{\sqrt{s_1^2+s_2^2}}-E\\&\quad\equiv-\frac{u_3^2}{u_1^2+u_2^2}\left[\frac{1}{2D^2}(p_{u_1}^2+p_{u_2}^2+p_{u_3}^2)-\left(i\frac{\mu}{D^2}+E\right)+\frac{D^2}{u_3^2}\left(E-\frac{i\mu}{D^2}\right)\right]=0,\end{aligned} \tag{31}$$

together with the substitutions $p_{u_i}=\partial S/\partial u_i$ and $p_{s_j}=\partial S/\partial s_j$ to obtain the Hamilton–Jacobi equations

$$\left(\frac{\partial S}{\partial s_1}\right)^2+\left(\frac{\partial S}{\partial s_2}\right)^2+\left(\frac{\partial S}{\partial s_3}\right)^2-\frac{2\mu}{R}\frac{s_3}{\sqrt{s_1^2+s_2^2}}-2E=0, \tag{32}$$

$$\left(\frac{\partial S}{\partial u_1}\right)^2+\left(\frac{\partial S}{\partial u_2}\right)^2+\left(\frac{\partial S}{\partial u_3}\right)^2-2D^2\left(\frac{i\mu}{D^2}+E\right)+\frac{2D^4}{u_3^2}\left(E-\frac{i\mu}{D^2}\right)=0. \tag{33}$$

This last equation can be solved by separation of variables in the spherical coordinates on the complex sphere S_{2C} (17).

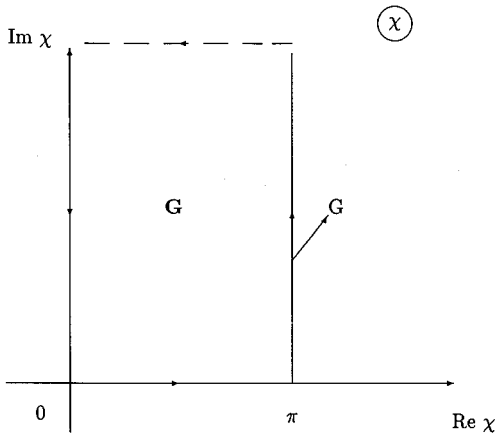

FIG. 1. Domain $\mathbf{G}=\{0\leqslant \mathrm{Re}\,\chi\leqslant\pi;0\leqslant \mathrm{Im}\,\chi<\infty\}$ on the complex plane of χ.

B. Quantum motion

If we write the Schrödinger equation on the sphere for the Coulomb potential (11)

$$\frac{1}{2}\Delta_s\Psi+\left(E+\frac{\mu}{R}\frac{s_3}{\sqrt{s_1^2+s_2^2}}\right)\Psi=0, \tag{34}$$

and use the transformation (12), we obtain [see formula (A3)]

$$\frac{1}{2}\Delta_u\psi+\left(\mathcal{E}-\frac{\omega^2D^2}{2}\frac{u_1^2+u_2^2}{u_3^2}\right)\psi=0, \tag{35}$$

where

$$\mathcal{E}=2i\mu,\quad \omega^2=2\left(E-\frac{i\mu}{D^2}\right). \tag{36}$$

Thus we see that the Coulomb problem on the real sphere S_2 is equivalent to the corresponding quantum mechanical problem on the complex sphere S_{2C} with the oscillator potential (Higgs oscillator[21,26,27]) and energy $2i\mu$, but with an altered inner product (see the Appendix).

Let us consider the Schrödinger equation (35). Using the complex spherical coordinates (17) we obtain

$$\frac{1}{\sin\chi}\frac{\partial}{\partial\chi}\sin\chi\frac{\partial\psi}{\partial\chi}+\frac{1}{\sin^2\chi}\frac{\partial\psi}{\partial\varphi}+\left\{\omega^2D^4-i\mathcal{E}D^2\frac{e^{i\chi}}{\sin\chi}\right\}\psi=0. \tag{37}$$

To solve Eq. (37) we first complexify the Coulomb coupling constant μ by setting $k=i\mu$ in the formulas for $\mathcal{E}$ and ω,

$$\mathcal{E}=2k,\quad \omega^2=2\left(E-\frac{k}{D^2}\right). \tag{38}$$

Further, we analytically continue the variable χ into the complex domain **G:** $0\leqslant\mathrm{Re}\,\chi\leqslant\pi$ and $0\leqslant\mathrm{Im}\,\chi<\infty$ (see Fig. 1) and pass from the variable χ to ϑ, defined by

$$e^{i\chi}=\cos\vartheta. \tag{39}$$

For real θ this substitution is possible if $\mathrm{Re}\,\chi=0$ or $\mathrm{Re}\,\chi=\pi$ and $\mathrm{Im}\,\chi\in(0,\infty)$, which corresponds to the motion on the upper $(0\leqslant\vartheta\leqslant\pi/2)$ or lower $(\pi/2\leqslant\vartheta\leqslant\pi)$ hemispheres of the real sphere. In any case conditions (39) and (38) translate the oscillator problem from the complex to the real sphere with spherical coordinates $(\vartheta,\varphi/2)$. In these coordinates we can rewrite (37) in the form

$$\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial\psi}{\partial\vartheta}+\frac{4}{\sin^2\vartheta}\frac{\partial^2\psi}{\partial\varphi^2}+\left\{(2\mathcal{E}D^2+\omega^2D^4)-\frac{\omega^2D^4}{\cos^2\vartheta}\right\}\psi=0. \tag{40}$$

Using the separation of variables ansatz

$$\psi(\vartheta,\varphi)=R(\vartheta)\frac{e^{im(\varphi/2)}}{\sqrt{2\pi}},\quad m=0,\pm1,\pm2,\ldots, \tag{41}$$

we obtain

$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\sin\vartheta\frac{dR}{d\vartheta}+\left\{(2\mathcal{E}D^2+\omega^2D^4)-\frac{\omega^2D^4}{\cos^2\vartheta}-\frac{m^2}{\sin^2\vartheta}\right\}R=0. \tag{42}$$

The corresponding solution regular at the points $\vartheta=0,\pi/2$ takes the form[7]

$$\begin{aligned}R_{n_rm}(\vartheta)&=C_{n_rm}(\nu)(\sin\vartheta)^{|m|}(\cos\vartheta)^{\nu+1/2}{}_2F_1(-n_r,n_r+\nu+|m|+1;|m|+1;\sin^2\vartheta)\\&=C_{n_rm}(\nu)\frac{(n_r)!|m|!}{(n_r+|m|)!}(\sin\vartheta)^{|m|}(\cos\vartheta)^{\nu+1/2}P_{n_r}^{(|m|,\nu)}(\cos2\vartheta)\end{aligned} \tag{43}$$

with energy spectrum given by

$$\mathcal{E}=\frac{1}{2D^2}[(n+1)(n+2)+(2\nu-1)(n+1)],\quad \nu=\left(\omega^2D^4+\frac{1}{4}\right)^{1/2}, \tag{44}$$

where $C_{n_rm}(\nu)$ is the normalization constant, $P_n^{(\alpha,\beta)}(x)$ is a Jacobi polynomial, $n_r=0,1,2,\ldots$ is the ''radial,'' and $n=2n_r+|m|$ is the principal quantum number.

To compute the normalization constant $C_{n_rm}(\nu)$ for the reduced system we require that the wave function (41) satisfy the normalization condition (see the Appendix):

$$-\frac{D^2}{2}\int_{S_{2C}}\psi_{n_rm}\psi^{\diamond}_{n_rm}\frac{u_1^2+u_2^2}{u_3^2}dv(u)=D^4\int_0^{\pi}R_{n_rm}R^{\diamond}_{n_rm}\sin\chi\,d\chi=1, \tag{45}$$

where the open diamond $(\diamond)$ means the complex conjugate together with the inversion $\chi\to-\chi$, i.e., $\psi^{\diamond}(\chi,\varphi)=\psi^*(-\chi,\varphi)$. [We choose the scalar product as $\psi^{\diamond}\psi$ because for real ω^2 and $\mathcal{E}$ the function $\psi^{\diamond}(\chi,\vartheta)$ also belongs to the solution space of (37).]

Consider now the integral over contour G in the complex plane of variable χ (see Fig. 1),

$$\begin{aligned}\oint R_{n_rm}R^{\diamond}_{n_rm}\sin\chi\,d\chi&=\int_0^{\pi}R_{n_rm}R^{\diamond}_{n_rm}\sin\chi\,d\chi+\int_{\pi}^{\pi+i\infty}R_{n_rm}R^{\diamond}_{n_rm}\sin\chi\,d\chi\\&\quad+\int_{\pi+i\infty}^{i\infty}R_{n_rm}R^{\diamond}_{n_rm}\sin\chi\,d\chi+\int_{i\infty}^{0}R_{n_rm}R^{\diamond}_{n_rm}\sin\chi\,d\chi.\end{aligned} \tag{46}$$

Using the facts that the integrand vanishes as $e^{2i\nu\chi}$ and that $R_{n_rm}(\chi)$ is regular in the domain **G** (see Fig. 1), then according to the Cauchy theorem we have

$$\int_0^{\pi} R_{n_r m} R^{\diamond}_{n_r m} \sin\chi \, d\chi = \int_0^{i\infty} R_{n_r m} R^{\diamond}_{n_r m} \sin\chi \, d\chi - \int_{\pi}^{\pi+i\infty} R_{n_r m} R^{\diamond}_{n_r m} \sin\chi \, d\chi$$

$$= \left[1 - e^{2i\pi(\nu+1/2)}\right] \int_0^{i\infty} R_{n_r m} R^{\diamond}_{n_r m} \sin\chi \, d\chi. \tag{47}$$

Making the substitution (17) in the right integral of Eq. (47), we find

$$\int_0^{\pi} R_{n_r m} R^{\diamond}_{n_r m} \sin\chi \, d\chi = [1 - e^{2i\pi(\nu+1/2)}] \int_0^{\pi/2} [R_{n_r m}]^2 \sin\vartheta \tan^2\vartheta \, d\vartheta. \tag{48}$$

Using the following formulas for integration of the two Jacobi polynomials:[33]

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}[P_n^{(\alpha,\beta)}(x)]^2 dx = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)},$$

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta-1}[P_n^{(\alpha,\beta)}(x)]^2 dx = \frac{2^{\alpha+\beta}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(\beta)n!\Gamma(n+\alpha+\beta+1)},$$

we find

$$C_{n_r m}(\nu) = \frac{2}{(|m|)!} \sqrt{\frac{-\nu(\nu+2n_r+|m|+1)\,(n_r+|m|)!\Gamma(|m|+n_r+\nu+1)}{D^4[1-2i\pi(\nu+1/2)](2n_r+|m|+1)\,(n_r)!\Gamma(n_r+\nu+1)}}. \tag{49}$$

The wave function $\psi(\vartheta,\varphi) \equiv \psi_{n_r m}(\vartheta,\varphi)$ is then given by Eqs. (41), (43), and (49).

Now we can construct the Coulomb wave functions and eigenvalue spectrum. From transformation

$$\psi_{n_r m}(\vartheta,\varphi+2\pi) = e^{im\pi}\psi_{n_r m}(\vartheta,\varphi) \tag{50}$$

and the requirement of 2π periodicity for the wave functions (41) we see that only even azimuthal angular momentum states of the oscillator correspond to the reduced system. Then, introducing new angular and principal quantum numbers M and N by the condition

$$n = 2n_r + |m| = 2n_r + 2|M| = 2N, \quad N = 0,1,2,\ldots, \quad |M| = 0,1,2,\ldots N, \tag{51}$$

comparing (38) with expression (44) for the oscillator energy spectrum, and putting $k = i\mu$, we find the energy spectrum for reduced systems,

$$E_N = \frac{N(N+1)}{2R^2} - \frac{\mu^2}{2(N+\frac{1}{2})^2}. \tag{52}$$

This formula coincides with that obtained from other methods in works Refs. 21, 26, and 27.

Transforming ϑ back to the variable χ by (39), we see that (44) and (38) imply

$$\nu = i\sigma - \left(N + \frac{1}{2}\right), \quad \sigma = \frac{\mu R}{N+\frac{1}{2}}.$$

Using

$$\frac{\Gamma(1/2+|M|+i\sigma)}{\Gamma(1/2-|M|+i\sigma)} = (-1)^{|M|}\frac{|\Gamma(1/2+|M|+i\sigma)|^2}{|\Gamma(1/2+i\sigma)|^2} = \frac{(-1)^{|M|}}{\pi}\cosh\sigma\pi\,|\Gamma(1/2+|M|+i\sigma)|^2, \tag{53}$$

we easily get from (41), (43), and (49) the eigenfunction of Schrödinger equation (34),

$$\Psi_{NM}(\chi,\varphi)=C_{NM}(\sigma)e^{-i\chi(N-|M|-i\sigma)}(\sin\chi)^{|M|}$$
$$\times{}_2F_1\left(-N+|M|,|M|+i\sigma+\frac{1}{2};2|M|+1;1-e^{2i\chi}\right)\frac{e^{iM\varphi}}{\sqrt{2\pi}},\tag{54}$$

where now

$$C_{NM}(\sigma)=\frac{2^{|M|}}{R(2|M|)!}\sqrt{\frac{[(N+\frac{1}{2})^2+\sigma^2](N+|M|)!}{\pi(N+\frac{1}{2})(N-|M|)!}}e^{\sigma\pi/2}|\Gamma(|M|+1/2+i\sigma)|\,.\tag{55}$$

By direct calculation it may be shown that the Coulomb wave function (54) satisfies the normalization condition

$$\int_0^{\pi}\sin\chi\,d\chi\int_0^{2\pi}d\varphi\,\Psi_{NM}\Psi^*_{NM}=\delta_{NN'}\delta_{MM'}\,.$$

Thus, by reduction from the two-dimensional quantum oscillator on the complex sphere we have constructed the wave function and energy spectrum for the Coulomb problem on the two-dimensional real sphere S_2. Formula (54) for Coulomb wave functions on the two-dimensional sphere is new.

Now let us consider the flat space contraction. In the contraction limit $R\to\infty$ the energy spectrum for finite N goes to the discrete energy spectrum of the two-dimensional hydrogen atom[8,9]

$$\lim_{R\to\infty}E_N(R)=-\frac{\mu^2}{2(N+\frac{1}{2})^2},\quad N=0,1,\ldots\,.$$

In the limit $R\to\infty$, putting $\tan\chi\sim\chi\sim r/R$, where r is the radius-vector in the two-dimensional tangent plane and using the asymptotic formulas[34]

$$\lim_{\substack{R\to\infty\\ \chi\to0}}{}_2F_1\left(-N+|M|,|M|+i\sigma+\frac{1}{2};2|M|+1;1-e^{2i\chi}\right)={}_1F_1\left(-N+|M|,2|M|+1;\frac{2\mu r}{N+\frac{1}{2}}\right),$$

$$\lim_{|y|\to\infty}|\Gamma(x+iy)|e^{(\pi/2)y}|y|^{1/2-x}=\sqrt{2\pi},\quad \lim_{z\to\infty}\frac{\Gamma(x+\alpha)}{\Gamma(x+\beta)}=z^{\alpha-\beta},\tag{56}$$

we obtain the well-known Coulomb wave function with correct normalization factor[9]

$$\lim_{\substack{R\to\infty\\ \chi\to0}}\Psi_{NM}(\chi,\varphi)=\frac{\mu\sqrt{2}}{(N+\frac{1}{2})^{3/2}}\sqrt{\frac{(N+|M|)!}{N-|M|)!}}\left(\frac{2\mu r}{N+\frac{1}{2}}\right)^{|M|}\frac{\exp[-\mu r/(N+1/2)]}{(2|M|)!}$$
$$\times{}_1F_1\left(-N+|M|,2|M|+1;\frac{2\mu r}{N+\frac{1}{2}}\right)\frac{e^{iM\varphi}}{\sqrt{2\pi}}.\tag{57}$$

In the case for large R and N such that $N\sim kR$, (where k is constant) we obtain the formula for continuous spectrum: $E=k^2/2$. Now taking into account that $\sigma\sim\mu/k$ and using the asymptotic relation (56), we have

$$\lim_{\substack{R\to\infty\\ \chi\to 0}} \sqrt{R}\Psi_{NM}(\chi,\varphi)=\sqrt{\frac{k}{\pi}}e^{\pi\mu/2k}|\Gamma(|M|+1/2+i\mu/k)|\frac{(2kr)^{|M|}}{(2|M|)!}e^{-ikr}$$

$$\times {}_1F_1\left(|M|+\frac{i\mu}{k}+\frac{1}{2};2|M|+1;2ikr\right)\frac{e^{iM\varphi}}{\sqrt{2\pi}}, \tag{58}$$

which coincides with the formula for the two-dimensional Coulomb scattering wave function in polar coordinates.[35]

III. THE THREE- AND FIVE-DIMENSIONAL KEPLER–COULOMB PROBLEMS

In complete analogy with the three- and five-dimensional Euclidean case, the corresponding regularizing transformations exist for the Kepler and Coulomb problems in spheres of dimension three and five. Indeed if we consider motion on the sphere of dimension n then the classical equations of motion in the presence of a potential are just (23), (24) again, where now

$$\mathbf{s}=(s_1,\ldots,s_{n+1}), \tag{59}$$

subject to the constraints

$$\mathbf{s}\cdot\mathbf{s}=R^2 \tag{60}$$

and its differential consequences

$$\mathbf{s}\cdot\dot{\mathbf{s}}=0, \quad \mathbf{s}\cdot\ddot{\mathbf{s}}+\dot{\mathbf{s}}\cdot\dot{\mathbf{s}}=0.$$

If we choose our potential to be

$$V=-\frac{\mu}{R}\frac{s_{n+1}}{\sqrt{s_1^2+\cdots+{}^2_n}}, \tag{61}$$

these equations assume the form

$$\frac{d^2}{dt^2}s_j=-s_j\dot{\mathbf{s}}\cdot\dot{\mathbf{s}}-\frac{\mu}{R}\frac{s_j s_{n+1}}{(\mathbf{s}\cdot\mathbf{s})^{3/2}}, \quad j=1,\ldots,n, \tag{62}$$

$$\frac{d^2}{dt^2}s_{n+1}=-s_{n+1}\dot{\mathbf{s}}\cdot\dot{\mathbf{s}}+\frac{\mu}{R(\mathbf{s}\cdot\mathbf{s})^{1/2}}. \tag{63}$$

The energy integral again has the form (26).

We are particularly interested in dimensions $n=3,5$. We deal with each of these cases separately.

A. Generalized KS transformation

For $n=3$ we choose the u_j coordinates in five-dimensional space according to

$$\begin{aligned} s_1 &= i\sqrt{u_1^2+u_2^2+u_3^2+u_4^2+u_5^2}\cdot\frac{u_1u_3+u_2u_4}{u_5},\\ s_2 &= i\sqrt{u_1^2+u_2^2+u_3^2+u_4^2+u_5^2}\cdot\frac{u_2u_3-u_1u_4}{u_5},\\ s_3 &= i\sqrt{u_1^2+u_2^2+u_3^2+u_4^2+u_5^2}\cdot\frac{u_1^2+u_2^2-u_3^2-u_4^2}{2u_5},\\ s_4 &= \sqrt{u_1^2+u_2^2+u_3^2+u_4^2+u_5^2}\cdot\left(u_5+\frac{u_1^2+u_2^2+u_3^2+u_4^2}{2u_5}\right). \end{aligned} \tag{64}$$

The basic identity is

$$s_1^2+s_2^2+s_3^2+s_4^2=(u_1^2+u_2^2+u_3^2+u_4^2+u_5^2)^2,$$

and the basic relationship for the infinitesimal distances is

$$\begin{aligned} ds_1^2+ds_2^2+ds_3^2+ds_4^2=-\frac{D^2}{u_5^2}\{&(u_1^2+u_2^2+u_3^2+u_4^2)[du_1^2+du_2^2+du_3^2+du_4^2+du_5^2]\\ &+(u_4\,du_3-u_3\,du_4+u_2\,du_1-u_1\,du_2)^2\}, \end{aligned} \tag{65}$$

where the constraint for mapping between the three-sphere: $\Sigma_{i=1}^4 s_i^2=R^2$ and the complex four-sphere: $\Sigma_{i=1}^5 u_i^2=D^2$ is clearly

$$u_4\,du_3-u_3\,du_4+u_2\,du_1-u_1\,du_2=0. \tag{66}$$

In this section we will use the Eulerian spherical coordinates on the complex four-sphere S_{4C},

$$\begin{aligned} u_1 &= D\sqrt{1-e^{2i\chi}}\cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2},\quad u_2=D\sqrt{1-e^{2i\chi}}\cos\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2},\\ u_3 &= D\sqrt{1-e^{2i\chi}}\sin\frac{\beta}{2}\cos\frac{\alpha-\gamma}{2},\quad u_4=D\sqrt{1-e^{2i\chi}}\sin\frac{\beta}{2}\sin\frac{\alpha-\gamma}{2},\\ u_5 &= De^{i\chi}, \end{aligned} \tag{67}$$

where the ranges of the variables are given by

$$0\leqslant\chi\leqslant\pi,\quad 0\leqslant\beta\leqslant\pi,\quad 0\leqslant\alpha<2\pi,\quad 0\leqslant\gamma<4\pi.$$

The corresponding spherical coordinates on S_3 are

$$s_1=R\sin\chi\sin\beta\cos\alpha,\quad s_2=R\sin\chi\sin\beta\sin\alpha,$$

$$s_3=R\sin\chi\cos\beta,\quad s_4=R\cos\chi.$$

1. Classical motion

In analogy with our previous analysis we choose a new variable τ according to

$$\frac{d\tau}{dt}=\frac{1}{D^2}\frac{u_5^2}{u_1^2+u_2^2+u_3^2+u_4^2}.$$

In the u coordinates the equations of motion can be written as

$$(u_1')^2+(u_2')^2+(u_3')^2+(u_4')^2+(u_5')^2-2D^2\left(E+\frac{i\mu}{D^2}\right)+\frac{2D^4}{u_5^2}\left(E-\frac{i\mu}{D^2}\right)=0,$$

$$u_j''+2\left(E+\frac{i\mu}{D^2}\right)u_j=0,\quad j=1,2,3,4, \tag{68}$$

$$u_5''+2\left(E+\frac{i\mu}{D^2}\right)u_5-\frac{2D^4}{u_5^3}\left(E-\frac{i\mu}{D^2}\right)=0,$$

subject to the constraints

$$\sum_{k=1}^{5}u_k^2=D^2,\quad \sum_{k=1}^{5}u_ku_k'=0,$$

$$\sum_{k=1}^{5}(u_ku_k''+(u_k')^2)=0,\quad u_4u_3'-u_3u_4'+u_2u_1'-u_1u_2'=0.$$

Note that Eq. (68) is compatible with these constraints. Here, the Kepler problem on the sphere in three dimensions is equivalent to choosing a Hamiltonian

$$H=\frac{1}{2}(p_{u_1}^2+p_{u_2}^2+p_{u_3}^2+p_{u_4}^2+p_{u_5}^2)-\left(E+\frac{i\mu}{D^2}\right)(u_1^2+u_2^2+u_3^2+u_4^2+u_5^2)+\frac{D^4}{u_5^2}\left(E-\frac{i\mu}{D^2}\right), \tag{69}$$

regarding the variables u_j as independent and τ as time. The only difference is that there is now the constraint

$$u_4p_{u_3}-u_3p_{u_4}+u_2p_{u_1}-u_1p_{u_2}=0.$$

In terms of the Hamilton–Jacobi formulation we have the relation

$$\begin{aligned}&\frac{1}{2}(p_{s_1}^2+p_{s_2}^2+p_{s_3}^2+p_{s_4}^2)-\frac{\mu}{R}\frac{s_4}{\sqrt{s_1^2+s_2^2+s_3^2}}-E\\&\quad=-\frac{u_5^2}{u_1^2+u_2^2+u_3^2+u_4^2}\left[\frac{1}{2D^2}(p_{u_1}^2+p_{u_2}^2+p_{u_3}^2+p_{u_4}^2+p_{u_5}^2)\right.\\&\qquad\left.-\left(E+\frac{i\mu}{D^2}\right)(u_1^2+u_2^2+u_3^2+u_4^2+u_5^2)+\frac{D^2}{u_5^2}\left(E-\frac{i\mu}{D^2}\right)\right]=0.\end{aligned}$$

With the usual substitutions, the corresponding Hamilton–Jacobi equations are

$$\sum_{k=1}^{4}\left(\frac{\partial S}{\partial s_k}\right)^2-\left(2E+\frac{2\mu}{R}\frac{s_4}{\sqrt{s_1^2+s_2^2+s_3^2}}\right)=0, \tag{70}$$

or

$$\frac{1}{2D^2}\sum_{k=1}^{5}\left(\frac{\partial S}{\partial u_k}\right)^2+\left[\frac{D^2}{u_5^2}\left(E-\frac{i\mu}{D^2}\right)-\left(E+\frac{i\mu}{D^2}\right)\right]=0, \tag{71}$$

and the constraint has become

$$\mathcal{L}\cdot S=0, \tag{72}$$

where operator $\mathcal{L}$ is

$$\mathcal{L}=u_2\frac{\partial}{\partial u_1}-u_1\frac{\partial}{\partial u_2}+u_4\frac{\partial}{\partial u_3}-u_3\frac{\partial}{\partial u_4}. \tag{73}$$

Equation (71) can be solved by separation of variables in the spherical coordinates (67) on the complex sphere S_{4C}.

2. Quantum motion

The associated quantum Kepler–Coulomb problem on the sphere corresponding to the potential (61),

$$\frac{1}{2}\Delta_s^{(3)}\Psi+\left(E+\frac{\mu}{R}\frac{s_4}{\sqrt{s_1^2+s_2^2+s_3^2}}\right)\Psi=0, \tag{74}$$

translates directly to [see formula (A10)]

$$\frac{1}{2}\Delta_u^{(4)}\Phi+\left(\mathcal{E}-\frac{\omega^2D^2}{2}\frac{u_1^2+u_2^2+u_3^2+u_4^2}{u_5^2}\right)\Phi=0 \tag{75}$$

with the constraint

$$\mathcal{L}\cdot\Phi=0, \tag{76}$$

where $\mathcal{L}$ is given by (73),

$$\Psi=u_5^{1/2}\Phi, \tag{77}$$

and

$$\mathcal{E}=2i\mu-\frac{1}{D^2},\quad \omega^2D^2=2ED^2-2i\mu+\frac{3}{4D^2}. \tag{78}$$

Here $\Delta_s^{(3)}$ and $\Delta_u^{(4)}$ are Laplace–Beltrami operators on the spheres S_3 and S_{4C}, respectively.

Consider the Schrödinger equation (75) in complex spherical coordinates (67). We have

$$\frac{e^{-i\chi}}{\sin^2\chi}\frac{\partial}{\partial\chi}e^{i\chi}\sin^2\chi\frac{\partial\Phi}{\partial\chi}+\left[\omega^2D^4-i\mathcal{E}D^2\frac{e^{i\chi}}{\sin\chi}+\frac{\vec{L}^2}{\sin^2\chi}\right]\Phi=0, \tag{79}$$

where the operator $\vec{L}^2$ is defined in (A8). We complexify the angle χ to the domain $\mathbf{G}$ (see Fig. 1) by the transformation (39), such that $\vartheta\in[0,\pi/2]$ and also complexify μ by setting $k=i\mu$ in expression for $\mathcal{E}$ and ω^2. Then Eq. (79) transforms to the Schrödinger equation for the oscillator problem on real sphere S_4.

We make the ansatz

$$\Phi(\vartheta,\alpha,\beta,\gamma)=(\sin\vartheta)^{-3/2}Z(\vartheta)\mathcal{D}^{\ell}_{m_1,m_2}(\alpha,\beta,\gamma), \tag{80}$$

where

$$\mathcal{D}^{\ell}_{m_1,m_2}(\alpha,\beta,\gamma)=e^{im_1\alpha}d^{\ell}_{m_1,m_2}(\beta)\,e^{im_2\gamma} \tag{81}$$

is the Wigner function,[36] satisfying the eigenvalue equation

$$\vec{L}^2\mathcal{D}^{\ell}_{m_1,m_2}(\alpha,\beta,\gamma)=\ell(\ell+1)\mathcal{D}^{\ell}_{m_1,m_2}(\alpha,\beta,\gamma), \tag{82}$$

and normalization condition

$$\int \mathcal{D}^{\ell'*}_{m_1',m_2'}(\alpha,\beta,\gamma)\mathcal{D}^{\ell}_{m_1,m_2}(\alpha,\beta,\gamma)\frac{1}{8}\sin\beta d\beta\, d\alpha\, d\gamma=\frac{2\pi^2}{2\ell+1}\delta_{\ell\ell'}\delta_{m_1m_1'}\delta_{m_2m_2'}. \tag{83}$$

Then the function $Z(\vartheta)$ satisfies

$$\frac{d^2Z}{d\vartheta^2}+\left[\left(2\mathcal{E}D^2+\omega^2D^4+\frac{9}{4}\right)-\frac{\omega^2D^4}{\cos^2\vartheta}-\frac{(2\ell+1)^2-\frac{1}{4}}{\sin^2\vartheta}\right]Z=0. \tag{84}$$

The corresponding solution regular at $\vartheta=0,\pi/2$ and energy spectrum are given by

$$Z_{n_r\ell}(\vartheta)=\text{const}(\sin\vartheta)^{2\ell}(\cos\vartheta)^{\nu+1/2}{}_2F_1(-n_r,n_r+2\ell+\nu+2;2\ell+2;\sin^2\vartheta), \tag{85}$$

$$\mathcal{E}=\frac{1}{2D^2}[(n+1)(n+4)+(2\nu-1)(n+2)], \tag{86}$$

where $\nu=(\omega^2D^4+\frac{1}{4})^{1/2}$, $n=2n_r+2\ell=0,1,2,...$ is the principal quantum number. The other quantum numbers are

$$n_r=0,1,...,n,\quad 2\ell=0,1,\ldots,n,\quad m_1,m_2=-\ell,-\ell+1,\ldots,\ell-1,\ell.$$

Thus the wave function $\Phi(\vartheta,\alpha,\beta,\gamma)$ normalized under the condition (see the Appendix)

$$-\frac{iD^2}{2\pi}\int_{S_{4C}}\Phi_{n_r\ell m_1m_2}\Phi^{\diamond}_{n_r\ell m_1m_2}(u_1^2+u_2^2+u_3^2+u_4^2)\frac{dv(u)}{u_5^2}=1 \tag{87}$$

has the form

$$\Phi_{n_r\ell m_1m_2}(\vartheta,\alpha,\beta,\gamma)=C_{n_r\ell}(\nu)\sqrt{\frac{2\ell+1}{2\pi^2}}R_{n_r\ell}(\vartheta)\mathcal{D}^{\ell}_{m_1,m_2}(\alpha,\beta,\gamma) \tag{88}$$

with

$$R_{n_r\ell}(\vartheta)=(\sin\vartheta)^{2\ell}(\cos\vartheta)^{\nu+1/2}{}_2F_1(-n_r,n_r+2\ell+\nu+2;2\ell+2;\sin^2\vartheta), \tag{89}$$

$$C_{n_r\ell}(\nu)=\frac{\sqrt{\pi}}{D^{7/2}}\sqrt{\frac{[(-i\nu)(\nu+2\ell+2n_r+2)](2\ell+n_r+1)!\Gamma(2\ell+\nu+n_r+2)}{(1-e^{2i\pi\nu})(\ell+n_r+1)[(2\ell+1)!]^2(n_r)!\Gamma(\nu+n_r+1)}}. \tag{90}$$

We now construct the wave function and energy spectrum for the Schrödinger equation (74). The corresponding wave function $\Psi(s)$ connecting with $\Phi(u)$ by formula (77) is independent of the variable γ and 2π periodic in α (the transformation $\alpha\to\alpha+2\pi$ is equivalent to the inversion $u_i\to-u_i$, $i=1,2,3,4$). The constraint (76) in the spherical coordinate (67) is equivalent to

$$\frac{\partial}{\partial\gamma}\cdot\Phi_{n_r\ell m_1m_2}(\vartheta,\alpha,\beta,\gamma)=m_2\Phi_{n_r\ell m_1m_2}(\vartheta,\alpha,\beta,\gamma)=0$$

and we have $m_2=0$. From 2π periodicity we get that ℓ and m_1 are integers. Then, upon introducing the principal quantum number $N=(n_r+\ell)+1=n/2+1$ and using expression (78), we obtain the energy spectrum of the reduced system

$$E=\frac{N^2-1}{2R^2}-\frac{\mu^2}{2N^2},\quad N=1,2,\ldots, \tag{91}$$

where $k=i\mu$. This spectrum coincides with that obtained from other methods.[20,24,23]

Returning from ϑ to the variable χ, observing that

$$\nu=i\sigma-N,\quad \sigma=\frac{\mu R}{N},$$

and using the relations ($m_1\equiv m$)

$$\mathcal{D}^{\ell}_{m,0}(\alpha,\beta,\gamma)=(-1)^m\sqrt{\frac{4\pi}{2\ell+1}}Y_{\ell m}(\beta,\alpha), \tag{92}$$

we obtain the wave functions (with correct normalization) for the reduced system in the form

$$\Psi_{N\ell m}(\chi,\beta,\alpha)=\sqrt{D}\,e^{i\chi/2}\Phi_{n_r\ell m0}(\chi,\alpha,\beta,\gamma)=\frac{(-1)^m}{\sqrt{R^3}}\mathrm{C}_{n_r\ell}(\sigma)\,(\sin\alpha)^{\ell}\,\mathrm{e}^{-i\alpha(N-\ell-i\sigma)}$$
$$\times\,{}_2F_1(-N+\ell+1,1+\ell+i\sigma;2\ell+2;1-\mathrm{e}^{-2i\chi})\,Y_{lm}(\beta,\alpha), \tag{93}$$

where

$$\mathrm{C}_{n_r\ell}(\sigma)=\frac{2^{\ell+1}\mathrm{e}^{\pi\sigma/2}}{(2\ell+1)!}\sqrt{\frac{(N^2+\sigma^2)(N+\ell)!}{2\pi N(N-\ell-1)!}}\,|\Gamma(1+\ell+i\sigma)|.$$

This solution is identical to that given for the Coulomb eigenfunction on S_3 in Refs. 23 and 25. Note that in Ref. 25 it already has been shown that the function (93) contracts as $R\to\infty$ into the flat space Coulomb wave function for discrete and continuous energy spectrum.

B. Generalized Hurwitz transformation

The analogous problem in five dimensions can be realized via the variables

$$\begin{aligned}
s_1&=\left(\sum_{k=1}^{9}u_k^2\right)^{1/2}\frac{i}{u_9}(u_1u_5+u_2u_6-u_3u_7-u_4u_8),\\
s_2&=\left(\sum_{k=1}^{9}u_k^2\right)^{1/2}\frac{i}{u_9}(u_1u_6-u_2u_5+u_3u_8-u_4u_7),\\
s_3&=\left(\sum_{k=1}^{9}u_k^2\right)^{1/2}\frac{i}{u_9}(u_1u_7+u_2u_8+u_3u_5+u_4u_6),\\
s_4&=\left(\sum_{k=1}^{9}u_k^2\right)^{1/2}\frac{i}{u_9}(u_1u_8-u_2u_7-u_3u_6+u_4u_5),
\end{aligned} \tag{94}$$

$$s_5=\left(\sum_{k=1}^{9} u_k^2\right)^{1/2}\frac{i}{2u_9}(u_1^2+u_2^2+u_3^2+u_4^2-u_5^2-u_6^2-u_7^2-u_8^2),$$

$$s_6=\left(\sum_{k=1}^{9} u_k^2\right)^{1/2}\left(u_9+\frac{1}{2u_9}\sum_{k=1}^{8} u_k^2\right),$$

which satisfy

$$\sum_{j=1}^{6} s_j^2=\left(\sum_{\not=1}^{9} u_{\not}^2\right)^2. \tag{95}$$

The relation between the infinitesimal distances on the five-dimensional sphere S_5: $\Sigma_{i=1}^{6} s_i^2=R^2$ and the eight-dimensional complex sphere S_{8C}: $\Sigma_{i=1}^{9} u_i^2=D^2$, $(R=D^2)$ is

$$\frac{1}{R}\sum_{j=1}^{6} ds_j^2=\frac{-1}{u_9^2}\left[\left(\sum_{k=1}^{8} u_k^2\right)\sum_{\not=1}^{9} du_{\not}^2+\omega_1^2+\omega_2^2+\omega_3^2\right], \tag{96}$$

where

$$\omega_1=u_4\,du_1+u_3\,du_2-u_2\,du_3-u_1\,du_4-u_8\,du_5-u_7\,du_6+u_6\,du_7+u_5\,du_8,$$

$$\omega_2=u_3\,du_1-u_4\,du_2-u_1\,du_3+u_2\,du_4-u_7\,du_5+u_8\,du_6+u_5\,du_7-u_6\,du_8,$$

$$\omega_3=u_2\,du_1-u_1\,du_2+u_4\,du_3-u_3\,du_4+u_6\,du_5-u_5\,du_6+u_8\,du_7-u_7\,du_8,$$

and the constraint for mapping $S_{8C}\to S_5$ corresponds to

$$\omega_i=0,\quad i=1,2,3.$$

Following Ref. 16 (see also Ref. 18) we can supplement the transformation (94) with the angles

$$\alpha_H=\frac{1}{2}\left[\arctan\frac{2u_1u_2}{u_1^2-u_2^2}+\arctan\frac{2u_3u_4}{u_3^2-u_4^2}\right]\in[0,2\pi),$$

$$\beta_H=2\arctan\left(\frac{u_3^2+u_4^2}{u_1^2+u_2^2}\right)^{1/2}\in[0,\pi], \tag{97}$$

$$\gamma_H=\frac{1}{2}\left[\arctan\frac{2u_1u_2}{u_1^2-u_2^2}-\arctan\frac{2u_3u_4}{u_3^2-u_4^2}\right]\in[0,4\pi).$$

The transformations (94) and (97) correspond to $S_{8C}\to S_8=S_5\otimes S_3$. If we now choose the spherical coordinates on S_5 as

$$s_1+is_2=R\sin\chi\sin\vartheta\cos\frac{\beta}{2}e^{i(\alpha+\gamma)/2},\quad s_5=R\sin\chi\cos\vartheta,$$

$$s_3+is_4=R\sin\chi\sin\vartheta\sin\frac{\beta}{2}e^{i(\alpha-\gamma)/2},\quad s_6=R\cos\chi.$$

then the corresponding (nonorthogonal) spherical coordinates on the eight-dimensional complex sphere take the form $(D^2=R)$

$$u_1 = D\sqrt{1-e^{2i\chi}}\cos\frac{\vartheta}{2}\cos\frac{\beta_H}{2}\cos\frac{\alpha_H+\gamma_H}{2},$$

$$u_2 = D\sqrt{1-e^{2i\chi}}\cos\frac{\vartheta}{2}\cos\frac{\beta_H}{2}\sin\frac{\alpha_H+\gamma_H}{2},$$

$$u_3 = D\sqrt{1-e^{2i\chi}}\cos\frac{\vartheta}{2}\sin\frac{\beta_H}{2}\cos\frac{\alpha_H-\gamma_H}{2},$$

$$u_4 = D\sqrt{1-e^{2i\chi}}\cos\frac{\vartheta}{2}\sin\frac{\beta_H}{2}\sin\frac{\alpha_H-\gamma_H}{2},$$

$$u_5 = D\sqrt{1-e^{2i\chi}}\sin\frac{\vartheta}{2}\left(\cos\frac{\beta}{2}\cos\frac{\beta_H}{2}\cos\frac{\alpha+\gamma+\alpha_H+\gamma_H}{2}+\sin\frac{\beta}{2}\sin\frac{\beta_H}{2}\cos\frac{\alpha-\gamma-\alpha_H+\gamma_H}{2}\right), \tag{98}$$

$$u_6 = D\sqrt{1-e^{2i\chi}}\sin\frac{\vartheta}{2}\left(\cos\frac{\beta}{2}\cos\frac{\beta_H}{2}\sin\frac{\alpha+\gamma+\alpha_H+\gamma_H}{2}-\sin\frac{\beta}{2}\sin\frac{\beta_H}{2}\sin\frac{\alpha-\gamma-\alpha_H+\gamma_H}{2}\right),$$

$$u_7 = D\sqrt{1-e^{2i\chi}}\sin\frac{\vartheta}{2}\left(\sin\frac{\beta}{2}\cos\frac{\beta_H}{2}\cos\frac{\alpha-\gamma+\alpha_H+\gamma_H}{2}-\cos\frac{\beta}{2}\sin\frac{\beta_H}{2}\cos\frac{\alpha+\gamma-\alpha_H+\gamma_H}{2}\right),$$

$$u_8 = D\sqrt{1-e^{2i\chi}}\sin\frac{\vartheta}{2}\left(\sin\frac{\beta}{2}\cos\frac{\beta_H}{2}\sin\frac{\alpha-\gamma+\alpha_H+\gamma_H}{2}+\cos\frac{\beta}{2}\sin\frac{\beta_H}{2}\sin\frac{\alpha+\gamma-\alpha_H+\gamma_H}{2}\right),$$

$$u_9 = De^{i\chi},$$

where $\chi\in[0,\pi]$, $\vartheta\in[0,\pi]$, $\alpha\in[0,2\pi]$, $\beta\in[0,\pi]$, and $\gamma\in[0,4\pi]$.

1. Classical motion

The Kepler–Coulomb potential on the five-dimensional sphere S_5 has the form

$$V = -\frac{\mu}{R}\frac{s_6}{\sqrt{s_1^2+s_2^2+s_3^2+s_4^2+s_5^2}}. \tag{99}$$

As before we can define a new coordinate τ such that

$$\frac{d\tau}{dt} = \frac{1}{D^2}\frac{u_9^2}{\Sigma_{k=1}^8 u_k^2}.$$

The corresponding equations of motion are given by

$$\sum_{\ell=1}^{9}(u_\ell')^2 - 2\left(E+\frac{i\mu}{D^2}\right) - \frac{2D^2}{u_9^2}\left(E-\frac{i\mu}{D^2}\right) = 0,$$

$$u_k'' + 2\left(E+\frac{i\mu}{D^2}\right) = 0, \quad k=1,\ldots,8, \tag{100}$$

$$u_9'' + 2\left(E+\frac{i\mu}{D^2}\right)u_9 - \frac{2D^2}{u_9^3}\left(E-\frac{i\mu}{D^2}\right) = 0,$$

subject to the constraints

$$\sum_{\ell=1}^{9} u_\ell^2 = D^2, \quad \sum_{\ell=1}^{9} u_\ell u_\ell' = 0, \quad \sum_{\ell=1}^{9} (u_\ell u_\ell'' + (u_\ell')^2) = 0,$$

$$u_4u_1' + u_3u_2' - u_2u_3' - u_1u_4' - u_8u_5' - u_7u_6' + u_6u_7' + u_5u_8' = 0,$$

$$u_3u_1' - u_4u_2' - u_1u_3' + u_2u_4' - u_7u_5' + u_8u_6' + u_5u_7' - u_6u_8' = 0,$$

$$u_2u_1' - u_1u_2' + u_4u_3' - u_3u_4' + u_6u_5' - u_5u_6' + u_8u_7' - u_7u_8' = 0.$$

These equations of motion are equivalent to what we would obtain by choosing the Hamiltonian

$$H = \frac{1}{2}\sum_{\ell=1}^{9} p_{u_\ell}^2 - \left(E + \frac{i\mu}{D^2}\right)\sum_{\ell=1}^{9} u_\ell^2 + \frac{D^2}{u_9^2}\left(E - \frac{i\mu}{D^2}\right), \tag{101}$$

regarding the variables u_i as independent and using τ as time. The associated constraints are

$$u_4p_1 + u_3p_2 - u_2p_3 - u_1p_4 - u_8p_5 - u_7p_6 + u_6p_7 + u_5p_8 = 0,$$

$$u_3p_1 - u_4p_2 - u_1p_3 + u_2p_4 - u_7p_5 + u_8p_6 + u_5p_7 - u_6p_8 = 0, \tag{102}$$

$$u_2p_1 - u_1p_2 + u_4p_3 - u_3p_4 + u_6p_5 - u_5p_6 + u_8p_7 - u_7p_8 = 0.$$

If we wish to solve this problem from the point of view of the Hamilton–Jacobi equation we use the relation

$$\frac{1}{2}\sum_{j=1}^{6} p_{s_j}^2 - \frac{\mu}{R^2}\frac{s_6}{\sqrt{s_1^2+s_2^2+s_3^2+s_4^2+s_5^2}} - E$$

$$= -\frac{u_9^2}{\Sigma_{k=1}^{8} u_k^2}\left\{\frac{1}{2D^2}\sum_{\ell=1}^{9} p_{u_\ell}^2 - \left(\frac{i\mu}{D^2} + E\right)\sum_{\ell=1}^{9} u_\ell^2 + \frac{D^2}{u_9^2}\left(E - \frac{i\mu}{D^2}\right)\right\} = 0.$$

The corresponding Hamilton–Jacobi equations are

$$\frac{1}{2}\sum_{j=1}^{6}\left(\frac{\partial S}{\partial s_j}\right)^2 - \frac{\mu}{R^2}\frac{s_6}{\sqrt{s_1^2+s_2^2+s_3^2+s_4^2+s_5^2}} - E = 0, \tag{103}$$

$$\frac{1}{2D^2}\sum_{\ell=1}^{9}\left(\frac{\partial S}{\partial u_\ell}\right)^2 - \left(\frac{i\mu}{D^2} + E\right)\sum_{\ell=1}^{9} u_\ell^2 + \frac{D^2}{u_9^2}\left(E - \frac{i\mu}{D^2}\right) = 0, \tag{104}$$

subject to the constraints

$$u_4\frac{\partial S}{\partial u_1} + u_3\frac{\partial S}{\partial u_2} - u_2\frac{\partial S}{\partial u_3} - u_1\frac{\partial S}{\partial u_4} - u_8\frac{\partial S}{\partial u_5} - u_7\frac{\partial S}{\partial u_6} + u_6\frac{\partial S}{\partial u_7} + u_5\frac{\partial S}{\partial u_8} = 0,$$

$$u_3\frac{\partial S}{\partial u_1} - u_4\frac{\partial S}{\partial u_2} - u_1\frac{\partial S}{\partial u_3} + u_2\frac{\partial S}{\partial u_4} - u_7\frac{\partial S}{\partial u_5} + u_8\frac{\partial S}{\partial u_6} + u_5\frac{\partial S}{\partial u_7} - u_6\frac{\partial S}{\partial u_8} = 0,$$

$$u_2\frac{\partial S}{\partial u_1}-u_1\frac{\partial S}{\partial u_2}+u_4\frac{\partial S}{\partial u_3}-u_3\frac{\partial S}{\partial u_4}+u_6\frac{\partial S}{\partial u_5}-u_5\frac{\partial S}{\partial u_6}+u_8\frac{\partial S}{\partial u_7}-u_7\frac{\partial S}{\partial u_8}=0.$$

2. Quantum motion

The Schrödinger equation for the five-dimensional quantum Coulomb problem

$$\frac{1}{2}\Delta_s^{(5)}\Psi+\left(E+\frac{\mu}{R}\frac{s_6}{\sqrt{s_1^2+s_2^2+s_3^2+s_4^2+s_5^2}}\right)\Psi=0 \tag{105}$$

transforms to the eight-dimensional oscillator equation (see the Appendix)

$$\frac{1}{2}\Delta_u^{(8)}\Phi+\left(\mathcal{E}-\frac{\omega^2D^2}{2}\frac{1}{u_9^2}\sum_{i=1}^{8}u_i^2\right)\Phi=0 \tag{106}$$

with constraints

$$T_i\Phi=0, \tag{107}$$

where the operator $\vec{T}$ is given by formula (A13),

$$\mathcal{E}=\left(2i\mu-\frac{6}{D^2}\right),\quad \omega^2D^2=2\left(D^2E-2i\mu+\frac{15}{8D^2}\right), \tag{108}$$

and

$$\Psi=(u_9)^{3/2}\Phi. \tag{109}$$

Considering the oscillator equation (106) in complex spherical coordinates (98) we get (see the Appendix)

$$\frac{e^{-3i\chi}}{\sin^4\chi}\frac{\partial}{\partial\chi}e^{3i\chi}\sin^4\chi\frac{\partial\Phi}{\partial\chi}+\left[\omega^2D^4-i\mathcal{E}D^2\frac{e^{i\chi}}{\sin\chi}+\frac{\vec{M}^2}{\sin^2\chi}\right]\Phi=0, \tag{110}$$

where the operator $\vec{M}^2$ has the form

$$\vec{M}^2=\frac{1}{\sin^3\theta}\frac{\partial}{\partial\theta}\sin^3\theta\frac{\partial}{\partial\theta}-\frac{\vec{L}^2}{\sin^2\frac{\theta}{2}}-\frac{\vec{J}^2}{\cos^2\frac{\theta}{2}}, \tag{111}$$

and

$$\vec{J}=\vec{L}+\vec{T},\quad \vec{J}^2=\vec{L}^2+\vec{T}^2+2\vec{L}\cdot\vec{T}. \tag{112}$$

As before, we make the complex transformation (39) and also complexify parameter μ by putting $k=i\mu$. We make the separation ansatz[18]

$$\Phi=R(\vartheta)Z(\theta)G(\alpha,\beta,\gamma;\alpha_H,\beta_H,\gamma_H), \tag{113}$$

where G is an eigenfunction of operators $\vec{L}^2$, $\vec{T}^2$, and $\vec{J}^2$ with eigenvalues $L(L+1)$, $T(T+1)$, $J(J+1)$, respectively. Correspondingly the wave function $Z(\theta)$ is the eigenfunction of operator $\vec{M}^2$ with eigenvalue $\lambda(\lambda+3)$. Because there is $\vec{L}\cdot\vec{T}$ interaction the eigenvalue equation

$$\vec{J}^2G(\alpha,\beta,\gamma;\alpha_H,\beta_H,\gamma_H)=J(J+1)G(\alpha,\beta,\gamma;\alpha_H,\beta_H,\gamma_H) \tag{114}$$

cannot be separated in variables $(\alpha,\beta,\gamma;\alpha_H,\beta_H,\gamma_H)$ but we can apply the rules for the addition of angular momenta $\vec{L}$ and $\vec{T}$ and, following Ref. 18 express G as a Clebsch–Gordan expansion

$$G^{JM}_{Lm;Tt}=\sum_{M=m'+t'}(J,M|L,m';T,t')\mathcal{D}^{L}_{m,m'}(\alpha,\beta,\gamma)\mathcal{D}^{T}_{t,t'}(\alpha_H,\beta_H,\gamma_H), \tag{115}$$

where $(JM|Lm;Tt)$ are the Clebsch–Gordan coefficients. Note that the functions $G^{JM}_{Lm;Tt}$ satisfy the normalization condition

$$\int_{\Omega}d\Omega\int_{\Omega_H}d\Omega_H G^{JM}_{Lm;Tt}G^{J'M'*}_{L'm';T't'}=\left(\frac{2\pi^2}{2L+1}\right)\left(\frac{2\pi^2}{2T+1}\right)\delta_{JJ'}\delta_{LL'}\delta_{TT'}\delta_{MM'}\delta_{mm'}\delta_{tt'}. \tag{116}$$

If we substitute ansatz (113) into the Schrödinger equation (110), then after separation of variables we obtain the differential equations

$$\frac{1}{\sin^3\theta}\frac{d}{d\theta}\sin^3\theta\frac{dZ}{d\theta}+\left[\lambda(\lambda+3)-\frac{2L(L+1)}{1-\cos\theta}-\frac{2J(J+1)}{1+\cos\theta}\right]Z=0, \tag{117}$$

$$\frac{1}{\sin^7\vartheta}\frac{d}{d\vartheta}\sin^7\theta\frac{dR}{d\vartheta}+\left[(2D^2\mathcal{E}+\omega^2D^4)-\frac{4\lambda(\lambda+3)}{\sin^2\vartheta}-\frac{\omega^2D^4}{\cos^2\vartheta}\right]=0, \tag{118}$$

with real parameters

$$\mathcal{E}=\left(2k-\frac{6}{D^2}\right),\quad \omega^2D^2=2\left(D^2E-2k+\frac{15}{8D^2}\right). \tag{119}$$

Consider Eq. (117). Taking the new function by $v(\theta)=(\sin\theta)^{3/2}Z(\theta)$ we obtain the Pöschl–Teller equation. Then the solution $Z(\theta)\equiv Z^{JL}_{\lambda}(\theta)$ orthonormalized by the condition

$$\int_0^{\pi}Z^{JL}_{\lambda}(\theta)Z^{JL}_{\lambda'}{}^{*}(\theta)\sin^3\theta\,d\theta=\delta_{\lambda\lambda'} \tag{120}$$

has the form

$$Z^{JL}_{\lambda}(\theta)=\sqrt{\frac{(2\lambda+3)(\lambda+J+L+2)!(\lambda-L-J)!}{2^{2J+2L+2}(\lambda-L+J+1)!(\lambda-J+L+1)!}}(1-\cos\theta)^J \times(1+\cos\theta)^L P^{(2L+1,2J+1)}_{n_\theta}(\cos\theta),\quad n_\theta=0,1,2,\ldots, \tag{121}$$

where λ is quantized as $\lambda-L-J=n_\theta$.

Let us now turn to the quasiradial equation (118). Setting $w(\vartheta)=(\sin\vartheta)^{-7/2}R(\vartheta)$, we can rewrite this equation in the Pöschl–Teller form

$$\frac{d^2w}{d\vartheta^2}+\left[\left(2D^2\mathcal{E}+\omega^2D^4+\frac{49}{4}\right)-\frac{(2\lambda+3)^2-\frac{1}{4}}{\sin^2\vartheta}-\frac{\omega^2D^4}{\cos^2\vartheta}\right]w=0. \tag{122}$$

Solving this equation we have following expression for quasiradial functions $R(\vartheta)\equiv R_{n_r\lambda}(\theta)$:

$$R_{n_r\lambda}(\theta)=(\sin\vartheta)^{2\lambda}(\cos\vartheta)^{\nu+1/2}{}_2F_1(-n_r,n_r+\nu+2\lambda+4;2\lambda+4;\sin^2\vartheta),\quad n_r=0,1,2,\ldots \tag{123}$$

with energy levels given by

$$\mathcal{E}=\frac{1}{2D^2}[(n+1)(n+8)+(2\nu-1)(n+4)], \quad n=0,1,2,\ldots, \tag{124}$$

where $\nu=(\omega^2D^4+\frac{1}{4})^{1/2}$, and principal quantum number

$$n=2(n_r+\lambda)=2(n_r+n_\theta+L+J).$$

Thus, the full wave function Φ is the simultaneous eigenfunction of the Hamiltonian and commuting operators M^2, $\vec{J}^2$, $\vec{L}^2$, $\vec{T}^2$, J_3, L_3, and T_3. The explicit form of this function satisfying the normalization condition (see the Appendix)

$$-\frac{iD^5}{32\pi^2}\int_{S_{8C}}\Phi^{JLT}_{n_r\lambda Mmt}\Phi^{JLT\,\diamond}_{n_r\lambda Mmt}\sum_{i=1}^{8}u_i^2\frac{dv(u)}{u_9^2}=1$$

is

$$\Phi^{JLT}_{n_r\lambda Mmt}=C_{n_r\lambda}(\nu)\frac{\sqrt{(2L+1)(2T+1)}}{2\pi^2}R_{n_r\lambda}(\vartheta)\,Z^{JL}_{\lambda}(\theta)\,G^{JM}_{Lm;Tt}(\alpha,\beta,\gamma;\alpha_H,\beta_H,\gamma_H), \tag{125}$$

where $R_{n_r\lambda}(\vartheta)$ is given by formula (123) and

$$C_{n_r\lambda}(\nu)=\frac{4}{(2\lambda+3)!}\sqrt{\frac{i\nu(\nu+2\lambda+2n_r+4)\Gamma(2\lambda+\nu+n_r+4)(n_r+2\lambda+3)!}{D^{13}\pi^2(1-e^{2i\pi\nu})(\lambda+n_r+2)(n_r)!\Gamma(\nu+n_r+1)}}. \tag{126}$$

Let us now construct the five-dimensional Coulomb system. The constraints tell us

$$\vec{T}^2\Phi(u)=T(T+1)\Phi(u)=0 \tag{127}$$

and therefore the oscillator eigenstates span the states with $T=0$ and $L=J$. For $L=J$ the Jacobi polynomial in (121) is proportional to the Gegenbauer polynomial[34]

$$P^{(2L+1,2L+1)}_{\lambda-2L}(\cos\theta)=\frac{(4L+2)!(\lambda+1)!}{(2l+1)!(2L+\lambda+2)!}C^{2L+3/2}_{\lambda-2L}(\cos\theta), \tag{128}$$

and we obtain

$$Z^{JL}_{\lambda}(\theta)\equiv Z_{l\lambda}(\theta)=2^{2L+1}\Gamma\left(2L+\frac{3}{2}\right)\sqrt{\frac{(2\lambda+3)(\lambda-2L)!}{\pi(\lambda+2L+2)!}}(\sin\theta)^{2L}\,C^{2L+3/2}_{\lambda-2L}(\cos\theta). \tag{129}$$

Then from properties of Clebsch–Gordan coefficients $(JM|Lm';00)=\delta_{JL}\delta_{Mm'}$ and using $\mathcal{D}^0_{0,0}(\alpha_H,\beta_H,\gamma_H)=1$ we see that the expansion (115) yields

$$G^{JM}_{Lm;00}(\alpha,\beta,\gamma;\alpha_H,\beta_H,\gamma_H)=\mathcal{D}^L_{m,m'}(\alpha,\beta,\gamma)\,\delta_{JL}\,\delta_{Mm'}. \tag{130}$$

Thus, the function Φ now depends only on variables $(\vartheta,\theta,\alpha,\beta,\gamma)$. Observing that $\lambda=n_\theta+2L=0,1,2,\ldots,n$, introducing the new principal quantum number $N=(n_r+\lambda)=n/2=0,1,2,...$, and setting $k=i\mu$, we easily get from the oscillator energy spectrum (124) the reduced system energy levels

$$E_N=\frac{N(N+4)}{2R^2}-\frac{\mu^2}{2(N+2)^2}. \tag{131}$$

Noting that $\nu=i\sigma-(N+2)$ and taking into account the formulas (123) and (125)–(130), we finally have the solution of the Schrödinger equation (105) as

$$\Psi^{L\lambda}_{n_r mm'}(\chi,\theta;\alpha,\beta,\gamma)=D^{3/2}\,e^{3/2\,i\chi}\,\Phi^{L\lambda}_{n_r mm'}(\chi,\theta;\alpha,\beta,\gamma)$$

$$=N^{L\lambda}_{n_r}(\sigma)\,R_{n_r\lambda}(\vartheta)\,Z_{L\lambda}(\theta)\,\sqrt{\frac{2L+1}{2\pi^2}}\,\mathcal{D}^{L}_{m,m'}(\alpha,\beta,\gamma), \tag{132}$$

where $Z_{L\lambda}(\theta)$ is given by (129) and

$$R_{n_r\lambda}(\chi)=(\sin\chi)^{\lambda}\,e^{-i\chi(N-\lambda-i\sigma)}\,{}_2F_1(-N+\lambda,\,\lambda+2+i\sigma;2\lambda+4;1-e^{2i\chi}), \tag{133}$$

$$N^{L}_{n_r\lambda}(\sigma)=\frac{2^{\lambda+2}e^{\pi\sigma/2}}{(2\lambda+3)!}\,\sqrt{\frac{[(N+2)^2+\sigma^2](N+\lambda+3)!}{2R^5\pi(N+2)(N-\lambda)!}}\,|\Gamma(\lambda+2+i\sigma)|. \tag{134}$$

Thus, we have constructed the wave function and energy spectrum for the five-dimensional Coulomb problem. In the contraction limit $R\to\infty$ for finite N we get the formula for the discrete energy spectrum of the five-dimensional Coulomb problem,[37]

$$\lim_{R\to\infty} E_N(R)=-\frac{\mu^2}{(N+2)^2},\quad N=0,1,\ldots .$$

Taking the limit $R\to\infty$ and using asymptotic formulas as in (56) we get from (132) to (134)

$$\lim_{R\to\infty}\Psi^{L\lambda}_{n_r mm'}(\chi,\theta;\alpha,\beta,\gamma)=R_{N\lambda}(r)\,Z_{L\lambda}(\theta)\,\sqrt{\frac{2L+1}{2\pi^2}}\,\mathcal{D}^{L}_{m,m'}(\alpha,\beta,\gamma) \tag{135}$$

with

$$R_{N\lambda}(r)=\frac{4\mu^{5/2}}{(N+2)^3}\,\sqrt{\frac{(N+\lambda+3)!}{(N-\lambda)!}}\left(\frac{2\mu r}{N+2}\right)^{\lambda}\frac{e^{-\mu r/(N+2)}}{(2\lambda+3)!}\,{}_1F_1\left(-N+\lambda;2\lambda+4;\frac{2\mu r}{N+2}\right),$$

which coincides with the five-dimensional Coulomb wave function obtained in Ref. 37.

IV. SUMMARY AND DISCUSSION

In this paper we have constructed a series of mappings $S_{2C}\to S_2$, $S_{4C}\to S_3$, and $S_{8C}\to S_5$, that generalize those well known from the Euclidean space Levi-Civita, Kustaanheimo–Steifel, and Hurwitz transformations. We have shown, that as in case of flat space, these transformations permit one to establish the *correspondence* between the Kepler–Coulomb and oscillator problems in classical and quantum mechanics for the respective dimensions. We have seen that using these generalized transformations (12), (64), and (94) we can completely solve the quantum Coulomb system on the two-, three-, and five-dimensional sphere, including eigenfunctions with correct normalization constant and energy spectrum.

For the solution of the quantum Coulomb problem, we first transformed the Schrödinger equation to the equation with oscillator potential on the complex sphere. Then, via complexification of the Coulomb coupling constant μ ($\mu=Ze^2$) and the quasiradial variable χ this problem was translated to the oscillator system on the real sphere and solved.

It is interesting to note that the complexification of constant Ze^2/R and the quasiradial variable were first used by Barut, Inomata, and Junker[24] in the path integral approach to the Coulomb system on the three-dimensional sphere and hyperboloid, and further were applied to two- and three-dimensional superintegrable systems on spaces with constant curvature.[27,30] The substitution used in Ref. 24,

$$e^{i\chi}=-\coth\beta,\quad \beta\in(-\infty,\infty), \tag{136}$$

is correct as an analytic continuation to the region $0 \leqslant \mathrm{Re}\,\chi \leqslant \pi$ and $-\infty < \mathrm{Im} \leqslant 0$ and translates the Coulomb quasiradial equation with variable χ to the modified Pöschl–Teller equation with variable β. It is possible to show that there exists a connection between (136) and generalized Levi-Civita transformations on constant curvature spaces. Indeed, for instance, along with the mapping $S_{2C} \to S_2$ we can determine a mapping $H_{2C} \to S_2$, i.e., from the two-dimensional complex hyperboloid to the real sphere:

$$s_1^2 + s_2^2 + s_3^2 = (u_3^2 - u_1^2 - u_2^2)^2.$$

This transformation has the form

$$s_1 = i\sqrt{u_3^2 - u_1^2 - u_2^2} \cdot \frac{u_1^2 - u_2^2}{2u_3},$$

$$s_2 = i\sqrt{u_3^2 - u_1^2 - u_2^2} \cdot \frac{u_1 u_2}{u_3}, \tag{137}$$

$$s_3 = \sqrt{u_3^2 - u_1^2 - u_2^2} \cdot \left(u_3 - \frac{u_1^2 + u_2^2}{2u_3} \right),$$

and translates the Schrödinger equation for the Coulomb problem on the sphere to the oscillator problem on the complex hyperboloid. Then the substitution (136) transforms the oscillator problem from the complex to the real hyperbolid, a solution well known from Refs. 30 and 29.

The method described in this paper can be applied not just to (11) but to many Coulomb-type potentials. In particular the generalized two-dimensional Kepler–Coulomb problem may be transformed to the Rosokhatius system on the two-dimensional sphere.[28]

As we have seen, in spite of the similarity of transformations (4) and (12) on the sphere and Euclidean space there exist essential differences. Equations (12), (64), and (94) determine the transformations between complex and real spheres or in ambient spaces a mapping $C_{2p+1} \to R_{p+2}$ for $p = 1,2,4$. Evidently these facts are closely connected to Hurwitz theorem,[38] according to which the nonbijective bilinear transformations satisfy the identity

$$s_1^2 + s_2^2 + \cdots + s_f^2 = (u_1^2 + u_2^2 + \cdots + u_n^2)^2 \tag{138}$$

only for four pair of dimensions: $(f,n) = (1,1),(2,2),(3,4)$, and $(5,8)$, which corresponds to a mapping $R_{2p} \to R_{p+1}$ for $p = 1,2,4$, respectively.

For transformations between real spaces of constant curvature the situation is more complicated, and more interesting. For example, the two-dimensional transformation on the hyperboloid is

$$s_1 = \sqrt{u_3^2 \pm u_1^2 \pm u_2^2} \cdot \frac{u_1^2 - u_2^2}{2u_3},$$

$$s_2 = \sqrt{u_3^2 \pm u_1^2 \pm u_2^2} \cdot \frac{u_1 u_2}{u_3}, \tag{139}$$

$$s_3 = \sqrt{u_3^2 \pm u_1^2 \pm u_2^2} \cdot \left(u_3 \pm \frac{u_1^2 + u_2^2}{2u_3} \right),$$

and

$$s_3^2 - s_1^2 - s_2^2 = (u_3^2 \pm u_1^2 \pm u_2^2)^2. \tag{140}$$

Thus, the upper and lower hemispheres of the real sphere or the upper and lower sheets of the two-sheet hyperboloid in u-space map to the upper and lower sheets, respectively, of the two-sheet hyperboloid in s-space.

The next example is the transformation

$$s_1=\sqrt{u_1^2+u_2^2-u_3^2}\cdot\frac{u_1^2-u_2^2}{2u_3},$$

$$s_2=\sqrt{u_1^2+u_2^2-u_3^2}\cdot\frac{u_1u_2}{u_3}, \tag{141}$$

$$s_3=\sqrt{u_1^2+u_2^2-u_3^2}\cdot\left(u_3-\frac{u_1^2+u_2^2}{2u_3}\right),$$

and

$$s_1^2+s_2^2-s_3^2=(u_1^2+u_2^2-u_3^2)^2. \tag{142}$$

Here the one-sheet hyperboloid in u-space maps to the one-sheet hyperboloid in s-space. From transformations (139) and (141) (using the methods as in Sec. II) it is easy to show that in the contraction limit $D\to\infty$ this transformation goes to the real Levi-Civita transformation (up to the translation $\overline{u_i}\to\sqrt{2}\overline{u_i}$) (4). This shows that the method of this article can be adapted to treat a Kepler–Coulomb system on the two- and one-sheet hyperboloids.

Finally, note that in this article we do not discuss two important questions. First is the correspondence between integrals of motion for Kepler–Coulomb and oscillator systems. Second is the connection between separable systems of coordinates (not only spherical) under mappings (12), (64), and (94). This investigation will be carried out elsewhere.

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APPENDIX: TRANSFORMATION FORMULAS

We present some differential aspects of the generalized Levi-Civita, KS, and Hurwitz transformations. These calculations are related to those in Refs. 15 and 18 for flat space.

1. Transformation $S_{2C}\to S_2$

The Laplace–Beltrami operator on the $\underline{u}$-sphere in complex spherical coordinates (17) is

$$\Delta_u^{(2)}=\frac{1}{D^2}[(u_1\partial_{u_2}-u_2\partial_{u_1})^2+(u_3\partial_{u_2}-u_2\partial_{u_3})^2+(u_3\partial_{u_1}-u_1\partial_{u_3})^2]$$
$$=\frac{2i}{D^2}\sin\chi e^{-i\chi}\left\{\frac{1}{\sin\chi}\frac{\partial}{\partial\chi}\sin\chi\frac{\partial}{\partial\chi}+\frac{1}{\sin^2\chi}\frac{\partial^2}{\partial\varphi^2}\right\}, \tag{A1}$$

while the usual Laplace–Beltrami operator on the s-sphere in spherical coordinates (χ,φ) has the form

$$\Delta_s^{(2)}=\frac{1}{R^2}[(s_1\partial_{s_2}-s_2\partial_{s_1})^2+(s_3\partial_{s_2}-s_2\partial_{s_3})^2+(s_3\partial_{s_1}-s_1\partial_{s_3})^2]$$
$$=\frac{1}{R^2}\left\{\frac{1}{\sin\chi}\frac{\partial}{\partial\chi}\sin\chi\frac{\partial}{\partial\chi}+\frac{1}{\sin^2\chi}\frac{\partial^2}{\partial\varphi^2}\right\}. \tag{A2}$$

The two Laplacians are connected through

$$\Delta_s^{(2)}=-\frac{u_3^2}{u_1^2+u_2^2}\frac{1}{D^2}\Delta_u^{(2)}. \tag{A3}$$

The volume elements in u- and s-spaces are

$$dv(u)=-\frac{iD^2}{2}e^{i\chi}\,d\chi\,d\varphi,\quad dv(s)=R^2\sin\chi\,d\chi\,d\varphi \tag{A4}$$

and

$$\frac{1}{R}dv(s)=-\frac{u_1^2+u_2^2}{u_3^2}dv(u). \tag{A5}$$

We have (the variable φ runs the from 0 to 4π)

$$\int_{S_2}\cdots\,dv(s)=-\frac{D^2}{2}\int_{S_{2C}}\cdots\,\frac{u_1^2+u_2^2}{u_3^2}dv(u). \tag{A6}$$

2. Transformation $S_{4c}\rightarrow S_3$

The Laplace–Beltrami operator on the u-sphere in $(\chi,\alpha,\beta,\gamma)$ coordinates is

$$\Delta_u^{(4)}=\frac{2i}{D^2}\sin\chi e^{-i\chi}\left[\frac{e^{-i\chi}}{\sin^2\chi}\frac{\partial}{\partial\chi}e^{i\chi}\sin^2\chi\frac{\partial}{\partial\chi}+\frac{\vec{L}^2}{\sin^2\chi}\right], \tag{A7}$$

where

$$L_1=i\left(\cos\alpha\cot\beta\frac{\partial}{\partial\alpha}+\sin\alpha\frac{\partial}{\partial\beta}-\frac{\cos\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right),$$
$$L_2=i\left(\sin\alpha\cot\beta\frac{\partial}{\partial\alpha}-\cos\alpha\frac{\partial}{\partial\beta}-\frac{\sin\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right), \tag{A8}$$
$$L_3=-i\frac{\partial}{\partial\alpha},$$

and

$$\vec{L}^2=\left[\frac{\partial^2}{\partial\beta^2}+\cot\beta\frac{\partial}{\partial\beta}+\frac{1}{\sin^2\beta}\left(\frac{\partial^2}{\partial\gamma^2}-2\cos\beta\frac{\partial}{\partial\gamma}\frac{\partial}{\partial\alpha}+\frac{\partial^2}{\partial\alpha^2}\right)\right], \tag{A9}$$

while the usual Laplace–Beltrami operator on the s-sphere in (χ,β,α) coordinates is

$$\Delta_s^{(3)}=\frac{1}{R^2}\left[\frac{1}{\sin^2\chi}\frac{\partial}{\partial\chi}\sin^2\chi\frac{\partial}{\partial\chi}+\frac{1}{\sin^2\chi}\left(\frac{\partial^2}{\partial\beta^2}+\cot\beta\frac{\partial}{\partial\beta}+\frac{1}{\sin^2\beta}\frac{\partial^2}{\partial\alpha^2}\right)\right].$$

The two Laplace–Beltrami operators are connected by

$$\Delta_u^{(4)}=\frac{2i}{D^2}\sin\chi e^{-(3i/2)\chi}\left[D^4\Delta_s^{(3)}+\left(\frac{1}{4}-i\cot\chi\right)+\frac{1}{\sin^2\chi}\frac{1}{\sin^2\beta}\frac{\partial}{\partial\gamma}\left(\frac{\partial}{\partial\gamma}-2\cos\beta\frac{\partial}{\partial\alpha}\right)\right]e^{(i/2)\beta},$$

and the operator acting on functions of variables (χ,β,α) is

$$\Delta_s^{(3)}=-u_5^{1/2}\left\{\frac{u_5^2}{u_1^2+u_2^2+u_3^2+u_4^2}\left[\frac{1}{D^2}\Delta_u^{(4)}-\frac{1}{D^4}\left(2+\frac{3}{4}\frac{u_1^2+u_2^2+u_3^2+u_4^2}{u_5^2}\right)\right]\right\}u_5^{-1/2}. \quad \text{(A10)}$$

The volume elements on S_{4c} and S_3 are given by

$$dv(u)=-\frac{D^4}{4}e^{2i\chi}\sin\chi\sin\beta d\chi\, d\beta\, d\alpha\, d\gamma, \quad dv(s)=R^3\sin^2\chi\sin\beta\, d\chi\, d\beta\, d\alpha,$$

where

$$\frac{u_1^2+u_2^2+u_3^2+u_4^2}{u_5^3}dv(u)=\frac{i}{2D^2}dv(s)\,d\gamma. \quad \text{(A11)}$$

Integration over $\gamma\in[0,4\pi]$ gives

$$\int_{S_3}\cdots\, dv(s)=-\frac{iD^2}{2\pi}\int_{S_{4C}}\cdots\,\frac{u_1^2+u_2^2+u_3^2+u_4^2}{u_5^3}dv(u).$$

3. Transformation $S_{8C}\to S_5$

The Laplace–Beltrami operator on the u-sphere in $(\chi,\vartheta;\alpha,\beta,\gamma,\alpha_H,\beta_H,\gamma_H)$ coordinates is

$$\Delta_u^{(8)}=\frac{2i}{D^2}\sin\chi e^{-i\chi}\left\{\frac{e^{-3i\chi}}{\sin^4\chi}\frac{\partial}{\partial\chi}e^{3i\chi}\sin^4\chi\frac{\partial}{\partial\chi}+\frac{1}{\sin^2\chi}\left[\frac{1}{\sin^3\vartheta}\frac{\partial}{\partial\vartheta}\sin^3\vartheta\frac{\partial}{\partial\vartheta}\right.\right.$$
$$\left.\left.-\frac{4\left(\vec{L}^2+2\vec{L}\cdot\vec{T}\sin^2\frac{\vartheta}{2}+\vec{T}^2\sin^2\frac{\vartheta}{2}\right)}{\sin^2\vartheta}\right]\right\}, \quad \text{(A12)}$$

where operator $\vec{L}$ is given by (A8) and $\vec{T}$ is

$$T_1=i\left(\cos\alpha_H\cot\beta_H\frac{\partial}{\partial\alpha_H}+\sin\alpha_H\frac{\partial}{\partial\beta_H}-\frac{\cos\alpha_H}{\sin\beta_H}\frac{\partial}{\partial\gamma_H}\right),$$

$$T_2=i\left(\sin\alpha_H\cot\beta_H\frac{\partial}{\partial\alpha_H}-\cos\alpha_H\frac{\partial}{\partial\beta_H}-\frac{\sin\alpha_H}{\sin\beta_H}\frac{\partial}{\partial\gamma_H}\right), \quad \text{(A13)}$$

$$T_3=-i\frac{\partial}{\partial\alpha_H}.$$

The Laplace–Beltrami operator on the five-dimensional sphere in $(\chi,\vartheta;\alpha,\beta,\gamma)$ coordinates is

$$\Delta_s^{(5)}=\frac{1}{R^2}\left[\frac{1}{\sin^4\chi}\frac{\partial}{\partial\chi}\sin^4\chi\frac{\partial}{\partial\chi}+\frac{1}{\sin^2\chi}\left(\frac{1}{\sin^3\vartheta}\frac{\partial}{\partial\vartheta}\sin^3\vartheta\frac{\partial}{\partial\vartheta}-\frac{4\vec{L}^2}{\sin^2\vartheta}\right)\right]. \quad \text{(A14)}$$

The Laplace–Beltrami operators are related by

$$\Delta_u^{(8)}=\frac{2i}{D^2}\sin\chi \mathrm{e}^{-(5i/2)\,\alpha}\left[D^4\Delta_s^{(5)}+\left(\frac{9}{4}-6i\cot\chi\right)-\frac{1}{\sin^2\chi}\frac{2\vec{L}\cdot\vec{T}+\vec{T}^2}{\cos^2\frac{\vartheta}{2}}\right]\mathrm{e}^{(3i/2)\,\chi}$$

and the operator acting on a function of variables $(\chi,\vartheta;\alpha,\beta,\gamma)$ is

$$\Delta_s^{(5)}=-u_9^{3/2}\left\{\frac{1}{D^2}\frac{u_9^2}{\Sigma_{i=1}^8 u_i^2}\left[\Delta_u^{(8)}-\frac{1}{D^2}\left(12+\frac{15}{4}\frac{1}{u_9^2}\sum_{i=1}^8 u_i^2\right)\right]\right\}u_9^{-3/2}. \tag{A15}$$

The volume elements on S_{8C} and S_5 have the form

$$dv(u)=-8D^8e^{4i\chi}\sin^3\chi\sin^3\theta\,d\chi\,d\theta\,d\Omega\,d\Omega_H,$$

$$dv(s)=R^5\sin^4\chi\sin^3\theta\,d\chi\,d\theta\,d\Omega,$$

where

$$d\Omega=\tfrac{1}{8}\sin\beta d\alpha\,d\beta\,d\gamma. \tag{A16}$$

We have

$$\frac{1}{u_9^5}\sum_{i=1}^8 u_i^2\,dv(u)=\frac{16i}{D^5}dv(s)\,d\Omega_H \tag{A17}$$

and integration over the variables $(\alpha_H,\beta_H,\gamma_H)$ gives the formula

$$\int_{S_5}\cdots dv(s)=-\frac{iD^5}{32\pi^2}\int_{S_{8C}}\cdots\sum_{i=1}^8 u_i^2\frac{dv(u)}{u_9^5}. \tag{A18}$$

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