Essays on the use of commitment and tough negotiation tactics in bargaining

by

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Declaration

I declare that the thesis is my own work and has not been submitted for a degree at another university.

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Abstract

This thesis analyses the role of commitment in bargaining. Chapter 1 looks at how players could use finite length commitment to affect the bargaining model in a multi-period model. The idea of this is to complement the existing literature on infinite length commitment. In line with the infinite commitment literature, a rational player can mimic a commitment type to gain a considerable advantage, although, as will be seen, there are key differences.

Chapter 2 analyses whether one should take the opportunity to commit oneself when the opponent does not perfectly observe the decision taken. Logically, if one's opponent sees no difference between a bluff and actual commitment then one may as well bluff, since the opponent acts the same and committing is a needless sacrifice of freedom. When the opponent may discover a bluff as such, the situation is far less clear and this Chapter analyses when a commitment outcome is likely to prevail.

Chapter 3 takes a rather different approach and analyses how hard one should negotiate when there are other parties who may enter the deal. The general finding is that one should follow the crowd and act the same way as everyone else. All three chapters heavily use the mathematical tool of game theory. However, while Chapter 1 uses non-cooperative game theory, the analysis of Chapters 2 and 3 primarily use evolutionary game theory.

Abbreviations

NE - Nash Equilibirum

SPE - Subgame Perfect Equilibrium

PBE - Perfect BayesianEquilibrium

LRE - Long Run Equilibrium

ESS - Evolutionarily Stable Strategy

KMR - The journal paper "Learning, mutation, and long run equilibria in games" in Econometrica written by Kandori, Mailath and Rob

DTWM - Direction of travel without mutations

Introduction

The human race as a whole relies greatly on interactions with others. This is especially true of the way modern man lives. Very few of us would be able to live a self-sufficient life, and even those who could generally would not want to. From the basics such as food, clothing and shelter to all the modern technology and gadgets which enrich our lives, nearly all of us are reliant on interacting with wider society. In addition to the mutually beneficial trade of goods and services, we are also a sociable species which rely on one another for love, friendship and companionship.

When two people co-operate in some form, they generally do so to become better off than they were before and so a surplus is created. There are countless traditional economic examples such as a consumer who needs groceries interacting with a seller of groceries, as well as social examples such as two people who enjoy spending time together. In the first example, the typical solution is for the consumer to hand some money to the grocer in exchange for the groceries he wants. The question then becomes how much money he should hand over for the groceries he buys. Obviously the consumer prefers less, while the grocer prefers more and the answer to this question determines how they share the surplus they have created by coming together to interact. In the second example, while both people want to spend time together, they will likely have different preferences as to how. This is epitomized in the well-known "battle of the sexes" game where the woman wants to go to the ballet, while the man wants to go to the football. While they both gain from spending time together, this decision of where to go determines who gains more from this interaction.

In both, of these examples it is in the parties' mutual interests to interact, but they have diverging interests on the terms of this interaction. This is the embodiment of the bargaining problem. In some circumstances how they divide the surplus is clear. If one goes to a supermarket to buy groceries, the terms of any trade are clear: the supermarket, in effect, offers the consumer a take-it-or-leave-it offer on every item in store. Of course, there are questions about what prices the supermarket should offer but these questions are best answered by models of competition between firms.

Where the literature on bargaining enters the fray is where the terms of any trade are uncertain. Take for example the buying and selling of a house: the seller may set an asking price, but trade will not necessarily take place at this price, because unlike the supermarket, the house seller has a flexibility. If one offers a supermarket one penny less than what it is demanding for a bundle of goods, the supermarket will refuse. If one offers the house seller one penny less, the offer would almost certainly be accepted, as would an offer of one further penny less. This then begs the question of how far the buyer can haggle down the seller's asking price and hence at what price trade will occur. In an example like this, the question is not an easy one and many of the greatest minds in the discipline of Economics have proposed models to try to answer it. Particularly notable examples are [34, 39].

The first two chapters of this thesis add to this literature, focusing on the issue that commitment may have on the outcome. From a non-technical perspective, Schelling [40] articulated the effect that commitment can have on the bargaining outcome by constraining one of the parties' available actions. Schelling expressed this as follows:

> The essence of these tactics is some voluntary but irreversible sacrifice of freedom of choice. They rest on the paradox that the power to constrain an adversary may depend on the power to bind oneself; that in bargaining weakness is often strength, freedom may be freedom to capitulate, and to

burn bridges behind one may suffice to undo an opponent.

The models of the first two chapters assume that a commitment, once made is irreversible. This is in line with most of the literature, although a notable exception which allows for players to revoke their commitments at a cost is [31]. The power commitment can have is obvious: if one party is committed to not compromising from their stated position, then in order to reach what is a mutually beneficial agreement, the other party is the one who must compromise. Thus by committing to a certain demand, one can induce the other party to concede to that demand, provided that the other party is aware of the commitment and not restricted by any commitments of their own. However if there is uncertainty whether a party is committed or merely bluffing then the result one would expect is far less clear.

In Chapter 1, this uncertainty is captured by the modeling of two different types: one who is committed and another who is merely bluffing, but may find it in their interests to mimic commitment. There is already much literature on this where players, once committed stay committed forever, see [1, 42] for two of the better known papers. Chapter 1 departs from this literature by assuming that commitments are finite. This means that once the commitment period expires without agreement, a player of commitment type must select another demand to commit to. In particular, such a player is now required to act strategically. This represents a significant break from the literature, which assumed that model committed players as unthinking automata. The question asked here is *what effect do such commitment tactics have on the bargaining outcome*? Although there are subtle differences, the broad conclusion, in line with the literature on infinite commitment, is that even when the probability of a player being of this committed type is small, it can have a large effect.

Chapter 2, while still on the role of commitment in bargaining asks quite different questions. Here a player has a choice of whether to actually commit or merely bluff at commitment. Clearly if the opposing player knows that a commitment has been made, then the best response is to give in to the demands, whereas after a bluff, the best response is to ignore it and not concede. However I assume that the opposing player has imperfect information about whether the first player actually made the commitment or not. This structure naturally gives rise to two Nash Equilibria: the first being commitment followed by concession to the commitment; the second being a bluff followed by non-concession. I use the evolutionary game theory technique of stochastic stability to argue under which circumstances each is likely to prevail. The results support Schelling's statement that "in bargaining weakness is often strength". A player is more likely to be able to commit if a bluff is more likely to be discovered as such. Also, in the long run, the other player may be disadvantaged by technology which allows a bluff to be discovered.

The first two chapters focus on bargaining between two parties who must deal with each other and so it is rational to use commitment tactics or any other tactics to try to negotiate as hard as possible. In some scenarios such as negotiations between a firm and a union they can only reasonably bargain with each other. The firm is unable to hire a whole new non-unionized workforce while the union can only deal with that firm. However, in other circumstances, one or both parties may have a choice of who to deal with. Consider once again the house buyer and seller example. A house buyer would likely be looking at more than just one house, and similarly the house seller is likely to consider more than just one buyer. This means that if one buyer is negotiating particularly hard and offering the seller a poor deal, then the seller may be able to find another buyer who offers a better deal. So a buyer, while wanting to negotiate more of the surplus for himself, also has to be wary that he faces competition from other buyers. Similarly, the seller has the same dilemma. The model of Chapter 3 looks at precisely this issue of how tough agents should be in an environment where they face such competition. Thus Chapter 3 can be thought of as sitting in the void between models of bargaining and models of competition and auctions. The model of the third chapter predicts a tendency for players to herd toward homogenoeus behaviour.

As well as asking different questions the models of the three papers also use different techniques to predict the outcome. Chapter 1 makes the hyper-rationality assumptions of non-cooperative game theory and applies Perfect Bayesian Equilibrium. By contrast Chapters 2 and 3 replace this assumption by the evolutionary game theoretic perspective of players looking around at the wider environment and learning to adapt their play based on what others are doing. Chapter 2 in particularly, heavily relies upon the stochastic stability approach pioneered by [24, 45]. All three chapters here analyze the questions around bargaining from a technical approach. For a less technical approach and greater detail on the importance of the bargaining question see [32, 40]

Chapter 1

Bargaining with strategically irrational types

1.1 Introduction

This chapter looks at the effect of introducing one period commitment on the classical alternating offers bargaining model. I modify the classic Rubinstein-Stahl alternating offers model by allowing players to threaten to reject certain offers. Players can be one of two types: a "commitment" type who becomes committed to such threats and a "rational" type for whom such threats are meaningless. I find that introducing only a small probability of a player being the commitment type can have a large effect on the equilibrium payoffs when the discount factor is sufficiently large. The reason for this is rational players looking to build a reputation for being committed. There is already a reasonably large literature exploring these reputation effects when commitment is infinite. The approach taken here is very similar in spirit to much of this literature but with one period commitment instead of infinite commitment. One particularly notable consequence of this is to make the commitment type a strategic player, which adds complexity as well as a large multiplicity of equilibria. I will argue that some of these equilibria are more compelling than others. The more compelling equilibria yield results very similar to those in the world of infinite commitment with one sided uncertainty, but not two sided uncertainty.

Imagine a firm negotiating with a union over wage demands. As the firm is deciding what to offer, the union makes the statement "we will not accept less than y". The firm, unsure whether or not the union is really committed to this position must decide what offer to make. How should the firm respond to this threat? How high should the union set the threshold y? What impact does allowing for such statements have on the outcome of the negotiations? This chapter aims to answer these questions.

Previous papers have asked similar questions, but with the condition that a commitment, if made, is forever. So in the above example, if the union is committed to not accepting any less than y next period, it is also committed to this position for all future periods. This chapter drops that assumption. Here commitments are only made for one period. In many applications, I would contend that this is more realistic, or robust to the realities of the real world. In the example above, things may change during the bargaining process to either strengthen or weaken the union's negotiating position, or the firm's negotiating tactics may change, thus altering the dynamic of the bargaining game and rendering previously held positions obsolete. Thus even if commitment was intended to be forever, realities may render such commitments finite. So it would seem useful to have an understanding of the effects of finite commitment to complement the infinite commitment analysis.

Without commitment, Rubinstein [39] found that the alternating offers bargaining game has a unique subgame perfect equilibrium. Adding commitment dsirupts the equilbirium. Clearly, if a player knows his opponent is committed, then the only sensible option becomes to concede to this commitment, but if this player is unsure about his opponent's commitment status then the situation is less straightforward.

This is where the work on reputational bargaining enters the fray, with the complication that a rational player may look to build a reputation for being committed. The inaugural work on reputational bargaining is from Myerson [33] who introduced a small probability of an α- insistent type. That is, one player, say player 1, could be of this type who is committed to always insisting on α in every subgame, whether proposing or responding. Myerson showed that as players become increasingly patient, player 1 will demand α and get it almost immediately, regardless of whether he is the α- insistent type or not, and even if the opposing player puts very low probability on him being the α- insistent type. The reason is that when players are very patient, the rational player 1 sees it in his interests to mimic the α- insistent type with high probability for a long time. The other player, knowing this concedes almost immediately to avoid a long and costly period of delay. There has since been much further research allowing players to mimic α- insistent types [1, 2, 3, 9].

In a slightly different direction, Kambe [23] and Wolitzky [42] allowed players to choose which commitment they announce, which can be seen as endogenising the α of the α- insistent type. In their models, a player announces a "posture "[1] and then finds out whether they are committed to this posture. The alternative assumption is that a player knows whether he will become committed prior to anouncing his posture. That is, to introduce two types: a "commitment type" who becomes committed to the posture he announces, and a "rational type" who does not become committed but may choose to mimic the commitment type. This assumption leads to the commitment type becoming a strategic player. As will be discussed in Section 2 making the commitment type into a "strategically irrational" player who decides on what commitments to make complicates the analysis, and introduces a large multiplicity of equilibria.

[1]This is the term used by Wolitzky. Unlike Kambe, in his model, players can announce a posture which changes the share of the surplus they demand over time. Its also worth noting that unlike the other papers, Wolitzky looks for maxmin strategies and payoffs under common knowledge of rationality instead of equilibrium.

By making the assumption that players only realise whether they are committed after announcing their posture, Kambe and Wolitzky both remove the need for the commitment type to act strategically. If a player is subsequently found to be the commitment type, his moves are forced for the remainder of the game, thus he becomes an automaton in exactly the same way as Myerson's α- insistent type. Kambe found a unique equilibrium. If on the other hand Kambe had assumed that a player knows whether he will become committed prior to announcing the posture, then while this equilibrium still holds, i is accompanied by a continuum of others which occur because of the freedom in updating beliefs off the equilibrium path when a player of unknown type makes a strategic decision.

This chapter allows players to choose which levels they are committed to, but unlike [23, 42] the attention here is on finite commitment. This means that if no agreement is reached before the commitment period expires, then the committed player must act again. This makes it impossible to avoid the commitment type becoming a strategic player, in the way that the other reputational bargaining papers have done. As we shall see, this makes the multiplicity of Perfect Bayesian Equilibria (henceforth PBE) unavoidable. In essence, the problem is that a player can be forced into particular actions by being "threatened" by the beliefs stipulating that he is believed to be rational should he try to deviate to make more effective use of his commitment. This issue is discussed in Section 2. While this multiplicity might at first be regarded as a curse, it also has its benefits, as it allows the construction of equilibria for what would otherwise be an almost unsolvable model without such threatening by beliefs. I characterise a large chunk of the set of possible PBE payoffs, and point towards some of these as being more compelling than others.

One alternative approach would be to follow Ellingsen and Miettenen [12, 13] by supposing that every period the player announces his position, there is a probability p that he becomes committed to that position and the commitment decays with the remaining probability $1 - p$. This solves the problem of having two types of strategic players by amalgamating the commitment and rational types into a single stochastically committed type. However, this assumption means that every period, regardless of past events, the probability a player is committed is p and so it completely eliminates a player's ability to build a reputation for being of commitment type. I solve the model under the alternative assumption in order to illuminate the reputation building effects driving this model. I show that when p is small, the direct effect of the opposing player being committed to rejecting some offers with probability p is also small. The big effect which dramatically shifts the equilibrium outcome comes from a player's ability to build a reputation for being the commitment type.

For much of the intuition behind the results in this paper, the key concept in this paper will be the idea of proposer power. In the basic alternating offers bargaining model solved in [39] players' power, and hence their equal equilibrium shares comes from their equal opportunity to propose. By contrast, in an infinite horizon bargaining game in which the same player proposes all the time, that player holds all the power and, as a result, will take all the surplus. This argument was made quite forcibly by Yildiz [43] who looked at what happens when players hold different beliefs about who will propose in the future. He found that the more optimistic a player is about how often he will propose in the future, the more he will demand, a phenomenon strong enough to make the players demand incompatible shares when both are optimistic and so cause delay to agreement.

The model in this paper is very different, but still, it will be convenient to interpret the results in terms of players' proposer power. The threat that a player makes before the opponent's offer can be seen as an attempt to steal some of the proposer power for that period. How successful the attempt is depends on the probability which the proposing player places on the threatening player being the commitment type. For example if the proposer believed that the threatening player definitely is not the commitment type, then the proposer will simply view the threat as being utterly irrelevant and ignore it. At the other extreme, if the proposer believes the threatening player is the commitment type for sure then in order to reach agreement this period, the proposer must accommodate the threat. So in this latter case, the threat supersedes the proposer's offer as the proposing player is left with the choice of either accepting the terms in the opponent's threat or rejecting them. Hence the threatening player has stolen all the proposer power for that period.

The driving force behind many of the results in this paper is that the threatening player, even if rational, will want to mimic the commitment type and so gain a reputation for being of commitment type. As players become increasingly patient, it becomes less costly to mimic the commitment type, and so by the Coasean dynamics[2], that player manages to steal almost all the proposer power and does almost as well as if he was known to be the commitment type for sure. If only one player might be of commitment type then that player steals the opponent's proposer power, while keeping all his own, and hence takes almost all the surplus. However when both players could be of commitment type, they steal each others' proposer power, with the result that both players still effectively propose half the time, and so we get back to an equal split of the surplus as in [39].

Throughout the paper, it is assumed players have a common discount rate δ which could be thought of as $\delta = e^{-r\triangle}$ where r is the rate of time preference and $\triangle$ is the time that elapses between offers. In common with most of the other literature on bargaining I consider the $\delta \to 1$ case of increasing patience or increasingly frequent offers as being salient and much of the analysis is with respect to this case.

The rest of the chapter proceeds as follows: Section 2 gives the model and explains the multiplicity of equilibria issue. Sections 3 analyses the one sided uncertainty case, where the other player is rational for sure and offers an easy to understand

[2] See [1] for more discussion on this

decomposition of the reputation effects. Section 4 deals with two sided uncertainty where both players could be the committed type. Section 5 concludes, while Section 6 discusses some potential further research in this area. The majority of the proofs, along with some other technical material are in the appendix, Section 7.

1.2 The Model

The general framework is the same as in the basic alternating offers model: player A is the proposer in even periods, starting at period 0 and player B is the proposer in odd periods, until a proposal is accepted. Players discount the future according to the common discount factor $\delta \in (0,1)$ and so if agreement is reached in period t giving player $i \in \{A,B\}$ share s_i, this generates payoff $U^i = \delta^t s_i$. If players never reach agreement then both receive payoff of 0.

The difference from the standard model is that at the start of the game, each player $i \in \{A,B\}$ is endowed with a type $\theta \in \{C,R\}$ where C denotes "commitment" and R denotes "rational". Let i^θ denote player i of type θ. The significance of these types is in periods when this player is the responder. Before receiving an offer, player i announces a threat level y, which has the interpretation "I will only accept if you offer me at least y." Player i becomes committed to this statement if the commitment type, while the rational type faces a free decision of whether or not to accept. Notice that a player of commitment type temporarily becomes an automaton when responding to his opponent's offer, but acts strategically in deciding which offers and threats to make. This is why I call the commitment type "strategically irrational".

Section 3 considers the one sided uncertainty case where B is rational for sure, while A is the commitment type with probability $p_A \in [0,1]$ and rational with remaining probability $1 - p_A$. This means that in even periods, the games is as in the standard model: A makes an offer $x \in [0,1]$, which if accepted gives shares of x to the responder (B) and $1-x$ to the proposer (A); if rejected, play proceeds to the next period. In odd periods before B proposes, A announces a threat $y \in [0,1]$. Player B then makes an offer $x \in [0,1]$ which A^R decides whether or not to accept, while the commitment type A^C accepts if and only if $x \geq y$. If the offer is accepted, this gives shares of x to the responder (A) and $1-x$ to the proposer (B); if rejected, play proceeds to the next period. Section 4 considers the case of two sided uncertainty. So in even periods B is the commitment type with probability $p_B \in [0,1]$ and makes a threat y before A offers, in exactly the same way as A does in odd periods and the commitment type B^C is compelled to stick to this threat.

Note that the threat a player makes is only valid for that one period. If the game is still going in two periods time then the player announces a new threat level, which may be different from the previous one. If $p_A = p_B = 0$ then the threats are irrelevant and we have the basic alternating offers model, for which Rubinstein proved there is a unique subgame perfect equilibrium. More generally, whenever there is no uncertainty, that is $(p_A, p_B) \in \{0,1\}^2$, the equilibrium is unique. However when there is uncertainty about the type of at least one player, the question of how to update beliefs causes multiplicity of equilibria.

Multiplicity of PBE and slackness

Throughout this chapter, the equilibrium concept used is Perfect Bayesian Equilibrium (henceforth PBE). This means that at every information set players act sequentially rationally given their beliefs, and that these beliefs are consistent with the strategies played. While the PBE concept requires that on the equilibrium path, beliefs are consistent with Bayes' rule, it places almost no restrictions on beliefs off the equilibrium path. As a result it allows a great multiplicity of equilibria including some with questionable off equilibrium path beliefs.

The work here is not the first to encounter this problem. Kambe [23] in Section

4.2 of his paper considers an extension to see what happens when players know their type before announcement of commitment level. He calls this "inborn stubbornness." To illustrate the multiplicity problem, the subsection below considers the one period version of the game. For more on the multiplicity problem, refinement of PBE and Kambe's solution to these issuues see Appendix, Section 7.1

One period game

The game runs as follows: Player A knows his type, which is A^C with probability p and A^R with probability $1-p$. However A's type is unknown to B. Player A announces a threat, to reject less generous offers than y, then B makes an offer x and A decides whether to accept or reject. A^R has a free choice of whether to accept or not, while A^C abides by the threat, thus accepting if and only if $x \geq y$. If A accepts then the pie is divided $(x, 1-x)$, while if A rejects both get 0.

The first thing to note is that A^R will accept any $x > 0$ and is indifferent about whether to accept $x = 0$. The first part of this statement implies that B can guarantee a share of the pie abritraily close to $1-p$, by offering $x = \varepsilon$, where ε is small. The question is what happens to the remaining share of the surplus, p? There are many answers which are consistent with the PBE concept including some or all of it being lost[3] but the focus here is on the Pareto efficient PBE, where agreement is certain. The following Lemma shows that any split of this surplus can result.

Lemma 1. *For each $s \in [0, p]$ there exists a PBE with $U^A = s$, $U^B = (1-p)+(p-s)$.*

Proof. Fix $s \in [0, p]$ and consider the following strategies and beliefs. Both types of A set threat $y = s$, B sets $x = y$ iff $y \leq s$ or $x = 0$ otherwise, and A^R accepts all offers. The beliefs of B are unchanged after $y \leq s$, but after any $y > s$ B believes A is A^R for sure. This clearly defines a PBE: beliefs are updated via Bayes' rule

[3] Appendix, Section 7.1 shows this with a simpler informed seller sender-receiver game which I call the informed seller game. The same idea applies in the game here.

where possible. Working backwards: firstly A^R accepting all offers is a best response. Secondly, given this startegy and beliefs, the offering strategy of B is optimal at every information set, and thirdly given B's offering strategy A acts optimally in the choice of threat. □

Basically, given a threat y player B has two possibly optimal moves: either accommodate the threat, setting $x = y$, or set $x = 0$ in which case only A^R accepts. If A did not know his type before making the offer then the equilibrium would be unique and would have A making the highest threat such that B should accommodate, this being $y = p$. However in this game A does know his type and so the threat A makes could influence B's beliefs about A's type. This is how we sustain PBE with $s < p$. Player A is prevented from making higher threats by B interpreting such threats as coming from the rational type.

To help describe the set of efficient PBE outcomes of Lemma 1, I introduce the following notion of slackness.

Definition 2. An equilibrium offer exhibits **slackness** if the responder strictly prefers accommodating the threat to not. An equilibrium offer exhibits **no slackness** if it leaves the B indifferent between accommodating and not. For a PBE σ, let $S(\sigma)$ denote the difference in B's payoff between accommodating and making the next best offer. An equilibrium offer exhibits **complete slackness** if it exhibits slackness and maximises $S(\sigma)$ across the set of efficient PBE.

It is immediate to see that the no slackness PBE has A taking p of the pie, while B takes only his guaranteed $1 - p$. The complete slackness PBE has beliefs restrincting A to setting threat $y = 0$, giving B the contested proportion p of the pie, leading to payoffs $U^A = 0$, $U^B = 1$.

A key observation here is that a threat is optimal for one type of A if and only if it is also optimal for the other. This implies that in any PBE $U^A = U^{A^R} =$

U^{A^C} and that both types have the same incentive to make any deviation, suggesting that posterior beliefs should equal prior beliefs off the equilibrium path. While this argument supports the no slackness PBE, the search for a formal refinement is not an easy one. See Section 7.1 for more detail.

In the multi-period bargaining model

The multi-period model is slightly more complicated, since after rejection of an offer, the game continues. However the idea behind the multiplicity of PBE is exactly the same. Take an even period in which A is the proposer. Player A can be forced into making a more generous offer than he otherwise would by beliefs stipulating that he is believed to be of type A^R otherwise. Taken to the extreme, we can describe the PBE with complete slackness: This is where both players play as if A was rational for sure, and if A tried to deviate from this, would be believed to be rational for sure. Thus the complete slacknes PBE erodes away all the power A has from possibly being the commitment type. Such slackness considerations play a role whenever both types of A are required to move. That is at the start of any even period whn making an offer, or at the start of any odd period when making a threat.

In addition to the efficient PBE where immediate agreement is certain, there are also many inefficient PBE, which arise from two different sources. The first source is the existence of inefficient PBE in the one period game (see Appendix, Section 7.1 for details), while the second is due to the multi-period horizon. We can use the fact that there are multiplicity of equilibria in every continuation subgame to use the threat of an unfavourable equilibrium in the continuation game to force players to play strategies resulting in considerable delay.

Given the huge multiplicity of PBE, my solution is to focus on the more compelling of these, which I judge to be the PBE with certain immediate agreement. Of the efficient PBE, I judge those with less slackness to be more compelling than those with more, for reasons given in Section 7.1 and so pay particular attention to these.

1.3 With one sided uncertainty

1.3.1 Background

Throughout this section I assume that the only uncertainty is on the type of player A while B is rational for sure, that is $p_B = 0$ and $p_A \in [0,1]$. Before studying the more complicated cases, I first consider what happens when A is also of known type. When $p_A = 0$, we get the following very well known result:

Fact 3. *(Rubinstein) With $p_A = 0$ the unique Subgame Perfect equilibrium yields shares*

$$U^A = \frac{1}{1+\delta},\ U^B = \frac{\delta}{1+\delta}$$

As $\delta \to 1$, these shares each converge to a half, a result which makes intuitive sense since each player proposes half the time and so holds half the proposer power. In this model, players making a threat before the opponent makes an offer can be seen as an attempt to steal some of the opponents proposer power. As the next result shows, when player A is believed to be the commitment type for sure, that is $p_A = 1$, he successfully steals all the proposer power and so takes the entire surplus.

Fact 4. *With $p_A = 1$ the unique Subgame Perfect equilibrium yields shares*

$$U^A = 1,\ U^B = 0$$

When B is the proposer A can demand any threat which B knows A is committed to. So in response to a threat, B can either accommodate the threat or delay the game to the next period, by offering something lower. However this is simply analogous to A making the offer and B deciding whether to accept or reject. So the game is as if

A is proposing in every period, which is well known to result in A taking all the pie. The SPE strategy for A is to demand the whole pie in even periods and threaten to accept no less than the whole pie in odd periods. In response to this, B can do no better than to accept the demand or accommodate the threat immediately.

It will be shown that with uncertainty, that is for any $p_A \in (0, 1)$, the payoffs in our favoured PBE approach this outcome as $\delta \to 1$. Furthermore, I show that this is not due to the often slight chance of A being the commitment type per se, but due to the rightful fear B has of A delaying in order to build a reputation for being of commitment type. To make this point, I consider what would happen without any updating of beliefs. That is I tweak the game slightly by assuming each period that player A makes a threat, this commitment sticks with probability p and decays with probability $1 - p_A$; the stochastic commitment asumption of [13]. With this assumption, there is no point in A voluntarily rejecting an offer to build a reputation for being committed in the responder subgame since the probability of commitment is subsequently reset to p.

Lemma 5. *With no updating there is an SPE with shares:*

$$U^A = 1 - \frac{\delta\,(1 - p_A)}{1 + \delta},\; U^B = \frac{\delta\,(1 - p_A)}{1 + \delta}$$

Furthermore, if we do not allow an uncommitted A to reject offers that the committed type would accept, then the SPE shares are unique.

The proof is in the appendix. The reason for stipulating that an uncommitted A not, in effect, change his mind, is that this is required for claim 4 of the proof.

The intuition for Lemma 5 is as follows: In odd periods B knows that A's threat will be carried through with probability p_A, so B has to possibly optimal strategies: to accommodate the threat so to be accepted by both players or to gamble that A is not committed and make the lowest offer that the non-committed A accepts, while accepting the consequences of delay if A is committed. Knowing this, player A sets the highest threat which B should accommodate, which results in him stealing proportion p_A of player B's proposer power. Since this happens every odd period, while A proposes as per normal in even periods, B's share of the bargaining power falls from $1/2$ to $\frac{1-p_A}{2}$, which is his equilibrium share in the limiting case as $\delta \to 1$. A formal proof is in the appendix.

1.3.2 The PBE with one sided uncertainty

I now return to the original model where A can build a reputation for being of commitment type. Define $\psi(p_A, \delta) = \frac{(1-\delta)(1-p_A)}{p_A}$

Theorem 6. *Given parameters $(p_A, \delta) \in (0,1)^2$ satisfying $\psi(p_A, \delta) \leq \frac{\delta}{1+\delta}$[4] there exists a PBE which gives payoffs $U_A = 1 - \psi$ and $U_B = \psi$.*

Noticing that for any $p_A > 0$, $\lim_{\delta \to 1} \psi(p_A, \delta) = 0$ gives the following:

Theorem 7. *For any $\varepsilon > 0$ and $p_A > 0$, there exists $\bar{\delta} \in (0,1)$ such that for any discount factor $\delta > \bar{\delta}$, there is a PBE with payoffs $U^A > 1 - \varepsilon$ and $U^B < \varepsilon$.*

This means that, no matter how small the initial probability that A is the commitment type, for high enough discount factors, A takes almost the whole pie.

The equilibrium constructed here has immediate agreement at period 0, and also immediate agreement in any subgame starting from A making an offer or threat. That is, in even periods A makes an offer which B accepts, while in odd periods A makes a threat which B accommodates. The offers in even periods satisfy the no slackness condition; while there is a bit of slackness in odd periods but this is disappearing in the limit as $\delta \to 1$.

Finding the no slackness equilibrium for the complete range of parameter values is

[4] This holds for δ close enough to 1

not practical[5]. However, logically we can make some conclusions about it. Slackness reduces the payoffs of player A and increases the payoffs of player B, since for every PBE with slackness, there exists another PBE in which A demands slightly more and B accommodates the demand. Therefore the conclusion of Theorem 7 must also hold for the no slackness equilibrium.

The construction of the PBE of Theorem 6 relies upon the following property in odd periods: if B was to deviate away from accommodating the offer, then his next best action involves making an offer which A always rejects for sure would be better for him than making an offer A might accept. This property makes it far easier to calculate continuation payoffs off the equilibrium path, and hence avoids the difficulties encountered when trying to calculate the no slackness equilibirium. This property is shared by the no slackness PBE in the finite horizon version of this game and consequently much of the analysis is shared with this case.

Consider the finite horizon ($N+1$ period) version: If by the end of period N agreement still has not been reached then the game ends with both players receiving payoff of zero[6], otherwise the model is the same as laid out at the start of Section 2.

Lemma 8. *Consider the $N+1$ period finite horizon version with parameters (p_A, δ, N) satisfying $\frac{\delta^N}{1-\delta} \geq \frac{1}{p_A}$. The PBE satisfying the no slackness condition has payoffs*

$$U^A = 1 - \delta^N (1 - p_A), \; U^B = \delta^N (1 - p_A)$$

The proof is by backwards induction argument and is relegated to the appendix, but the idea is as follows: In even periods, there are no surprises as A makes the highest offer which B should accept - that is the offer equal to the continuation payoff if he rejects. The key to calculating this equilibrium is the analysis in odd

[5]There is a sequence $1 > a_1(\delta) > a_2(\delta) > a_3(\delta) > \ldots > 0$ such that the equilibrium strategies constitute a different function of parameters for each interval $(a_{i+1} a_i)$ which the belief may fall into

[6]Considering different disagreement payoffs does not change the structure of the equilibrium found here

periods. Consider what would have happened had A set the threat $y = 1$, so that B must optimise by offering something A^C rejects. The big question is then what offers A^R accepts. This depends on his continuation payoff which is an increasing function of the belief B attaches to A being of commitment type. If A^R accepts with probability 1 then after a rejection B believes A is of commitment type for sure, meaning that A takes the whole pie in the continuation game. So any offer below δ must be rejected by A^R with positive probability, and lower offers are accepted with lower probability.

The proof shows that after an unreasonable threat like $y = 1$, given a high enough discount factor so that $\frac{\delta^N}{1-\delta} \geq \frac{1}{p_A}$, player B does best by offering something that A^R rejects for sure. This means that any threat made by A which leaves B with more than his continuation payoff from causing a one period delay should be accommodated. Therefore in equilibrium A sets a threat leaving B with pecisely this amount, and B accommodates leading to immediate agreement.

The reason that for such a high discount factor B is better off offering something A is certain to reject than offering something which elicits a mixed response from A^R is the following: If B offers something below the threat and A^R accepts for sure then after a rejection A is known to be the commitment type and so gains a payoff of 1 in the continuation game. Thus any offer below δ must be rejected by A^R with some probability. If B offers something to which A^R mixes between acceping and rejecting, then the posterior beliefs of the commitment type increase, which lowers B's continuation payoffs after a rejection. Given the paramter values assumed in Lemma 8, this reduction in continuation payoffs after a rejection outweighs the gain from the possible immediate agreement. Player B would be better off $\delta^N (1 - p_A)$ deliberately delaying agreement until the last period, and then offering 0 which A accepts with probability $(1 - p_A)$. Lemma 8 says that for sufficiently high discount factor, player B cannot do any better than this and so in effect, has no proposer power until the last period. This enables player A to take almost the entire pie in the first period.

With Lemma 8 in place, the proof of Theorem 6 becomes a lot simpler. We can construst an equilibrium with payoffs as follows: in any odd period in which the belief of A being the commitment type is q_o, the continuation payoff to B is $\delta^{K-1}(1-q_o)$; while in even periods with belief q_e, the continuation payoff to B is $\delta^{K}(1-q_e)$ for some $K \in \mathbb{R}$ satisfying $\frac{\delta^K}{1-\delta} \geq \frac{1}{p_A}$. That B cannot do better follows the same logic as Lemma 8, while a form of threatening by beliefs is used to ensure that A cannot profitably deviate - a deviation by A would be interpreted as coming from the rational type.

Discussion of reputation effects

The result of Theorem 7, based on the PBE of Theorem 6 is that for any positive chance of A being the commitment type, as players become increasingly patient, A takes the whole pie, whereas with no probability of commitment type, the pie is shared equally. Where does this dramatic shift come from which enables A to take B's half of the pie? As shown in Lemma 5, the direct effect of enabling A to become committed to a threat with probability p_A every odd period has an effect only in proportion to the size of p_A. So for small probabilities the effect is fairly negligible.

The explanation lies in the difference between the models studied in Lemma 5 and Theorem 6 - that is the ability of A to build a reputation for being of commitment type. When p_A is small and $\delta \to 1$ it is this reputation effect which accounts for almost the entire half of the pie which would otherwise remain with B. Under the stochastic commitment asumption if A^R rejects a reasonable offer from B then this does not gain him anything. In future periods B still attaches the same probability to A being committed. However, in the main model, where players can be of commitment type, they can crucially build a reputation for being of commitment type. So rejecting a seemingly reasonable offer can have the consequence that in future periods B thinks it is more likely that A will be of commitment type. This gives A^R an incentive to reject even very high offers with some probability, and lower offers with far greater probability. Suppose A sets an unreasonable high threat and so B makes an offer A^C rejects, but that A^R should accept with mixed probability. Now there are two effects working against B: firstly there is a significant probability that the offer is rejected, thus eroding his bargaining power, and secondly if rejection occurs there is a higher subsequent belief that A is of commitment type and so the first problem intensifies for future periods. Of course player A will not set such unreasonable threats in equilibrium, but knowing the problem B faces, can exploit this by making the threat as tough as B is willing to accommodate.

1.3.3 Other PBE

There are two classes of other equilibria. The first is efficient PBE in which there is a greater level of slackness imposed on the offers and threats of A. Allowing for such equilibria gives a continuum of outcomes from those discussed above to the players receiving equal shares. A consequence of this multiplicity of efficient PBE is the existence of a class of inefficient PBE, many of which have considerable delay with up to half the pie being wasted as a result. This remains true in both the finite and infinite horizon model. The results are presented in the infinite horizon model.

Equilibria with greater slackness

The PBE of Theorem 6 was just one of many efficient PBE. There are others in which player A does slightly better (although these are hard to characterise) and many in which A does much worse. These are constructed by increasing the slackness on the threats or offers of player A - that is to force him to play a more generous strategy towards B by stipulating that B believes A to be the rational type otherwise. The most extreme example of this is the PBE with complete slackness (see appendix). In this PBE play simply proceeds according to the Rubinstein equilibrium, so in even periods, regardless of type, A offers $\frac{\delta}{1+\delta}$, keeping $\frac{1}{1+\delta}$ for himself and in odd periods threatens to reject offers below $\frac{\delta}{1+\delta}$. Player A is prevented from deviating to harsher offers or threats by stipulating that any such deviation would be met by the belief that he is rational for sure. In a similar way, any payoff between that in Theorem 6 and the Rubinstein outcome can be supported as a PBE as the following theorem states.

Theorem 9. *Let parameters* $(p_A, \delta) \in (0,1)^2$ *satisfy* $\psi(p_A, \delta) \leq \frac{\delta}{1+\delta}$. *Consider the interval*

$$J = \left[\psi(p_A, \delta), \frac{\delta}{1+\delta}\right]$$

For all $s \in J$, *there exists a PBE with payoffs*

$$U^A = 1 - s \; U^B = s$$

Inefficient PBE

The idea of these PBE is to use the above multiplicity of PBE to force players to play strategies resulting in delay using the threat of a disadvantageous PBE should they deviate. Doing this we can restrict A to a payoff close to the Rubinstein payoff of one half and restrict B to a payoff close to 0.

Theorem 10. *Let parameters* $(p_A, \delta) \in (0,1)^2$ *satisfy* $\psi(p_A, \delta) \leq \frac{\delta}{1+\delta}$. *For any* $(\alpha, \beta) \in [0,1]^2$ *and* $\tau \in \mathbb{N} \cup \{0\}$ *satisfying*

$$\alpha \geq \frac{1}{1+\delta}, \; \beta \geq \psi(p_A, \delta), \; \alpha + \beta = \delta^\tau$$

there exists a PBE with payoffs $U^A = \alpha$, $U^B = \beta$.

See appendix for proof. Note that in these PBE, agreement takes place in period τ and the set of PBE in Theorem 9 correspond to the $\tau = 0$ case.

Taking everything together, the message is that a wide variety of outcomes are consistent with the PBE solution concept. Some equilibria are efficient, others are not. All that is known is that player A is guaranteed half the pie, while player B is guaranteed almost nothing as players become increasingly patient. The intuition behind this is that player A retains all the proposer from even periods when he is the proposer, and can thus guarantee himself half the pie. However, depending on the chosen equilibrium, the proposer power in odd periods could stay with B (as in high slackness PBE), be stolen by A (as in low slackness PBE), or disappear (as in inefficient PBE). Among these various PBE, for the reasons given in Section 2 and Section 7.1, I find the most compelling to be those with little or no slackness. These have the property that as players become increasingly patient, the reputation effects allow A to steal almost all of B's propser power, and hence take almost the entire pie.

1.4 With two sided uncertainty

I now look at the game where both A and B might be of commitment type, that is to consider $(p_A, p_B) \in [0, 1]^2$.

By the same logic as before, we get multiplicity of PBE. In fact the multiplicity of PBE now is even greater because we can get threatening by beliefs whenever two types of player B are required to act strategically as well as player A. To start with, it is worth thinking about what would happen if both players were known to be the commitment type for sure. Here there is only one type of each player and so the multiplicity described above disappears. In this game, in even periods A knows the only chance of reaching agreement that period is to accommodate B's threat, so B

steals all the proposer power, with the converse happening in odd periods. Thus the game is practically identical to the game where both types are rational for sure but their identities are swapped over. As such, in even periods the threat B makes coincides exactly with the demand A makes in Rubinstein SPE for the game with both players being rational for sure. This gives:

Lemma 11. *If $p_A = p_B = 1$ then the unique SPE has payoffs*

$$U^A = \frac{\delta}{1+\delta},\ U^B = \frac{1}{1+\delta}$$

1.4.1 With B being the commitment type for sure

The next step is to look at what happens when B is the commitment type for sure, that is to consider $p_B = 1$. The case of $p_A = 1$ is covered above, while if $p_A = 0$ the whole pie goes to B as discuused in Section 3, so I now restict attention to $p_A \in (0,1)$. Now once again we have multiplicity of equilibria due to the possibility of threatening by beliefs whenever player A makes any offers or threats. I start by showing that with little or no slackness player A manages to get half the pie as $\delta \to 1$.

Once again calculating the no slackness PBE is problematic for the same reason as in Section 3. For this reason, the PBE I give here has a little slackness in odd periods, although the amount of slackness tends to 0 as $\delta \to 1$.

Theorem 12. *Let $p_B = 1$. Define*

$$\psi\left(p_A,\delta\right) = \frac{1-\delta^2\left(1-p_A\right)}{p_A\left(1+\delta\right)}$$

Given parameters $(p_A,\delta) \in (0,1)^2$ such that $\psi\left(p_A,\delta\right) \in [1/2, 1]$ (which holds for δ large enough[7]) *there exists a PBE which gives payoffs $U_A = 1-\psi$ and $U_B = \psi$.*

The proof is in the appendix.

[7] $\psi\left(p_A,\delta\right)$is a decreasing function of δ and $\lim_{\delta\to 1}\psi\left(p_A,\delta\right) = 1/2$

Noticing that for any $p_A > 0$, $\lim_{\delta \to 1} \psi(p_A, \delta) = 1/2$ gives the following:

Theorem 13. *For any $\varepsilon > 0$ and $p_A > 0$, there exists $\bar{\delta} \in (0, 1)$ such that for any discount factor $\delta > \bar{\delta}$, there is a PBE with payoffs $U^A > 1/2 - \varepsilon$ and $U^B < 1/2 + \varepsilon$.*

The construction of the the equilibrium in Theorem 12 relied on having a little bit of slackness in odd periods, so that by accommodating A's threat B gets a strictly higher payoff than any alternative offer. Player A is prevented from increasing the threat slightly by beliefs off the equilibrium path that stipulate A would be believed to be rational for sure after any such deviation. The no slackness equilibrium would allow A to make slightly more aggressive demands without changing B's beliefs about his type and would thus reuslt in a slighly higher payoff for A. But as Theorem 13 shows, this equilibrium is sufficient for showing that, for any positive prior probability of A being the commitment type, A can get arbitrarily close to half the pie as $\delta \to 1$. The intuition is that as $\delta \to 1$ the cost of mimicking the commitment type decreases, meaning that A finds it less costly to build a reputation for being the commitment type. This allows him to do just as well as when he is believed to be the commitment type for sure, when he gets half the pie as in Lemma 11. In terms of proposer power, B steals all of A's proposer power from even periods and as $\delta \to 1$ player A steals (almost) all of B's proposer power in odd periods.

If we allow slackness then there is a wide range of efficient PBE outcomes. In the most extreme of these, where we have complete slackness, player B gets the whole surplus. We achieve this by the following play in period 0: B sets the threat $y = 1$ and A offers $x = 1$. We sustain this by stipulating that beliefs after any other offer would be that A is rational for sure, thus meaning any other offer would also give him payoff of zero. Just as in Section 3, there is a whole continuum of other efficient PBE and using the multiplicity, a large range of inefficient PBE. Theorem 14 characterises some of these:

Theorem 14. *Let* $p_B = 1$ *and* $(p_A, \delta) \in (0,1)^2$*. For any* $(\alpha, \beta) \in [0,1]^2$ *and* $\tau \in \mathbb{N} \cup \{0\}$ *satisfying*

$$\alpha \geq 0,\ \beta \geq \psi(p_A, \delta),\ \alpha + \beta = \delta^\tau$$

there exists a PBE with payoffs $U^A = \alpha$, $U^B = \beta$.

We conclude from this, that as $\delta \to 1$, B is guaranteed half the surplus, while the other half could go anywhere. In the no slackness equilibrium A takes the proposer power from odd periods and so takes the other half of the surplus. In the complete slackness equilibrium B holds on to the proposer power in odd periods and so takes the whole surplus. There are also ineffecient PBE where some or all of the unclaimed half is wasted through delay.

1.4.2 With uncertainty about the type of both players

Now I assume $(p_A, p_B) \in (0,1)^2$. This creates a greater multiplicity of PBE, so great in fact that almost any share of the pie can be sustained as a PBE. The reason is the potential range in ability to steal the opponents proposer power: if we impose little or no slackness on a player's offers and threats, then that player can steal almost all their opponents proposer power as players become increasingly patient. Whereas by imposing complete slackness on a player's threats and offers, we restrict that player to act as if he was rational for sure and so be unable to steal any proposer power when it is the opponent's turn to offer. This means that if we impose little slackness on A and complete slackness on B then A steals almost all B's share of the proposer power, while hanging on to his own, and so receives almost the entire surplus. Likewise, we can construct PBE in which B receives almost the entire surplus, or use this multiplicity to construct inefficient PBE in which both players receive close to 0, or virtually anything else in between.

Theorem 15. *Let* $(\delta, p_A, p_B) \in (0,1)^3$ *satisfy* $\frac{(1-\delta)(1-p_i)}{p_i} \leq \frac{\delta}{1+\delta}$ $i \in \{A, B\}$*. For any*

$(\alpha, \beta) \in [0,1]^2$ and $\tau \in \mathbb{N} \cup \{0\}$ satisfying

$$\alpha \geq \frac{(1-\delta)(1-p_B)}{p_B}, \; \beta \geq \frac{(1-\delta)(1-p_A)}{p_A}, \; \alpha + \beta = \delta^\tau$$

there exists a PBE with payoffs $U^A = \alpha$, $U^B = \beta$.

See Apppendix for proof.

Using the reults shown throughout this paper, for the limiting case of $\delta \to 1$, we can summarise the range of admissible outcomes under PBE as follows:

1. If a player is the commitment type for sure then he is guaranteed a payoff of at least one half.

2. If a player is the rational type for sure then the opponent is guaranteed a payoff of at least one half.

3. Almost any payoffs which satisfy the above two conditions are possible.

The intuition behind the first statement is that if player i is known to be the commitment type for sure then every time player $j \neq i$ comes to make an offer, player i can steal all the proposer power every time. Since this happens every other period, player i holds at least half the bargaining power and so receives payoff at least half. On the other hand if player i is rational for sure, then he has no ability whatsoever to steal any proposer power from j aand so player j retains his half of the bargaining power, which explains the second statement. The third statement follows because when there is uncertainty about the type of player i there is such great multiplicity of equilibria. In PBE with high levels of slackness on i's decisions, player j retains his share of the proposer power, while with low levels of slackness i steals the proposer power from j. Or we can construct inefficient PBE where j loses his proposer power but not to i. In these equilibria both players are forced into actions causing delay because failure to play along would result in an undesirable equilibrium in the continuation game.

The more compelling PBE

Returning to the case of $(p_A, p_B) \in (0,1)^2$, Theorem 15 shows that almost any payoffs can result, however I would argue that some PBE are more natural than others. Unfortunately once again, constructing a PBE with no slackness is too problematic in the infinite horizon game, however we can gain some insight by looking at Theorem 13. If player B is the commitment type for sure then in a PBE with little slackness, player A can gain half the pie. Now reducing the probability of B being the commitment type increases player A's bargaining power at the expense of player B, and so A should still be able to take half the pie. A symmetric argument implies that B can gain half the pie when there is little slackness on his threats and offers. Hence we get the result that for any $(p_A, p_B) \in (0,1)^2$, as $\delta \to 1$, we would expect an equal split of the surplus.

This is in stark contrast to the infinite horizon results of [23] and [1] which showed that the expected divisions depends critically on the players' relative probabilities of being the commitment type. The intuition for the difference is as follows: Consider the infinite commitment model in which players have made incompatible demands, player A has a low probability of being the commitment type and B has a high one. Player A has little incentive to delay and build a reputation for being the commitment type, since there is a high probability that B will be unable to concede to his demand anyway. Whereas with finite commitment, even if B is the commitment type, he can concede to A's demand and is likely to do so if he believes A is likely to be the commitment type.

1.4.3 Other PBE

In the finite horizon version of the game it is possible to construct the no slackness PBE using backwards induction. However in the infinite horizon case, while such a

PBE will exist, it is not practical to construct it for a wide range of parameter values.[8] So in the $p_B = 0$ case of Section 3, there are PBE giving A more of the pie than in Theorem 6, which I have been unable to characterise, which is unfortunate, since these seem the most compelling. However on the bright side, as $\delta \to 1$, the payoff to A in the Theorem 6 PBE approaches 1, and so the distance in payoffs between this PBE and the no slackness PBE must tend to 0. Similar conclusions hold with respect to the PBE in Theorem 12 for the $p_B = 1$ case.

With two sided information, as already mentioned, the outcome of the no slackness PBE gives half the pie to each player and so is included in the set of PBE payoffs in Theorem 15. However, the set of PBE outcomes in Theorem 15 still is not the complete set of PBE outcomes. For example, the equilibrium with complete slackness on B and none on A would give B a payoff less than β. Although as $\delta \to 1$, the set described in Theorem 15 approaches $\left\{(\alpha, \beta) \in (0,1)^2 \mid \alpha + \beta \leq 1\right\}$, and so must approximate the complete set of PBE outcomes.

There is another source of additional PBE. Throughout I have only considered equilibria whereby both players' equilibrium payoff is independent of type. However, especially with uncertainty on the type of both players, it is possible to construct PBE whereby for one of the two players, the payoff to the rational type is higher than that to the irrational type. In order to construct such a PBE, it must be that the threatening player (say A) should make a high threat, which is achievable by stipulating that he gets punished by a bad equilibrium if he doesn't. The proposing player then makes a reasonably generous offer (has to be over half) which is accepted by A^R but not A^C. Then the PBE played in the continuation game gives A^C less

[8] In the one sided uncertainty case of Section 3 ($p_B = 0$) I can construct it with , $p_A \geq \delta^6$, however this is of little interest when we are generally concerned with the case of $\delta \to 1$. To construct the no slackness PBE for a larger range of parameters becomes extremely challenging as the PBE takes the following form (where q is the belief on A being the commitment type): For $q \geq q_1 = \delta^2$ we can find formulae for the continuation payoffs, depending on the identity of the proposer, we then use this to construct formulae for $q \in [q_2, q_1] = \left[\delta^6, \delta^2\right]$. The next step is then to construct formulae for $q \in [q_3 q_2]$ for some q_2 and keep proceeding in the same manner. However at each step the formulae become increasingly complex.

than A^R has already accepted.[9] When one player's type is known, it is far harder to construct such PBE. Take for instance $p_B = 0$ then in any such PBE, the continuation payoff A^C receives next period is 1 as he will be known to be commitment type for sure. Thus the offer made by B which only A^R can accept must have been at least δ for A^R to accept for sure. In addition this means that as $\delta \to 1$ the difference in possible payoffs between commitment and rational types tends to zero. Similar logic holds for $p_B = 1$ (as $\delta \to 1$) player B is guaranteed half the pie and the continuation payoff to A if revealed to be the commitment type tends to $1/2$.

1.5 Conclusion

There is a significant literature on how introducing commitment affects the outcome of bargaining. However, in the main part, this literature assumes that players, once they become committed to a position, stay committed until the end of the game. By making this assumption and assuming that either commitment types have a pre-determined demand [33, 1] or that players do not know their type when making a demand [42, 23], these papers produced nice clean results. This paper drops the assumption of finite commitment and replaces it with finite commitment, which means that the commitment type necessarily becomes a *strategically irrational* player as opposed to the unthinking automaton in previous papers. This creates the problem that when a player who could be of either type acts, we can use threatening by beliefs to create an awkward multiplicity of PBE. Section 2 discussed this problem and suggested that attention should be focused on the PBE exhibiting no slackness. Sections 3 and 4 have found PBE approximating the payoffs in the no slackness euilibrium for one and two sided uncertainty and also characterised the other PBE.

[9] As an example of this consider the following: Both types of A make threat $y = 1$ and then B offers $x = 0.7$, which A^R accepts if either player deviates, he is punished by a PBE giving the deviating player utility of 0.1. In the continuation game A is known to be commitment type for sure and the PBE played gives A^C utility of 0.6.

In the one sided uncertainty case it is well established that the possibly committed player uses the reputation effects to take all the pie. In the finite commitment case, restricting attention to the no slackness PBE this is still true. Although in the finite commitment case there will also be many other PBE giving a large range of outcomes. Depending on how we specify the infinite commitment game, these can also persist in the infinite commitment case, but previous authors have understandably made assumptions that eliminate this multiplicity. If we take the no slackness equilibrium to be the most compelling one, then we can conclude that the predictions under finite commitment coincide with those under infinite commitment.

If there is uncertainty about the type of players, again a similar statement is true: with both finite and infinite commitment, almost any division of the pie can result, including inefficient ones in the finite commitment case. More interesting is what happens when we restrict attention to the equilibrium without threatening by beliefs. With finite commitment as $\delta \to 1$ the PBE with little or no slackness have players sharing the pie evenly, regardless of the exact probabilities of commitment, as long as both are positive. That is, as players become increasingly patient, the exact probabilities of being the commitment type at the start of the game have decreasing significance, so long as both are positive. By contrast, with infinite commitment, Kambe found that the shares both players receive depends critically on the relative probabilities of the two players being the commitment type. Thus with two sided uncertainty, we have a subtle but significant difference between the results of finite and infinite commitment.

In my finite commitment model, we can describe any equilibrium as $\delta \to 1$ in terms of adding up proposer power. In odd periods: if $p_A = 0$ then player B keeps all his proposer power; if $p_A \in (0, 1)$ then in the no slackness equilibrium A steals all the proposer power, while in the wider set of PBE any distribution of proposer power can result, including those where some proposer is lost (inefficient equilibria);

if $p_A = 1$ then A takes all the proposer. An analogous analysis describes B's ability to steal proposer power in even periods. The resulting PBE payoffs for each player is then the average of his proposer power in even and odd periods.

1.6 Further research

The first area I see is to tackle the multiplicity of PBE issue better. The concept introduced here of slackness has not been formally defined mathematically. It would be good to do so and construct a sensible measure for slackness. None of the PBE constructed here in the infinite horizon satisfy no slackness, however some of the PBE constructed do approach the payoffs that the no slackness PBE must give as $\delta \to 1$. Thus it would seem logical that in such a PBE the slackness measure would tend to 0 as $\delta \to 1$. A second aim in this area is to construct a sensible refinement to justify the no slackness PBE.

The second area is to investigate what happens as we vary the length of commitment, and if we allow players to have different lengths of commitment. Throughout this paper, I assumed that the commitment type is a one period commitment type, which means that in every period players can agree even if both are the commitment type. Crucially it means that the cost of building a reputation for the commitment type, is small since the size of the pie will only have shrunk to δ before players get another chance to negotiate. By contrast, consider the case where players' commitment lasts for k periods and $p_B = 1$. Now if player A wishes to build a reputation for being of commitment type, he must wait k periods before B can accept, by which time the pie will have shrunk to δ^k, which is in effect the new discount factor. This makes it more expensive for A to build a reputation for being of commitment type. This leads me to conjecture that as the length of commitment increases, the prior probabilities with which each player is the commitment type become increasingly important. Although this logic suggests that as $\delta \to 1$, it would require k to increase to ∞ sufficiently quickly to keep δ^k bounded away from one to have a significant effect.

As already argued, I would expect that as the length of commitment increases, the probabilities of commitment become more relevant. When players have differing lengths of commitment, I would expect the advantage to lie with the player who has the larger commitment length, although how large that commitment would be is not obvious. Another thing to consider is the timing of commitment as well as the length. For example, if both players can commit for a week, then the player who commits every monday is at an advantage against the player who commits every tuesday. The reason is the following: The monday player, by re-committing instead of giving in to the opponenent's demand only delays agreement by one day, whereas the tuesday player doing the same would cause a six day delay before agreement can be struck again.

1.7 Appendix

1.7.1 For Section 2

To simplify matters, I introduce a slightly simpler sender-receiver game which has a multiplicity of PBE problem caused by the same phenomenon as in our bargaining game.

Informed seller game:

A seller, player A, owns some product, which can be high quality (A^H) or low quality (A^L). A buyer, player B, values the product to be worth 1 if it is high quality and to be worth 0 if it is low quality. A has no value for the product. The quality of the product is known to A but not B, who believes it is the high quality with probability p, with all this being common knowledge. A announces a price and B decides whether or not to buy. Naturally, the seller's payoff is the price he receives if the buyer buys, while the buyer gets payoff of the expected value of the good minus expenditure.

In solving this game, we can proceed via backwards induction, noticing that B should buy if and only if the probability of the high type is at least the price, i.e. $\mu(z) \geq z$, where $\mu(z)$ is the probability of A^H in the updated belief after announcement of price z. Combine this with the observation that both types of player A have the same payoff function, so if a strategy is optimal for one type then it must be optimal for the other type, and we are naturally led to the following equilibrium: both types of seller always set price $z = p$ and B buys after any $z \leq p$. Beliefs following any price are the same as the prior beliefs i.e. $\mu(z) = p$ for any offer z. This gives (ex ante) expected payoffs of p to the seller and 0 to the buyer.

However, there are also many other PBE with certain trade, hence Pareto efficiency, but with a different split of the surplus, as well as ineficient PBE[10].The idea is to threaten with beliefs so that both types set a low price, because a higher price would be interpreted by B as meaning the seller has the low quality product. Note that $U^{A^H} = U^{A^L}$ in any PBE, since if $U^{A^H} > U^{A^L}$ then A^L could profitably deviate by copying the strategy of A^H, and vice-versa if $U^{A^H} < U^{A^L}$. Thus I write $U^A = U^{A^H} = U^{A^L}$.

Lemma 16. *The above game has the following set of pure strategy pooling PBE outcomes, in which trade always takes place: For each $s \in [0, p]$ there exists a PBE with $U^A = s$, $U^B = p - s$.*

Proof. Fix $s \in [0, p]$ and consider the following strategies and beliefs. Both types of A set price $z = s$, B buys iff $y \leq s$ and beliefs are $\mu(z) = p$ when $z \leq s$ and $\mu(z) = 0$ otherwise. This clearly defines a PBE since B acts optimally given beliefs, A acts optimally given B's strategy and beliefs are updated via Bayes' rule on the equilibrium

[10]Consider a pure strategy pooling PBE where both types of seller sets a price above p and any other price is believed to have come from A^L. Here the buyer refuses to buy and no trade occurs. There is also a wide spectrum of mixed PBE in which trade sometimes occurs - see appendix 7.1.

path. There are no equilibria whatsoever with $U^A > p$ because the aggregate utility in the game is equal to p, and B's utility in any equilibrium is bounded from below by zero, the payoff from rejecting all offers. □

The expected value of the gains from trade in this problem is p. Notice that these PBE admit any split in the gains from trade, despite A having all the bargaining power from being the sole proposer. This is in contrast to when A is of known type or when A only discovers his type after setting a price. In both of these instances, the outcome is unique: player A would take the entire surplus.

With the above notion of splitting the surplus in mind, I introduce the idea of slackness and a measurement of it for pure strategy pooling PBE:

- Stage 0: Nature selects the type of player A: The "high quality" type, A^H, has probability p; the "low quality" type, A^L, has probability $1-p$.

- Stage 1: A selects price $z \in [0,1]$

- Stage 2: B (after observing z and updating belief of A^H to some $\mu(z) \in [0,1]$) chooses buy (b) or not buy (nb)

- Payoffs are $U^A(z,b) = z$, $U^A(z,nb) = 0$, $U^B\left(z,b,A^H\right) = 1-z$, $U^B\left(z,b,A^L\right) = -z$, $U^B\left(z,nb,A^H\right) = U^B\left(z,nb,A^L\right) = 0$.

To see this suppose that the two types of seller play mixed strategies over some set of prices Z in such a way that for any price $z \in Z$ the posterior probability of the seller being high quality is z. This leaves the seller indifferent between buying or not, and so allows for a PBE whereby for, each price, the buyer mixes betweeen buying and not in such a way that all prices give the seller the same expected revenue. As an example, let $p = \frac{1}{2}$ and consider the following strategies where A mixes over $Z = \left\{\frac{1}{3}, \frac{2}{3}\right\}$: A^H sets price $\frac{1}{3}$ twice as frequently as price $\frac{2}{3}$, while A^L does the opposite. This gives $\mu\left(\frac{1}{3}\right) = \frac{1}{3}$ and $\mu\left(\frac{2}{3}\right) = \frac{2}{3}$ and B responds by always buying after price of $\frac{1}{3}$ and half the time after price of $\frac{2}{3}$. After any other price A is believed to be A^L for sure, ie $\mu(z) = 0$ for all $z \notin Z$ and so B never buys. It is easy to see this constitutes a PBE. In fact, using similar logic, it is even possible to construct PBE where Z is the entire price space and so there are no prices off the equilibrium path, and so no hope of applying refinements on beliefs off the equilibrium path.

On the search for refinement

Due to the observation about the seller's payoff function being type independent, it is not possible to refine the set of equilibria using standard refinements such as the intuitive criterion [8] or divinity [4] which rely on some actions being more suitable for some types than others.

Kambe's solution was to use a modification of perfect sequential equilibrium (PSE) [19] which he clled semiperfect sequential equilibrium. The idea here behind PSE is that when the seller announces an unexpected price, the buyer should ask "what is the seller trying to tell me? Who benefits by setting this price?" Roughly speaking, the buyer should look for a consistent belief: that is a subset of types K such that the set of types who gain from this deviation is precisely K given that the seller believes K is the set of types who would make this deviation. However things are greatly complicated by [19] having two defeinitions of PSE. Following their language (p.101-103 of their paper) I call these the "rough" and the "formal" definition. [11] As will be seen, the difference between the two is important. I am of the firm opinion that the "formal" definition is the more proper one, since the "rough" definition allows for strange inconsistencies as the following example shows.

Example 17. Investment Game

[11] The Grossman and Perry paper (1986) is less than clear on this point. In defining perfect sequential equilibrium for sender-receiver games, they first define it roughly, whereby those types who are indifferent must either always take part in the deviation or never do so (p.101). They then define it more formally, allowing indifferent types to take part in the deviation with any probability between 0 and 1 (p102-103). Then they define perfect sequential equilibrium in multi-period games building on the "rough" definition (p.115).

Entrepreneur (E) has two equally probable types $\{G, B\}$, meaning Good and Bad. He offers a financier (F) either $1/3$ or $1/2$ of the company for an investment of 1. If F invests, then the good entrepreneur uses the money well and the company becomes worth 4 units, whereas a bad entrepreneur fails to make any profitable use of the money and the company's value remains at 1. There are two pooling PBE:

1. Entrepreneur pools on $1/3$ and financier invests.

2. Entrepreneur pools on $1/2$ and financier invests. This requires F to believe that E is probably[12] of Bad type if he offered $1/3$.

The "rough" definition of PSE eliminates the second PBE because after the offer $1/3$ the only consistent belief places probability $1/2$ on each type, supported by $K = \{G, B\}$. With these beliefs, both types of E deviate to offer $1/3$.

Now consider a slight adaptation of the above game: the Good type is split in two so that the types are now $\{G_1, G_2, B\}$ which occur with probabilities $(1/6, 1/3, 1/2)$. Now the "rough" definition of PSE fails to remove the second PSE, because after an offer of $1/3$, a belief of $2/5$ on G_2 and $3/5$ on B is consistent, supported by $K = \{G_2, B\}$ and the financier investing with probability $3/4$ after an offer of $1/3$.

Thus the "rough" definition of PSE gives different results for two games which are strategically equivalent. The "formal" definition of PSE is not subject to this criticism. It never eliminates the second PBE.

Returning to the informed seller game, consider a PBE where both types of seller pool on the price $s < p$. Does PSE refine away such a PBE? To answer this, consider what the buyer should believe after observing a price $z \in (s, p)$. It is a consistent belief to keep the posterior beliefs supported by the assumption that both types of seller make this deviation. Under the "rough" definition of PSE this is the only consistent belief and hence the seller has an incentive to deviate to this price z. However, when the formal definition of PSE is applied, it is also a consistent belief to believe that

[12] with probability at least $3/5$

the probability of high quality is z (the buyer buys with probability s/z), and under this belief neither type of seller gains by deviating to price z. Hence the "rough" definition of PSE rules out this equilibrium, while the formal definition does not. Kambe's semiperfect sequential equilibrium does refine away pooling equilibria with prices below p and so is not a weakening of the "formal" definition of PSE.

In my opinion, the most compelling argument for refinement of those PBE with slackness comes from the work on Neologism proofness, due to Farrell [15, 16]. This was introduced to refine the set of PBE in cheap talk games, although it can be extended in a routine manner for more generall games as discussed in [19]. This allows a player to make a statement of the form "my type is in the set K" and if this statement is credible then it should be believed and beliefs updated accordingly. Farrell says a statement should be believed if the set of types who gain from this message being believed is precisely K, although other papers ([28], [37]) have disagreed over which statements should be accepted as credible. However, in this specific setting here, we have the seller's payoff function being independent of type, and so the disagreements over which statements should be viewed as credible disappear.

Neologism proofness refines away all PBE except the no slackness PBE. The reason is straightforward: the no slackness PBE gives the seller a payoff of p, while all other PBE give the seller a smaller payoff. Consider another PBE giving the seller payoff $s < p$. Now the seller deviates to offering a price of $\frac{s+p}{2}$, accompanied by the message "I would make this move regardless of which type I am". Such a statement is credible and so the buyer believes that the probability of high quality is still p and so buys at this price. This refinement is also supported by [27] who define the notion of "undefeated equilibrium" which refines away all the pure strategy PBE except the no slackness one. There is no refinement or no argument that I am aware of that questions the validity of the no slackness PBE.

Period	Player	Action
Even	A	offers x_A
	B	accepts iff the offer is $\geq x_A$
Odd	A	threatens to reject any offers $< y_A$ (A1)
		(when not committed) accepts iff the offer is $\geq \hat{x}_B$ (A2)
	B	after any threat $y \in [\hat{y}, y_A]$ offers y (B1)
		otherwise offers $\hat{x}_B$ (B2)

Table 1.7.1: Equilibrium strategies with no updating of beliefs

1.7.2 For Section 3

I will prove Lemma 8 with quite detailed explanation before Theorem 6 since it gives some insight into the proof of the latter. Theorem 7 follows immediately from Theorem 6 and so its proof is omitted.

Proof of Lemma 5

Proof. Let $x_A = \frac{\delta(1-p_A)}{1+\delta}$, $y_A = 1 - \frac{1-p_A}{1+\delta}$, $\hat{y} = \frac{1-\delta}{p_A} + \delta x_A$, $\hat{x}_B = \delta\left(1 - x_A\right)$ and consider the strategies represented in Table 1.7.1 on page 41.

These strategies constitute a SPE.

To check this is a SPE is reasonably straightforward. In even periods this is clearly optimal for B since his next period continuation payoff from rejecting is $\frac{1-p_A}{1+\delta}$. Likewise it is clearly optimal for A since larger offers would leave him a lower share of the pie and smaller offers would be rejected leading to a payoff of $\delta\left(1 - \frac{1-p_A}{1+\delta}\right)$ which is less than $\frac{\delta(1-p_A)}{1+\delta}$, the payoff from offering x_A.

In odd periods the clause (A2) is clearly optimal for A since rejecting the offer would lead to him receiving $(1 - x_A)$ next period, giving payoff $\delta\left(1 - x_A\right)$. Given (A2), a straightforward calculation of B's payoffs shows that clauses (B1) and (B2) are optimal for B. Given (B1) and (B2), it is then clear that clause (A1) is the best strategy for A.

The proof of uniqueness is as follows: Let M_A (m_A) be the supremum (infimum) payoff for A in any subgame starting from an even period (where A is the proposer). Let M_B (m_B) be the supremum (infimum) payoff for B in any subgame starting from an odd period (where B is the proposer). I show that $m_A = M_A = 1 - \frac{\delta(1-p_A)}{1+\delta}$ and $m_B = M_B = \frac{1-p_A}{1+\delta}$, in agreement with the SPE in the table above. First I show that $M_A = 1 - \frac{\delta(1-p_A)}{1+\delta}$ by the showing the following two claims:

Claim 1: $m_B \geq (1-p_A)(1-\delta M_A) + p_A\delta(1-M_A)$. Consider an odd period. Note that the uncommitted A accepts offers above δM_A and the continuation payoff for B after a rejection is bounded below by $\delta(1-M_A)$. This means that for any $\varepsilon > 0$, B can guarantee himself $(1-p_A)(1-\delta M_A-\varepsilon) + p_A\delta(1-M_A)$ by offering $(1-\delta M_A-\varepsilon)$

Claim 2: $M_A \leq 1-\delta m_B$. Consider an even period. If B is guaranteed at least m_B next period, then his payoff this period is at least δm_B. Since the sum of players' payoffs is bounded above by 1, this implies that A cannot get more than $1-\delta m_B$.

Claim 3: $m_A \geq 1-\delta M_B$. Consider an even period. B must accept any offer of the form $\delta M_B + \varepsilon$, where $\varepsilon > 0$. This shows the claim since if $m_A < 1-\delta M_B$, it would be possible to find an $\varepsilon > 0$ such that A would get more than m_A by offering $\delta M_B + \varepsilon$.

Claim 4: $M_B \leq (1-p_A)(1-\delta m_A)+p_A\delta(1-m_A)$. Consider an odd period. Note that the uncommitted A rejects offers below δm_A and the continuation payoff for B after a rejection is bounded above by $\delta(1-m_A)$. This means that if A sets a threat y such that $1-y = (1-p_A)(1-\delta m_A) + p_A\delta(1-m_A) + \varepsilon$, where $\varepsilon > 0$ then B should accommodate the threat, setting $x = y$, giving B a payoff of $1-y$. Furthermore, if $M_B > (1-p_A)(1-\delta m_A)+p_A\delta(1-m_A)$ then for small enough ε it is in the interests of A to do this, since at this SPE, A is getting no more than $1-M_B$ which for small enough ε is less than $y = 1-(1-p_A)(1-\delta m_A) - p_A\delta(1-m_A) - \varepsilon$. Thus for each $\varepsilon > 0$, $M_B < (1-p_A)(1-\delta m_A) + p_A\delta(1-m_A) + \varepsilon$ and hence the claim follows.

Claims 1 and 2 taken together imply that $M_A \leq 1 - \frac{\delta(1-p_A)}{1+\delta}$ and $m_B \geq \frac{1-p_A}{1+\delta}$.

Claims 3 and 4 imply that $m_A \geq 1 - \frac{\delta(1-p_A)}{1+\delta}$ and $M_B \leq \frac{1-p_A}{1+\delta}$. Furthermore, by the SPE given in the table above $m_A = M_A = 1 - \frac{\delta(1-p_A)}{1+\delta}$ and $m_B = M_B = \frac{1-p_A}{1+\delta}$, thus the unique SPE shares are as claimed. □

Notation: since only A can be committed, I shorten p_A to p in the following proofs.

Proof of Lemma 8

First, a quick note on notation: In any period t subgame, I let x_t, y_t and ρ_t be the offers made players, the threats made by A and the probability with which A^R accepts an offer. At the start of a period t subgame, I let q_t be the probability B attaches to type A^C, and$U_t^i(q)$ be the continuation payoff to player i at the start of a period t subgame with beliefs q. When I talk about payoffs at a period t subgame, it is always in period t units, not the total utility from the game as a whole, which would be obtained by multiplying these payoffs by δ^t. Also note that, in any PBE, when a player is faced with the choice of accepting or rejecting an offer where he is indifferent between both actions, he will accept for sure. If there was a positive probability of rejection then the other player would be best off offering slightly more but the problem $\max_x \{1 - x : x > k\}$ has no solution and hence no equilibria of this form can exist. I proceed by backwards induction.

Proof. (Lemma 8) Period N: We first calculate the best option for B given that A^C always rejects (equivalent to when A sets $y_N = 1$), and then use this to determine the highest y that A can get away with setting. When A sets $y_N = 1$, the best option for B is $x_N = 0$, the lowest offer A^R accepts, which generates B utility of $1 - q_N$. So A should set y such that the payoff to B from accommodating, $1 - y$, equals this. That is, A sets $y_N = 1 - (1 - q_N) = q_N$, which generates

$$U_N^A(q) = q,\ U_N^B(q) = (1-q)$$

Period $N-1$: Both types of A offer B the lowest offer B will accept. This is $x_{N-1} = \delta U_N^B(q_{N-1}) = \delta(1-q_{N-1})$, generating payoffs

$$U_{N-1}^A(q) = 1 - \delta(1-q), \; U_{N-1}^B(q) = \delta(1-q)$$

Period $N-2$: Things start getting more complicated as we begin to see reputation effects. This is because A^R has an incentive to mimic A^C by rejecting offers that he would accept if B knew his type. As above, the method is to find the optimal strategies if A had set $y=1$ so that A^C rejects all offers less than one and use this to determine what level A should set y at. Given offer x, the payoff from accepting is simply x, but the payoff from rejecting is $\delta U_{N-1}^A\left(q_{N-2}^u\right)$ where q_{N-2}^u is the updated belief from q_{N-2} depending on the acceptance function A^R uses. The updated belief is calculated via Bayes' rule, and so for any $x_{N-2} < 1$ (since A^C rejects), must satisfy $q_{N-2}^u = 1$ if A^R always accepts and $q_{N-2}^u = q_{N-2}$ if A^R always rejects. Also note that $\delta U_{N-1}^A\left(q_{N-2}^u\right)$ is strictly increasing in q_{N-2}^u. So the payoff to A^R from rejecting is in the interval

$$I_{N-2} = \left[\delta U_{N-1}^A(q_{N-2}), \delta U_{N-1}^A(1)\right] = \left[\delta(1-\delta(1-q_{N-2})), \delta\right]$$

So A^R will accept for sure any offers above this interval and reject all offers below. For $x \in I_{N-2}$, A^R will mix between accepting and rejecting; accepting $x \in I_{N-2}$ with probability $\rho_{N-2}(x) \in [0,1]$. When A^R accepts for sure, that is $\rho_{N-2} = 1$, beliefs following a rejection are updated to $q_{N-2}^u = 1$ so A^R could get payoff δ by rejecting. So for any $x \in I_{N-2} \setminus \delta$ player A reduces ρ_{N-2}, lowering q_{N-2}^u and hence also $\delta U_{N-1}^A\left(q_{N-2}^u\right)$ until $\delta U_{N-1}^A\left(q_{N-2}^u\right) = x$, the point at which A^R is indifferent between accepting and rejecting. Putting all this together gives

$$\rho_{N-2}(x) = \begin{cases} 0 & x < \inf I_{N-2} \\ \frac{x-\delta+\delta^2(1-q_{N-2})}{(1-q_{N-2})(x-\delta+\delta^2)} & x \in I_{N-2} \\ 1 & x > \sup I_{N-2} \end{cases}$$

Note this is only the probability with which A^R accepts an offer $x < 1$, since A^C must reject. The total probability an offer is accepted is $a_{N-2} = (1 - q_{N-2})\,\rho_{N-2}$. Now we can write the expected utility to B from offering x. This is

$$u^B(x) = a_{N-2}(1-x) + (1-a_{N-2})\,\delta U^B_{N-1}\left(q^u_{N-2}\right)$$

where a_{N-2} and $U^B_{N-1}\left(q^u_{N-2}\right)$ have already been described as functions of ρ_{N-2}, which is in turn a function of x. Now B has to maximise $u^B(x)$ over x, which yields the following solution:

$$x = \begin{cases} \delta & \frac{\delta^2}{1-\delta} < q_{N-2} \\ \delta - \delta^2 + \delta\sqrt{q_{N-2}(1-\delta)} & q_{N-2} \leq \frac{\delta^2}{1-\delta} \leq \frac{1}{q_{N-2}} \\ < \delta(1-\delta(1-q_{N-2})) & \frac{\delta^2}{1-\delta} > \frac{1}{q_{N-2}} \end{cases}$$

which in turn generates the following payoffs for B:

$$u^B_{N-2}(q_{N-2}) = \begin{cases} (1-q_{N-2})(1-\delta) & \frac{\delta^2}{1-\delta} < q_{N-2} \\ 1-\delta+\delta^2-2\delta\sqrt{q_{N-2}(1-\delta)} & q_{N-2} \leq \frac{\delta^2}{1-\delta} \leq \frac{1}{q_{N-2}} \\ \delta^2(1-q_{N-2}) & \frac{\delta^2}{1-\delta} > \frac{1}{q_{N-2}} \end{cases}$$

Note that when $\frac{\delta^2}{1-\delta} > \frac{1}{q_{N-2}}$, B maximises his payoff by offering something which will be rejected for sure, and so any $x < \inf I_{N-2}$ does the trick. Now, these were the best strategies and payoffs for B if A had set $y = 1$. With this knowledge, A sets the highest y_{N-2} which B should accommodate. This entails making B indifferent between accommodating and following the above strategy and so setting $y_{N-2} = 1 - u^B_{N-2}(q_{N-2})$, which generates payoffs

$$\hat{U}^A_{N-2}(q) = 1 - \delta^2 (1 - q)\,,\ \hat{U}^B_{N-2}(q) = \delta^2 (1 - q)$$

Previous periods: Having three different cases complicates analysis for previous periods. Fortunately, with players playing the strategies prescribed at the start, every time beliefs are updated, the probability of type A^C can only increase. So for any t, $p < q_t$. This means that, provided the discount factor is high enough so that the (δ, p) combination at the start of the game satisfies $\frac{\delta^2}{1-\delta} > \frac{1}{p}$, it must also be the case that $\frac{\delta^2}{1-\delta} > \frac{1}{q_{N-2}}$. So I concentrate on the third case and obtain solutions which work for discount factors close to 1. In period $N-3$, I use the same logic as in period $N-1$ to say that both types of A offer $x_{N-3} = \delta \hat{U}^B_{N-2}(q_{N-3})$, where $\hat{U}^B_{N-2}(q) = \delta^2 (1 - q)$. Since B accepts this, for $\frac{\delta^2}{1-\delta} > \frac{1}{q_{N-2}}$ we have

$$\hat{U}^A_{N-3}(q) = 1 - \delta^3 (1 - q)\,,\ \hat{U}^B_{N-3}(q) = \delta^3 (1 - q)$$

Similarly at period $N-4$ we can apply the same logic as in period $N-2$. This would show that after A sets $y = 1$ in period $N-4$, provided that $\frac{\delta^4}{1-\delta} > \frac{1}{q_{N-4}}$, B's optimal strategy is to offer something that is rejected for sure, giving B a continuation payoffs of

$$u^B_{N-4}(q_{N-4}) = \delta^4 (1 - q_{N-2})$$

Observe that if $\frac{\delta^4}{1-\delta} > \frac{1}{p}$ then both $\frac{\delta^4}{1-\delta} > \frac{1}{q_{N-4}}$ and $\frac{\delta^2}{1-\delta} > \frac{1}{q_{N-2}}$ are also satisfied. Knowing this, A sets $y_{N-4} = 1 - \delta^4 (1 - q_{N-4})$ and continuation payoffs are

$$\hat{U}^A_{N-4}(q) = 1 - \delta^4 (1 - q)\,,\ \hat{U}^B_{N-4}(q) = \delta^4 (1 - q)$$

Proceeding in this way we find that for any odd t, so that period $N-t$ is even, both types of A pool on $x_{N-t} = \delta^t(1-q_{N-t})$; B accepts this offer and all higher offers, but would reject any lower offer. For even t so that $N-t$ is odd, both types of A pool on $y_{N-t} = 1-\delta^t(1-q_{N-t})$. In reply to this, and for any lower threats, B accommodates the threat by setting $x_{N-t} = y_{N-t}$, while for higher threats, B sets $x_{N-t} < \delta(1-\delta^{t-1}(1-q_{N-t}))$ which A^R would reject. Formally, the acceptance strategy of A^R to general offers x is described by$\rho_{N-t}(x)$, calculated in the same way as $\rho_{N-2}(x)$ above. These strategies together with beliefs that update according to Bayes' rule after offers from B, and leave beliefs unchanged after A makes offers or threats off the equilibrium path constitutes a PBE. Furthermore at each period the equilibrium actions were unique, hence this is the unique PBE outcome under the no slackness assumption. Rolling back to period A offers $x_0 = \delta^N(1-p)$ and B accepts. □

The complete slackness equilibrium

Before giving the proofs of Theorems 7 and 11, it is a good idea to define the complete slackness equilibrium. This is the PBE which erodes all power from A from possibly being the commitment type every period. This is defined by the following strategies. Player A: in even periods A (both types) offer $x = \frac{\delta}{1+\delta}$; in odd periods sets threat $y = \frac{\delta}{1+\delta}$ and A^R accepts offer x if and only if $x \geq \frac{\delta}{1+\delta}$. Beliefs are updated via Bayes' rule on the equilibrium path, but after offers or threats off the equilibrium path, B believes A is rational for sure. Clearly beliefs are consistent with PBE and the strategies are the same as the Rubinstein strategies in the standard model, so neither player has an incentive to deviate. Hence this defines a PBE.

Proof of Theorem 6

Proof. Given p and δ, let $s = \frac{(1-\delta)(1-p)}{p}$. The object is to show that there is a PBE giving $U^B = s$. Define $K = \frac{\log s - \log(1-p)}{\log \delta}$, which implies $s = \delta^K (1-p)$ and also $\frac{\delta^K}{1-\delta} = \frac{1}{p}$. Consider the following strategies and beliefs:

In any even period with belief $q \geq p$ that A is the commitment type, both types of A set $x = \delta^K (1-q)$; B accepts an offer x iff $x \geq \delta^K (1-q)$ and rejects otherwise; given x, beliefs are updated to $q^u(x)$ given by $q^u(x) = q$ if $x \geq \delta^K (1-q)$ and $q^u(x) = 0$ otherwise. In any even period with $q_e < p$ play proceeds according to the complete slackness equilibrium: A offers $x = \frac{\delta}{1+\delta}$ and any other offer generates beliefs that A is rational for sure. B accepts x if and only if $x \geq \frac{\delta}{1+\delta}$

In any odd period with belief $q \geq p$ that A is the commitment type, both types of A set $y = 1 - \delta^{K-1}(1-q)$; for any $y \leq 1 - \delta^{K-1}(1-q)$, B accommodates, setting $x = y$.[13] After any $y > 1 - \delta^{K-1}(1-q)$, B sets $x = \frac{\delta}{1+\delta}$; after $y = 1 - \delta^{K-1}(1-q)$, the acceptance function of A^R is ρ_o to be defined below, while after any other y the acceptance function for A^R is whatever the PBE concept requires it to be.[14] Beliefs after a threat are $q^u(y) = q$ if $y \leq 1 - \delta^{K-1}(1-q)$ and $q^u(y) = 0$ after $y > 1 - \delta^{K-1}(1-q)$; beliefs after A's acceptance decision are updated via Bayes' rule. In any odd period with $q < p$ play proceeds according to the complete slackness equilibrium: A sets $y = \frac{\delta}{1+\delta}$ and B would believe that A is rational for sure following any other threat. B offers $x = \frac{\delta}{1+\delta}$ and A accepts x if and only if $x \geq \frac{\delta}{1+\delta}$

According to these strategies, given that A has not deviated, so $q \geq p$ continuation

[13] Strictly speaking this might not be true if A sets $y < \frac{\delta}{1+\delta}$ - see A^R's acceptance function in next footnote. But A setting such a low y is clearly so far off the equilibrium path that the details do not seem important.

[14] This is far off the equilibrium path and so I won't bother calculating it precisely, but do give some idea. After $y \in \left[\frac{\delta}{1+\delta}, 1 - \delta^{K-1}(1-q_o)\right]$, this is similar to ρ_o, for $y < \frac{\delta}{1+\delta}$ or $y > 1 - \delta^{K-1}(1-q_o)$ this is to accept x iff $x \geq \frac{\delta}{1+\delta}$.

payoffs from even periods will be

$$U_e^A(q) = 1 - \delta^K(1-q) \;\; U_e^B(q) = \delta^K(1-q)$$

and payoffs in odd periods will be

$$U_o^A(q) = 1 - \delta^{K-1}(1-q) \;\; U_o^B(q) = \delta^{K-1}(1-q)$$

In even periods clearly beliefs are updated via Bayes' rule where possible and given these beliefs and future play, B is behaving optimally. It is then easy to check that A cannot do any better than offering $x = \delta^K(1-q)$ given future play. In even periods, there is no slackness, since A makes B an offer he is indifferent between accepting and rejecting.

Consider what happens in odd periods if A^C rejects every offer and beliefs are $q \geq p$. What should B offer? An offer of x from B would be met by the following acceptance function from A^R:

$$\rho(x) = \begin{cases} 0 & x \leq \delta f_A(q) \\ \frac{x-\delta+\delta^K(1-q)}{(1-q)(x-\delta+\delta^K)} & x \in (\delta f_A(q), \delta) \\ 1 & x \geq \delta \end{cases}$$

Since $q \geq p$ and $\frac{\delta^K}{1-\delta} \geq \frac{1}{p}$, we have that $\frac{\delta^K}{1-\delta} \geq \frac{1}{q}$, and so given the acceptance function above, it can be shown in the same way as in Lemma 9 that the best strategy for B is to offer $x \leq \delta f_A(q)$ so that A^R always rejects. Given that B follows this strategy, his expected payoff would be

$$U_{na}^B = \delta f_B(q) = \delta^{K+1}(1-q)$$

so if there is no updating of beliefs, A could set any threat up to

$$y = 1 - U_{na}^{B} = 1 - \delta^{K+1}(1-q)$$

and still have B accommodate. However, by the way beliefs are updated the maximum threat A can set is $y = 1 - \delta^{K-1}(1-q)$ because our updating rule specifies that B believes A is rational for sure after any higher threat, and play would proceed according to the unique SPE giving A payoff of $\frac{\delta}{1+\delta}$. So, clearly the best strategy for A is to set $y = 1 - \delta^{K-1}(1-q)$ which causes us to alter the acceptance strategy of A^R to

$$\rho_o(x) = \begin{cases} 0 & x \leq \delta f_A(q) \\ \frac{x-\delta+\delta^K(1-q_o)}{(1-q_o)(x-\delta+\delta^K)} & x \in (\delta f_A(q), y) \\ 1 & x \geq y \end{cases}$$

Given these strategies, B is strictly better off accommodating the threat, giving payoff $g_B(q) = \delta^{K-1}(1-q)$ than he would be after making the best non-accommodating offer, which would give utility $U_{na}^{B} = \delta^{K+1}(1-q)$. In this equilibrium there is this slackness in every odd period. The slackness measure for this equilibrium σ is

$$SM_{A,t}(\sigma) = \begin{cases} 0 & t = 0, 2, 4, ... \\ \delta^{K+1}(1-q) - \delta^{K-1}(1-q) & t = 1, 3, 5, \ldots \end{cases}$$

Note that this tends to 0 as $\delta \to 1$. □

The result of Theorem 9 is contained in Theorem 10, so its proof omitted.

Proof of Theorem 10

Proof. Given $(p_A, \delta) \in (0,1)^2$ satisfying $\psi(p_A, \delta) \leq \frac{\delta}{1+\delta}$. Take $(\alpha, \beta) \in [0,1]^2$ and $\tau \in \mathbb{N} \cup \{0\}$ satisfying

$$\alpha \geq \frac{1}{1+\delta},\ \beta \geq \psi(p_A, \delta),\ \alpha + \beta = \delta^\tau$$

I construct an equilibrium with agreement in period τ giving payoffs of α and β to A and B respectively. In each period $t < \tau$ we require that players choose actions resulting in delay. This uses the existence of the complete slackness equilibrium and the Theorem 6 PBE. Throughout this equilibrium, beliefs are updated as specified already if players play strategies which take us to either the aforementioned PBE. Otherwise, beliefs are updated via Bayes' rule on the equilibrium path and remain unchanged after actions off the equilibrium path.

For each $t < \tau$ players act as follows: in even periods A offers $x = 0$, which B rightfully rejects. In odd periods A sets the threat $y = 1$, B sets $x = 0$ and A rejects. If player A deviates to either a different threat or offer in any period $t < \tau$ then play thereafter proceeds according to the complete slackness equilibrium, which results in A getting a continuation payoff of no more than $\frac{1}{1+\delta}$. If player B deviates to make a different offer then play proceeds according to the equilibrium of Theorem 7. Thus neither player has an incentive to make a different offer or threat given that they get payoffs $\alpha \geq \frac{1}{1+\delta}$ and $\beta \geq \psi(p_A, \delta)$ by sticking to the prescribed strategy. Also, clearly neither player gains from accepting the offer of the other since that would lead to a payoff of 0.

Suppose τ is even then we can construct a PBE with agreement in period τ as follows: At any even period $t \geq \tau$ player A makes offer $x^A = \frac{\beta}{\delta^\tau}$ which B accepts. In odd periods $t > \tau$ player A sets the threat $y^A = 1 - \frac{\beta}{\delta^{\tau+1}}$ to which B accommodates, setting $x^B = y^A$ and both types of A accept. If A deviates from this offer in even periods or threat in odd periods, play proceeds according to the complete slackness equilibrium thereafter. If B deviates in odd periods by making a different offer then play proceeds according to the Theorem 6 equilibrium. It is clear to see that neither player has an incentive to deviate when making a threat or offer. In even periods the payoff (in period t units) to B from accepting x^A is $\frac{\beta}{\delta^\tau}$ which equals $\delta\left(1 - x^B\right)$ which his payoff from accepting and so accepting this offer is a best response. Likewise, in odd periods the payoff (in period t units) to A^R from accepting x^B is $1 - \frac{\beta}{\delta^{\tau+1}}$ which is greater than$\frac{\delta}{1+\delta}$, the payoff from rejecting. The payoff to A is $U^A = \delta^\tau\left(1 - \frac{\beta}{\delta^\tau}\right) = \alpha$ and to B is $U^B = \delta^\tau\left(\frac{\beta}{\delta^\tau}\right) = \beta$.

Suppose τ is odd then we can construct a PBE with agreement in period τ as follows: At any odd period $t \geq \tau$ player A sets the threat $y^A = \frac{\alpha}{\delta^\tau}$ to which B accommodates, setting $x^B = y^A$ and both types of A accept. In even periods $t > \tau$ player A makes offer $x^A = \delta\left(1 - \frac{\alpha}{\delta^\tau}\right)$ which B accepts. If A deviates from this offer in even periods or threat in odd periods, play proceeds according to the complete slackness equilibrium thereafter. If B deviates in odd periods by making a different offer then play proceeds according to the Theorem 6 equilibrium. Clearly neither player has an incentive to make a different offer or threat, and as above, each player is best responding by accepting the offers of the other. The payoff to A is $U^A = \delta^\tau\left(\frac{\alpha}{\delta^\tau}\right) = \alpha$ and to B is $U^B = \delta^\tau\left(1 - \frac{\alpha}{\delta^\tau}\right) = \beta$. □

1.7.3 For Section 4.1

This is where $p_A \in [0,1]$, $p_B = 1$. Lemma 11 is obvious and so its proof is omitted. I prove Theorem 12 and Theorem 14. Theorem 13 follows immediately from Theorem 12. First I define the complete slackness equilibrium which is useful for these proofs.

Complete slackness equilibrium

This is the PBE which imposes complete slackness on the threats and offers of player A resulting in player B taking the whole pie.

In every even period B sets threat $y^B = 1$ and A accommodates with $x^A = y^B$ after any threat $y^B \in [0,1]$. Since $p_B = 1$, it is pre-determined that B will accept x^A if and only if $x^A \geq y^B$. The beliefs on the type of player A remain unchanged after $x^A = y^B$ (by Bayes' rule), however after any other offer B believes A is rational for sure. In every odd period A sets $y^A = 0$; player$B$ offers $x^B = 0$; A^R (as well as A^C) accepts any offer $x^B \in [0,1]$. Beliefs following $y_A = 0$ remain unchanged, but after any different threat B believes A is rational for sure.

Clearly beliefs are consistent with Bayes' rule and both players are best responding given these beliefs so this is a PBE.

Proof of Theorem 12

Proof. Here $p_B = 1$, so we only have uncertainty about the type of player A, thus I let q denote the probability with which A is the commitment type at any given stage of the game. Let $\psi(q) = \frac{1-\delta^2(1-q)}{q(1+\delta)}$. I find a PBE giving payoffs in even periods of $U_e^A(q) = 1 - \psi(q)$, $U_e^B(q) = \psi(q)$ and in odd periods of $U_o^A(q) = \frac{1-\psi(q)}{\delta}$, $U_o^B(q) = 1 - \frac{1-\psi(q)}{\delta}$, which substituting p_A for q as the the belief in period 0 gives the desired result. When $q = 0$ or $q = 1$ play proceeds as per the unique SPE in either case. When $q \in (0,1)$ the equilibrium play is as follows:

In even periods B sets $y^B = \psi(q)$; player A accommodates any $y^B \leq \psi(q)$, setting $x^A = y^B$, while after $y^B > \psi(q)$, he makes any offer $x^A < y^B$. Since $p_B = 1$, it is pre-determined that B will accept x^A if and only if $x^A \geq y^B$. The beliefs on the type of player A remain unchanged after any offer.

In odd periods A sets threat $y_A = \frac{1-\psi(q)}{\delta}$, B accommodates this threat setting $x^B = y^A$, and A accepts. If A had set a different threat, he would be believed to be the rational type for sure and play would proceed according to the complete slackness equilibrium defined above. If B had made a different offer, A^R would use the following acceptance strategy:

$$\rho(x) = \begin{cases} 1 & x \geq \min\left\{\frac{\delta^2}{1+\delta}, \frac{1-\psi(q)}{\delta}\right\} \\ 0 & x \leq \delta(1-\psi(q)) \\ \frac{qx+\delta(1-q(1-x))-\delta^2 q-\delta^3(1-q)}{\delta(1-\delta^2)(1-q)} & otherwise \end{cases}$$

The reason for this acceptance strategy is the same as in the Lemma 9 proof - A^R should always accept when offered something high enough that A^C accepts or which gives greater utility than he could achieve next period; he should reject if offered something beneath his continuation payoff; for offers in between he should mix such that given the updated beliefs (by Bayes' rule), he is indifferent between accepting and rejecting x.

To check this defines a PBE: Clearly beliefs are updated in accordance with Bayes' rule where possible. In even periods, A is acting optimally by only accommodating threats which leave him with at least his continuation payoff. Player B can do no better than making the threat $y^B = \psi(q)$ since lower threats yield lower payoffs and higher threats would not be accommodated, leading to delay and payoff $\delta U_o^B(q) < U_e^B(q)$. In odd periods, A cannot deviate from the threat, since the no slackness equilibrium would then take charge in which he gets 0, and as already covered, the acceptance strategy defined for A^R is optimal. For player B, offering $x \geq \min\left\{\frac{\delta^2}{1+\delta}, \frac{1-\psi(q)}{\delta}\right\}$ generates payoff $(1-q)(1-x) + \frac{q\delta}{1+\delta}$ which is less than $U_o^B(q)$, the payoff from complying with the prescribed strategy. Offering $x \leq \delta(1-\psi(q))$ generates payoff $\delta U_e^B(q) < U_o^B(q)$. The payoff from offering some x which A^R accepts with positive probability is $a(x)(1-x) + (1-a(x))\delta U_e^B(q)$, where $a(x) = (1-q)\rho(x)$ is the probability that x is accepted by A. This function has no turning points in the required interval and so is less than the payoff from one of $x \leq \delta(1-\psi(q))$ and $x \in \left\{\frac{\delta^2}{1+\delta}, \frac{1-\psi(q)}{\delta}\right\}$ and so in turn must be less than $U_o^B(q)$. $\square$

Proof of Theorem 14

Proof. This is very similar to the proof of Theorem 10. Take any α, β, τ satisfying the conditions of the Theorem. We need disagreement until period τ and then agreement in this period. For each $t < \tau$ the threatening player sets threat $y = 1$, the offering player offers $x = 0$, which is rejected. If player A deviates by making a different offer or threat the complete slackness equilibrium is played thereafter, while if B deviates the PBE of Theorem 13 is played thereafter.

Let $\kappa = \frac{\beta}{\psi(p_A,\delta)}$ (note that $\kappa \geq 1$ by assumption on β). Note that by using the same structure of equilibrium as in Theorem 12, and imposing more slackness on the threat of A we can construct a PBE with payoffs $U_e^A(q) = 1-\nu(q)$, $U_e^B(q) = \nu(q)$ and in odd periods of $U_o^A(q) = \frac{1-\nu(q)}{\delta}$, $U_o^B(q) = 1 - \frac{1-\nu(q)}{\delta}$ for any $\nu(q)$satisfying $\nu(q) = c\psi(q)$ for any $c \geq 1$. If τ is even then we use this to construct a PBE such that $f(q) = \frac{\kappa\psi(q)}{\delta^\tau}$, while if τ is odd we construct a PBE such that $g(q) = 1-\delta\left(1-\frac{\kappa\psi(q)}{\delta^\tau}\right)$. This gives agreement in period τ and payoffs to B of $\delta^\tau f(q) = \kappa\psi(q)$ if τ is even and $\delta^\tau\left(1-\frac{1-g(q)}{\delta}\right) = \kappa\psi(q)$. If both players follow this equilibrium, the belief on A's type in period τ will still be p_A and so this gives B a payoff of $\kappa\psi(p_A) = \beta$ with player A getting $\delta^\tau - \beta = \alpha$. $\square$

1.7.4 For Section 4.2

This is where $(p_A, p_B) \in (0,1)^2$. In order to prove Theorem 15, I will first describe two PBE which may be of some interest in themselves. In one A gets almost everything, while in the other B gets almost everything.

Equilibrium A1 - A gets almost everything

This PBE is formed by putting complete slackness on the offers and threats of B, while very little slackness on those of A. This means that B acts like the rational player in Theorem 6. Let δ be sufficiently large so that $\frac{(1-\delta)(1-p_A)}{p_A} < \frac{\delta}{1+\delta}$, then we can construct a PBE giving $U^B = \frac{(1-\delta)(1-p_A)}{p_A}$ and $U^A = 1 - U^B$. The strategies and beliefs for player A are exactly the same as in Theorem 6. In even periods B makes the threat equal to the offer A makes in Theorem 6. Had he made any other threat, he would be believed to be rational for sure and play continues exactly as in Theorem 6. In odd periods, the offer B makes is exactly the same as in Theorem 6 and after any other offer he would be believed to be rational for sure and play continues exactly as in Theorem 6.

Equilibrium B1 - B gets almost everything

The idea is very similar to the above but with the roles of A and B reversed. By putting complete slackness on the actions of A and very little on B, we generate payoffs $U^A = \frac{(1-\delta)(1-p_B)}{p_B}$ and $U^B = 1 - U^A$.

Proof of Theorem 15

Proof. Again the idea is similar to that of Theorem 10. Take any α, β, τ satisfying the conditions of the Theorem. We need disagreement until period τ and then agreement in this period. For each $t < \tau$ the threatening player sets threat $y = 1$, the offering player offers $x = 0$, which is rejected. If player A deviates by making a different offer or threat then equilibrium B1 is played thereafter, while if B deviates the equilibrium A1 is played thereafter.

In period τ: if τ is even B sets threat $\frac{\beta}{\delta^\tau}$, A accommodates, offering $\frac{\beta}{\delta^\tau}$ and B accepts, while if τ is odd A sets threat $\frac{\alpha}{\delta^\tau}$, B accommodates, offering $\frac{\alpha}{\delta^\tau}$ and A accepts. If player A deviates, he is punished by equilibrium B1 being played thereafter and if B

deviates he is punished by equilibrium A1. Clearly this is a PBE since neither player has incentive to deviate, and the payoffs to A and B are α and β respectively. $\square$

Chapter 2

Stochastic stability and the use of commitment in bargaining

2.1 Introduction

How two players will divide a given amount of surplus is one of the oldest questions in Economics. In this paper I present a model in which one player may try to increase his share of the surplus by committing himself to a favourable division. Schelling [40], in his Essay on Bargaining, says such a commitment is only effective if the other side realises this commintment is in place. In particular I find that the advantage one has by being able to use a commitment technology is eroded away by the ability to bluff at commitment. This is because we would prefer to bluff since it is safer, in case our opponent does not back down. However, knowing this, our opponent will then refuse to back down since he expects us to bluff. Using the evolutionary game theory technique of stochastic stability, I argue when commitments are likely to be used. Furthermore I find that the presence of either an outside option or an observation technology can alter the dynamics so that commitment is used in the long run, even though these options are not used in this long run equilibrium.

The question of how two players will split a surplus arises in many different contexts and is such a difficult question to answer due to the vast range of possible outcomes. Take, for example a firm negotiating with the union representing its workforce. If the two can come to an agreement, they will both be better off; if they feel to reach agreement and strikes occur then both lose out. Suppose that the workers' reservation wage is £8 per hour and the firm can make profit with any wage up to £13 per hour, then we have a range of possible agreement outcomes between the two which are of benefit to both parties. The question is how much of the economic surplus created by the two working together should go to the workers (in the form of wages) and how much should go to the firm (in the form of profits).

This chapter looks at the role of commitment on the outcome. The model here is best suuited to situations in which one party can threaten to do something mutually disadvantageous if his demands are not met. For example consider the workers' union who threaten to go on strike unless their wage is increased, or the employee who threatens to look for another job if he is not given a better office or other better working conditions. Or in a modern political context, the European country who threatens to leave the EU unless they get a better deal from it[1]. Schelling [40] in his well renowned essay on bargaing talks about many ways in which one of the two parties may commit themselves. Commitment works by tying the hands of the committed party so that the sole responsibility of avoiding disagreement now falls upon the other party.

Schelling talked about a few methods which one party, which I denote $\mathcal{A}$, could use to achieve commitment thus forcing the other party, denoted $\mathcal{B}$ to make the concession. One method is contracting with a third party. Schelling gives the example of $\mathcal{A}$ valuing a house at $\$20,000$ but wishes to commit to paying no more than $\$16,000$. To do this $\mathcal{A}$ signs a contract with a third party stipulating that $\mathcal{A}$ is to forfeit $\$5,000$

[1]In UK this is something the Conservative party are saying they will do if they win the 2015 general election.

to the third party should $\mathcal{A}$ pay more than $\$16,000$ for the house. As Schelling noted this might not necessarily work if the third party can be persuaded to release $\mathcal{A}$ from the contract for a lesser sum[2]. Although if $\mathcal{A}$ can find a third party who has their own reasons for wanting $\mathcal{A}$ to keep to the commitment, then this may be more successful. For example suppose that $\mathcal{A}$ is a supplier to $\mathcal{B}$ and the third party is a rival of $\mathcal{B}$, then it would be in the third party's interests to help $\mathcal{A}$ to commit to not selling goods to $\mathcal{B}$ below a certain price. Another example is for $\mathcal{A}$ to employ a bargaining agent who is given strict instructions. A particular example of this in the structure of a firm is a manager who has been given a budget for the purchase of some product by the board of directors. Another possible commitment device is staking one's reputation or staking one's reputation on not being prepared to give ground. This may be particularly possible if $\mathcal{A}$ has many other similar negotiaton with others and so could use the argument "If I did it for you, I'd have to do it for everyone else"

Returning to the firm union example, suppose that the current wage is £10 per hour and that the union is demanding an increase to £11 per hour. The firm could simply refuse since they know it is not in the workers' interests to withdraw their labour. On the other hand, suppose that the workers could somehow commit to not working for any less than £11 per hour. If this is truly an irrevocable commitment, and the firm knows this, then it will have to concede to this demand. By committing, the workers have displaced the responsibility of having the last chance to avoid the mutually disastrous consequence of non-agreement from themselves onto the firm. This forces the frim to concede to their demand.

This chapter makes the important point that, just because a commitment technology is available, it won't necessarily be used. Schelling, throughout his essay on bargaining, is careful to stress that in order for commitment to work, the other party

[2]Note that the third party does not expect to receive anything, since $\mathcal{A}$ will not be prepared to break his commitment given this side deal

must realise such a commitment has been made. In the above example, if the firm is unaware that the workers are committed to £11 per hour, the firm will simply refuse this demand, believing that there is no chance of the workers deciding to withdraw their labour. When the workers do in fact withdraw their labour, this hurts both the firm and themselves, thus rendering the workers' decision to commit unwise.

This example shows that the key is not commitment itself, but the opponent believing that you are committed. One might think this to be the easy part: if the workers are in fact committed to £11 per hour, then it is in the interests of both the workers to pass this information to the firm and of the firm to receive this information. However, in reality this is often not so easy. Suppose that there is one day left for the union and the firm to reach agreement before the strikes begin. The union could commit itself by sending into the negotiation a low-level official with the instruction to demand £11 per hour and no authority to make any other decisions, and no means of contacting any higher ranking union officials. Now all the union has to do is tell the firm this is what it has done and the firm will have no choice but to concede to the union's demand. However, the problem with the union's masterplan is this: for all the firm knows, the low ranking union official may have been sent into the negotiation with the instruction to demand £11 per hour until the very last moment of the day and then settle for £10 per hour if the firm doesn't give in.

To be more precise, when the firm hears the union's message of "we are committed to £11 per hour", the firm does not know whether the union has actually committed itslef or whether it is merely bluffing. If it is bluffing then the firm should take the hardline stance of not making any concessions; whereas if the union is actually committed, the firm should give in and accept the union's demand. It is exactly this choice of whether the party with the commitment technology should really use it, or just bluff, and how the other party should repond that we model here. In the base model, I suppose there is some allocation, $(w, 1-w)$, that can be thought of as a status quo allocation as it would pertain in the case that neither player has a commitment technology. Now let one player, which I denote as $\mathcal{A}$, have a commitment technology allowing him to commit himself to receiving no less than $c > w$ which he must decide whether or not to use. Whether he uses the commitment technology or not, he will try to convince the other player, denoted $\mathcal{B}$ that he is in fact committed. Unsure about whether $\mathcal{A}$ is committed to c or not, $\mathcal{B}$ must decide whether to respond by conceding to the alleged commitment by offering c, or resisting it by sticking to the offer of w.

Note that whether $\mathcal{A}$ is committed or not, it is in the interests of $\mathcal{A}$ to convince $\mathcal{B}$ believe that he is indeed committed, so that $\mathcal{B}$ will give up a larger proportion of the surplus. If $\mathcal{A}$ is able to be just as convincing whether committed or not, so that the action taken by $\mathcal{A}$ conveys no information to $\mathcal{B}$, the action chosen by $\mathcal{B}$ will then be completely independent of whether $\mathcal{A}$ is actually committed or not. In this circumstance, one may wonder why $\mathcal{A}$ would ever bother to commit, since bluffing is just as likely to induce $\mathcal{B}$ to offer c, while at the same time ensuring he has the back up option of being able to settle for w if $\mathcal{B}$ refuses to offer c. Indeed I find that $\mathcal{A}$ does not commit and so $\mathcal{B}$ responds by offering w, thus we get the non-commitment outcome $(w, 1-w)$. However, one might ask whether this is really a fair assumption. Schelling [40] argued that it is easier to convince your opponent that you are committed if you actually are committed:

> How does one person make another believe something? The answer depends importantly on the factual question, "Is it true?" It is easier to prove the truth of something that is true than of something false. To prove the truth about our health we can call on a reputable doctor; to prove the truth about our costs or income we may let the person look at books that have been audited by a reputable firm or the Bureau of Internal Revenue. But to persuade him of something false we may have no such convincing

evidence.

In order to address this point, I assume that $\mathcal{B}$ may discover a bluff with some probability, in which case, as one would expect, he automatically restricts his offer to w. If $\mathcal{B}$ always discovers bluffs then we have perfect information and so $\mathcal{A}$ will commit, knowing that $\mathcal{B}$ will observe his commitmnet and hence concede to the commitment and so we end up with the commitment outcome $(c, 1-c)$. In general, when the probability of bluffs being discovered is in the open interval $(0, 1)$, we have two strict Nash Equilibria and using the evolutionary game theory technique of stochastic stability to determine what will be termed the *long run equilibrium*, I make an argument as to which outcome is likely to predominate. I find the nice intuitive result that the greater the ability of $\mathcal{A}$ to bluff (that is the smaller the likelihood of a bluff being discovered as such), the more likely we are to end up with the non-commitment outcome.

This means that as long as the chance of a bluff being discovered is not too large, the non-commitment outcome still predominates. One may wonder how robust this result is. Section 4 shows that if we combine the assumption of a bluff being discovered with either the presence of an outside option for $\mathcal{A}$ or an opportunity for $\mathcal{B}$ to observe the commitment stauts of $\mathcal{A}$ then we may get the opposite result. As long as the cost of observation is sufficiently small, or the outside option is in the relevant range (between £10 and £11 per hour in the example above) then the long run equilibrium is for $\mathcal{A}$ to commit and $\mathcal{B}$ to concede to this commitment. That is, neither the outside option or the commitment technology are used in the long run equilibrium, but their mere presence as alternatives means that the long run equilibrium switches from the bluffing equilibrium to the commitment equilibrium.

The idea that a player in a bargaining situation may pretend to be committed even when he is not, in order to induce the other player to concede a larger chunk of the surplus is well-established. Indeed Chapter 1 presents a model of precisely this occurrence, as do many of the other papers referenced there. However what we model here is rather different. This chapter analyses whether a player with a commitment technology should actually commit, or merely pretend to be committed. I am not aware of any other papers investigating this issue.

In general we find more than one Nash Equilibrium, so to make a prediction about which is likely to predominate, the solution concept used is stochastic stability. This works by considering an evolutionary dynamic in which players switch from less to more successful strategies. Such a dynamic will generally lead us to an equilibrium. To choose among equilibria when more than one exists, we perturb the dynamic by introducing mutations, and roughly speaking stochastic stability finds the equilibrium which is most resistant to these mutations.

There is a reasonable sized literature applying evolutionary game theory techniques to bargaining. Particularly relevant in terms of the solution cocept used is [44] where Young applies his version of staochastic stability to a two population bargaining model in which agents play a finite strategy version of the Nash Demand game. He found that, if all agents have the same sample size, then agents will split the surplus equally, and allowing for heterogeneous sample sizes, agents split the surplus according to a generalization of the Nash Bargaining Solution. Also worth mentioning are Ellingsen [11] and following on from this paper, Poulsen [36], who like here, use evolutionary game theory to investigate the impact of commitment on bargaining. Although their model considers a single population of agents interacting in a symmetric role game and they use Evolutionary Stable Strategy (ESS) as the solution concept.

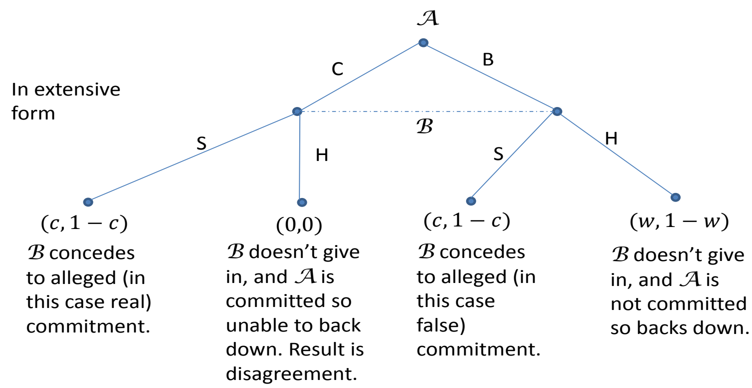
I begin the analysis in Section 2 with an outline of the model and a discussion of the solution concept, stochastic stability, in the context of asymmetric games. There are many different specifications of stochastic stability in the literature, depending on when and how agents revise their strategy. I give a brief account of this literature and explain where the specification I use sits in comparison to others. Sections 3 and 4 apply stochastic stability to the bargaining situation described above to get clear predictions of when $\mathcal{A}$ will be able to make use of his commitment technology in the long run equilibrium. Section 5 has a brief look into a more complicated scenario, where $\mathcal{B}$ receives a random, partially informative signal about the action taken by $\mathcal{A}$.

2.2 Basic model and Solution concept

There will be a few different commitment stage games considered in this chapter. Here I present the most basic one, that future stage games are built upon.

Two players, labelled $\mathcal{A}$ and $\mathcal{B}$ are bargaining over a surplus of size normalized to 1. Let $0 < w < c < 1$. There is a status quo share $(w, 1-w)$ which represents the shares that $\mathcal{A}$ and $\mathcal{B}$ respectively would obtain absent any commitments. Now player A is given a commitment technology which allows him to commit to accepting no less than c (binding himself to being unable to accept the status quo offer w). The fact that this commitment technology is available is common knowledge.

Player $\mathcal{A}$ moves first and has two actions to choose from $\{C, B\}$. Action C means to employ the commitment technology, often termed "to commit". Action B means not to employ the commitment technology. It is labelled B for "bluff" since player $\mathcal{A}$ will clearly find it in his interests to pretend to be committed even if he is not. For interpretation purposes we can think of player $\mathcal{A}$ sending the costless message "I will accept no less than c" regardless of whether he is committed or not. Once player $\mathcal{A}$ has chosen his action, player $\mathcal{B}$, unaware of the action chosen by $\mathcal{A}$, must decide whether to concede to the alleged commitment or not. More formally, he has two actions: $\{S, H\}$, where S (standing for "Soft") means to concede to the alleged commitment by offering c, while H (standing for "Hard") means to refuse to give in to the alleged commitment and continue offering $\mathcal{A}$ the status quo share of w.



In normal form

		$\mathcal{B}$	
		S	H
$\mathcal{A}$	C	$(c, 1-c)$	(0,0)
	B	$(c, 1-c)$	$(w, 1-w)$

$0 < w < c < 1$

Figure 2.2.1: Basic perfect disguise game

Payoffs are then determined by $\mathcal{A}$ responding optimally to the offer received, given his commitment position: If $\mathcal{B}$ chooses "Soft" then $\mathcal{A}$ accepts the offer of c, so that payoffs are $\left(U^{\mathcal{A}}, U^{\mathcal{B}}\right) = (c, 1-c)$. If $\mathcal{B}$ chooses "Hard" then $\mathcal{A}$ will accept the offer of w if he can. Thus payoffs are $(w, 1-w)$ if $\mathcal{A}$ is bluffing and $(0,0)$ if $\mathcal{A}$ was committed. This game is represented in Figure 2.2.1 on page 66

Note that the reaction of $\mathcal{A}$ to the offer from $\mathcal{B}$ is assumed rather than modeled, and the reason for this will be made clear. This point may at first seem inconsequential since restricting the game in this way has no effect on the Subgame Perfect Equilibria. However, when applying a technique like stochastic stability, allowing for mutations at such points can have an adverse effect. 27 and the discussion after explains why it is necessary to restrict the strategy space to what can be thought of as "sensible" strategies instead of considering all possible strategies.

This game can be thought of as the base game, on top of which other more complicated games will be built. In this game there are two Nash Equilibria: (C, S) and (B, H), although only (B, H) is a strict Nash Equilibrium and C is a weakly dominated strategy. However more complicated games will follow in which committing is not weakly dominated and so both (C, S) and (B, H) are strict Nash Equilibria. To obtain predictions for such games I apply the solution concept of stochastic stability

Solution concept

Instead of assuming only a single player $\mathcal{A}$ playing a game with a single player $\mathcal{B}$, we now assume that there is a large population of agents in the player $\mathcal{A}$ role and another large population of agents playing the player $\mathcal{B}$ role. Now agents can use the precedent of how the game has been played in the wider population to help them decide how to play. I present two ways of doing this, all of which give similar results. The first follows Young [45], one of the pioneers of the stochastic stability literature. This model has an infinite sequence of agents in the roles of $\mathcal{A}$ and $\mathcal{B}$, and each period the players who are playing the game base their play on a random sample which is a subset of the last m observations. He called this an "adaptive" dynamic.

The other route builds on the idea of [24], a paper so widely cited it is now known in the literature by the abbreviation KMR. To do this we set up a population game, in which we assume two distinct populations of agents, one for each of the $\mathcal{A}$ and $\mathcal{B}$ roles. Agents' payoffs are determined by a round-robin format, in which each period every agent plays against every other agent of the opposing population. Players then adapt their behaviour either by copying the most successful strategies in their own population (imitative dynamic) or by playing a best response to the current distribution of play among agents in the other population (best response dynamic).

1. Young's adaptive dynamic

Consider a 2 player fixed finite game, with players $\{\mathcal{A}, \mathcal{B}\}$, where S_i with typical element s_i denotes the set of pure strategies available to player $i \in \{\mathcal{A}, \mathcal{B}\}$ and payoff functions $\{U^i(s_{\mathcal{A}}, s_{\mathcal{B}})\}_{i \in \{\mathcal{A}, \mathcal{B}\}}$. Let time be discrete, indexed by $t \in \{0, 1, 2, \ldots\}$. In each time period t two new players enter the scene, one in the role of player $\mathcal{A}$ and one in the role of player $\mathcal{B}$ to play the game once, against each other. Each pair of players plays the game only once and is replaced by a new pair of players who play the game the following period. At period $t > m$, a *state* $h(t)$ is defined as the record of play in the last m periods, i.e.

$$h(t) = ((s_{\mathcal{A}}(t-1), s_{\mathcal{B}}(t-1)), \ldots, (s_{\mathcal{A}}(t-m), s_{\mathcal{B}}(t-m)))$$

After two players have played the game in period t, the state for period $t+1$ is then updated to include $(s_{\mathcal{A}}(t), s_{\mathcal{B}}(t))$, while $(s_{\mathcal{A}}(t-m), s_{\mathcal{B}}(t-m))$, the play from period $t-m$ drops out. The process by which players choose their strategies is time ivariant and so a state will simply be an m-dimensional vector of pairs $(s_{\mathcal{A}}, s_{\mathcal{B}})$, with typical element denoted h, and the set of states is the set of all such m-dimensional vector pairs, denoted H.

Definition 18. A *successor* to $x \in X$ is any state $\hat{h} \in H$ obtained by deleting the right-most strategy pair and adjoining a new left-most strategy pair.

Note that a state $\hat{h}$ can only follow h if it is a successor of h.

The adaptive dynamic is then defined as follows: Each player samples a randomly chosen subset of the last m periods played. A player in the role of player $i \in \{\mathcal{A}, B\}$ will inspect k_i of the previous m periods drawn randomly without replacement, where $1 \leq k_{\mathcal{A}}, k_{\mathcal{B}} \leq m$ in which he observes how the game was played. Using this information each player plays a best response[3] to the plays of the game which he has sampled. So a player in the $\mathcal{A}$-role will get $k_{\mathcal{A}}$ observations from the set $S^{\mathcal{B}}$, which I label $\left\{s_{\mathcal{B}}^{(1)}, s_{\mathcal{B}}^{(2)}, \ldots, s_{\mathcal{B}}^{(k_{\mathcal{A}})}\right\}$ and will play $\hat{s}_{\mathcal{A}}$ with positive probability if and only if

$$\hat{s}_{\mathcal{A}} \in \arg\max_{s_{\mathcal{A}} \in S_{\mathcal{A}}} \frac{1}{k_{\mathcal{A}}} \sum_{i=1}^{k_{\mathcal{A}}} U^{\mathcal{A}}\left(s_{\mathcal{A}}, s_{\mathcal{B}}^{(i)}\right) \tag{2.2.1}$$

[3]if there is more than one best response then each is chosen wth positive probability

While a player in the $\mathcal{B}$-role reacts analogously to his $k_{\mathcal{B}}$ observations from the set $S^{\mathcal{A}}$. We can thus calculate the probability of state $\hat{h}$ following h, denoted $P_{h\hat{h}}$, under the adaptive dynamic, which defines a Markov chain on the state space H. Moreover, for each state $h \in H$ we can find which successors have a positive probability of following h. Suppose $\hat{h}$ is obtained from x by the addition of the new left-most pair $(\hat{s}_{\mathcal{A}}, \hat{s}_{\mathcal{B}})$. The state h lists the last m plays in each of the $\mathcal{A}$ and $\mathcal{B}$ roles and so for the $\mathcal{A}$-role player to play $\hat{s}_{\mathcal{A}}$, we require that there exists some subset $\left\{s_{\mathcal{B}}^{(1)}, s_{\mathcal{B}}^{(2)}, \dots, s_{\mathcal{B}}^{(k_A)}\right\}$ of the $\mathcal{B}$-role plays for which (2.2.1) holds. Similarly for the $\mathcal{B}$-role player to play $\hat{s}_{\mathcal{B}}$ we need that there exists some $k_{\mathcal{B}}$ size subset of the last m plays in the $\mathcal{A}$-role such that the analog of this equation for the $\mathcal{B}$-role player holds.

Definition 19. An *absorbing state* is a state that, once entered, the process will never leave. In other words h is an absorbing state if and only if $P_{hh} = 1$

Clearly h is an absorbing state if and only if it consists of a strict Nash Equilibrium played m times in succession. To select among the set of absorbing states, Young then perturbs this dynamic with mutations. This means that each period, there is an $\varepsilon > 0$ probability of making a mistake, meaning the player picks a startegy at random instead of following the process of the adaptive dynamic. This gives us an alternate Markov chain defined by transition probabilities P^{ε}. Now, for any $\varepsilon > 0$, we have $P^{\varepsilon}_{h\hat{h}} > 0$ for any $\hat{h}$ which is a successor of h. This ensures that the Markov chain is ergodic and so will have unique invariant distribution μ^{ε} over H which solves

$$\sum_{h \in H} \mu^{\varepsilon}_h P^{\varepsilon}_{hh'} = \mu^{\varepsilon}_{h'} \quad \forall h, h' \in H$$

or more succinctly, $\mu^{\varepsilon} P^{\varepsilon} = \mu^{\varepsilon}$. We then define states as stochastically stable if they survive with positive probability in the limit as $\varepsilon \to 0$ and as the Long Run Equilibrium (LRE) if it is the only stochastically stable state.

Definition 20. A state $h \in H$ is stochastically stable relative to the process P^{ε} if

$\lim_{\varepsilon \to 0} \mu_h^\varepsilon > 0$ and the LRE if $\lim_{\varepsilon \to 0} \mu_h^\varepsilon = 1$.

I now summarise how Young calculates the stochastically stable states. The basic idea is to count the number of mistakes needed to nove between absorbing states. To do this we introduce the following definitions:

Definition 21. For any two states $h, h' \in H$ the *resistance* $r(h, h')$ is the minimum number of mistakes involved in the transition form h to h' if h' is a successor of h; otherwise $r(h, h') = \infty$.

Note that in our two population setting $r(h, h') \in \{0, 1, 2, \infty\}$. Let us now view the state space H as the vertices of a directed graph. For every pair of states (h, h'), insert a dirceted edge $h \to h'$ if $r(h, h')$ is finite and call $r(h, h')$ its resistance. Now let $\Omega_1, \Omega_2, \ldots, \Omega_J$ be the *recurrent communication classes* of P^ε. These classes are disjoint and characterized by the following three properties: (i) From every state there is a path of zero resistance to at least one of the classes Ω_i; (ii) within each class Ω_i there is a path of zero resistance from every state to every other; (iii) Every edge exiting any Ω_i has positive resistance.

Often, especially in the examples considered here, a recurrent communication class will simply consist of a singleton absorbing state. Noting property (ii) of recurrent classes we can define the resistance between any two communication classes. Young then applies the Friedlin and Wentzell [17] tree algorithm to find which recurrent class contains the stochastically stable states (Theorem 2 of [45]). Much of the analysis can be captured by Ellison's [14] rather simpler radius-coradius theorem, which relies on the Strong and Weak Basins of Attraction of a recurrent class, where these are defined as follows:

Definition 22. The Strong Basin of Attraction of a recurrent class Ω is those states from which the unperturbed Markov process, P^0 converges to Ω with probability one. While the Weak Basin of Attraction of a recurrent class Ω is those states from which the unperturbed Markov process, P^0 might converge to Ω. These are given respectively by the formulae:

$$SB(\Omega) = \left\{ h \in H \mid \forall \zeta > 0 \, \exists T \; s.t \, \forall t > T \; \Pr\left(\left(P^0 \right)^t (h) \in \Omega \right) > 1 - \zeta \right\}$$

$$WB(\Omega) = \left\{ h \in H \mid \exists T \; s.t \, \forall t > T \; \Pr\left(\left(P^0 \right)^t (h) \in \Omega \right) > 0 \right\}$$

Note that given a stochastic adaptive dynamic like the one here, a state can be in the Weak Basin of several recurrent classes, whereas if it is in a Strong Basin of a recurrent class, then this is the only Basin it can be in. The Strong Basin is important for determining how hard it is to escape Ω, while the Weak Basin determines how hard it is to enter Ω. More formally, letting Ω be a union of recurrent classes, Ellison [14] defines the Radius $R(\Omega)$ to be the minimum resistance for any path from Ω to $H \setminus \Omega$ and the coradius $CR(\Omega)$ is the maximum resistance among states outside Ω for minimum resistance paths from $H \setminus \Omega$ to Ω. Ellison then proves the following:

Theorem 23. *If $R(\Omega) > CR(\Omega)$ then the set of stochastically stable states is contained in Ω.*

Note that adaptive play requires a sample of m periods, so we suppose random play in the first m periods and start in period $m + 1$. While this initial draw effects which recurrent communiation class the process will most likely enter at the start of the game (depending on which basin of attraction(s) it is in), it does not affect the invariant distribution of the process.

2. Population game

The other route is to introduce a large population $N^{\mathcal{A}}$ of $\mathcal{A}$-role players and another large population $N^{\mathcal{B}}$ of $\mathcal{B}$-role players and to form a population game in which agents utility is determined by their average payoff against the strategy mix in the opposing population. Agents switch to more successful strategies via an evolutionary dynamic which is then perturbed by mutations. The state space then measures the strategy mix within each population. The pioneering paper of KMR was principally concerned with a single population of agents playing a symmetric game. Although they did mention the two population case in section 9 and there have been subsequent papers to take up the two player case. Two of the papers to really focus on this, whose models are particularly close to mine are Hehenkamp [22] and Staudigl [41].

More formally, there are two distinct populations, labeled as $p \in \{\mathcal{A}, \mathcal{B}\}$, where population p has N^p agents, each of has with pure strategy set $S^p = \{p_1, p_2, \ldots, p_{n^p}\}$. Let x_i^p be the fraction of agents in population p playing strategy $i \in S^p$. We thus describe a state of population p by a vector $x^p = (x_1^p, x_2^p, \ldots x_{n^p}^p)$ which keeps track of the fraction of agents playing each strategy. The set of *population-p states* is then given by

$$X^p = \left\{ x^p = (x_1^p, x_2^p, \ldots x_{n^p}^p) \mid x_i^p \in \left\{0, \frac{1}{N^p}, \ldots, 1\right\}, \sum_{i=1}^{n^p} x_i^p = 1 \right\}$$

and the set of *social states* is

$$X = X^{\mathcal{A}} \times X^{\mathcal{B}} = \left\{ x = \left(x^{\mathcal{A}}, x^{\mathcal{B}}\right) \mid x^{\mathcal{A}} \in X^{\mathcal{A}},\ x^{\mathcal{B}} \in X^{\mathcal{B}} \right\}$$

This is quite a different state space than in Young's model, but the definitions of absorbing states, recurrent classes, basins of attraction and Ellison's radius-coradius Theorem can equally well be applied here.

Agents in population $\mathcal{A}$ are continually randomly matched to play agents in population B, so that the current expected utility of an agent in population p playing strategy $i \in S_p$ given population state x^{-p} of the opposing population is simply the average utility from pairwise matches. Payoffs are given by the formula:

$$U^p\left(i \mid x^{-p}\right) = \sum_{j=1}^{n} x_j^{-p} u^p\left(i, j\right) \tag{2.2.2}$$

where x_j^{-p} is the proportion of agents in the other population playing strategy j and agents in population p get utility $u^p(i, j)$ when playing strategy i against startegy j. This can have two interpretations: the first is that every agent plays against every other agent in the opposing population in a round-robin format; the second is that are continually randomly matched to play agents in the opposing population. Either way, (2.2.2) captures the average payoff per interaction.

Within the population game approach, there are two main classes of dynamic to consider. The first is the best response dynamic in which agents, knowing the proportion of agents in the opposing population using each strategy, update their own strategy to a current best response. The second type is the imitative dynamic in which agents copy the most successful agents in their own population. With two populations instead of a single population, these two dynamics are virtually identical[4], since a strategy which is a best reply to the strategy mix in the opposing population will have a higher payoff than one that isn't. The only issue arises when a strategy in one population is extinct, that is, has nobody playing it and so there is no payoff from this strategy to compare. Here an imitative dynamic like the one suggested by Hehenkamp [22] would require a mutation to introduce a new strategy into the population even if that strategy was a best response; whereas under a best response dynamic, like in Staudigl [41], this is not the case. This distinction is not big enough to have much effect on the results. In the interests of making the analysis slightly simpler, I will use a best response dynamic, although the results would be very similar with an imitation dynamic.

[4]This is not necessarily the case in a single population when agents must factor in that they do not interact with themselves.

The best response dynamic used is the following indivdualistic best response dynamic as described in Example 1 of Staudigl [41]. Every period a single randomly drawn agent, which could be from either population, is selected to revise his strategy. If this agent is currently playing a best reply to the strategy mix in the opposing population, that agent does not change his strategy. If he is not currently playing a best reply he switches to a best reply, picking each best reply with equal probability if there is more than one. Note that this ensures that the absorbing states correspond to Nash Equilibria, and that due to the stochastic nature of this model, a state can be in the Weak Basin of Attraction of more than one absorbing state. This is in contrast to the deterministic dynamic used in KMR. Section 9 of KMR and Hahn [20] discuss the trouble with using deterministic dynamics in a two population model.

Note that here only one agent is allowed to change strategy at a time. This ensures that only transitions to neighbouring states are possible, these are states connected by dotted lines in Figure 2. This simplifies the analysis and isn't quite as large a restriction as one may at first think. Another common method is to suppose agents are given opportunities to update via a stochastic alarm clock as in [6], where at the end of each period of length τ, each agent updates with probability τ. As Binmore and Samuelson noted, as $\tau \to 0$, as in their model, the occurrence that tow or more agents update strategy simultaneously becomes very rare.

The process is now disturbed by the presence of mutations: with a small probability $\varepsilon > 0$, the agent chosen to update his strategy makes a mistake. That is, instead of choosing a best reply, that agent picks a strategy at random. Given $\varepsilon > 0$, this process defines an irreducible Markov chain on X, where, for $x, y \in X$, P^{ε}_{xy} is the probability of moving from x to y in a single period.

This process defines a Markov chain on the state space X, where, for $x, y \in X$, P^{ε}_{xy} is the probability of moving from x to y in a single period. Notice that the set of transitions with positive probability are precisely those between neighbouring states.

	B_1	B_2
A_1	(a_{11}, b_{11})	(a_{12}, b_{12})
A_2	(a_{21}, b_{21})	(a_{22}, b_{22})

where $a_{11} > a_{21}$, $b_{11} > b_{21}$,$a_{22} > a_{12}$ and $b_{22} > b_{12}$.

Table 2.2.1: General 2x2 game with two pure strategy NE

This is sufficient to ensure that any state has a positive probability of being reached from any other state in at most $N^{\mathcal{A}} + N^{\mathcal{B}}$ periods. So given $\varepsilon > 0$, the Markov chain , P^{ε} is irreducible and hence admits a unique invariant distribution, that is a probability distribution μ^{ε} over X such that $\mu^{\varepsilon} P^{\varepsilon} = \mu^{\varepsilon}$.

Once again we will be interested in the limit, $\varepsilon \to 0$. We use the above analysis of basins of attraction and resistance to find the stochastically stable states and LRE. This is now presented below for 2x2 games.

Consider the class of 2x2 games with two strict, pure strategy Nash Equilibria. Without loss of generality we may write the payoff matrix as in Table 2.2 on page 75

where the two strict, pure strategy strict Nash Equilibria are (A_1, B_1) and (A_2, B_2). In addition there is also a mixed strategy Nash Equlibirum at $(\beta A_1 + (1-\beta) A_2, \alpha B_1 + (1-\alpha) B_2)$, where

$$\alpha = \frac{(a_{22} - a_{21})}{(a_{11} - a_{12}) + (a_{22} - a_{21})}$$

$$\beta = \frac{(b_{22} - b_{21})}{(b_{11} - b_{12}) + (b_{22} - b_{21})}$$

There are two[5] absorbing states corresponding to the two pure strategy Nash Equilibria

$$x^{(1)} = ((1,0), (1,0)) \quad x^{(2)} = ((0,1), (0,1))$$

The mixed strategy equilibrium is unstable, but the numbers α and β will have a crucial role to play in determining the basins of attraction as seen in Figure 2.2.2 on page 76.

[5]Depending on population size, it can also be the case that the state$((\alpha, 1-\alpha), (\beta, 1-\beta))$ is also an absorbing state, although generically this will not be the case

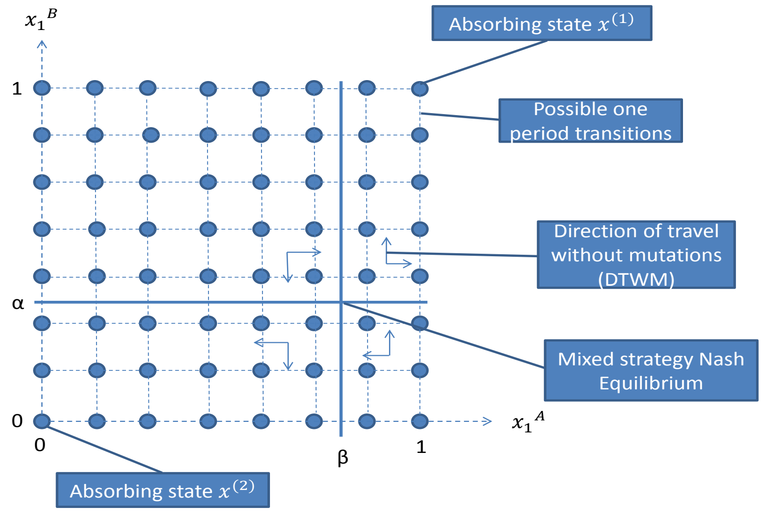


Figure 2.2.2: Basin of attraction diagram for 2x2 games with two pure strategy NE

Figure 2.2.2 on page 76gives an example with $N^{\mathcal{A}} = N^{\mathcal{B}} = 7$ where $5/7 < \beta < 6/7$ and $2/7 < \alpha < 3/7$. As can be seen above, the absorbing states have the following Basins of Attraction:

$$SB\left(x^{(1)}\right) = \left\{x \in X \mid x_1^{\mathcal{A}} > \beta,\, x_1^{\mathcal{B}} > \alpha\right\}, \quad WB\left(x^{(2)}\right) = X \setminus SB\left(x^{(1)}\right)$$

In other words the Strong Basin of $x^{(1)}$ is the those states to the northeast of mixed strategy Nash Equilibrium, while the Weak Basin of $x^{(2)}$ is the entire state space minus this segment. Similarly the Weak Basin of $x^{(1)}$ is precisely those states not in the bottom left segment, the Strong Basin of $x^{(2)}$.

$$SB\left(x^{(2)}\right) = \left\{x \in X \mid x_1^{\mathcal{A}} < \beta,\, x_1^{\mathcal{B}} < \alpha\right\}, \quad WB\left(x^{(1)}\right) = X \setminus SB\left(x^{(2)}\right)$$

Figure 2 also allows us to see the resistance of any transition between a pair of states. This is displayed by the arrows showing the direction of travel without mutations (DTWM). A transition in one of the two directions obeying the DTWM can be achieved without mutations and so has cost 0. While a transition going against the DTWM requires a mutation and so has resistance of 1. With this in mind, we can see that leaving the Strong Basin of $x^{(1)}$ requires two mutations in population $\mathcal{A}$, while leaving the Strong Basin of $x^{(2)}$ requires three mutations in population $\mathcal{B}$. This means that transitioning from $x^{(1)}$ to $x^{(2)}$ can be done with only two mutations while the reverse transition requires three mutations and so as $\varepsilon \to 0$, the latter becomes infinitely less likely and so $x^{(2)}$ is the LRE in this example. This can also be seen applying Ellison's radius-coradius Theorem since $R\left(x^{(2)}\right) = 3$ and $CR\left(x^{(2)}\right) = 2$.

More generally, for population $\mathcal{A}$, A_1 is the best reply if and only if $x_1^{\mathcal{B}} > \alpha$; while for population $\mathcal{B}$, B_1 is a best reply if and only if $x_1^{\mathcal{A}} > \beta$. This means that from state $x \in X$, moving towards $x^{(1)}$ requires mutations if $x_1^{\mathcal{B}} \leq \alpha$ and $x_1^{\mathcal{A}} \leq \beta$. On the other hand, if $x_1^{\mathcal{B}} > \alpha$, then $x_1^{\mathcal{A}}$ can increase without the need for any mutations, and once $x_1^{\mathcal{A}}$ is high enough that $x_1^{\mathcal{A}} > \beta$, strategy B_1 becomes a best reply for population $\mathcal{B}$, and we reach $x^{(1)}$ without the need for any mutations. Similarly there exists a mutationless path to $x^{(1)}$ from any $x \in X$ with $x_1^{\mathcal{A}} > \beta$. From this it is clear that the path from $x^{(2)}$ to $x^{(1)}$ requiring fewest mutations is either by having either $\left\lceil \beta N^{\mathcal{A}} \right\rceil$ consecutive population $\mathcal{A}$ mutations or $\left\lceil \alpha N^{\mathcal{B}} \right\rceil$ consecutive population $\mathcal{B}$ mutations, where $\lceil x \rceil$ denotes the smallest integer strictly greater than x. This leads to the following formula:

$$M\left(x^{(2)} \to x^{(1)}\right) = \min\left\{\left\lceil \beta N^{\mathcal{A}} \right\rceil, \left\lceil \alpha N^{\mathcal{B}} \right\rceil\right\}$$

where $M\left(x \to y\right)$ is defined as the minimum number of mutations needed in any path from state x to state y. Similar logic shows that

$$M\left(x^{(1)} \to x^{(2)}\right) = \min\left\{\left\lceil \left(1-\beta\right) N^{\mathcal{A}} \right\rceil, \left\lceil \left(1-\alpha\right) N^{\mathcal{B}} \right\rceil\right\}$$

This leads us to the following result:

Proposition 24. If $M\left(x^{(2)} \to x^{(1)}\right) > M\left(x^{(1)} \to x^{(2)}\right)$ then $x^{(2)}$ is uniquely stochastically stable. If $M\left(x^{(2)} \to x^{(1)}\right) < M\left(x^{(1)} \to x^{(2)}\right)$ then $x^{(1)}$ is uniquely stochastically stable. If $M\left(x^{(2)} \to x^{(1)}\right) = M\left(x^{(1)} \to x^{(2)}\right)$ then both $x^{(1)}$ and $x^{(2)}$ are stochastically stable.

Hehenkamp [22] showed something almost identical[6]. With this Proposition in place, the following Corollary becomes obvious.

Corollary 25. *(i) As long as both population sizes are even or sufficiently large if odd, if α and β are both greater than $1/2$ then $x^{(2,2)}$ is the LRE. Similarly $x^{(1,1)}$ is the LRE if α and β are both less than $1/2$. While if α and β are either side of $1/2$, relative population sizes matter.*

(ii) For equal population sizes, $N^{\mathcal{A}} = N^{\mathcal{B}} = N$, in the limit as $N \to \infty$, stochastic stability coincides with risk dominance. In other words, if $\min\{\alpha, \beta\} > \min\{1-\alpha, 1-\beta\}$ *then $x^{(2,2)}$ is the LRE*; while if $\min\{\alpha, \beta\} < \min\{1-\alpha, 1-\beta\}$ *then $x^{(1,1)}$ is the LRE. However, this does not extend beyond the 2x2 case.*

We obtain very similar results in Young's model. Young's original model [45] only considered the two populations having equal sample sizes, that is $k_A = k_B = k$, and found that given sample size $m \geq 3k$ stochastic stability coincided with risk dominance as suggested above. When Young applied this model to a bargaining scenario [44] he allowed for different sample sizes. Here this difference in sample size has as a very similar effect to the difference in population size. As Hehenkamp noted, we can associate the sample size of one population with the population size of the other in the population game, when considering the effect it has. To demonstrate this, suppose we are at the absorbing state of (A_2, B_2) having been played in the last m periods, which I denote $h^{(2)}$. Then to escape this state and move to m repetitions of (A_1, B_1), which I denote $h^{(1)}$, would require $\lceil \beta k_B \rceil$ consecutive mutations of the $\mathcal{A}$

[6]They obtain the same formula, the difference being that they define $\lceil x \rceil$ as the smallest integer greater than or equal to x. I actually believe their Theorem is incorrect for reasons related to this

player playing A_1. From here, if all $\lceil \beta k_{\mathcal{B}} \rceil$ of these observations make their way into the sample of $\mathcal{B}$ for the next $\lceil \alpha k_A \rceil + \lceil \beta k_{\mathcal{B}} \rceil$ then $\mathcal{B}$ plays B_1 throughout this period. After $\lceil \alpha k_A \rceil$ periods of this, it is then possible that the A player then samples at least $\lceil \alpha k_A \rceil$ observations of B_1 in the next $\lceil \beta k_B \rceil$ periods inducing him to play A_1 in each of these. Thus from here we can get to the state $h^{(1)}$ if each i-player simply samples the most recent k_i plays. For this to happen requires $m \geq 2 \lceil \beta k_B \rceil + \lceil \alpha k_A \rceil$. Similarly if $m \geq \lceil \beta k_B \rceil + 2 \lceil \alpha k_A \rceil$, we can have $\lceil \alpha k_A \rceil$ population $\mathcal{B}$ mutations shifting the state from $h^{(2)}$ to $h^{(1)}$. Thus, provided $m \geq 3k$, we obtain

$$M\left(h^{(2)} \to h^{(1)}\right) = \min\left\{\lceil \beta k_B \rceil, \lceil \alpha k_A \rceil\right\}$$

I work with the population game model, but the analysis above shows that very similar results will hold in Young's model. I now discuss a few critiques of the model. The first and most common critique is that we are declaring one state stochastically over another because it requires slightly less highly improbable mutations. For example if the transition from $x^{(1)}$ to $x^{(2)}$ requires 1000 highly improbable mutations will be judged infinitely more likely than the transition from $x^{(2)}$ to $x^{(1)}$ requiring 1001 mutations as $\varepsilon \to \infty$. However, whichever state's basin of attraction we start in, is likely to persist for a very long period of time. Thus the model perhaps works better when the population sizes are relatively small. Robson and Vega-Redondo [38] presented a similar model which requires fewer mutations, although produces different results, with the payoff dominant, instead of risk-dominant equilibrium being selected.

A second critique, due to Bergin and Lipman [5], is that the model is sensitive to the specification of mutations. By allowing mutations to be state dependent they show that any invariant distribution of the mutationless process is close to an invarint distribution of the process with appropriately chosen small mutation rates. Their analysis works by allowing mutations to occur with probability ε^{κ} where κ is higher for less likely mutations. A third critique is from Kim and Wong [25] who show that the outcomes is sensitive to the addition and deletion of strictly dominated strategies which should be completely irrelevant. They did this in the one population symmetric game model, but their logic nevertheless carries over to the two population case. Intuitively, the reason is that a mutation towards a crazy strategy can impact which of the sensible strategies should be played in the opposing population and so provide a shortcut to an absorbing state requiring fewer mutations than would be the case otherwise. This demonstrates that to obtain sensible results, it is necessary to restrict the strategy space to prevent this. Another alternative would be to follow Bergin and Lipman's approach of making crazy mutations far less likely than more sensible ones. With the right specification this would do the trick, but given the great many crazy strategies to consider, I think it is cleaner to restrict the strategy space to sensible strategies as I argue in Section 3.

Interpretation of stochastic stability in the bargaining game

Stochastic stability relies upon agents being able to learn from how the game has been played in the past, so it is most applicable to scenarios which are often repeated, preferably with a large pool of agents in each role. For example suppose one population is the set of upstream firms and the other population the set of downstream firms in an industry. The upstream firms which apply the most sucessful negotiation tactics are set to make more profit and so grow faster, gaining market share, while those who apply unsuccesful tactics will see the amount of business they do diminish. So as time progresses, those strategies which have been more succesful in the past will be used a greater proportion of the time.

Or in a more local setup, consider, a firm interacting with a union, who interact often, and over a range of different issues both big and small: one day they could be discussing what powers the firm should have to discipline workers who fail to arrive on time, the next day they might be discussing employee perks. Both sides have a pool of "agents" they could send to the negotiations whenever an issue arises. So that we can ignore the longer term reputation effects, suppose that both parties are sufficiently impatient or that the agents at both have reasonably short expected tenures in the current job, so that at any point in time, each is more concerned about the present deal than building a reputation for toughness which might be advantageous in future negotiations. If a particular agent has achieved a good outcome for the side he was representing in the past then he is more likely to be asked to represent his side again in the future and so the strategy that agent was using is set to grow. Alternatively we might think that the union's agents discuss strategy and hence learn off each other, and similarly with the firm's agents. Either way, strategies which have done better in the past are more likely to grow in popularity in the future.

2.3 The failure of commitment

In this Section I show that the result of our commitment game displayed in Figure 2.2.1 on page 66 is the non-commitment outcome, that is population $\mathcal{A}$ players will choose to bluff instead of commit and this is met with the Hard response by population $\mathcal{B}$ players. Furthermore this result is reasonably robust to a change in the model, allowing $\mathcal{B}$ to discover bluffs with small probability.

Perfect disguise

To start with I assume *perfect disguise*. That is $\mathcal{A}$ is perfectly adept at bluffing commitment so that $\mathcal{B}$ receives no information whatsoever about the action chosen by $\mathcal{A}$. This has already been discussed in Section 2 and fully represented in Figure 2.2.1 on page 66.

Note that under the perfect disguise assumption there is no drawback of bluffing instead of committing and so B weakly dominates C. In addition to the obvious Nash Equilibrium (B,H), there is also a connected component of Nash Equilibria giving the commitment outcome $(c,1-c)$: $\mathcal{B}$ plays S and $\mathcal{A}$ plays C with probability at least $\frac{c-w}{1-w}$. The absorbing states of the population game correspond to these Nash Equilibria:

$$\left\{\left(x^A,x^B\right)=\left(\left(x,1-x\right),\left(1,0\right)\right):\,x\in\left[\frac{c-w}{1-w},1\right],\,N^Ax\in\mathbb{Z}\right\}\cup\left\{\left(\left(0,1\right),\left(0,1\right)\right)\right\}$$

If the process is in any one of these states, it will require a mutation to leave that state. The connected component with S being played are linked by a chain of single population $\mathcal{A}$ mutations. From any member of this component, we require just one population $\mathcal{B}$ mutation to get to the (B,H) equilibrium of $((0,1),(0,1))$. However, as long as population $\mathcal{A}$ is large enough, it requires more than one mutation to get from $((0,1),(0,1))$ to a member of the connected component. This gives the following result:

Theorem 26. *For any population sizes* $\left(N^{\mathcal{A}},N^{\mathcal{B}}\right)$ *satisfying* $N^{\mathcal{A}}\geq\frac{1-w}{c-w}$ *and* $N^{\mathcal{B}}\geq 1$, *the LRE is* $x^{(B,H)}=((0,1),(0,1))$

Proof. The result follows from Ellison's radius-coradius theorem. $CR\left(x^{(B,H)}\right)=1$ since a single Hard agent in population $\mathcal{B}$ is enough to induce all population $\mathcal{A}$ agents to bluff, which in turn induces all population $\mathcal{B}$ to to act hard. While $R\left(x^{(B,H)}\right)>1$, since $N^{\mathcal{A}}\geq\frac{1-w}{c-w}$ ensures that one population $\mathcal{A}$ mutation is not enough to entice agents in population $\mathcal{B}$ to switch to soft. □

An important note, in light of the Kim and Wong [25] critique, is that these are the only strategies considered. Consider the following Example:

Example 27. Suppose that the method $\mathcal{A}$ agents use to claim commitment is employing a bargaining agent allegedly under contract not to accept less than c, where

	S	H	bribe
C	$(c, 1-c)$	$(0, 0)$	$(2+w, -1-w)$
B	$(c, 1-c)$	$(w, 1-w)$	$(w, -1-w)$

Table 2.3.1: Adding "bribe" strategy

the terms of the contract compel the bargaining agent to pay 2 (in payoff units) to $\mathcal{A}$ should he break this. Now, one option available to $\mathcal{B}$, albeit a deliberately nonsensical one, is to bribe the bargaining agent 2 units to accept w. Then denoting this new strategy by "bribe", the new payoff table is given in Table 27 on page 83

Note that this option to bribe the bargaining agent with 2 units will always give $\mathcal{B}$ a payoff of $-1-w<0$ and so is strictly dominated and hence should never be played. But the fact that agents in population $\mathcal{B}$ could mutate to this strategy affects the results by reducing $R\left(x^{(B,H)}\right)$. If $N^{\mathcal{B}} \leq \frac{2+w}{w}$ then from $x^{(B,H)}$ a single mutation of a population $\mathcal{B}$ agent from "H" to "bribe" is sufficient to make C the best response for population $\mathcal{A}$ agents and thus allow us to reach state $x^{(C,S)} = ((1,0),(1,0))$. So now we can connect the set of absorbing states via a cycle with resistance 1 between any two adjacent states. Hence[7] they are all stochastically stable.

I take the view that in this scenario the correct solution is obtained by not allowing the strategy "bribe" since it clearly is not a sensible strategy for $\mathcal{B}$ to consider thinking about players mutating through making mistakes, it is really easy to comprehend how an agent might mutate between "Soft" and "Hard" since both strategies are in some contexts sensible. Indeed, such a mutation could be from a mis-reading of the current population state. However it is far harder to make an argument for why an agent would mutate to "bribe", since this is guauranteed to worse than either of the other two strategies. One way to model this is to allow for state dependent mutation rates as in [5] and make such mutations far less likely, but then that raises the question how unlikely to make them. Then there is also the same issue with any other dominated

[7]This is fairly obvious. Either directly apply the Freidlin and Wentzell tree surgery or see for example Lemma 3 and 4 of [35]

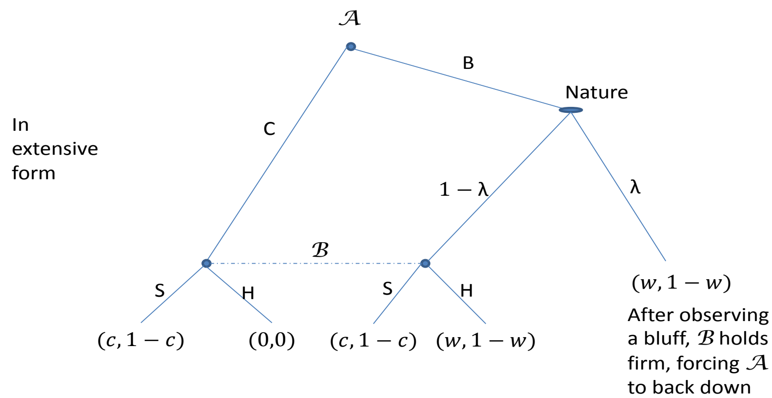
strategies that players have. So, with this in mind, I think it is more appropriate to proceed by restricting the strategy space to "sensible" strategies.

This consideration of restricting the strategy sapce also came into consideration when forming this game displayed in Figure 1. As noted in Section 2 the game represented in Figure 1 can be thought of as a truncation since it was assumed automatically that $\mathcal{A}$ would respond optimally to the offer of $\mathcal{B}$. There is a literature applying stochastic stability to extensive form games, see [10], [18], [21], [26], [35] and generally they show that the presence of such suboptimal replies can make a difference if agents are allowed to mutate toward such suboptimal replies. Once again I argue for restricting the strategy space based on what strategies are sensible. It seems hard to justify $\mathcal{A}$ wanting to turn down an offer from $\mathcal{B}$ in payoff terms, but also when we consider the context of what the strategies "Commit" and "Bluff" mean. The whole point of bluffing instead of committing is to then be able to accept w if $\mathcal{B}$ acts Hard. So in light of this, it would seem very odd behaviour to then turn w down.

Imperfect disguise

Now I drop the asssumption that a bluff is never discovered, so that now there is some incentive for $\mathcal{A}$ to commit rather than bluff against an opponent choosing a soft strategy. If we suppose a bluff is discovered with probability $\lambda \in (0, 1)$ then the game is represented in Figure 2.3.1 on page 85.

Note that once again this could be viewed as a slightly truncated version of the game. On top of the truncation described in Figure 2.2.1 on page 66, I have also truncated the game slightly by assuming that after $\mathcal{A}$ bluffs and nature reveals the bluff, $\mathcal{B}$ automatically chooses Hard and $\mathcal{A}$ accepts w, that is players follow the backwards induction equilibrium in this subgame. I argue that it is fairly obvious given the interpretation of the game that this is the only sensible play for both players. The reason for truncating the game in this way is the same as before.



In normal form

$\mathcal{A}$ \ $\mathcal{B}$	S	H
C	$(c, 1-c)$	$(0,0)$
B	$(r, 1-r)$	$(w, 1-w)$

$r = \lambda w + (1-\lambda)c$

Figure 2.3.1: Game under imperfect disguise assumption

There is a second interpretation of this game, which has the same game tree structure as Figure 2.2.1 on page 66. Suppose that post-agreement, $\mathcal{B}$ might discover $\mathcal{A}$ was bluffing and be able to use this to renegotiate. For example, returning to the firm-union example, suppose the union bluffs and the firm concedes to the bluff, but later discovers this was a bluff. Although the union gets its way with the current pay deal, the firm may be able to use the knowledge of the bluff to its advantage during future negotiations.

Notice that in this new payoff matrix given in Figure 2.3.1 on page 85, B no longer weakly dominates C and so both (C, S) and (B, H) are now strict Nash equilibria. This means that $x^{(C,S)} = ((1,0),(1,0))$ and $x^{(B,H)} = ((0,1),(0,1))$ are the only absorbing states[8], and for large populations, both states will require several mutations to escape from and so we must apply the mutation counting methods discussed in Section 2. We find $\alpha = \frac{w}{c-r+w}$ and $\beta = \frac{r-w}{1-c+r-w}$ are the cutoffs determining the basins

[8]It is posible to have a mixed strategy Nash Equilibrium be an absorbing state if the equilibrium mixtures are compatible with the integer problem from the population sizes. But generically this will not be the case, and even when such an absorbing state does exist it will be very unstable to mutations and not a candidate for being stochastically stable.

of attraction and so applying Theorem 23, we obtain

Theorem 28. *If* $r > \max\{c-w, 1-(c-w)\}$ *(ie* $\lambda < \min\left\{\frac{w}{c-w}, \frac{2c-w-1}{c-w}\right\}$*) and both populations have either an even or sufficiently large number of agents, then* $x^{(B,H)}$ *is the LRE.*

If $r < \min\{c-w, 1-(c-w)\}$ *(ie* $\lambda > \max\left\{\frac{w}{c-w}, \frac{2c-w-1}{c-w}\right\}$*) and both populations have either an even or sufficiently large number of agents, then* $x^{(C,S)}$ *is the LRE.*

Proof. Note that $\alpha > 1/2 \iff r > c-w$ and $\beta > 1/2 \iff r > 1-(c-w)$ and the result follows from 25 □

If the population sizes are equal and arbitrarily large so that stochastic stability coincides with risk dominance.

Theorem 29. *Let* $N^1 = N^2 = N \to \infty$*. If* $r > \frac{c-c^2+w^2}{1-c+w}$ *(ie* $\lambda < \frac{w}{1-c+w}$*) then* $x^{(B,H)}$ *is the LRE. If* $r < \frac{c-c^2+w^2}{1-c+w}$ *(ie* $\lambda > \frac{w}{1-c+w}$*)* then $x^{(C,S)}$ *is the LRE.*

Proof. Some elementary algebra shows that $\min\{\alpha,\beta\} > \min\{1-\alpha, 1-\beta\} \iff r > \frac{c-c^2+w^2}{1-c+w}$. Then the result follows from 25 □

This means that, as long as $\mathcal{A}$ is sufficiently adept at bluffing so that the probability λ of the bluff being discovered is reasonably small, the outcome (B,H) will still prevail. On the other hand, if the probability of a bluff is likely to be discovered as such then the commitment outcome of (C,S) prevails. This makes good intuitive sense, as when $\lambda = 1$ we are in the perfect information case where $\mathcal{B}$ knows whether $\mathcal{A}$ is bluffing or committed. In this case it is well established that $\mathcal{A}$ can commit. Note that while, the ability to bluff well may at first sight seem a strength, it is in fact a weakness since $\mathcal{B}$ then expect $\mathcal{A}$ to use this ability and thus chooses the Hard strategy.

I provide an illustrative numerical example:

		$\mathcal{B}$	
		S	H
$\mathcal{A}$	C	$(0.8, 0.2)$	$(0, 0)$
	B	$(0.75, 0.25)$	$(0.5, 0.5)$

Table 2.3.2: Imperfect disguise example with $x^{(C,S)}$ being the LRE

Example 30. Let payoffs be given by Table 2.3.2 on page 87.

The only candidates for stochastically stable sets are the absorbing states of the process without mutations, these $x^{(C,S)}$ and $x^{(B,H)}$. The process will naturally move away from any other state very quickly, since doing so does not require any mutations. So we are interested in the relative time spent in the $x^{(C,S)}$ state compared to that in the $x^{(B,H)}$ state. This is determined by the number of transitions needed to transition between the two.

To get from the $x^{(C,S)}$ state to the $x^{(B,H)}$ state requires either $4/9$ of population $\mathcal{A}$ mutating from C to B or $1/11$ of population $\mathcal{B}$ mutating from S to H.

To get from the $x^{(B,H)})$ state to the $x^{(C,S)}$ state requires either $5/9$ of population $\mathcal{A}$ mutating from B to C or $10/11$ of population $\mathcal{B}$ mutating from H to S.

So as $\varepsilon \to 0$, it becomes infinitely more likely to transition from $x^{(C,S)}$ to $x^{(B,H)}$ state than vice-versa, and thus $x^{(B,H)}$ is the uniquely stochastically stable state.

2.4 The return of commitment

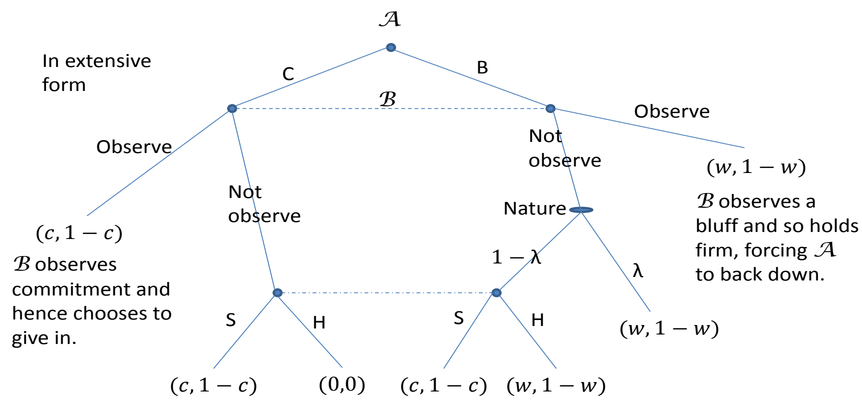
So far I have argued that for a commitment technology to be useful, the committed player must be able to tell the other player that he is committed. In particular, it is important that the message the committed player sends cannot be replicated by a non-committed player who is simply bluffing. Section 3 showed that this result is reasonably robust to the relaxation of the perfect disguise assumption. However in this Section I show that if we combine imperfect disguise with either an observation technology or an outside option, this can reverse the result and leave us with the commitment outcome.

With observation

Now I assume that $\mathcal{B}$ has a third strategy, this being to pay amount $k > 0$ to observe whether $\mathcal{A}$ is actually committed. As an example of how this might work in practice, consider the case of the firm and union again, which was mentioned at the end of Section 2. Suppose the firm is able to make contact with an individual in the union camp who would be prepared to act as a spy and they could bribe to inform the firm about the union's position. Would this be beneficial for the firm? In the short term the answer may be yes. However in the long run, the answer is no. If the union becomes aware that the firm is using this tactic, it can exploit this by committing itself to higher demands, knowing that the firm will observe the commitment. Even if the union is unaware such a tactic is being used, union negotiators may notice that those who commit themselves to high demands do better than those who do not, and so start using this tactic. Thus in the long run, we would expect the commitment outcome $(c, 1-c)$ to prevail. The model below will confirm this logic.

I introduce the following slight variation to the game previously discussed in Figure 3. Now $\mathcal{B}$ has the option to use an observation technology at cost k (the sum of the bribe in the example above), which enables B to discover whether or not $\mathcal{A}$ is indeed committed to demanding c. If $\mathcal{B}$ chooses not to use the observation technology, he faces the same dilemma as in Figure 2.3.1 on page 85. The new game, with this observation techonolgy is given in Figure 2.4.1 on page 89.

The game can thus be described as follows: after the decision of $\mathcal{A}$ whether or not to commit, $\mathcal{B}$ decides whether or not to use the observation technology with cost k. If $\mathcal{B}$ uses this observation technology then players play the backwward induction equilibrium from this point. As well as making sense from a payoff perspective, this is also the only sensible outcome given the meaning of the strategies in the



In normal form

$\mathcal{B}$

$\mathcal{A}$	S	H	O
C	$(c, 1-c)$	$(0,0)$	$(c, 1-c-k)$
B	$(r, 1-r)$	$(w, 1-w)$	$(w, 1-w-k)$

Figure 2.4.1: Imperfect disguise and observation technology

game. The fact that $\mathcal{B}$ pays k in order to observe whether or not $\mathcal{A}$ is committed suggests that he plans to use this information optimally. On the other hand, if $\mathcal{B}$ does not use the observation technology, he faces the same game as in Figure 2.3.1 on page 85. Although this generates an extensive form game, I justify the normal form representation by arguing that if $\mathcal{B}$ chooses not to observe he knows he is heading into the game of Figure 2.3.1 on page 85 and so should know what strategy he plans to use in this game, when deciding whether or not to deploy the observation technology. Therefore we can think of $\mathcal{B}$ making a choice between his three sensible strategies all at once. These three sensible strategies being: to "not observe" and play Soft (denoted S); "not observe" and play Hard (denoted H); to observe and respond optimally to this observation (denoted O).

Here it was assumed that $\mathcal{B}$ must decide whether to use the observation technology prior to discovering whether $\mathcal{A}$ was bluffing with probability λ through "Nature's" move. This allows the payoff matrix to be compatible with the other interpretation of the payoffs in the imperfect disguise game (the Normal form representation in Figure 2.3.1 on page 85), that $\mathcal{B}$ may discover a bluff post-agreement and be able to renegotiate the agreement. The alternative assumption of allowing Nature's move prior to the decision of $\mathcal{B}$ whether to use the observation technology would not change the analysis greatly[9].

In order for the observation technology to conceivably be useful, it must not be too expensive. If $k > \frac{(1-c)(r-w)}{1+c+r-w}$ then it can never be a best response to use it: Let p_c be the proportion of agents in population $\mathcal{A}$ choosing C, then agents in population $\mathcal{B}$ would always do better choosing either S or H, depending on whether p_c is relatively high or low. For this reason I henceforth assume $0 < k < \frac{(1-c)(r-w)}{1+c+r-w}$ which means that for "small" p_c the best reply is H, for "medium" p_c the best reply will be O and for "large" p_c the best reply will be S.[10] Given $k > 0$, the only pure strategy Nash Equilibria are (C, S) and (B, H) and so $x^{(C,S)} = ((1,0), (1,0,0))$ and $x^{(B,H)} = ((0,1), (0,1,0))$ are still the only two candidates for being stochastically stable[11].

As k decreases the size of the middle interval, where O is the best reply, increases at the expense of the other two. In particular this means that from the state $x^{(B,H)}$ fewer population $\mathcal{A}$ mutations are required to make O the best reply. Once O is the best reply, the proportion of population $\mathcal{B}$ playing O can then expand, followed by the proportion of population $\mathcal{A}$ playing C expanding, followed by the proportion of population $\mathcal{B}$ playing S expanding, all without the need for further mutations. Thus the observation technology acts as a conduit to allow easier passage from $x^{(B,H)}$ to $x^{(C,S)}$. If k is small enough so that the number of mutations required to reach $x^{(C,S)}$ from $x^{(B,H)}$ is less than the number of mutations to move in the opposite direction then $x^{(C,S)}$ replaces $x^{(B,H)}$ as the stochastically stable state.

[9] Strategy "O" would then mean to observe if unsure. The payoff matrix would be virtually identical to Table 3, the only difference being that the payoff to strategies (B, O) increases by λk to $1 - w - (1 - \lambda) k$. The impact of this change on the analysis is negligible and it has no effect on ny of the conclusions.

[10] Small, medium and large are relative terms here. The parameters (c, r, w, k) determine the sizes of these 3 intervals and hence what is meant by small, medium and large

[11] It is also possible to have a mixed strategy Nash Equilibrium with $\mathcal{B}$ mixing between O and H, but due to populations being finite, generically there is no absorbing state corresponding to this, and even if there is, this we would be just one mutation from the Weak Basins of the other two absorbing states.

The path decsribed above took us to $x^{(C,S)}$ from $x^{(B,H)}$ via the following four step process:

1. $\frac{k}{(1-c)}$ proportion of population $\mathcal{A}$ mutate from B to C.

2. Selection increases the proportion of O in population $\mathcal{B}$.

3. Selection increases the proportion of C in population $\mathcal{A}$.

4. Selection increases the proportion of S in population $\mathcal{B}$.

An alternative way to make this journey from $x^{(B,H)}$ to $x^{(C,S)}$ is for $\frac{w}{c}$ proportion of population $\mathcal{B}$ to mutate from H to O. This cuts out the first step of the above four step process and replaces selection by mutation in the second step. Although, if we assume r is relatively close to c then $\frac{(c-r)}{w+c-r} < \frac{w}{c}$ and so population $\mathcal{B}$ mutations are more likely to lead us from $x^{(C,S)}$ to $x^{(B,H)}$ than vice-versa. Hence the more likely way the observation technology is going to make a difference is via the four step process described above. One may even think it to be the salient case to consider k small relative to $1-c$, making the four stp process described above quite likely. For instance, in the firm-union example, it is quite conceivable that it be a lot cheaper for a firm to bribe a union official than to suffer the consequences of disagreement.

Theorem 31. *Let $N^A = N^B = N \to \infty$ and $r > \frac{c-c^2+w^2}{1-c+w}$ (so that $x^{(B,H)}$ is the LRE without an observation technology).*

If $\min\left\{\frac{k}{(1-c)}, \frac{w}{c}\right\} < \min\left\{\frac{(c-r)}{w+c-r}, \frac{(1-c)}{1-c+r-w}\right\}$, then $x^{(C,S)}$*is the LRE.*

If $\min\left\{\frac{k}{(1-c)}, \frac{w}{c}\right\} > \min\left\{\frac{(c-r)}{w+c-r}, \frac{(1-c)}{1-c+r-w}\right\}$, *then $x^{(B,H)}$ is the LRE.*

Proof. From $x^{(B,H)}$ to escape its Strong Basin, $SB\left(x^{(B,H)}\right)$, we need one of the following three things to happen: (i) $\left\lceil\frac{Nk}{(1-c)}\right\rceil$ of population $\mathcal{A}$ mutate from B to C and we follow the four step process described above; (ii) $\left\lceil\frac{Nw}{c}\right\rceil$ of population $\mathcal{A}$ mutate from B to C and followed by steps 3 and 4 of the process described above; (iii) $\left\lceil\frac{N(r-w)}{1-c+r-w}\right\rceil$ of population $\mathcal{A}$ mutate from B to C, so that S becomes the best response for population $\mathcal{B}$. Also, $x^{(B,H)}$ is the furthest state away from the Weak Basin of $x^{(C,S)}$, thus

$$CR\left(x^{(C,S)}\right) = R\left(x^{(B,H)}\right) = \min\left\{\left\lceil \frac{Nk}{(1-c)} \right\rceil, \left\lceil \frac{Nw}{c} \right\rceil, \left\lceil \frac{N\,(r-w)}{1-c+r-w} \right\rceil\right\}$$

The addition of the observation technology makes no difference to the shortest path from $x^{(C,S)}$ to $x^{(B,H)}$. Thus it is still the case that

$$CR\left(x^{(B,H)}\right) = R\left(x^{(C,S)}\right) = \min\left\{\left\lceil \frac{N\,(c-r)}{c-r+w} \right\rceil, \left\lceil \frac{N\,(1-c)}{1-c+r-w} \right\rceil\right\}$$

Note that it is not necessary to consider the $\left\lceil \frac{N(r-w)}{1-c+r-w} \right\rceil$ because of the assumption that $r > \frac{c-c^2+w^2}{1-c+w}$. Then we can see that for N large enough $\min\left\{\frac{k}{(1-c)}, \frac{w}{c}\right\} < \min\left\{\frac{(c-r)}{w+c-r}, \frac{(1-c)}{1-c+r-w}\right\}$ implies that $R\left(x^{(C,S)}\right) > CR\left(x^{(C,S)}\right)$ and thus using Ellison's Radius-Coradius Theorem $x^{(C,S)}$is the LRE. Similarly for N large enough, $\min\left\{\frac{k}{(1-c)}, \frac{w}{c}\right\} > \min\left\{\frac{(c-r)}{w+c-r}, \frac{(1-c)}{1-c+r-w}\right\}$ implies $R\left(x^{(B,H)}\right) > CR\left(x^{(B,H)}\right)$ and thus the LRE is $x^{(B,H)}$. □

This shows that the combination of a relaxation of the perfect disguise assumption and a cheap observation technology is enough to cause the commitment outcome $x^{(C,S)}$ to prevail. At first it may seem surprising that the addition of an observation option which doesn't get used in the long run should harm $\mathcal{B}$ like this. The intuition is the following: when population $\mathcal{A}$ is split between committing and bluffing, using the observation technology is a good idea and increases the instantaneous payoff for the player using it. However, the fact that O increases in popularity will increase the fitness of C for population $\mathcal{A}$ agents and so this proportion rises. So while playing O may be good at the time for the individual population $\mathcal{B}$ agent, the fact that he and others act in this way has negative long term consequences for the agents in population $\mathcal{B}$ as a whole.

I present an easier to follow numerical example:

Example 32. Appending Table 2.3.2 on page 87 with an observation column, where $k = 0.01$ gives the following payoffs:

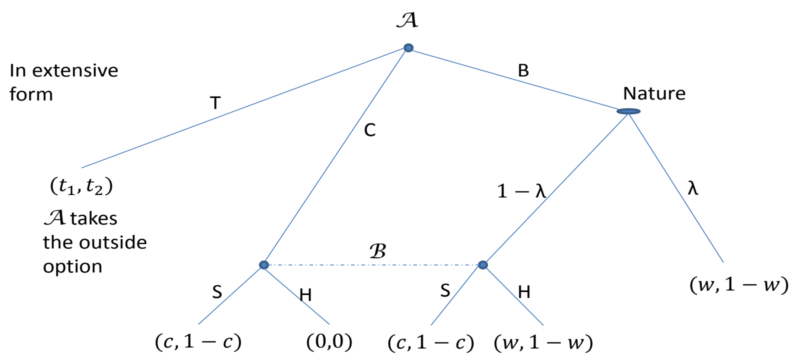
	S	H	O
C	$(0.8, 0.2)$	$(0, 0)$	$(0.8, 0.19)$
B	$(0.75, 0.25)$	$(0.5, 0.5)$	$(0.5, 0.49)$

Just as in the previous Table 2.3.2 on page 87, where it was found that $x^{(B,H)}$ is the LRE, it takes either $4/9$ of population $\mathcal{A}$ or $1/11$ of population $\mathcal{B}$ mutating to travel from $x^{(C,S)}$ to $x^{(B,H)}$. However, now the introduction of the observation option makes it much quicker to travel from the $x^{(B,H)}$ to $x^{(C,S)}$. If $1/20$ of population $\mathcal{A}$ mutating from B to C then O becomes a best response for agents in population $\mathcal{B}$. So the proportion of population $\mathcal{B}$ playing O expands without any further mutations. Once this proportion reaches $3/8$, C becomes a best response for agents in population $\mathcal{A}$ and so without any further mutations, the proportion playing C expands until it reaches one. This in turn makes S a best response instead of O for population $\mathcal{B}$ agents and so we end up with all of population $\mathcal{B}$ playing S and so we reach the $x^{(C,S)}$ state. So by going via states in which some agents to observe, we have found a way to travel from $x^{(B,H)}$ to $x^{(C,S)}$ which only requires $1/20$ of population $\mathcal{A}$ to mutate. Thus when the two populations are similar sizes and large enough so that $\left\lceil \frac{N_1}{20} \right\rceil < \left\lceil \frac{N_2}{11} \right\rceil$, we find that $x^{(C,S)}$ is the LRE.

With outside options

Now we return to the imperfect disguise game of Figure 2.3.1 on page 85 and instead of giving $\mathcal{B}$ an observation technology, give $\mathcal{A}$ an outside option (action T) which he can decide to take instead of heading into the bargaining game. This is represented in Figure 2.4.2 on page 94.

When $\mathcal{A}$ takes the outside option, we may think it natural to set $\mathcal{B}$'s payoff $t_2 = 0$, if for example, the outside option is doing a deal with an outside party. However the



$\mathcal{B}$

$\mathcal{A}$		S	H
	C	$(c, 1-c)$	$(0,0)$
	B	$(r, 1-r)$	$(w, 1-w)$
	T	(t_1, t_2)	(t_1, t_2)

$(w < t_1 < c)$

Figure 2.4.2: Imperfect disguise with outside option

results will not depend on t_2.

As an example, consider the house buyer scenario and suppose $\mathcal{A}$ has the option of purchasing a different house for a set price. In this instance it is important that the house buyer must make a choice between which of the two houses to pursue, which is true if for example the decision on whether to buy this other house has to be made before he will hear back from $\mathcal{B}$, the seller. If the converse was true, $\mathcal{A}$ could commit in his dealings with $\mathcal{B}$, and if $\mathcal{B}$ does not give in, use the outside option as a backup. In the firm-union example, the outside option might be for the union to bring in an independent adjudicator who will decide on what a "fair" rate of pay would be. In this instance, a deal is struck between the two parties and so $t_2 > 0$.

I assume that $t_1 \in (w, c)$ so that the problem is not trivial. If the outside option was any less attractive, it would make no difference, as it would not be used; any more attractive and it would always be used. Now the Nash Equilibrium set changes: (C, S) is still a Nash Equilibrium and there is a connected componenet of Nash Equilibria where $\mathcal{A}$ plays T and $\mathcal{B}$ plays S with probability of no more than $\bar{p} = \min\left\{\frac{t-w}{r-w}, \frac{t}{c}\right\}$. Once again, the absorbing states of the population game correspond to these Nash

Equilibria, however only $x^{(C,S)}$ is stochastically stable.

Theorem 33. *For sufficiently large population sizes* $x^{(C,S)} = ((1,0,0),(1,0))$ *is the LRE.*

Proof. (C,S) is a strict Nash Equilibrium, so for sufficiently large population sizes, it is clear that it requires more than two mutations to exit the Strong Basin of Attraction of $x^{(C,S)}$. Hence $R\left(x^{(C,S)}\right) > 2$. I now argue that $CR\left(x^{(C,S)}\right) \leq 2$ (in fact, generically is just 1) and hence the result follows from the Radius-Coradius Theorem.

If we are at a state in which the current best reply for population $\mathcal{A}$ is C then no mutations are needed to reach $x^{(C,S)}$, since the proportion playing C increases by selection, and once this is high enough, the proportion of S in population $\mathcal{B}$ increases by selection. If the current best reply in population $\mathcal{A}$ is T then the proportion playing T expands until all of population $\mathcal{A}$ are playing it. From here, suppose that one member of population $\mathcal{A}$ mutates to C. Then the agents in population $\mathcal{B}$ will find S the best reply and so the proportion playing S expands via selection until it reaches the entire population, from which point C also becomes a best reply for population A and we reach $x^{(C,S)}$ without any further mutations. If we are at a state where agents in population $\mathcal{A}$ find B is the best reply then the proportion playing B increases, thus making H the best reply for population $\mathcal{B}$. Once the proportion of population $\mathcal{B}$ playing H is large enough, T then becomes the best reply for population $\mathcal{A}$ and we are in the same case as before.

So if there is a unique best reply in population A then it only takes one mutation to reach $x^{(C,S)}$. If[12] we happen to be in a state where there are two best replies for population $\mathcal{A}$, then we introduce a mutation in population $\mathcal{B}$ so that there is a unique Best Reply and use the result above. This shows that $CR\left(x^{(C,S)}\right) \leq 2$. □

This result holds for any $t \in (w, c)$. The intuition is a forward induction type

[12] This is very unlikely. Indeed given the finite population assumption requires a specific mix of population size and payoff parameters to make this possible.

argument: The decsion of $\mathcal{A}$ to turn down the outside option means that he expects to do better in the imperfect disguise game than he would by taking the outside option. This means that (C, S) instead of (B, H) should be expected to prevail in this game. It is well known, for example [35] that stochastic stability has forward inducton properties.

2.5 Continuous signalling space

This Section briefly explores the impact of introducing a far richer signalling space. I now dispense with the population game setup and introduce a model which shares some similarities with Young's adaptive dynamic [45], in the sense that the role of the state space is to give a signal to the players of how the game has been played in the past, and how they expect the game to be played in the future. Whereas in the population game model, the state represents how players are actually playing the game.

The imperfect disguise assumption used in the previous two sections only allowed for two possible signals; one revealing the action of $\mathcal{A}$ to be B and the other leaving both possibilities open. Now I replace this with a continuous signal space so that after $\mathcal{A}$ plays $i \in \{C, B\}$, $\mathcal{B}$ receives a signal $\sigma \in [0, 1]$ with cumulative distribution function $F_i(\sigma)$ and probability distribution function $f_i(\sigma)$. I interpret σ as agent's instinctive probability that $\mathcal{A}$ has played C. $\mathcal{B}$ uses this instinctive probability and knowledge about past history to form a belief about how likely it is $\mathcal{A}$ played C, and then best responds given this belief. This Section represents quite a large deviation from the literature on stochastic stability discussed above.

Formally the model is as follows: There are N agents in population $\mathcal{B}$ and an infinite stream of agents who appear in the $\mathcal{A}$ role, one after the other, play against a randomly selected population $\mathcal{B}$ agent. The state space is $X = \{0, 1, 2, \ldots N\}$ where state $x \in X$ means that x of these N agents played S the last time they played. In each period, given the state $x \in X$, the $\mathcal{A}$-role player plays C with probability $g(x)$ and B with probability $1-g(x)$. Player $\mathcal{B}$ is aware of this, and so taking into account both the state x and signal σ, forms the following posterior belief that $\theta(x,\sigma)$ is the probability of C having been played.

$$\theta(x,\sigma) = \frac{g(x) f_C(\sigma)}{g(x) f_C(\sigma) + (1-g(x)) f_B(\sigma)}$$

Player $\mathcal{B}$ then best responds given this belief $\theta(x,\sigma)$[13]. This updates the state and the process moves to the following period where it repeats. If we assume $g(x) \in (0,1)$ for all $x \in X$ and conditions F1-F3 (below) on the signal functions to ensure that the signal has the power to be sufficiently informative, we get an irreducible Markov chain on X. In fact, we get a *birth-death chain*, since each period, the state will either increase by one, decrease by one or remain the same. To ensure the signal is sufficiently informative, I assume

F1: $f_i(\sigma) > 0$ for all $\sigma \in (0,1)$ and $i \in \{C, B\}$.

F2: $\lim_{\sigma\to 0} \frac{f_C(\sigma)}{f_B(\sigma)} = 0$

F3: $\lim_{\sigma\to 1} \frac{f_C(\sigma)}{f_B(\sigma)} = \infty$

The frist condition says that all signals are possible, in particular there is a chance of $\mathcal{B}$ receiving a signal close to 0 when $\mathcal{A}$ is committed and receiving a signal close to 1 when $\mathcal{A}$ is bluffing. The second (third) condition ensures that such events are sufficiently unlikely that when a high (low) enough signal is received $\mathcal{B}$ will believe that it is almost certain that $\mathcal{A}$ is committed (bluffing) regardless of the state $x \in X$, and thus play Soft (Hard).

[13] To simplify notation I will say that if indifferent between H and S, he plays S. Although the results are not dependent on any assumption here, since $f_i(\sigma)$ for $i \in \{C, B\}$ will be chosen so that getting the exact signal σ to leave agent 2 indifferent is a zero-probability event.

While the payoffs agents receive at different outcomes has not changed, and is still represented by Table 1, one should note that this is now a very different game to that of Sections 3 and 4. Now one must take into account how the action of $\mathcal{A}$ will influence the beliefs and hence the action of $\mathcal{B}$. The expected utilities to $\mathcal{A}$ of playing C and B are given by:

$$U_1 (C \mid x) = \Pr (S \mid C, x)\, c$$

$$U_1 (B \mid x) = \Pr (S \mid B, x)\, c + \Pr (H \mid B, x)\, w$$

From Table 1, it is clear that the tipping point in terms of the beliefs of $\mathcal{B}$, determining his strategy is $\beta = \frac{c-w}{1-w}$ chance of C. A posterior belief $\theta (x, \sigma)$ which puts a higher weight on C makes S the best reply, while a belief below β makes H the best reply. Therefore $\Pr (S \mid i, x)$ is the probability that given strategy $i \in \{C, B\}$, agent 2 receives a signal high enough to cause $\theta (x, \sigma) \geq \beta$. Now I introduce the following four assumptions on $g (x)$.

G1: $g (0) = \varepsilon$

G2: $g (N) = 1 - \varepsilon$

G3: $g (x) = 1 - g (N - x)$ for all $x \in X$

G4: $g (x)$ is an increasing funtion of x

Consider the state $x = 0$, in which all of the agents in population 2 played H last. Assumption G1 says that $\mathcal{A}$ should then play B, barring a probability ε mistake or mutation. This would seem wise assuming that playing C would not induce $\mathcal{B}$ to play S enough of the time, which is indeed the case as $\varepsilon \to 0$ Similarly at state $x = N$, where all of population $\mathcal{B}$ last played S, assumption G2 says that $\mathcal{A}$ should play C, barring a probability ε mistake. Given this is the expectation of play it seems wise to continue this way, since as $\varepsilon \to 0$, both $\Pr(H \mid C, N)$ and $\Pr(H \mid B, N)$ but $\frac{\Pr(H|C,N)}{\Pr(H|B,N)} \to \infty$. This logic is formalized below. Assumption G3 says is a symmetry condition, which says that the probability of playing C when only x agents last played S, is the same as the probability of playing B when only x agents last played H. Assumption G4 says that, the greater the number of population 2 agents who last played S, the more likely agent 1 is to play C. The logic for this is that the two outcomes which are likely to predominate are (C, S) and (B, H) and the state is taken as a signal is to which one is supported by precedent and so is more likely to predominate in the future. I think the hardest of the four to justify is G3. Nevertheless there are examples of sensible procedures satisfying all four assumptions. One such procedure is for $\mathcal{A}$ to randomly sample one agent from population $\mathcal{B}$, and best respond to that agent's last action with probability $1 - \varepsilon$.

From here on, I will work with the following signal distribution functions:

$$F_C(\sigma) = \sigma^2 \quad f_C(\sigma) = 2\sigma \tag{2.5.1}$$

$$F_B(\sigma) = 2\sigma - \sigma^2 \quad f_B(\sigma) = 2(1 - \sigma) \tag{2.5.2}$$

Note that these do satisfy F1-F3. Also note the implied symmetry in these functions: $\mathcal{B}$'s signal after C is distributed around 1 in the same way that the signal after B is distributed around 0, and so there is no bias induced by these functions.

To get a feel for how the dynamics work here, suppose we are at a state with low x and so the probability of $\mathcal{A}$ playing C is small. There are two things which could happen to induce $\mathcal{B}$ to play S (and so move the state to $x + 1$ if that agent previous played H). Firstly, since $g(x) > 0$, agent 1 might play C, and if this happens there is a decent chance that the signal $\mathcal{B}$ receives is high enough to induce him to play S. Secondly, there is a chance that, as predicted $\mathcal{A}$ plays B but the signal $\mathcal{B}$ receives is sufficiently high as to convince him to play S. This is the case of $\mathcal{B}$ dramatically mis-reading $\mathcal{A}$. Likewise, for high x and $g(x)$, there are two ways in which $\mathcal{B}$ could be induced to play H: firstly since $g(x) < 1$, $\mathcal{A}$ might actually bluff and $\mathcal{B}$ recognise the bluff through receiving a low signal, or secondly $\mathcal{B}$ may mis-read $\mathcal{A}$ and think he is bluffing when he is in fact committed. As $g(x) \to 0$ the probability of both events which could cause S to be played also diminish to zero; and similarly as $g(x) \to 1$ the probability of both events which could cause H to be played diminish to zero. Therefore we can expect the process, in the $\varepsilon \to 0$ limit to spend most of its time either at state $x = 0$ or state $x = N$, since these are the absorbing states.

The effect of the state space is on the players' expectations of future play, and we have a self-confirming equilibrium type argument. When x is low, so that action H is most prevalent, there is an expectation that $\mathcal{A}$ should play B and that $\mathcal{B}$ should again play H. $\mathcal{A}$ is discouraged from switching to C because agent 2 expects B to be played, and the signal he receives is likely not to be powerful enough to overcome his prior belief, meaning that H is likely to be played even if $\mathcal{A}$ chooses C. This is the logic behind assumptions G1 and G2. To see this, suppose that we are in state $x = 0$, so that the prior probability $\mathcal{B}$ attaches to C is ε. In order to induce agent 2 to play S, the signal agent 2 receives must induce a sufficiently high posterior belief of C. That is, we need

$$\theta(0, \sigma) = \frac{2\varepsilon\sigma}{2\varepsilon\sigma + 2(1-\varepsilon)(1-\sigma)} \geq \beta$$

which requires

$$\sigma \geq \frac{\beta(1-\varepsilon)}{\varepsilon + \beta - 2\beta\varepsilon}$$

Even if agent 1 chooses C, under our signal functions, a high enough signal to induce

S only happens with probability

$$1 - \left(\frac{\beta(1-\varepsilon)}{\varepsilon + \beta - 2\beta\varepsilon}\right)^2$$

which tends to 0 as $\varepsilon \to 0$, making action C a bad choice, and justifying assumption G1 that, barring mistakes, $\mathcal{A}$ should choose B. Now suppose the state is $x = N$, so that the prior probability $\mathcal{B}$ attaches to C is $1 - \varepsilon$. $\mathcal{B}$ plays S if

$$\theta(N, \sigma) = \frac{(1-\varepsilon)\sigma}{(1-\varepsilon)\sigma + \varepsilon(1-\sigma)} \geq \beta$$

which requires signal

$$\sigma \geq \hat{\sigma} = \frac{\beta\varepsilon}{1 - \varepsilon - \beta + 2\beta\varepsilon}$$

and therefore

$$Pr(H \mid C, N) = \hat{\sigma}^2,\ Pr(H \mid B, N) = 2\hat{\sigma} - \hat{\sigma}^2$$

which implies

$$\lim_{\varepsilon \to 0} \frac{Pr(H \mid B, N)}{Pr(H \mid C, N)} = \infty$$

Rearranging the formulae for expected utility, it can be shown that

$$U_1(C \mid N) - U_1(B \mid N) = Pr(H \mid B, N)(c - w) - Pr(H \mid C, N)c$$

which implies that as $\varepsilon \to 0$, playing C gives a higher expected utility than playing B, thus supporting assumption G2.

In general the invariant distribution of the process will depend on the population size N, as well as the mutation rate ε, and will be denoted $\mu^{\varepsilon,N} = \left(\mu_0^{\varepsilon,N}, \mu_1^{\varepsilon,N}, \ldots, \mu_N^{\varepsilon,N}\right)$. Given population size N , define the *limit invariant distribution* as $\mu^{*,N} = \lim_{\varepsilon \to 0} \mu^{\varepsilon,N}$ and apply the usual definition of stochastic stability and LRE. I now state the result:

Theorem 34. *If $\beta > 1/2$, then as $N \to \infty$ the LRE is $x = 0$ and if $\beta < 1/2$ the LRE* *is* $x = N$.

This result says that for a sufficiently large number of agents in population $\mathcal{B}$, if c is sufficiently high relative to w so that $\beta = \frac{c-w}{1-w} > 1/2$, we get (B, H) as the unique outcome; whereas, if the commitment level c is relatively undemanding compared to w so that $\beta < 1/2$, we get (C, S) as the unique outcome.

Proof. I show the following: If $\beta > 1/2$ then for any $\xi > 0$, there exists $\bar{N}$ such that for all $N \geq \bar{N}$ and $\varepsilon_N > 0$ with $\mu_0^{\varepsilon_N, N} > 1 - \xi$. An almost symmetric proof would show that if $\beta < 1/2$, then for any $\xi > 0$, there exists $\bar{N}$ such that for all $N \geq \bar{N}$ and $\varepsilon_N > 0$ with $\mu_N^{\varepsilon_N, N} > 1 - \xi$.

Fix N and let u_x be the probability of moving from x to $x + 1$, and d_x be the probability of moving from x to $x - 1$. The key lies in proving that

$$\beta > 1/2 \Rightarrow \exists k < 1 \; s.t. \; \frac{u_x}{d_{N-x}} < k \qquad \forall x \in X \tag{2.5.3}$$

Let the state be $x \in X$. Then u_x is the probability of selecting a player who previously played Hard ($\Pr(Select\ Hard \mid x)$), multiplied by the probability of the selected player playing Soft ($\Pr(Play\ Soft \mid x)$). The former is easy, $\Pr(Select\ Hard \mid x) = \frac{N-x}{N}$, while the latter requires more calculation:

$$\Pr(Play\ Soft \mid x) = \Pr\left(\theta(x, \sigma) = \frac{2\sigma g(x)}{2\sigma g(x) + 2(1-\sigma)(1-g(x))} \geq \beta\right)$$

which requires

$$\sigma \geq \bar{\sigma} = \frac{\beta(1-g(x))}{g(x) + \beta - 2\beta g(x)} \tag{2.5.4}$$

Similarly, from state $N - x \in X$, the probability of moving down one S player, d_{N-x}, is the the probability of selecting a Soft player multiplied by the probabilty of the selected player playing Hard. Once again, the former is easy, $\Pr(Select\ Soft \mid N - x) =$

$\frac{N-x}{N}$, while the latter requires more calculation:

$$\Pr\left(Play\ Hard \mid x\right) = \Pr\left(\theta\left(N-x, \sigma_{N-x}\right) = \frac{2\sigma g\left(N-x\right)}{2\sigma g\left(N-x\right) + 2\left(1-\sigma\right)\left(1-g\left(N-x\right)\right)} \leq \beta\right)$$

which requires

$$\sigma \leq \hat{\sigma} = \frac{\beta\left(1-g\left(N-x\right)\right)}{g\left(N-x\right) + \beta - 2\beta g\left(N-x\right)} \tag{2.5.5}$$

Comparing u_x and d_{N-x}, we see that the $\frac{N-x}{N}$ terms cancel and so the comparison is between the relative likelihoods of (2.5.4) and (2.5.5), that is $\Pr\left(\sigma \geq \bar{\sigma} \mid x\right)$ and $\Pr\left(\sigma \leq \hat{\sigma} \mid N-x\right)$. If $\beta = 1/2$ then (2.5.4) simplifies to $1 - g\left(x\right)$ and (2.5.5) simplifies to $1 - g\left(N-x\right)$. Combining assumption G3 with the signal functions (2.5.4) and (2.5.5), we then have $\Pr\left(\sigma \geq \bar{\sigma} \mid x\right) = \Pr\left(\sigma \leq \hat{\sigma} \mid N-x\right)$. Now, observe that both $\bar{\sigma}$ and $\hat{\sigma}$ are increasing in β, hence $\Pr\left(\sigma \geq \bar{\sigma} \mid x\right)$ is decreasing and $\Pr\left(\sigma \leq \hat{\sigma} \mid N-x\right)$ is increasing in β. This shows that for any $\beta > 1/2$, $\Pr\left(\sigma \geq \bar{\sigma} \mid x\right) < \Pr\left(\sigma \leq \hat{\sigma} \mid N-x\right)$ and hence $d_{N-x} > u_x$. Noting that, for each N, u_x and d_x are only defined at finitely many points gives equation (2.5.3).

Since we have a birth death chain, the weight on state 0 in the invariant distribution is given by

$$\mu_0 = \left(1 + \frac{u_0}{d_1} + \frac{u_0 u_1}{d_1 d_2} + ... + \frac{u_0 u_1 \cdots u_{N-1}}{d_1 d_2 \cdots d_N}\right)^{-1} \tag{2.5.6}$$

After the "1" term, where the last term is by far the greatest. Since $\lim_{\varepsilon \to 0} \frac{u_{N-1}}{d_N} = \infty$ by G1, G3, G4, we can find ε_N such that $\frac{u_0 u_1 \cdots u_{N-1}}{d_1 d_2 \cdots d_N}$ is greater than the sum of the preceding $N-1$ terms and hence

$$\mu_0^{\varepsilon_N, N} > \left(1 + 2\frac{u_0 u_1 \cdots u_{N-1}}{d_1 d_2 \cdots d_N}\right)^{-1} \tag{2.5.7}$$

Note that

$$\frac{u_0 u_1 \cdots u_{N-1}}{d_1 d_2 \cdots d_N} = \frac{u_0}{d_N} \frac{u_1}{d_{N-1}} \frac{u_2}{d_{N-2}} \cdots \frac{u_{N-1}}{d_1}$$

This is the product of N terms all of which satisfy (2.5.3). So as $N \to \infty$, this product tends to 0 and hence for any $\xi > 0$, there exists $\bar{N}$ such that for all $N \geq \bar{N}$ and $\varepsilon_N > 0$ such that $\mu_0^{\varepsilon_N, N} > 1 - \xi$ (using (2.5.7)). □

2.6 Conclusion

This Chapter shows that, when looking at the effectiveness of commitment, an important consideration is the ability of a player to bluff at being committed. While on the face of it, one might expect the ability to bluff well to be useful, this is not the case since in the long run the other player will anticipate the bluff being used and so ignore the commitment. The intuition is that if a player can bluff well, then the opponent would expect the bluff to occur and act accordingly.

Increased sophistication can also be a disavantage for player $\mathcal{B}$. Section 4 shows that if $\mathcal{B}$ has the option of observing the commitmnet decision of $\mathcal{A}$ at relatively small cost then this flips long run equilibrium back to the commitmnet outcome. The basic intuition is that if $\mathcal{A}$ expects his decision to be observed by $\mathcal{B}$ then this gives him incentive to commit. In actual fact things are slightly more complicated since B does not actually use the observation technology in the long run equilibrium. However, its presence is enough to make the alternative equilibrium of $x^{(B,H)}$ very unstable.

Similarly a return to the commitment outcome can occur when $\mathcal{A}$ is given an outside option. Even if $t_1 < c$ so that it is not used in the long run equilibrium, its presence still makes a difference by unsettling the $x^{(B,H)}$ equilibrium.

Chapter 3

Toughness and goodwill in bargaining: following the crowd

3.1 Introduction

In most bargaining situations, with only two agents, it seems logical to think that each will negotiate as hard as they can to get as large a share of the surplus as possible. However, when there are other agents who can come into the deal then the situation becomes unclear. While a hardline bargaining position may give you more of the surplus if you do manage to interact, it will decrease the probability of others wanting to interact with you. Thus there is a tradeoff between taking a soft position, thus increasing the probability of interacting, and taking a hardline position to increase your share of the surplus when interacting. This chapter shows that often there will be a tendency for players to want to follow the crowd, that is take a soft position if and only if others do the same.

Why do some societies have a culture of really tough negotiating, while others have a culture of far softer negotiating? Why in some families is there an argument over every last pea on the dinner table while in other families such discussions are handled with politeness and goodwill? The answer I propose here is that it is in each person's interests to copy the behaviour of everyone else: if others are playing tough, then so must you to get your fair share; while if others play nice, then a heavy handed bargaining approach will see you effectively be shunned from the group as others don't want to interact with you.

In a business environment are you a tough or a gentle negotiator? In a joint project with a work colleague do you willingly make a large contribution or look to free-ride on your partner's efforts? When meeting with a friend who prefers to go to the opera, do you insist on going to watch the football or accommodate your friend's preference of opera? At the pub, how generous are you with buying drinks for friends? Business, and life in general, is full of relationships in which there has to be give and take on both sides. The question is: *how much do you give and how much do you take?* This is not a trivial question. The temptation to be tough and insist on getting one's own way looks good when the other player gives in to your demands, but may hurt your chances of interacting if the other is equally tough or finds finds another more amenable partner. Thus there is an obvious tradeoff between being tough so as to get as large a share from the interaction as possible, and being willing to compromise and interact more regularly.

One obvious and quite simple way to investigate this tradeoff is with a Hawk-Dove model [30]. "Hawk" would have the interpretation that the negotiator adopts a tough negotiating position, being unwilling to compromise, and relying on concessions from the other party to reach agreement, while "Doves" are willing to compromise in order to reach agreement. Each player is either a Hawk or Dove and randomly matched to interact with one another, under a veil of ignorance about each others type. The significance of this veil of ignorance is that agents cannot adapt their play based on their opponent, so if two Hawks meet they fail to reach agreement. This theory predicts a mixture of Hawks and Doves.

By contrast, the models in this Chapter drop the veil of ignorance assumption, so that all agents knows each others' type. If one agent is tougher than the other then that agent will get a larger share of any interaction between the two. However, unlike Hawk-Dove, tough agents are able to recognise one another, and it is assumed, will be able to reach a compromise. The disadvantage of a tough negotiating position comes from one's negotiating partner preferring to seek other, less demanding, trading partners.

When there are only two agents the situation is clear: each will try with all their might to take as much as possible from each interaction and there is a great deal of literature, including the two previous chapters on how agents might seek to do so. However, often in real world situations there are more than two relevant parties. Take for example a firm with a job vacancy to be filled. They might interview somebody for the post who is well suited and who can fill the post very comfortably. If we suppose that this firm would generate an extra £200,000 per year by employing this person, then how much should this person get paid? If this person is the only one who has the skills to do the job, then it would seem logical that he try to negotiate as much as possible for himself, both in terms of the £200,000 surplus and work conditions such as office size. However, if there are another hundred people waiting outside who could do the job just as well, then he would be well advised to think twice before demanding too much. If this worker was to ask for a lot, then the chances are the firm will instead hire somebody less demanding. Indeed, even if there are only two workers, who could both fill the role, then the literature on auction theory suggests that the workers will compete against each other and the firm will walk away with the bulk of the surplus.

However, in the example above there are many more potential complications. For instance it may well be the case that each worker has several firms who he could work for, thus preventing the firm from demanding too much of the surplus, for fear of a rival giving the worker a better offer. Additionally there may be some uncertainty as to whether each interaction can happen, since some of the parties may not always be available, or have the necessary skills or resources to interact. We incorporate this in to a stochastic payoff function where a player's payoff is the probability of interacting multiplied by his average share when interacting. A tough negotiating approach will sacrifice ground in the former to gain ground in the latter. This establishes the tradeoff between maximising the probability of interacting and average share when interacting.

Often in economics, it is assumed agents take as much as possible from each interaction, but this would be to neglect the consideration that your gain comes at the other person's loss and so taking a tough position may lead to other agents being less enthusiastic about interacting with you. Whereas, an agent who is more generous could be expected to be involved in more interactions since other agents will be more likely to want to trade with him. Thus there is a tradeoff between being tough in order to get a greater share from each interaction, and being kind so that you will be involved in more such possible interactions.

The approach taken in this chapter is to abstract away from many of the complexities of how agents reach agreement and model agents as having two choices: the first is to adopt what will be termed a *Soft* strategy, which is to be relatively undemanding so that others will be more willing to interact with you, but this inevitably means receiving less of the surplus when interacting. The second is to adopt what will be termed a *Hard* strategy, which has the opposite meaning, that of a player making relatively large demands. Interacting with such a player will be less desirable for others, and so this player will be less likely to interact, but when interacting, gains a greater share of the surplus. Payoffs are determined by assuming that each period one interaction takes place between those players that are available to interact and whose strategy is most conducive to agreement.

I find that in the base model of Section 2, there are strong pressures incentivising agents to act the same way as one another, which I term "herding" of behaviour. This will be the case in both a one group model and a two group model. In the two group model, we can think of the pressures from agents within one's own group, intra-group pressures, and from the other group, inter-group pressures. I find that both act to encourage herding of behaviour. In Section 3 I show that it is possible to break this intra-group pressure towards herding under different modeling assumptions. Section 4 concludes.

3.2 Base model: herding of behaviour

This Section shows the base model in which all players follow the actions of all others. In Section 2.1 I give the simplest version of this model where there are many agents in group A, each of whom looking to interact with one agent in group $\mathcal{B}$, who will henceforth be known as agent $\mathcal{B}$. For example, consider a firm with a job vacancy as the agent with an opportunity, who is looking to hire one of several job applicants. In Section 2.2, I allow for several agents in group $\mathcal{B}$ so that we have competition in both groups. Agreement then takes place between the most conducive agents in each group.

3.2.1 One sided competition

Recall the example consider of a firm looking for an employee, which would generate the firm an extra £200,000 profit from having that position filled, and a group of N workers who might be hired in the role. It will be in the firm's interests who picks the employee who demands the lowest share of the £200,000 surplus out of those that are available. Restricting workers to two types: those who drive a hard bargain (Hard) and those who take a softer, less demanding approach (Soft), leads us naturally to the following model:

Consider a group $\mathcal{A}$ of N agents, each of whom can be one of two behavioural types, $\{S, H\}$, standing for either Soft and Hard. Each of these agents could derive profit by interacting with another agent, who I call agent $\mathcal{B}$. The profits agents in $\mathcal{A}$ get from such an interaction are scaled to be $v_S = s$, $v_H = \frac{1}{2}$ for Soft and Hard agents respectively, where $s < \frac{1}{2}$. Given these behavioural types, payoffs are determined as follows: Each agent in $\mathcal{A}$ becomes available for interaction with probability $p \in (0, 1)$. In the above example, this could be caused by a range of factors such as agents not being deemed competent by the firm, or being unable to work in the times or locations the firm requires. Agent $\mathcal{B}$ picks amongst those agents available the one with the lowest demand. That is, he picks a Soft agent, if there is one available, randomizing amongst them if there is more than one available Soft agent. Or if there are no Soft agents available, he picks a Hard, again randomizing amongst the set of available Hard agents if there is more than one. The payoff to an agent in $\mathcal{A}$ from strategy $i \in \{S, H\}$ is then $\theta_i v_i$ where θ_i is the probability of interacting with this strategy. For simplicity, I will assume agent $\mathcal{B}$ is always available to interact, although this is not vital. If, like all agents in $\mathcal{A}$, agent $\mathcal{B}$ is only available with probability p, then all this does is scale down everybody's payoff by the multiple p, a change which has no bearing on any results.

An alternative and equivalent way of viewing this process would be for $\mathcal{B}$ to list the agents in $\mathcal{A}$ in some randomly chosen order such that all Soft agents appear before all Hard agents. Then starting with$\mathcal{B}$ seeks interaction with each of the agents in $\mathcal{A}$, starting at the top of the list and going down, until he finds an agent who is available for interaction.

The following example shows the $N = 2$ case:

Example 35. $N = 2$ payoffs

If both are Soft then each interacts with probability $\theta_S = p\,(1 - p) + \frac{p^2}{2}$.

		Player 2	
		S	H
Player 1	S	$\left(p\left(1-p\right)+\frac{p^2}{2}\right)s,\ \left(p\left(1-p\right)+\frac{p^2}{2}\right)s$	$ps,\ p\left(1-p\right)\frac{1}{2}$
	H	$p\left(1-p\right)\frac{1}{2},\ ps$	$\left(p\left(1-p\right)+\frac{p^2}{2}\right)\frac{1}{2},\ \left(p\left(1-p\right)+\frac{p^2}{2}\right)\frac{1}{2}$

Table 3.2.1: Payoff matrix for two player game

If both are Hard then each interacts with probability $\theta_H = p\left(1-p\right)+\frac{p^2}{2}$.

If one is Soft while the other is Hard then $\theta_S = p$ and $\theta_H = p\left(1-p\right)$.

Thus the agents' payoffs are given by 3.2.1

Solution Concept

We could think of players playing strategically as described above and apply the standard Nash Equilibrium definition: A **Nash Equilibrium** is a state in which, given the behaviours of all other agents, no Soft agent could get strictly higher payoff by switching to Hard and no Hard agent could get strictly higher payoff by switching to being Soft.

Instead we adopt an evolutionary game theory approach, which allows for more discussion of the stability of the equilibria. Now it makes more sense to think of Soft and Hard as being agents' behavioural types rather than strategies. A population state measures the number of agents of each type. Since the number who are Hard is simply N minus the number who are Soft, I characterise a state by $x \in \{0, 1, 2, \ldots, N\} = X$, the number who are Soft. It will also be useful to refer to an agent's perspective (of the state): pick an agent and let $n \in \{0, 1, 2, \ldots, N-1\}$ be the number of agents other than himself who are of the Soft type.[1] Agents' payoffs are a function of their behavioural type and perspective of the state. Agents' types may change over time through an evolutionary dynamic which encourages players to switch type if in their interests to do so. We are interested in the population state in

[1] It is useful to make this distinction between state and perspective since an important feature of this model is that agents do not interact with themselves. Note the relationship between state and perspective: $x = n + 1$ if the agent is Soft and $x = n$ if the agent is Hard.

the long run.

Each period one agent is randomly selected to revise strategy. With probability $1-\varepsilon$, this agent revises his strategy via a selection dynamic; with probability ε the revision happens via mutation. When $\varepsilon = 0$, this is a pure selection dynamic. In line with the stochastic stability literature I consider the case of unlikely mutations ($\varepsilon \to 0$). Under mutation, an agent switches strategy with probability $1/2$. The interpretation is that the agent dies or leaves the market, or social environment and is replaced by another agent who may be of either type with equal probability.

The interpretation of the selection dynamic is that an agent looks at the situation he faces and considers whether he is best off under his current behavioural type, or whether he would do better by changing his behaviour. If he believes the latter, he switches his behaviour. The selection dynamic used will be the Best Response dynamic: An agent switches strategy if by doing so he will get a strictly greater payoff given his perspective, otherwise his behaviour remains unchanged. This is a very common dynamic in the literature, eg [7]. The following definitions will be useful.

Definition 36. An **absorbing state** is a state which, once entered, the evolutionary process will never escape from without mutations. The **Weak basin of attraction** of an absorbing state x, denoted $WB_x \subseteq X$, is the set of states from which the absorbing state x can be reached without mutations. The **Strong basin of attraction** of an absorbing state x, denoted $SB_x \subseteq X$, is the set of states from which, without mutations x will be reached with probability one.

Clearly under the Best Response dynamic, the set of absorbing states coincides exactly with the Nash Equilibrium set.

When there are multiple absorbing states, we may also look to use stochastic stability to select among them, generally picking the one with the greater basin of attraction. When $\varepsilon > 0$ the above process defines an irreducible Markov Chain P^{ε} on the state space, with $P^{\varepsilon}_{xx'}$ denoting the probability of transitioning from state x to x'. This ensures that it will have unique invariant distribution μ^{ε} over X which solves

$$\sum_{x \in X} \mu^{\varepsilon}_x P^{\varepsilon}_{xx'} = \mu^{\varepsilon}_{x'} \quad \forall x, x' \in X$$

or more succinctly, $\mu^{\varepsilon} P^{\varepsilon} = \mu^{\varepsilon}$. We then define states as stochastically stable if they survive with positive probability in the limit as $\varepsilon \to 0$ and as the Long Run Equilibrium (LRE) if it is the only stochastically stable state.

Definition 37. A state $x \in X$ is **stochastically stable** relative to the process P^{ε} if $\lim_{\varepsilon \to 0} \mu^{\varepsilon}_x > 0$ and the **Long-run equilibrium** (LRE) if $\lim_{\varepsilon \to 0} \mu^{\varepsilon}_x = 1$.

The pioneering papers in the stochastic stability field were KMR [24] and Young [45], who brought the tree analysis of Freidlin and Wentzell [17] to the attention of economists to give us a tool to analyse these models. Although many of the results can be seen using the simpler radius-coradius Theorem of Ellison[14]. Intuitively, the analysis here lies on the sizes of the basins of attraction, with stochastic stability supporting those absorbing states with larger basins.

Result: Following the crowd (herding of behaviour)

It will be shown that either all agents in $\mathcal{A}$ act Hard or they all act Soft. For parameter choices (s, p) where s and p are relatively low, Hard will be a dominant strategy and so $x = 0$ will be the only absorbing state. When s and p are relatively high, Soft will be dominant and so $x = N$ will be the only absorbing state. For parameters (s, p) in between, both will be absorbing states but nothing in between will be. To get some intuition for this result, consider the $N = 2$ case once again:

Example 38. $N = 2$ analysis.

Let $q_1 = p$ be the probability that $\mathcal{B}$ interacts with his preferred choice and $q_2 = p(1 - p)$ be the probability that $\mathcal{B}$ interacts with his second choice agent. Now,

Other's strategy	θ_S	θ_H	$\theta_S - \theta_H$
Hard	q_1	$\frac{q_1+q_2}{2}$	$\frac{q_1-q_2}{2}$
Soft	$\frac{q_1+q_2}{2}$	q_2	$\frac{q_1-q_2}{2}$

Table 3.2.2: Probabilities of interacting when $N = 2$

consider how the chance of interacting for a group $\mathcal{A}$ agent depends on that agent's action and the other agent's action. This is shown in Table 3.2.2 on page 114

We see that the increase in interaction probability from adopting a Soft over a Hard behaviour is the same. However, since $q_2 < q_1$, this increase is rising from a lower base when the other agent is Soft, and so the proportional gain $\frac{\theta_S - \theta_H}{\theta_H}$ is higher when the other agent is Soft. Meanwhile the proportional increase in share of the surplus when interacting from being Hard over Soft is $\frac{v_H - v_S}{v_S}$ which is independent of the other's action. When $\frac{\theta_S - \theta_H}{\theta_H}$ is higher (lower) than $\frac{v_H - v_S}{v_S}$ the only absorbing state is both acting Soft (Hard). If $\frac{\theta_S - \theta_H}{\theta_H}$ is higher than $\frac{v_H - v_S}{v_S}$ when the other agent is Soft, but lower when the other agent is Hard, then both are absorbing states.

With more general N, it remains the case that the results remain: it is still the case that $\frac{\theta_S - \theta_H}{\theta_H}$ is increasing in the number of other players who are Soft. So there is some threshold number of Soft agents above which $\frac{\theta_S - \theta_H}{\theta_H} > \frac{v_H - v_S}{v_S}$, meaning that Soft is a best reply. This means that the only absorbing states are $x = 0$, $x = N$ or both.

The logic behind this runs as follows: for each $k \in \mathbb{N}$, define $q_k = p\,(1-p)^{k-1}$ be the probability of interaction between $\mathcal{B}$ and his kth choice, and define $s_k = \frac{q_1 + \ldots + q_k}{k}$ as the probability of a given Soft agent interacting with $\mathcal{B}$ when there are k Soft agents. Now, we can define $a_k = \frac{s_{k+1}}{s_k}$ to be the proportional decrease in a Soft agent's interaction probability from the arrival of a kth Soft agent. Note that this is the same as the proportional decrease to a Hard agent from kth Hard agent; while for a Hard agent, additional Soft agents decrease the probability of interacting by $(1-k)$ and additional Hard agents have no impact on a Soft agent's probability of interacting.

Let $\theta_i(n)$ the probability of interacting given behaviour $i \in \{S, H\}$ and perspective that $n \in \{0, 1, 2, \dots, N-1\}$ of the other agents are Soft (and so $N-n-1$ others are Hard) and $G(n) = \frac{\theta_S(n) - \theta_H(n)}{\theta_H(n)} = \frac{\theta_S(n)}{\theta_H(n)} - 1$ be the proportional increase in interaction probability from being Soft instead of Hard. The aim is to show that $G(n)$ is a strictly increasing function of n. One route would be to note that

$$\theta_S(n) = \frac{\sum_{j=1}^{n+1} q_j}{n+1} \quad \theta_H(n) = \frac{\sum_{j=n+1}^{N} q_j}{N-n}$$

$$G(n) = \frac{\left(1 - (1-p)^{n+1}\right)(N-n)}{(n+1)\left((1-p)^n - (1-p)^N\right)} - 1$$

and analyse $G(n)$ to confirm it is a strictly increasing function of n in the range $[0, N-1]$.

Instead I give a more detailed explanation which helps to explain the intuition for this result. With N players, a player with perspective $n \in \{0, 1, 2, \dots, N-1\}$ has the following probabilities of interacting:

$$\theta_S(n) = pa_1 \dots a_n$$

$$\theta_H(n) = p(1-p)^n a_1 \dots a_{N-n-1}$$

The formula for $\theta_S(n)$ comes from the following: if there are no other agents, the agent interacts if available, which happens with probability p. If we introduce one other Soft agent, this decreases the probability by a_1, the next Soft agent by a_2 and the ith by the multiple a_i. The $N-n-1$ Hard agents have no impact. The formula for $\theta_H(n)$ is derived in similar fashion: the initial probability p is discounted by $(1-p)$ for each of the n Soft agents while the effect of the $N-n-1$ other Hard agents is accounted for by $a_1 \dots a_{N-n-1}$.

Clearly showing that $G(n)$ is strictly increasing is equivalent to showing that $\frac{\theta_S(n+1)}{\theta_H(n+1)} > \frac{\theta_S(n)}{\theta_H(n)}$. To do this I show that

$$\frac{\frac{\theta_S(n+1)}{\theta_H(n+1)}}{\frac{\theta_S(n)}{\theta_H(n)}} = \frac{\theta_S(n+1)}{\theta_S(n)} \frac{\theta_H(n)}{\theta_H(n+1)} > 1$$

Now, using the formulae already obtained, we see a lot of terms cancel to leave:

$$\frac{\theta_S(n+1)}{\theta_S(n)} \frac{\theta_H(n)}{\theta_H(n+1)} = \frac{a_{n+1} a_{N-n-1}}{1-p}$$

At this point it is instructive to return to the $N = 2$ case and we see that (plugging $n = 0$ into the above)

$$\frac{\theta_S(1)}{\theta_S(0)} \frac{\theta_H(0)}{\theta_H(1)} = \frac{a_1 a_1}{1-p}$$

This leaves us to analyse the sequence a_1. Introducing a competing agent playing of the same type as the agent we are analysing has no effect when the competing agent is unavailable, which happens with probability $(1-p)$. While if the competing agent is available then half the time our agent interacts and half the time the competing agent interacts instead. Thus $a_1 = 1 - \frac{p}{2}$. The effect of further competing agents can be thought of similarly: Suppose our agent is currently interacting and introduce a second competing agent. Again with probability $(1-p)$ the agent is unavailable and so it makes no difference, while with probability p the agent is available and it comes down to whether our agent is above or below the competing agent in the order in which $\mathcal{B}$ ranks them. However unlike before, there is no longer an equal chance of each. Conditional on our agent interacting with only one competing agent there is a greater than half chance that our agent was ranked before the first competing agent, which implies a greater than half chance that our agent is also ranked above the second competing agent. Thus the effect of the second competing agent is slightly less than the effect of the first, and thus $a_2 > a_1$. Similarly the third competing agent has less effect than the second and so $a_3 > a_2$, and more generally $\{a_k\}_{k\in\mathbb{N}}$ is a strictly increasing sequence in the interval $\left[1-\frac{p}{2},1\right]$.[2]

Returning to the $N=2$ case, using $a_1 = 1-\frac{p}{2}$, we see that

$$\frac{\theta_S(1)}{\theta_S(0)}\frac{\theta_H(0)}{\theta_H(1)} = \frac{a_1 a_1}{1-p} = \frac{\left(1-\frac{p}{2}\right)^2}{2} > 1$$

This verifies that $G(1) > G(0)$. For the general N case the key is to note that for any $k\in\mathbb{N}$, $a_k > 1-\frac{p}{2}$. This implies that

$$\frac{\theta_S(n+1)}{\theta_S(n)}\frac{\theta_H(n)}{\theta_H(n+1)} = \frac{a_{n+1}a_{N-n-1}}{1-p} > \frac{\left(1-\frac{p}{2}\right)^2}{2} > 1$$

This verifies that for any N, the function $G(n)$ is strictly increasing as claimed.

Another, perhaps more intuitive way to think of this is the following: Our agent is better off being Soft when

$$G(n) = \frac{\theta_S(n)}{\theta_H(n)} - 1 > \frac{v_H - v_S}{v_S} = \frac{1/2 - s}{s}$$

If our agent was the only agent then $\theta_S(n) = \theta_H(n) = p$, but each of the other $N-1$ agents has an impact of increasing the $\frac{\theta_S(n)}{\theta_H(n)}$ ratio. If this ratio rises above the threshold $\frac{1/2}{s}$ then Soft becomes better. Introducing a Soft agent increases the ratio by $\frac{a_i}{1-p}$ and introducing a Hard agent increases the ratio by $\frac{1}{a_j}$ for some $a_i, a_j \in \left[1-\frac{p}{2},1\right]$. For any such a_i, a_j it is the case that $\frac{a_i}{1-p} > \frac{1}{a_j}$ and so, for fixed N, the $\frac{\theta_S(n)}{\theta_H(n)}$ ratio is increasing in the number of Soft agents, and thus $G(n)$ is strictly increasing, as claimed. This argument also tells us that for larger groups, Soft is more likely to be

[2] This can also be verified by directly analysing the formula $a_k = \frac{s_{k+1}}{s_k}$ with the aid of a computer program.

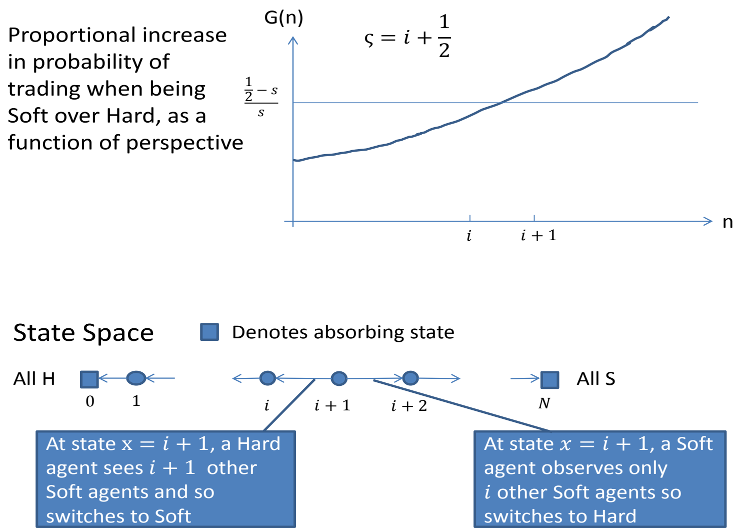


Figure 3.2.1: Base model: absorbing states

better, which is a statement I will return to in Section 2.2.

The fact that means that $G(n)$ is strictly increasing means that there exists a summary statistic statistic $\varsigma(N,p,s) \in \left\{-0.5, 0, 0.5, \ldots, N-1, N-\frac{1}{2}\right\}$, defined such that for all $n \in \{0,1,...,N-1\}$:

$$n < \varsigma \Rightarrow G(n) < \frac{1/2 - s}{s}$$

$$n = \varsigma \Rightarrow G(n) = \frac{1/2 - s}{s}$$

$$n > \varsigma \Rightarrow G(n) > \frac{1/2 - s}{s}$$

In other words $\varsigma(N,p,s)$ is a cutoff such that for perspectives above it Soft is better; for perspectives beneath it Hard is better. The effect this has on the dynamics can be displayed diagrammatically as below in Figure 3.2.1 on page 118:

Theorem 39. *No mixed population profile can be an absorbing state. Furthermore, there are three possible sets of absorbing states, corresponding to the following 3 pos-*

sibilities:

1. If $\varsigma(N,p,s) = N - \frac{1}{2}$ *then the only absorbing state is* $x = 0$.

2. If $\varsigma(N,p,s) = -0.5$ *then the only absorbing state is* $x = N$.

3. If $\varsigma(N,p,s)$ *is neither* $N - \frac{1}{2}$ *or* -0.5 *then both* $x = 0$ *and* $x = N$ *are absorbing states.*

Furthermore, if $\varsigma(N,p,s) < \frac{N-1}{2}$ *then the Long run equilibrium is* $x = N$*; if* $\varsigma(N,p,s) > \frac{N-1}{2}$ *then the Long run equilibrium is* $x = 0$*; if* $\varsigma(N,p,s) = \frac{N-1}{2}$ *then both* $x = 0$ *and* $x = N$ *are stochastically stable.*

Proof. The best response dynamic can take us from state $x \neq 0$ to $x - 1$ if in state x, a Soft agent wants to switch to Hard. A Soft agent has perspective $x - 1$, and so will switch to Hard if and only if $x - 1 < \varsigma(N,p,s)$. Similarly, for the best response dynamic to take us from state $x \neq N$ to $x + 1$ requires a Hard agent to switch to Soft. At this state a Hard agent has perspective x and so will switch to Soft if and only if $x > \varsigma(N,p,s)$.

From this it is clear that no mixed population profile can be an absorbing state, since for $x \in \{1, 2, \ldots, N-1\}$ at least one of $x - 1 < \varsigma(N,p,s)$ and $x > \varsigma(N,p,s)$ must hold. Considering specific values of $\varsigma(N,p,s)$ it is also clear that the absorbing states are as stipulated.

Consider $\varsigma(N,p,s) = \frac{N-1}{2}$. Then if N is odd the strong basins of attraction are $SB_0 = \left\{0, 1, \ldots, \frac{N-1}{2}\right\}$ and $SB_N = \left\{\frac{N+1}{2}, \ldots, N\right\}$. If N is even then $SB_0 = \left\{0, 1, \ldots, \frac{N}{2} - 1\right\}$ and $SB_N = \left\{\frac{N}{2} + 1, \ldots, N\right\}$ while state $\frac{N}{2}$ is in the weak basin of both. In both cases the two absorbing states have the same size basins of attraction and so are both stochastically stable. If we increase ς so that $\varsigma(N,p,s) > \frac{N-1}{2}$ then SB_0 is bigger, so $x = 0$ becomes the LRE, and similarly if $\varsigma(N,p,s) < \frac{N-1}{2}$ then SB_N is bigger, so $x = N$ becomes the LRE. □

3.2.2 Two sided competition

For example, suppose the government announces some building project, which will be given to the first partnership between a construction firm and supplier of building materials which puts a plan together to build it for £10 million, If a construction firm and supplier can come together to build it for £8 million, then there is a £2 million surplus to be split between the two. The building material suppliers most likely to be involved in the interaction are those who are more willing to give better prices to the construction firms and so concede a larger share of the surplus, and likewise for the construction firms. This situation is formally modeled as follows:

Model

There are $N^{\mathcal{A}}$ agents in group $\mathcal{A}$ and $N^{\mathcal{B}}$ agents in group $\mathcal{B}$. Every agent can adopt one of two behavioural types $\{S, H\}$ and so the state space is $\{0, 1, \ldots, N_{\mathcal{A}}\} \times \{0, 1, \ldots, N_{\mathcal{B}}\} = X$ with typical element $x = (x_{\mathcal{A}}, x_{\mathcal{B}})$ denoting that $x_{\mathcal{A}}$ of group $\mathcal{A}$ and $x_{\mathcal{B}}$ of group $\mathcal{B}$ are Soft, while the remaining agents are Hard. The state can change via an evolutionary dynamic which picks a random agent each period who revises strategy via the best response dynamic with probability $1 - \varepsilon$, and by mutation with probability ε.

The solution concepts used are the same as before as the notions of absorbing states, basins of attraction and Long run equilibria still apply to the new two dimensional state space. Once again, the set of Nash Equilibria, where no agent can increase payoff by switching strategy coincides with the set of absorbing states.

Payoffs

Given agents' behavioural types, the expected payoffs to agents are determined as follows: Each agent is available for interaction with probability $p \in (0, 1)$ and, as long as there is at least one agent in each group available, an interaction between two agents, one from each group will take place, benefiting both the agents involved. If a Soft agent interacts with a Hard agent then their respective shares are (s, h) where $s \in (0, 1/2)$ and $h = 1 - s$. From each group the agent selected to take place in the interaction is selected as follows: In group $i \in \{\mathcal{A}, \mathcal{B}\}$ if there are $m_i^S > 0$ Soft agents available then each is selected with probability $\frac{1}{m_i^S}$. If there are no Soft agents available then we move onto Hard agents: if there are $m_i^H > 0$ Hard agents available then each is interacts with probability $\frac{1}{m_i^H}$; while if there are no Soft or Hard agents available there is no interaction.

Clearly this can be thought of as a zero sum game, since the sum of all agents' expected payoffs is $\left(1 - (1-p)^{N_\mathcal{A}}\right)\left(1 - (1-p)^{N_\mathcal{B}}\right)$, the probability that an interaction takes place. Agents in the two groups play very different roles. The agents in your group are your competitors, and how they behave influences your probability of interacting; while the agents in the other group are your trading partners, and how they behave influences your expected share of the surplus when interacting. Thus one would like agents in one's own group to act Hard and agents in the other group to act Soft. Figure 3.2.2 on page 122 shows the state space for $N_\mathcal{A} = N_\mathcal{B} = 5$ and displays its key features:

When the best response dynamic allows for transition between two neighbouring states this is represented by an arrow. For example, the arrow from state $(4, 0)$ to $(5, 0)$ indicates that the Hard agent in group $\mathcal{A}$ would be better off becoming Soft and so would make this change given the opportunity, resulting in the state $(5, 0)$. Any state has up to four neighbours, those states which are horizontally or vertically adjacent, which are the states that can be reached by exactly one agent changing behaviour. The relationship between any two neighbouring states is determined by the best response of the agent who would be required to change behaviour between the two. This is displayed in Figure 2 for the transition between $(2, 4)$ and $(3, 4)$. Both states have four Soft and one Hard agent in group $\mathcal{B}$, two Soft and two Hard

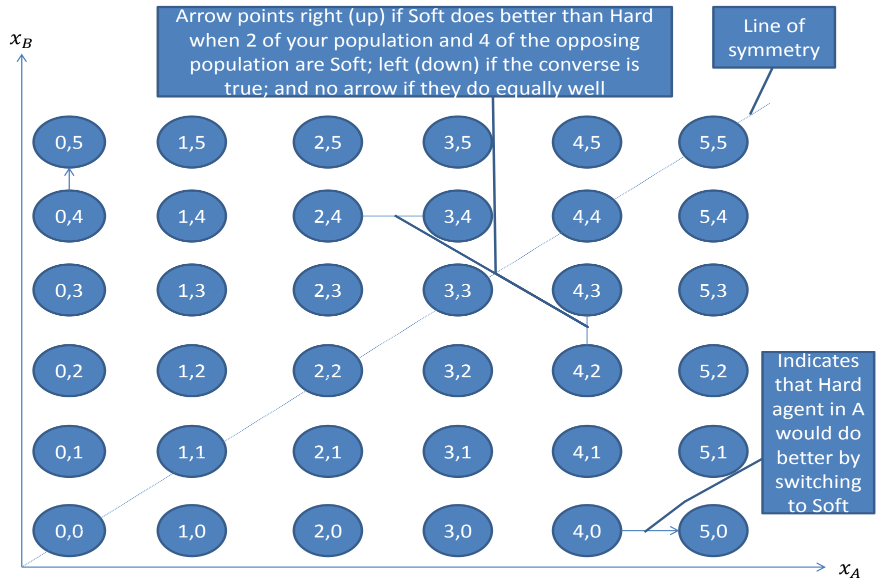


Figure 3.2.2: State space $N_{\mathcal{A}} = N_{\mathcal{B}} = 5$

agents in group $\mathcal{A}$. They only differ in the behaviour of the one remaining agent in group $\mathcal{A}$. So we consider this agents best response, having fixed the others' actions. If this agent does better by being Soft then we can transition from $(2,4)$ and $(3,4)$ via the best response dynamic so add a right-pointing arrow between the two states. Similarly, if Hard gives a higher payoff, we add a left-pointing arrow. If the agent is indifferent between Soft and Hard then, by our earlier description of the best response dynamic it is assumed that the agent sticks to its original strategy and so there is no arrow between the two states.

When $N_{\mathcal{A}} = N_{\mathcal{B}}$, there is no asymmetry between the two groups $\mathcal{A}$ and $\mathcal{B}$, and so the relationship between states $\{(x_1, x_2), (x_1 + 1, x_2)\}$ mirrors the relationship between states $\{(x_2, x_1), (x_2, x_1 + 1)\}$. In figure 2, this is explained between the pairs of states $\{(2,4), (3,4)\}$ and $\{(4,2), (4,3)\}$. For this reason, the presence of the arrow $(4,0) \rightarrow (5,0)$ implies there must also be an arrow $(0,4) \rightarrow (0,5)$.

Following the crowd result

Once again, we find that there is a tendency for homogeneous behaviour. For very similar reasons to the one side competition case analysed in Section 2.1, all agents in the same group will end up behaving the same way, implying that the only absorbing states are in the corners. It is possible that all agents in one group are Soft, while all agents in the other group are Hard. When one group is much larger than the other, it is quite possible that all agents in the larger group will be Soft, while all agents in the smaller group be Large, since there is added incentive to be of the more competitive, Soft type when you have more competitors. When the two groups are the same size we find that the most stable outcome is for everyone in both groups to behave the same way.

The analysis proceeds much as before. An agent's expected payoffs is now the product of three factors instead of two: as in Section 2.1, the first two factors are the probability of being selected from one's own group ($\theta_n(i)$, $i \in \{S, H\}$ is behaviour and $n \in \{0, 1, 2, \ldots, N-1\}$ is the number of other Soft agents in your group) and the average share when interacting (v_i, $i \in \{S, H\}$). The third factor is the probability of someone from the opposite group being available to trade, $(1-(1-p)^N$ if N agents in the opposing group). Since this third factor is independent of their actions of agents, it can be ignored in the analysis. So just as in Section 2.1, the analysis depends on the comparison of the proportional gain in probability of interacting from being Soft and the proportional gain in average share from being Hard. The former is determined by the behaviour of agents in your own group, who are your competitors; while the latter is determined by agents in the other group who are your trading partners.

Theorem 40. *All absorbing states $x = (x_{\mathcal{A}}, x_{\mathcal{B}})$ must have $x_{\mathcal{A}} \in \{0, N_{\mathcal{A}}\}$ and $x_{\mathcal{B}} \in \{0, N_{\mathcal{B}}\}$. If $N_{\mathcal{A}} = N_{\mathcal{B}} = N$ then either $x = (0,0)$ or $x = (N,N)$ is an absorbing state and stochastically stable. Furthermore, if $N_{\mathcal{A}} = N_{\mathcal{B}} = N$ is odd then $x = (0,0)$ or $x = (N,N)$ are the only possible stochastically stable states.*

Proof. Consider an agent in group $\mathcal{A}$ and call this agent a. Let $n_{\mathcal{A}}$ be the number of other agents in that group who are Soft in group. As already established in Section 2.1, the proportional increase in the probability of interacting from choosing Soft over Hard, $G(n_{\mathcal{A}})$ is

$$G(n_{\mathcal{A}}) = \frac{\left(1-(1-p)^{n_{\mathcal{A}}+1}\right)(N_{\mathcal{A}}-n_{\mathcal{A}})}{(n_{\mathcal{A}}+1)\left((1-p)^{n_{\mathcal{A}}}-(1-p)^{N_{\mathcal{A}}}\right)} - 1$$

which is an increasing function of $n_{\mathcal{A}}$. The behaviour of the agents in the opposing population solely influences the expected share of the surplus when interacting. Let $u_a(x_{\mathcal{B}})$ be the expected share of the surplus agent a gets when interacting, given that he is Soft and $x_{\mathcal{B}}$ of group $\mathcal{B}$ are Soft. Clearly $u_a(0) = s$ and $u_a(N_{\mathcal{B}}) = \frac{1}{2}$, and furthermore $u_a : \{0,1,\ldots,N_{\mathcal{B}}\} \to \left[s, \frac{1}{2}\right]$ is a strictly increasing function of the number of Soft agents in group $\mathcal{B}$. For any $x_{\mathcal{B}}$, the proportional increase in expected share when interacting is $\frac{\frac{1}{2}-s}{u_a(x_{\mathcal{B}})}$ which is independent of the behaviour of agents in group i. Since $G(n_{\mathcal{A}})$ is strictly increasing in $n_{\mathcal{A}}$, while $\frac{\frac{1}{2}-s}{u_a(x_{\mathcal{B}})}$ is independent of $n_{\mathcal{A}}$ we have the following:

$$G(n_{\mathcal{A}}) \geq \frac{\frac{1}{2}-s}{u_a(x_{\mathcal{B}})} \Longrightarrow G(\hat{n}_{\mathcal{A}}) > \frac{\frac{1}{2}-s}{u_a(x_{\mathcal{B}})} \; \forall \hat{n}_{\mathcal{A}} > n_{\mathcal{A}} \tag{3.2.1}$$

$$G(n_{\mathcal{A}}) \leq \frac{\frac{1}{2}-s}{u_a(x_{\mathcal{B}})} \Longrightarrow G(\hat{n}_{\mathcal{A}}) < \frac{\frac{1}{2}-s}{u_a(x_{\mathcal{B}})} \; \forall \hat{n}_{\mathcal{A}} < n_{\mathcal{A}} \tag{3.2.2}$$

This implies that, for any given $x_{\mathcal{B}}$, if the best response selection dynamic can make the transition $(x_{\mathcal{A}}, x_{\mathcal{B}}) \to (x_{\mathcal{A}}+1, x_{\mathcal{B}})$ then it can also make the transition $(x_{\mathcal{A}}+1, x_{\mathcal{B}}) \to (x_{\mathcal{A}}+2, x_{\mathcal{B}})$. Thus the only absorbing states have $x_{\mathcal{A}} \in \{0, N_{\mathcal{A}}\}$. A symmetric argument shows $x_{\mathcal{B}} \in \{0, N_{\mathcal{B}}\}$.

The other key thing to note is the effect of agents in the other group. Again consider our agent a from group $\mathcal{A}$ and recall that $u_a(x_{\mathcal{B}})$ is a strictly increasing function of x_B. This means that

$$G(n_{\mathcal{A}}) \geq \frac{\frac{1}{2}-s}{u_a(x_{\mathcal{B}})} \Longrightarrow G(n_{\mathcal{A}}) > \frac{\frac{1}{2}-s}{u_a(\hat{x}_{\mathcal{B}})} \; \forall \hat{x}_{\mathcal{B}} > x_{\mathcal{B}} \tag{3.2.3}$$

$$G(n_{\mathcal{A}}) \leq \frac{\frac{1}{2}-s}{u_a(x_{\mathcal{B}})} \Longrightarrow G(n_{\mathcal{A}}) < \frac{\frac{1}{2}-s}{u_a(\hat{x}_{\mathcal{B}})} \; \forall \hat{x}_{\mathcal{B}} < x_{\mathcal{B}} \tag{3.2.4}$$

(3.2.1)(3.2.2)(3.2.3)(3.2.4) tell us about horizontal arrows. Applying the same logic to a group B agent and gives the same formulae, where "$\mathcal{A}$" and "$\mathcal{B}$" are swapped over, which allows us to make the same conclusions about vertical arrows.

The main things to note are displayed on Figure 3.2.3 on page 126. The effect of these equations on the dynamic is illustrated in diagram (a). If there is a right pointing arrow between two states such as $(1,1) \to (2,1)$ (or more precisely, as long as there is no left-pointing arrow) then (3.2.1) and (3.2.3) imply that there must also be right pointing arrow between two states to the northeast of them. The vertical arrow analogues of these equations imply that the same conclusions hold for states to the northeast of any vertical arrows. Similarly, (3.2.2) and (3.2.4) and their vertical arrow analogues imply similar results for states to the southwest of any left or downward pointing arrows. In diagram (a) the arrow $(3,2) \to (3,1)$ implies the existence of another six downward arrows between states to the southeast.

From this diagram it is then clear that any absorbing states must be in the corners, that is $x_{\mathcal{A}} \in \{0, N_{\mathcal{A}}\}$ and $x_{\mathcal{B}} \in \{0, N_{\mathcal{B}}\}$ (in fact (3.2.1) and (3.2.2) together with their vertical arrow analogues suffice for this statement). To show that an absorbing state exists, look at diagram (b). In order for $(0,0)$ not to be absorbing, at least one of $(0,0) \to (0,1)$ or $(0,0) \to (1,0)$ must be possible under the best reply dynamic. Suppose it is the latter, $(0,0) \to (1,0)$ (the other case is very similar). Then, as seen on diagram (b), this implies that we have right arrows everywhere. Now, $(N_{\mathcal{A}}, 0)$ is an absorbing state unless $(N_{\mathcal{A}}, 0) \to (N_{\mathcal{A}}, 1)$ is possible, but as shown on the diagram, this implies $(N_{\mathcal{A}}, N_{\mathcal{B}})$ is an absorbing state. Hence at least one of the four corners

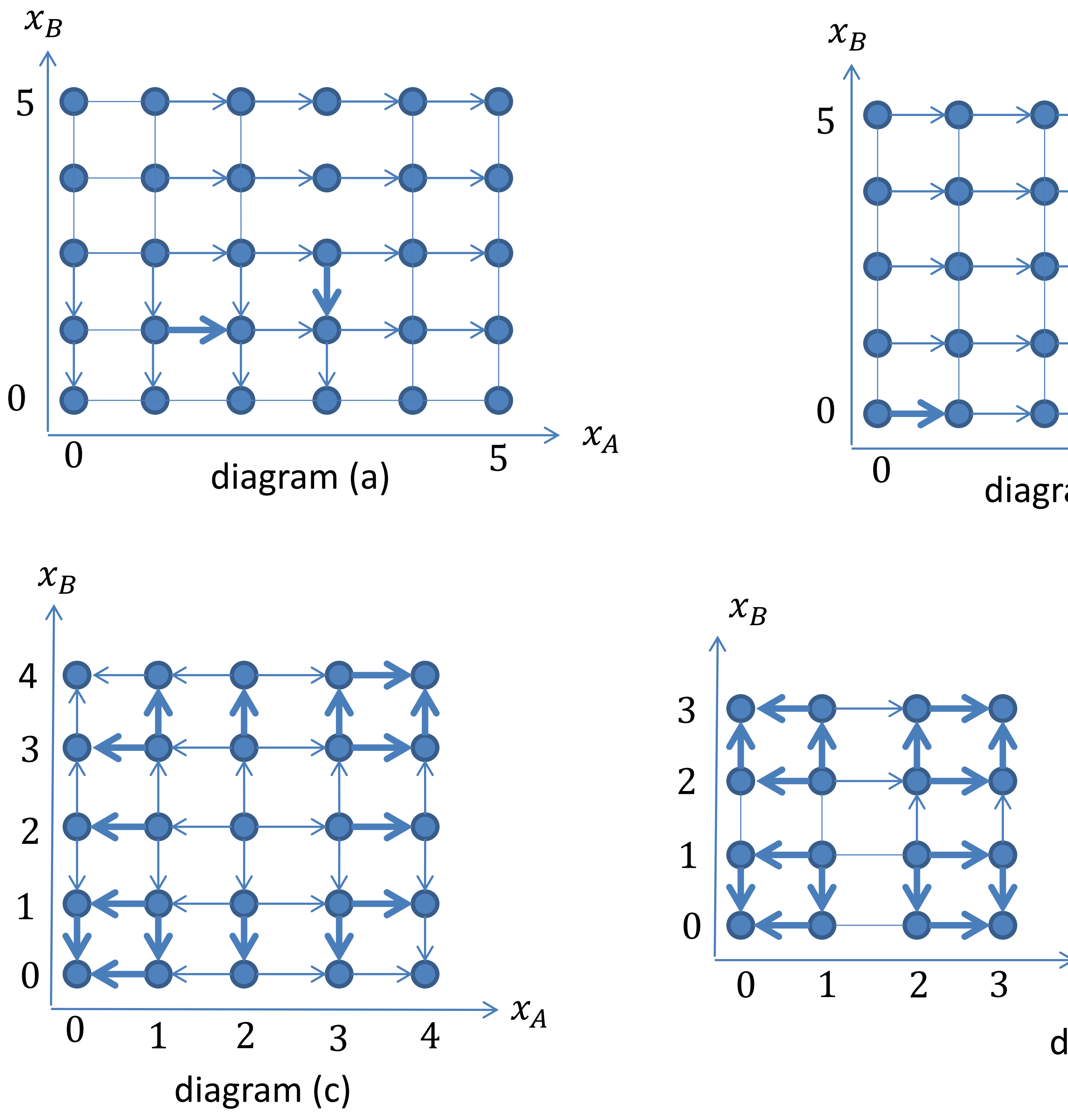


Figure 3.2.3: Possible dynamics

must be an absorbing state.

Now suppose $N_{\mathcal{A}} = N_{\mathcal{B}} = N$. Since groups $\mathcal{A}$ and $\mathcal{B}$ are now completely symmetric, $(0, N)$ is absorbing if and only if $(N, 0)$ is absorbing. If neither are absorbing then at least one of the other two corners $(0, 0)$ and (N, N) must be absorbing, and hence at least one is stochastically stable since the set of stochastically stable states is a subset of the set of absorbing states. If $(0, N)$ and $(N, 0)$ are absorbing, then, since there $(N, 0) \rightarrow (N-1, 0)$ is not a link, (3.2.3) implies the links $(N-1, i) \rightarrow (N-1, i)$ for any $i \in \{1, \ldots, N\}$. If we also apply (3.2.4), together with their vertical arrow analogues, we get all the arrows in bold in diagram (c) and hence we can see that $(0, 0)$ and (N, N) must both be absorbing.

Bearing in mind the symmetry discussed in Figure 3.2.2 on page 122, it is clear that the basin of attraction of $(0, N)$ is the same size as that of $(N, 0)$. Applying our four equations and their vertical arrow analogues, we see that the biggest we can make these basins of attraction of $(0, N)$ and $(N, 0)$ is by doing as in diagram (c), where each of the four corners has the same size basins: just under a quarter of states in their strong basins and just over a quarter in their weak basins. diagram (c) represents the case where $N = 4$, but we can generalise this diagram to the general N case very easily [3] By the symmetry it is obvious that all four corners are stochastically stable here. While any changes which are compatible with our four equations and their vertical arrow analogues, would see the basins of $(0, N)$ and $(N, 0)$ contract, to the benefit of either $(0, 0)$ or(N, N). This establishes that at least one of $(0, 0)$ or(N, N) must always be stochastically stable

Consider when N is odd, specifically $N = 3$ as in diagram (d). To give $(0, 3)$ and $(3, 0)$ maximum chance of being stochastically stable, suppose we have arrows

[3]One can think of $N/2$ corresponding to 2. The relationship between any pair of neighbouring states in the bottom left quarter, (marked out by $(0, 0) - (0, N/2) - (N/2, N/2) - (N/2, 0) - (0, 0)$) has all arrows pointing down or to the left; the bottom right quarter has all arrows pointing down or to the right, the top right quarter has all arrows pointing up or to the right; the top left quarter has all arrows pointing up or to the left.

heading towards $(0,3)$ and $(3,0)$. These are the bold arrows in diagram (d). Now, it must be the case that at least one of $(2,1) \to (1,1)$ or $(1,2) \to (2,2)$ are links. diagram (d) displays the case when $(1,2) \to (2,2)$ is a link. Using equation (2.3) and the symmetry between the two groups, we get another three arrows as displayed. These arrows ensure that moving from either $(0,3)$ or $(3,0)$ to $(3,3)$ requires one mutation, while moving in the opposite direction requires two. Applying the Freidlin and Wentzell tree analysis arguments (see [17, 24, 45]) it is clear that trees with root $(3,3)$ must have a lower cost in terms of mutations needed than trees with root either $(0,3)$ or $(3,0)$, and hence neither $(0,3)$ or $(3,0)$ can be stochastically stable

The same argument also holds for general odd N. It must be the case that at least one of $\left(\frac{N+1}{2}, \frac{N-1}{2}\right) \to \left(\frac{N-1}{2}, \frac{N-1}{2}\right)$ or $\left(\frac{N-1}{2}, \frac{N+1}{2}\right) \to \left(\frac{N+1}{2}, \frac{N+1}{2}\right)$ are links. In the former case a tree with root $(0,0)$ has a lower cost than any tree with root $(0,N)$ or $(N,0)$; In the latter case a tree with root (N,N) has a lower cost than any tree with root $(0,N)$ or $(N,0)$. Either way, neither $(0,N)$ or $(N,0)$ can be stochastically stable. □

Further Comments

In both the models of Sections 2.1 and 2.2, everybody in the same group will act the same way. The intuition for this result lies in the analysis of Section 2.1, which argued that a particular agent is better off being Soft if and only if

$$\frac{\theta_S(n) - \theta_H(n)}{\theta_H(n)} > \frac{\frac{1}{2} - s}{u} \Longleftrightarrow \frac{\theta_S(n)}{\theta_H(n)} > \frac{\frac{1}{2} - s}{u} + 1 \qquad (3.2.5)$$

The left hand side of the inequality, the ratio $\frac{\theta_S(n)}{\theta_H(n)}$, depends on the state within that agent's own group, which are the agents it is competing against. The right-hand side, $\frac{\frac{1}{2}-s}{u} + 1$, where u is the average share of a Soft agent when interacting depends on the state in the other group, which are that agent's potential partners. As the number of Soft agents in the other group increases, so does u, and hence this decreases the right-hand side of the inequality. Thus we see a force pulling agents toward copying the prevailing behaviour of their trading partners.

If the size of one's own group (the number of your competitors) is fixed, the left hand side of the inequality is increasing in the number of competitors who are Soft, which implies the follow the crowd result within one's own group. We can also use this equation to discuss how the preponderance of Soft or Hard behaviour depends on the models exogeneous parameters: group size and p, s. The addition of extra competitors, whether they are Hard or Soft increases the ratio $\frac{\theta_S(n)}{\theta_H(n)}$ and thus creates a pressure toward Soft behaviour. Increasing the probability of being available, p, also clearly increases $\frac{\theta_S(n)}{\theta_H(n)}$, and so creates a pressure toward Soft behaviour. Increasing s reduces the incentive to act Hard, decreasing $\frac{\frac{1}{2}-s}{u}+1$ and thus also creates a pressure toward Soft behaviour.

Intuitively if these parameters are all on the large side then the joint pressure towards being Soft would overwhelm any herding pressure towards Hard and so the only absorbing state would be everyone on the group being Soft. Conversely, if these parameters are all relatively small, everyone will act Hard. In between the two extremes, we get a range of parameter values for which both everyone acting Hard or Soft is an absorbing state. Which of these two states is likely to prevail in the short run depends on precedent, that is where we start from. While in the long run, applying stochastic stability, it is the size of the basins of attraction which matter.

Theorem 6 said that if both groups are the same size then, in addition to intra-group herding of behaviour, we should also expect inter-group herding of behaviour. The intuition following equation (3.2.5) suggests that this result should still pertain when the two groups are of similar size. However, when the two groups are of very different sizes this will no longer be the case. As already explained, the greater the number of competitors, the greater the pressure toward Soft behaviour, so if the group size difference is large enough, there are parameter choices for which everyone in the small group acts Hard and everyone in the larger group acts Soft.

Example 41. Let $N^A = 6$, $N^B = 2$, $p = 0.3$, $s = 0.3$

Then the only absorbing state is $x = (6, 0)$ where all of A are Soft and all of B are Hard. At this equilibrium $\mathcal{B}$-group agents are both three times more likely to be involved in an interaction and take the lion's share,0.7 from each interaction. This dual effect means that the expected payoff of $\mathcal{B}$-group agents is six times greater than $\mathcal{A}$-group agents, This example shows that having more competitors can cause agents the dual loss of being both less likely to interact, and be subject to less favourable terms of trade when interacting.

3.3 Adding distortions: the possibility of mixed behaviour

In Section 2, all agents always interacted in the same way because of the nice, smooth way in which their incentives worked: the probability of interacting was always higher when Soft than when Hard, and decreasing in a very smooth way in the number of competitors who are Soft. In this Section I add in distortions to alter the above set up. In Section 3.1, all agents come from the same group and interaction may occur between any two of them. This means that the previous separation between competitors and trading partners no longer exists. Furthermore the probability of interacting no longer decreases in the number of others who are Soft in the same smooth way as before. The result is that it is now possible to have mixtures of behaviour as opposed to the herding results of Section 2.

Sections 3.2 and 3.3 consider a model where one agent owns the opportunity to interact and so are guaranteed to be involved in that interaction. The possibility of being involved in the interaction regardless of your behaviour gives an incentive to act Hard. Section 3.2 considers the two group case, and Section 3.3 considers the one group case. In Section 3.2, it is found that this distortion may be big enough to overwhelm the pressures towards herding, while in Section 3.3, this is not the case and the herding results still persist.

3.3.1 One group, two spaces to be filled.

In this section I modify the Section 2.2 model by assuming that all agents are from one single group, any two of whom can interact to split the surplus of size one. Once again the two agents who do interact are the softest available, with ties broken randomly. Since there is one group of N agents, the state space is once again $X = \{0, 1, \dots, N\}$ where $x \in X$ means x Soft agents and $N - x$ Hard agents.

More formally, there is one group of N agents. Each agent is either Hard or Soft and payoffs are determined as follows: Each agent becomes available with probability $p \in (0, 1)$. If two or more agents are available for interaction, then an interaction takes place benefiting both agents involved, otherwise no interaction takes place and so all agents get zero payoff. Assuming two or more agents are available, let m_S be the number of available Soft agents and m_H the number of available Hard agents. If $m_S \geq 2$, then each Soft agent is involved in the interaction with probability $\frac{2}{m_S}$; if $m_S = 1$ then the available Soft agent is involved in the interaction, together with one of the available Hard agents, each of which are selected with probability $\frac{1}{m_H}$; if $m_S = 0$ then the interaction happens between two of the available Hard agents, each of which are selected with probability $\frac{2}{m_H}$. If two agents of the same type interact they split the surplus $(1/2, 1/2)$, while if two agents of different types interact the shares to the Soft and Hard agents respectively are $(s, 1 - s)$ where $s \in (0, 1/2)$.

Note that as in Section 2.2, this is a zero-sum game since the sum of all agents expected utilities equals the probability of an interaction taking place, which is $1 - (1 - p)^N$. Although since all agents are both competitors and trading partners to all

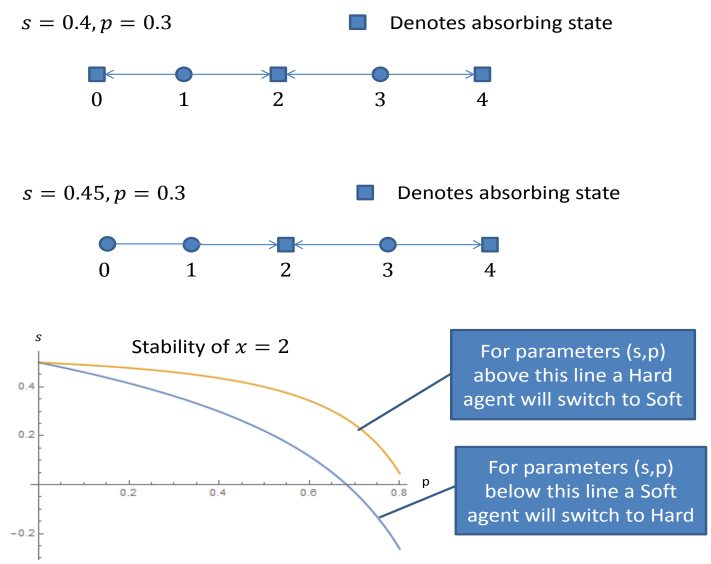


Figure 3.3.1: $N = 4$ example

other agents, it is no longer clear how one agent's behaviour will affect the payoff of another.

Result: Mixed population possibilities

In Section 2.2, every agent was either a trading partner or a competitor, and we found that there was a force pressuring agents to follow the actions of both. Now each agent is both a trading partners and competitor and so one might expect pressures to follow other agents again, leading to a herding of behaviour result. This is not the case. Unlike in Section 2, it is now possible to have a mixture of agent types in the population long term. I give an example of this:

Example 42. $N = 4$. See Figure 3.3.1 on page 132

1. If $p = 0.3$, $s = 0.4$ then the absorbing states are $\{0, 2, 4\}$.
2. If $p = 0.3$, $s = 0.45$ then the absorbing states are $\{2, 4\}$.

The reason for this change is that the probability of interacting functions do not behave in the same monotone, smooth manner as in Section 2. As in Section 2, if there is only one Soft agent, then this agent interacts whenever available, that is with probability p. However, unlike in Section 2, when there are two Soft agents, both still interact with probability p. In fact there is reason to think the state $x = 2$ as being quite stable for a large range of parameters. At this state, both Soft agents are sure to interact whenever available, and when interacting have a p chance of interacting with the other soft agent. Figure 4 (c) shows that there is a decent range of parameters (p, s) for which the state $x = 2$. Any parameter pair $(p, s) \in (0, 1)^2$ between the two curves has $x = 2$ as an absorbing state.

3.3.2 Owning opportunities: Two groups

In all three models so far it has been assumed that an opportunity for interaction is just presented and any pair of agents can interact. By contrast, in this Section and the next, I will assume that one randomly chosen agent discovers or is endowed with the possibility to interact. In other words, this agent owns the interaction opportunity and will search the set of his potential trading partners to find someone to interact with, in the same way as agent $\mathcal{B}$ did in Section 2.1

In this section I assume two groups of agents. As in Section 2.2, all interactions take place between agents from different groups. So if an agent in $\mathcal{A}$ ($\mathcal{B}$) discovers an interaction opportunity, he looks round group $\mathcal{B}$ ($\mathcal{A}$) asking each in turn, starting with the Soft agents, until he finds a partner who is available to interact. The assumption made here is that agents in both groups can discover opportunities and all agents have the same probability of being the one to discover an opportunity. If we made the converse assumption that only agents in one group, say $\mathcal{B}$ ever discovers the opportunities then this is equivalent to the model already analysed in Section 2.1, since, trivially, all agents in group $\mathcal{B}$ have the incentive to behave Hard.

Model

There are $N^{\mathcal{A}}$ agents in group $\mathcal{A}$ and $N^{\mathcal{B}}$ agents in group $\mathcal{B}$. Every agent can adopt one of two behavioural types $\{S, H\}$ and so the state space is $\{0, 1, \ldots, N_{\mathcal{A}}\} \times \{0, 1, \ldots, N_{\mathcal{B}}\} = X$ with typical element $x = (x_{\mathcal{A}}, x_{\mathcal{B}})$ denoting that $x_{\mathcal{A}}$ of group $\mathcal{A}$ and $x_{\mathcal{B}}$ of group $\mathcal{B}$ are Soft, while the remaining agents are Hard. The state can change via an evolutionary dynamic which picks a random agent each period who revises strategy via the best response dynamic with probability $1 - \varepsilon$, and by mutation with probability ε. All this is the same as in Section 2.2.

Payoffs

Given agents' behavioural types, the expected payoffs to agents are determined as follows: nature randomly selects an agent who becomes endowed with an opportunity, enabling him to split a surplus of size 1 with a partner. This agent looks for a partner amongst agents of the other group, asking each in turn, until he finds an agent with whom he can interact. As before, he asks all Soft agents before all the Hard ones, with the ordering between any two agents in the group of the same type being random. If the agent with the opportunity finds a partner from the other group to interact with, the two of them split the surplus according to their behaviours as before: equal split if behaviours are the same, or $(s, 1-s)$ where $s \in (0, 1/2)$ in favour of the Hard agent if their behaviours are different.

Clearly this can be thought of as a zero sum game, since the sum of all agents' expected payoffs is $\frac{N_{\mathcal{B}}}{N_{\mathcal{A}}+N_{\mathcal{B}}}\left(1-(1-p)^{N_{\mathcal{A}}}\right)+\frac{N_{\mathcal{A}}}{N_{\mathcal{A}}+N_{\mathcal{B}}}\left(1-(1-p)^{N_{\mathcal{B}}}\right)$, the probability that an interaction takes place. As in Section 2.2, the agents in your own group are your competitors, whom you wish would act Hard, while the agents in the opposing are your trading partners who you wish would act Soft.

Result: Mixed population possibilities

Agents have two chances to interact. The first comes from possibility of being the one to discover the opportunity, which happens with probability $\frac{1}{N_{\mathcal{A}}+N_{\mathcal{B}}}$. The second comes from the possibility that a member of the opposing group discovers the opportunity and selects you to interact with. The payoffs agents get is the sum of these two possibilities. The effects of the second have been discussed at length in Section 2, where it was found that there is pressure supporting both intra- and inter-group herding. While the effect of the first possibility is simply to create a pressure towards Hard behaviour, since given any behaviour mix in the opposing population, acting Hard adds an extra $\frac{1}{2} - s$ on to the average share.

Given this, one might logically expect the herding results of Section 2.2 to persist here. However this is not the case. As the following example demonstrates, the distortion from the first possibility has a large enough effect to move us away from even intra-group herding.

Example 43. Let $N_A = 4$, $N_B = 2$, $p = 0.9$, $s = 0.15$.

There are two Nash equilibria and absorbing states here. These are $\mathbf{x} = (x_A, x_B) = (2, 2)$ and $(0, 0)$. In other words, it is possible that half of population A are Soft and the other half are Hard. Figure 3.3.2 on page 136 illustrates the evolutionary dynamics over the state space.

Looking at the basins of attraction, we could apply stochastic stability. The transition $(0, 0) \rightarrow (2, 2)$ requires two mutations, whereas $(0, 0) \leftarrow (2, 2)$ only requires one. This shows that $(2, 2)$ is not stochastically stable, and $(0, 0)$ is the LRE. Although, in general it is possible for states without herding to be stochastically stable. If s is increased to 0.2, making Soft more attractive, we have a reversal of four arrows (in favour of more Soft agents, as one would expect). Under this new dynamic the absorbing states would be $\{(0, 0), (3, 1)\}$. Now, transitioning between the two in either direction would only require one mutation and so both absorbing states are

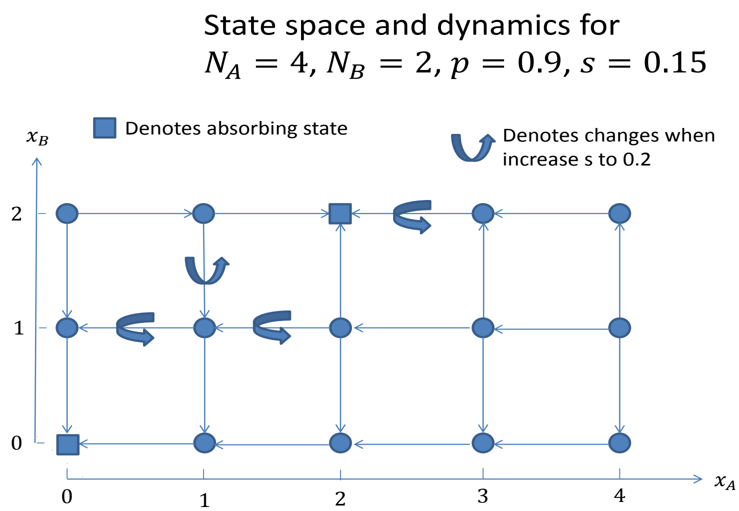


Figure 3.3.2: Example: $N_A = 4$, $N_B = 2$, $p = 0.9$.

stochastically stable.

The reasons for this break from herding behaviour may not at first be entirely clear. Indeed much of the intuition from Section 2.2 carries over: The effect of agents in the other group is unambiguous. Being Hard instead of Soft will always increase one's share when interacting by $\frac{1}{2} - s$. The relevance of the other group's behaviour is in determining the base from which this increase occurs. When the opposing population is mainly Soft, this increase is from a higher base and so represents a lower proportional change compared to when the opposing population is mainly Hard. Thus, just as in Section 2.1, there is a pressure toward inter-group herding of behaviour

The effect of other agents' behaviour in one's own population is more complicated. By the analysis of Section 2, taking into account only the expected utility from members of the other group, we get pressures toward intra-group herding. However, we also have to consider an agent's expected utility from the chance of being the one to discover the opportunity. This adds an extra $\frac{1}{N_A+N_B}\left(\frac{1}{2} - s\right)\left(1 - (1-p)^{N_i}\right)$ to the utility of a Hard type to an agent in group $j \neq i$, $i, j \in \{\mathcal{A}, \mathcal{B}\}$. While this is a constant, it still has an affect: The analysis of Section 2, shows that the ratio of the utilities

$U_n(S)/U_n(H)$ is increasing in n, the number of one's own group who are Soft. However as n becomes large, both $U_n(S)$ and $U_n(H)$ are decreasing and so their difference, $U_n(S) - U_n(H)$ can fall. If $U_n(S) - U_n(H)$ drops below $\frac{1}{N_A+N_B}\left(\frac{1}{2}-s\right)\left(1-(1-p)^{N_i}\right)$ then being Hard is preferable.

Intuitively, what is happening is the following: when enough other agents in one's group are Soft, the utility one gets from interacting when members of the other group discover opportunities diminishes, whether that agent try to compete by acting soft or not. So it becomes in the agent's interests to put all his eggs into the basket of maximising payoff from the times when he discovers the opportunity, which means being Hard.

3.3.3 Owning opportunities: one group

Model

There is a single group of N agents, each of whom can adopt one of two behavioural types $\{S, H\}$. Thus the state space is $X = \{0, 1, \ldots, N\}$ where $x \in X$ means x Soft agents and $N - x$ Hard agents. The state can change via an evolutionary dynamic which picks a random agent each period who revises strategy via the best response dynamic with probability $1 - \varepsilon$, and by mutation with probability ε.

Payoffs

Given agents' behavioural types, payoffs are determined as follows: nature randomly selects an agent who becomes endowed with an opportunity, enabling him to split a surplus of size 1 with a partner. This agent looks for a partner amongst the other agents of the group, asking each in turn, until he finds an agent with whom he can interact. As before, he asks all Soft agents before all the Hard ones, with the ordering between any two agents in the group of the same type being random. If the agent with the opportunity finds a partner from the other group to interact with, the two of them split the surplus according to their behaviours as before: equal split if behaviours are the same, or $(s, 1-s)$ where $s \in (0, 1/2)$ in favour of the Hard agent if their behaviours are different.

Clearly this can be thought of as a zero sum game, since the sum of all agents' expected payoffs is $1-(1-p)^{N-1}$, the probability that an interaction takes place. As in Section 3.1, the other members of the group are both your trading partners and competitors, so it is not immediately clear how the actions of one agent will change the payoff of another.

Result: return to herding

Pick an agent with perspective $n \in \{0, 1, ..., N-1\}$, meaning that n of the other $N-1$ agents are Soft. Then the agent's expected payoff from the two behaviours are

$$U_n(S) = \frac{1}{N}\left(\frac{1}{2}\sum_{j=1}^{n} q_j + s\sum_{j=n+1}^{N-1} q_j\right) + \frac{n}{N}\left(\frac{1}{2}\sum_{j=1}^{n} q_j\right) + \frac{N-n-1}{N}\left(\frac{s}{n+1}\sum_{j=1}^{n+1} q_j\right)$$

$$U_n(H) = \frac{1}{N}\left((1-s)\sum_{j=1}^{n} q_j + \frac{1}{2}\sum_{j=n+1}^{N-1} q_j\right) + \frac{n}{N}\left(\frac{1-s}{N-n}\sum_{j=n}^{N-1} q_j\right) + \frac{N-n-1}{N}\left(\frac{1}{2}\sum_{j=1}^{n+1} q_j\right)$$

where $q_j = p(1-p)^{j-1}$ is the probability that interaction takes place between the agent who owns the opportunity and the jth agent he asks. Letting $F(n) = N(U_n(S) - U_n(H))$, so that given perspective $n \in \{0, 1, ..., N-1\}$, this agent's optimal behaviour depends on the sign of $F(n)$. After some algebra we obtain

$$
\begin{aligned}
F(n) &= \frac{Ns}{n+1}\left(1-(1-p)^{n+1}\right)-\frac{n(1-s)}{N-n}p(1-p)^{n-1}- \\
&\quad sp(1-p)^{n}-\frac{N(1-s)}{N-n}\left((1-p)^{n}-(1-p)^{N-1}\right)
\end{aligned}
$$

Unfortunately, this function is not that easy to deal with. Unlike the function $G(n)$ from Section 2, it is not a monotone increasing function. It is mostly increasing, but for some parameters it will be decreasing for n close to 0 or $N-1$. Nevertheless, having, explored, this function for a very wide range of parameters using a computer program, I believe the following assumption to be true:

(A1): For any choice of parameters (N, p, s), and any $n \in \{0, 1, ..., N-2\}$, $F(n) \geq 0 \Longrightarrow F(n+1) > 0$

Assumption (A1) implies the existence of some cutoff, such that Hard is best when the number of agents choosing Soft is below, and Soft is best when the number of agents choosing Soft is above. This leads me to use the summary statistic $\tau(N, p, s) \in \left\{-0.5, 0, 0.5, \ldots, N-1, N-\frac{1}{2}\right\}$, defined such that for any $n \in \{0, 1, ..., N-2\}$:

$$n < \tau \Longleftrightarrow F(n) < 0$$

$$n = \tau \Longleftrightarrow F(n) = 0$$

$$n > \tau \Longleftrightarrow F(n) > 0$$

Note that $\tau(N, p, s) = N-\frac{1}{2}$ means that for parameters (N, p, s) Hard always does better than Soft, irrespective of the other agents' behaviour and so the only absorbing state is $x = 0$ where everybody is Hard. Similarly, $\tau(N, p, s) = -0.5$ implies that Soft always does better than Hard and so the only absorbing state is $x = N$.

$N=3$		s					
		0.2	0.25	0.3	0.35	0.4	0.45
p	0.3	2.5	2.5	2.5	2.5	2.5	2.5
	0.5	2.5	2.5	2.5	2.5	2	-0.5
	0.7	2.5	2.5	2.5	1.5	-0.5	-0.5
	0.8	2.5	2	1.5	0.5	-0.5	-0.5
	0.9	1.5	1.5	0.5	0.5	-0.5	-0.5

$N=40$		s				
		0.05	0.1	0.15	0.2	0.25
p	0.1	39.5	25.5	11.5	0.5	-0.5
	0.2	8.5	1.5	-0.5	-0.5	-0.5
	0.3	2.5	-0.5	-0.5	-0.5	-0.5
	0.4	0.5	-0.5	-0.5	-0.5	-0.5
	0.5	-0.5	-0.5	-0.5	-0.5	-0.5

$N=10$		s						
		0.1	0.15	0.2	0.25	0.3	0.35	0.4
p	0.1	9.5	9.5	9.5	9.5	9.5	9.5	2.5
	0.2	9.5	9.5	9.5	9.5	3.5	-0.5	-0.5
	0.3	9.5	8.5	4.5	0.5	-0.5	-0.5	-0.5
	0.4	5.5	2.5	0.5	-0.5	-0.5	-0.5	-0.5
	0.5	2.5	0.5	-0.5	-0.5	-0.5	-0.5	-0.5
	0.6	1.5	0.5	-0.5	-0.5	-0.5	-0.5	-0.5
	0.7	0.5	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5

Table 3.3.1: Some examples of $\tau(N,p,s)$ for different parameter values

Table 3.3.1 on page 140 gives a wide range of parameters for which assumption (A1) holds and finds $\tau(N,p,s)$ in each case.

As one would expect, the larger N, p or s are, the lower is $\tau(N,p,s)$ and hence the more attractive being Soft becomes.

Figure 3.3.3 on page 141 shows how the value of $\tau(N,p,s)$ determines the movement between states under the best response dynamics. For $n \in \{0,1,...,N-1\}$, the sign of $F(n)$ determines the relationship between states n and $n+1$: a left-pointing arrow if $F(n)<0$; no arrow if $F(n)=0$; a right-pointing arrow if $F(n)>0$.

With Figure 3.3.3 on page 141 in mind, the following results become obvious.

Theorem 44. *Assuming assumption (A1) holds, so that $\tau(N,p,s)$ is well defined:*

1. *If $\tau(N,p,s)=N-\frac{1}{2}$ then the only absorbing state is $x=0$.*
2. *If $\tau(N,p,s)=-0.5$ then the only absorbing state is $x=N$.*
3. *If $\tau(N,p,s)$ is neither $N-\frac{1}{2}$ or -0.5 then both $x=0$ and $x=N$ are absorbing states.*

Furthermore, if $\tau(N,p,s)<\frac{N-1}{2}$ then the Long run equilibrium is $x=N$; if

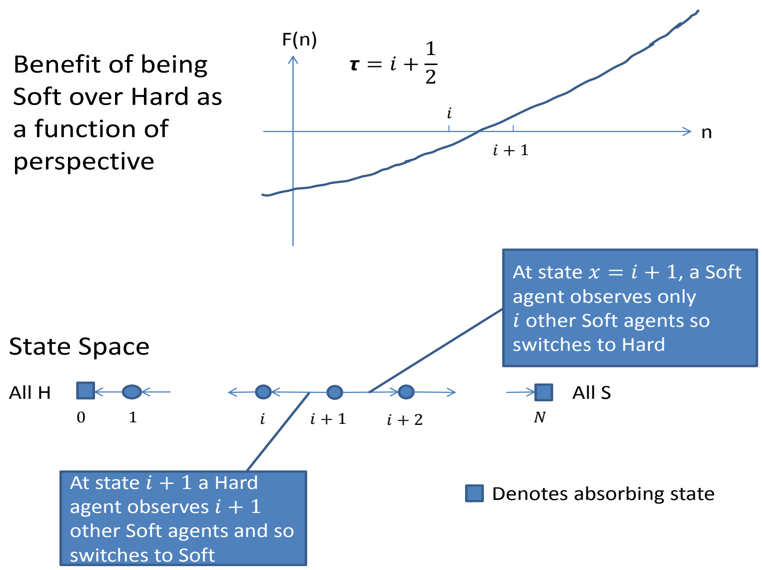


Figure 3.3.3: Dynamics for one group, stochastic owning of opportunity

$\tau(N,p,s) > \frac{N-1}{2}$ then the Long run equilibrium is $x = 0$; if $\tau(N,p,s) = \frac{N-1}{2}$ then both $x = 0$ and $x = N$ are stochastically stable.

This is the same result as was found in Section 2.1. The proof of this theorem is the same as the proof of Theorem 39 and hence is not repeated. This Theorem tells us that the herding behaviour seen in Section 2.1 should pertain once again here.

Indeed, if one compares the model here to that of Section 2.1, there are great similarities: for each agent in our current model, the agent takes the role of a group $\mathcal{A}$ agent in Section 2.1 with probability $\frac{N-1}{N}$ and takes the role of $\mathcal{B}$ in Section 2.1 with probability $\frac{1}{N}$. This change means that all other agents in the group are trading partners as well as competitors. But as Section 2.2 argues, this change in itself should not remove the herding result since it is in one's interests to adopt the same strategy of both competitors and trading partners. The one remaining factor that could cause a break with the herding behaviour is that seen in Section 3.2: the idea that if most others are Soft, one's chance of interacting is so small when someone else gets the opportunity, that it is not worth competing, and that one would be better off concentrating on maximising the share of the surplus when discovering the opportunity oneself. This turns out not to be the case.

One explanation for the difference in results between Section 3.2 and 3.3 lies in the construction of Example 43. Here the two groups were of different sizes. This dramatically reduced the incentive for group A agents to compete, since there were only two group $\mathcal{B}$ agents who may discover opportunities, and four group $\mathcal{A}$ agents competing to be picked. So for example, if over a period of time, every agent is expected to discover one opportunity, an average $\mathcal{A}$ agent only expects half an interaction from the source of being picked by group B agents. While in Section 3.3 there are $N-1$ other agents who may discover opportunities, and the same number competing, thus restoring the one-to-one balance. It remains an open question whether a mixing of intra-group behaviour is possible when $N^{\mathcal{A}} = N^{\mathcal{B}}$.

3.4 Conclusion

This chapter investigated the tradeoff between adopting Hard and Soft strategies. The trade-off is self evident: a Hard negotiating position has the advantage of taking a greater share of the surplus generated in each interaction, while having the disadvantage of being less likely to interact.

One way of analysing this tradeoff would be via the Hawk-Dove game where “Hawk” means to insist on a high portion of the surplus, while “Dove” means to be willing to compromise to reach agreement. It is well known [30, 29] that this theory predicts a mixture of behavioural types. If agents are matched together from a single population then the monomorphic outcomes of all Hawk or all Dove are not stable. The unique ESS predicts a certain mixture of Hawks and Doves. As this is ESS, it is locally stable in the sense that if more Doves were to appear, the Hawks would be getting a higher payoff than Doves and so would expand faster, returning us to our specified Hawk-Dove mix. In fact, this is globally stable. If there are two populations of agents, where all interactions take place between agents in opposing groups, then the predictions of the Hawk-dove model are one group of Hawks and another group of Doves, where the number of agents in each population is irrelevant.

The approach here is to drop the veil of ignorance assumed in the Hawk-Dove model by assuming that all agents know each others behavioural types and create a model where the probability of interacting depends on the behaviour of others as well as your own. This model generates almost polar opposite results to Hawk-Dove. With one group of agents, we find the absorbing states of the evolutionary dynamic being that everyone acts in the same way. This herding of behaviour is both apparent in the one population models of Sections 2.1 and 3.3, as well as the two group model of Section 2.2. There are both pressures toward intra-group herding, and in the two group models, inter-group herding of behaviour, in stark contrast to the results under Hawk-Dove.

Models of the kind studied here will not always deliver herding of behaviour. This is shown in Sections 3.1 and 3.2 which consider very similar models to Section 2. They show that a small change in the specifics of the model can cause distortions to the smooth analysis of Section 2 and break the herding pressures.

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