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On Delaunay Random Cluster Models

by

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Contents

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Declarations

All work thoughout this thesis is my own, unless stated otherwise, and was performed in the Mathematics Institute at the University of Warwick under the supervision of Dr. Stefan Adams during the period October 2011 to September 2014. This thesis has not been submitted for a degree at another university.

List of publications including submitted papers:

S. Adams and M. Eyers, *Phase transitions in Delaunay Potts models*, preprint – to be submitted to Journal of Statistical Physics (2015).

Abstract

We examine continuum percolative problems on the Delaunay hypergraph structure. In particular, we investigate the existence of a percolation transition for a class of Gibbsian particle systems with random hyperedges between groups of particles. Each such system will take the form of a random cluster representation of a corresponding continuum Potts model with geometric interactions on hyperedges of the Delaunay hypergraph structure. Any percolation results in the random cluster representation will lead to the existence of a phase transition for the continuum Potts model: that is, the existence of more than one Gibbs measure.

The original components of this research are as follows. After extending the random cluster representation of [GH96] to hypergraph structures, we achieve a phase transition for Delaunay continuum Potts models with infinite range type interactions – extending the work of [BBD03] in the process. Our main result is the existence of a phase transition for Delaunay continuum Potts models with no background interaction and just a soft type interaction. This is an extension of the phase transition results for the hardcore (resp. softcore) Widom–Rowlinson model of [R71] and later [CCK94], (resp. [LL72]). Our final piece of originality comes in the guise of an overview of the obstacles faced when investigating further percolative problems in the Delaunay hypergraph structure such as the Russo–Seymour–Welsh Theorem.

Chapter 1

Introduction

The Ising model, where each vertex of a finite graph is assigned a spin of -1 or $+1$, was first introduced in 1925 and has been essential in producing elegant mathematics that describes the underlying physics of a ferromagnet. The spins -1 and $+1$ correspond to the magnetic moments and the particle interaction is defined so that neighbouring particles with the same spin are favoured over ones with opposite spin. The interaction is stronger at lower temperatures and in particular, for temperatures below a critical value, is strong enough to result in the particles of one type dominating over the whole graph. In 1936, Peierls showed the existence of such a phase transition for the Ising model in two dimensions [Pe36], whereas an exact solution was given by Ontager in 1944, [On44]. The Potts model is a generalisation of the Ising model where vertices take spins (which we now call marks) from the finite set $1, \ldots, q$ instead of using -1 and $+1$. The Potts models behave in the same way to the Ising model: at low enough temperature, one mark will dominate.

Percolation for a disordered medium was first introduced in 1957 by Broadbent and Hammersley [BH57] using the example of a porous stone submerged in a volume of water. In the bond percolation model in particular, each edge of a graph is assigned, independently of the others, to be open, with probability p, or closed with probability $1 - p$. The percolation problem is to investigate the existence of an infinite connected component of open edges for different values of p. In the porous stone in water analogy, this corresponds to the question of whether or not water reached the centre of the stone. For any given p, the probability that an infinite connected component exists is either zero or one, and since this probability is increasing in p, there must exist $0 \leq p_c \leq 1$, such that for $p < p_c$ the probability is zero, and for $p > p_c$ it is one. Fast forward to today, and percolation is now a mature, wide reaching, well studied area of probability theory.

Although the random cluster model, invented around 1970 by Fortuin and Kasteleyn [FK69], [FK72], first had the purpose of unifying percolation with the Ising and Potts models, it quickly became a beautiful area of study in its own right. So too, it seems, is the case for the continuum random cluster representation of the continuum Potts models, first defined by [GH96], which continues to be the source of interesting problems. Whereas the (non-continuum) random cluster model, which we will call the discrete random cluster model throughout this thesis, has played a key role in many proofs and studies, including for example, the Wulff construction, the discontinuity of the phase transition for large cluster-factor, the Widom-Rowlinson two type lattice gas and the Edwards-Anderson spin-glass model, the situation in the continuum setting is quite different as results are far fewer. In [GH96], a phase transition is shown for a class of multi-type particle systems with a finite range repulsion between pairs of particles with different type. Later, [BBD03] used a similar proof to show the existence of a phase transition for Delaunay Potts models with hardcore repulsion between all pairs of particles and finite range type dependent interactions on the edges of the Delaunay graph.

In this study, we too will focus on Delaunay Potts models in $\mathbf{R}^2$, and indeed Delaunay random cluster models where interactions between particles occur on edges or triangles of the Delaunay graph. The Delaunay graph, in our two-dimensional setting, is the unique triangulation of the plane given a set of points. It satisfies the criteria that the circumcircles of all triangles have an empty intersection with the point set. The Delaunay graph corresponds to the dual graph of the Voronoi tessellation. It is straightforward to construct the Delaunay graph from the Voronoi tessellation: just join, with an edge, any two points whose Voronoi cells share a 1-dimensional face. Since the Voronoi cell of a point is the subset of the plane that is closer to this point than any other, the Delaunay graph is a nearest neighbour structure for a given set of points. This nearest neighbour graph gives rise to a prominent class of Gibbs measures with interactions that depend on the local geometry of point sets in $\mathbf{R}^2$, see [Der08] and [DDG10]. Systems like this are often studied in biology to model cell interactions [FRA07], where interactions depend on the area of the cells and the distance between them.

A thermodynamic system is said to have a phase transition if the system has more than one Gibbs measure/equilibrium state. Phase transitions are an important area of research in probability theory and statistical mechanics and aside from those described above, notable results in the continuum include the hardcore (resp. softcore) Widom-Rowlinson model of [R71] and later [CCK94], (resp.[LL72]). In this study we look at a phase transition from a probabilistic perspective through the use of Gibbs measures and point processes.

A realisation of such a point process is called a configuration. A configuration comprises of a number of particles (or points in $\mathbf{R}^2$), each with a mark assigned to it from a finite mark space. We consider two types of interaction between the particles: one which is mark dependent called the type interaction and one which is mark independent called the background interaction. All interactions occur between pairs of particles that share an edge in the Delaunay graph or triples of particles that comprise a triangle in the Delaunay graph. These pairs and triples are both called hyperedges of the Delaunay hypergraph structure. Interactions such as these are called geometric because they depend on the geometrical structure of the Delaunay triangulation. A Delaunay Potts model and its corresponding Delaunay random cluster representation are determined by the background and type interactions, together with a reference measure.

We draw your attention to some differences between geometric models on the Delaunay hypergraph structure and that of classical models such as the Widom-Rowlinson model and softcore variant of Lebowitz and Lieb [LL72]. The first is that edges and triangles in the Delaunay hypergraph are each proportional in number to the number of particles in the configuration. However, in the case of the complete hypergraph – on which interactions occur in the classical (non-geometric) models – the number of edges is proportional to the number of particles squared and the number of triangles is proportional to the number of particles cubed. Secondly, in the complete graph of the classical models, the neighbourhood of a given point depends only on the distance between points and so the number of neighbours increases with the activity parameter z of the underlying point process. This means that the system will become strongly connected for high values of z. This is not the case for the Delaunay hypergraphs which exhibit a self-similar property. Essentially, as the activity parameter z increases, the expected number of neighbours to a given point in the Delaunay hypergraph remains the same, see [Mø94]. Therefore, in order to keep a strong connectivity in our geometric models on Delaunay hypergraphs, we use a type interaction between particles of a hyperedge with a non-constant mark. Finally, and perhaps most importantly, is the question of additivity. Namely, suppose we have an existing particle configuration ω and we want to add a new particle x to it. In the case of classical many-body interactions, this addition will introduce new interactions that occur between x and the existing configuration ω. However, the interactions between particles of ω remain unaffected, and so classical many-body interactions are additive. On the other hand, in the Delaunay framework, the introduction of a new particle to an existing configuration not only creates new edges and triangles, but destroys some too. The Delaunay interactions are therefore not additive, and for this reason, attractive and repulsive interactions are indistinct. In the case of a hard exclusion interaction, we arrive at the possibility that a configuration ω is excluded, but for some x, $\omega \cup x$ is not. This is called the non-hereditary property, [DG09], which seems to rule out using techniques such as stochastic comparisons of point processes [GK97].

Although we show the existence of a Gibbs measure for each of our models, the primary focus of this thesis is that of continuum percolation. In showing the existence of percolation (for large activity and low temperature) of Delaunay random cluster representations, we also happen to show the non-uniqueness of Gibbs measures, and essentially, a phase transition for the Delaunay Potts models. In fact, one last piece of information is needed to show a phase transition. That is, for small activity and high temperature, the Gibbs measure is unique. By interpreting percolation in our Delaunay random cluster representation via a site percolation model, we prove our main results thanks to a discretization to the integer lattice, on which we show either site percolation, or mixed site-bond percolation, depending on the model. The main technical issue is to control the Papangelou conditional intensities of the marginal point distribution of the Delaunay random cluster measure with respect to the Gibbs process with background interaction. Heuristically, this is the conditional probability of there being a point of the process inside an infinitesimal neighbourhood of the location x, given the complete point configuration at all other locations ω. With a hardcore background interaction present, which we show to be equivalent to a classical hardcore short range repulsion between all pairs of particles, only a lower bound of the Papangelou conditional intensity is required. However, when we consider models with only a type interaction, an upper bound for the Papengelou conditional intensity will be needed as well.

To conclude the introduction, we remark on the makeup of the rest of the thesis. In Chapter 2, after introducing necessary notations and definitions that will see us through the rest of the thesis, we will put down an extension of the Fortuin–Kasteleyn (FK) representation of [GH96] to a hypergraph structure, show that percolation in the Delaunay random cluster models will imply the non-uniqness of Gibbs measures for the continuum Potts models and finally give some insight into our techniques to show percolation for certain parameter regimes. Although all models considered hereafter will be those on the Delaunay triangulations, all work in Chapter 2 will be done for more general hypergraph structures. Chapter 3 introduces our first class of Delaunay Potts models. These will exhibit infinite range type interactions on either triangles or edges of the Delaunay graph and will also have a hardcore background interaction. Percolation is first shown to occur for models displaying both the coarse-grain ready and bounded Papangelou properties – to be defined in Sections 3.5.1 and 3.5.2 respectively. We then look at three examples. Our proofs in Chapter 3 draw inspiration from [GH96] and [BBD03], although we do not benefit from the finite range property of the type interaction as in [BBD03], nor from the luxuries, described above, of classical multi-body interactions as in [GH96].

Chapter 4 introduces our second class of Delaunay Potts models. These will take the form of Widom-Rowlinson like models, i.e. without a background potential, and will be seen as an extension to the work of [CCK94] and [LL72]. The key difference is that the interspecies interaction will only exist on the Delaunay hyperedges. The proofs are much more complex compared to Chapter 3. This is because, without the crutch that is the hardcore background interaction, it becomes necessary to perform a much more in depth analysis of the underlying geometrical properties of the Delaunay triangulations in order to exhibit some control of the configurations. Chapter 5 sees an end to our investigations into phase transitions as we steer towards other percolative problems on the Delaunay hypergraph structure. These include the so-called Russo-Seymour-Welsh (RSW) Theorem which relates the probability of an open horizontal crossing of a square to that of an open horizontal crossing (the long way) of a rectangle with short side equal to the length of the square, see [Ru78] and [SW78]. There are only very few continuum RSW results in the literature, see [Al96], [Ro90], which is unfortunate, because they would give a greater insight into the percolative properties of a model and it is thought that an RSW theorem would provide an important step in a journey towards conformal invariance, similar to [Sm01], see [BS98]. Although a weak version is shown in [BR06a], the RSW theorem has yet to be proved for Voronoi percolation. We inform the reader of the major obstacles faced when trying to prove RSW for Voronoi percolation and variants. The final chapter (Chapter 6) consists of a discussion of the results given in the previous chapters and also gives an outlook, ideas and conjectures for the field of continuum percolation in the Delaunay setting.

Chapter 2

Continuum FK representation on hypergraph structures

In Section 2.1, we define the state spaces of marked and unmarked configurations of particles as well as other important notation before introducing hypergraph structures, both in the marked and unmarked cases, in Section 2.2. In Section 2.3 we deal with a deterministic particle configuration – we fix $\omega \in \Omega$ – and introduce the Potts model and the random cluster model in this discrete setting, similar to [G94]. Although, in this thesis, we are generally concerned with random structures, i.e. we do not fix $\omega \in \Omega$, but rather sample $\omega \in \Omega$ using a suitable Gibbs measure as described in Section 2.4, it is important to fully understand the discrete case as many of the results turn out to be useful to us later on. It is only then that we can define what has come to be known, from [GH96], as the continuum Potts model and continuum random cluster model – we do this in Sections 2.5 and 2.6. In Section 2.7 we discuss percolation in the continuum random cluster model and how this shows the existence of several distinct Gibbs measures for the corresponding continuum Potts model. Finally, in Section 2.8, we step away from considering general hypergraph structures, and formalise the Delaunay hypergraph structure that we will use throughout the remainder of the thesis.

2.1 Preliminaries

This introduction will be from the view point of continuum systems of particles that lie in the plane (we only work in two dimensions) and that interact with each other geometrically. That is, pairs, or groups of particles interact according to some underlying geometric hypergraph structure – quite unlike the classical approach of pair potentials. Each particle has a spatial location, but as we shall see, a spatial location does not always well define a particle. For instance, particles may be marked, and in this case, each particle will have a spatial location and a mark from some marked space.

Let us introduce the configuration spaces that we will use. We start off with the space of locally finite subsets of $\mathbf{R}^2$:

$$\Omega := \Big\{\omega \subset \mathbf{R}^2 : |\omega \cap \Lambda| < +\infty \text{ for all } \Lambda \in \mathcal{B}(\mathbf{R}^2)\Big\},$$

where $\mathcal{B}(\mathbf{R}^2)$ is the space of all bounded Borel sets of $\mathbf{R}^2$. Let $\mathcal{F}$ be the σ-algebra generated by the sets $\{\omega \in \Omega, N_\Lambda(\omega) = n\}$, where $n \in \mathbf{N}$, $\Lambda \in \mathcal{B}(\mathbf{R}^2)$ and $N_\Lambda(\omega)$ denotes the number of points of ω in Λ. More formally, $N_\Lambda : \Omega \to \mathbf{N}$ is defined as

$$N_\Lambda(\omega) = |\omega \cap \Lambda|, \tag{2.1}$$

and are often called the counting variables. As is usual, we take a Poisson point process with intensity $z > 0$ as the reference measure on $(\Omega, \mathcal{F})$.

Definition 2.1. *Let z be a positive real number. A Poisson point process with intensity z satisfies the following conditions:*

1. *For $\Lambda \in \mathcal{B}(\mathbf{R}^2)$, N_Λ is Poisson distributed with mean $z|\Lambda|$, where $|\Lambda|$ is the standard Lebesgue measure of Λ.*

2. *For any n disjoint sets $\Lambda_1, ..., \Lambda_n \in \mathcal{B}(\mathbf{R}^2)$, the random variables $N_{\Lambda_1}, ..., N_{\Lambda_n}$ are independent.*

We will denote a Poisson point process on Ω with intensity z by Π^z.

We also define the mark space. In our study, we restrict ourselves to the case of a finite mark space, as it will be used to attribute marks or spins to the particles. Let $\Sigma := \{1, \dots, q\}$, for a positive integer $q > 0$ denote the mark space. Denote λ as the reference probability measure on Σ, and set $\lambda(s) = 1/q$ for $s = 1, \dots, q$. We denote a marked point of the plane by $\overline{x} = (x, s)$, $x \in \mathbf{R}^2$, $s \in \Sigma$. Let $\overline{\Omega}$ be the space of marked configurations, i.e. the set of all pairs $\overline{\omega} = (\omega, \sigma_\omega)$ where $\omega \in \Omega$ is the spatial configuration and σ_ω is the mark vector in Σ^ω:

$$\overline{\Omega} := \Big\{\overline{\omega} : \overline{\omega} = (\omega, \sigma_\omega), \omega \in \Omega, \sigma_\omega \in \Sigma^\omega\Big\}.$$

We may write $\sigma_\omega := (\sigma_\omega(x) : x \in \omega)$ where $\sigma_\omega(x)$ is the unique mark of $x \in \omega$. Let $\rho : \overline{\Omega} \to \Omega$ be the projection of the marked point configurations to the positional point configurations defined by $(\omega, \sigma_\omega) \to \omega$. Let $\overline{\mathcal{F}}$ be the σ-algebra generated by the sets $\{\overline{\omega} \in \overline{\Omega}, N_\Lambda^i(\overline{\omega}) = n\}$, where $n \in \mathbf{N}$, $i \in \Sigma$, $\Lambda \in \mathcal{B}(\mathbf{R}^2)$ and $N_\Lambda^i(\overline{\omega})$ denotes the number of points of $\overline{\omega}$ in $\Lambda \times i$. Instead of the reference Poisson point process Π^z, we take the Poisson point process $\overline{\Pi}^z$ on $\overline{\Omega}$ with intensity measure $z\nu \otimes \lambda$. Here, ν is the Lebesgue measure on $\mathbf{R}^2$. The probability space of marked configurations is the following triple:

$$(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\Pi}^z).$$

Let $\Omega_f := \{\omega : |\omega| < \infty\} \subset \Omega$ denote the set of all finite configurations ω. Also let $\mathcal{F}_f$ be the trace sigma-algebra of $\mathcal{F}$ on Ω_f. For a bounded region $\Lambda \in \mathcal{B}(\mathbf{R}^2)$, we write $\omega_\Lambda := \omega \cap \Lambda$ and $\Omega_\Lambda := \{\omega \in \Omega : \omega \subset \Lambda\}$, with $\mathrm{pr}_\Lambda : \Omega \to \Omega_\Lambda, \omega \mapsto \omega_\Lambda$ the projection of configurations onto Λ. The trace σ-algebra $\mathcal{F}'_\Lambda := \mathcal{F}|_{\Omega_\Lambda}$ is the restriction of $\mathcal{F}$ to Ω_Λ and $\mathcal{F}_\Lambda := \mathrm{pr}_\Lambda^{-1}\mathcal{F}'_\Lambda \subset \mathcal{F}$ is the σ-algebra of all events that happen in Λ only.

In the case of marked configurations, $\overline{\Omega}_f$ and $\overline{\mathcal{F}}_f$ are defined analogously. However, $\Lambda \in \mathcal{B}(\mathbf{R}^2)$ will only refer to the positional part of a marked configuration. In particular, we set $\overline{\omega}_\Lambda := \overline{\omega} \cap (\Lambda \times \Sigma)$, $\overline{\Omega}_\Lambda := \{\overline{\omega} \in \overline{\Omega} : \rho(\omega) \subset \Lambda\}$, $\overline{\mathcal{F}}'_\Lambda := \overline{\mathcal{F}}|_{\overline{\Omega}_\Lambda}$ and $\overline{\mathcal{F}}_\Lambda := \overline{\mathrm{pr}}_\Lambda^{-1}\overline{\mathcal{F}}'_\Lambda \subset \overline{\mathcal{F}}$ where $\overline{\mathrm{pr}}_\Lambda : \overline{\Omega} \to \overline{\Omega}_\Lambda$ satisfying $\overline{\omega} \mapsto \overline{\omega}_\Lambda$ is the projection of marked configurations onto $\Lambda \times \Sigma$. Let $\Theta = (\vartheta_x)_{x \in \mathbf{R}^2}$ be the group of translations, where $\vartheta_x : \overline{\Omega} \to \overline{\Omega}$ is a translation by the vector $-x \in \mathbf{R}^2$ that only act on the positions of particles. They have no affect on their marks.

2.2 Hypergraph structures

In what follows, we are interested in geometrical interactions that act on hyperedges between points. Suppose $\mathcal{H} \subset \Omega_f \times \Omega$ is measurable. We call $\mathcal{H}$ a hyperedge structure if $\eta \subset \omega$ for all $(\eta, \omega) \in \mathcal{H}$. If $(\eta, \omega) \in \mathcal{H}$, we call η a hyperedge of ω. We will write $\eta \in \mathcal{H}(\omega)$, thereby defining $\mathcal{H}(\omega)$.

2.2.1 Marked hypergraph structures

The definition of a hypergraph structure is easily extended to the space of marked configurations. Suppose $\mathcal{H} \subset \overline{\Omega}_f \times \overline{\Omega}$ is measurable. We call $\mathcal{H}$ a marked hypergraph structure if $\overline{\eta} \subset \overline{\omega}$ for all $(\overline{\eta}, \overline{\omega}) \in \mathcal{H}$. Similar to the non-marked case, if $(\overline{\eta}, \overline{\omega}) \in \mathcal{H}$, we call $\overline{\eta}$ a hyperedge of $\overline{\omega}$ and write $\overline{\eta} \in \mathcal{H}(\overline{\omega})$. Note that the geometrical properties of a marked hyperedge $\overline{\eta}$ are still defined in terms of the underlying η. Given a marked hypergraph structure, $\mathcal{H}$, and an unmarked configuration $\omega \in \Omega$, define

$$\mathcal{H}(\omega) = \bigcup_{(\eta,\sigma_\eta)\in\mathcal{H}(\overline{\omega})} \{\eta\} \tag{2.2}$$

as the set of the images, under ρ, of the hyperedges of an arbitrary marked configuration $\overline{\omega}$ satisfying $\rho(\overline{\omega}) = \omega$. This is particularly useful as when working with marked hypergraph structures we can easily distinguish between $\mathcal{H}(\overline{\omega})$ and $\mathcal{H}(\omega)$. In fact, $\mathcal{H}(\omega)$ is the set of hyperedges of ω in the corresponding unmarked hypergraph structure $\mathcal{G} \subset \Omega_f \times \Omega$, defined by

$$\mathcal{G} := \bigcup_{((\eta,\sigma_\eta),(\omega,\sigma_\omega))\in\mathcal{H}} \{(\eta,\omega)\}.$$

2.3 Potts model and FK representation on an arbitrary graph

In this section we deal with a deterministic particle configuration – we fix $\omega \in \Omega$ – and introduce the Potts model and the random cluster model in this discrete setting. It is identical to that in [G94], except that we consider hyperedges rather than simply edges. Let $\mathcal{H} \subset \Omega_f \times \Omega$ be a hypergraph structure, but fix $\omega \in \Omega$. Denote by $G = (\omega_\Lambda, \mathcal{H}(\omega_\Lambda))$ the finite hypergraph in $\Lambda \in \mathcal{B}(\mathbf{R}^2)$ where ω_Λ is the vertex set and $\mathcal{H}(\omega_\Lambda)$ is the set of hyperedges of ω_Λ. Let p be a constant satisfying $0 \le p \le 1$. A Potts measure, $\nu_p^{(q)}$, is a measure on the sample space $\Sigma^{\omega_\Lambda} := \{1,\ldots,q\}^{\omega_\Lambda}$ where each vertex in ω_Λ is assigned a mark from the set $\{1,\ldots q\}$, with $q \ge 2$. Given a mark vector, or configuration $\sigma_{\omega_\Lambda} \in \Sigma^{\omega_\Lambda}$, we define

$$\nu_p^{(q)}(\sigma_{\omega_\Lambda}) = \frac{1}{Z_1} \prod_{\eta\in\mathcal{H}(\omega_\Lambda)} \exp\left[-J\left(1-\delta_{\sigma_{\omega_\Lambda}}(\eta)\right)\right], \tag{2.3}$$

where Z_1 is the normalisation constant given by

$$Z_1 := \sum_{\sigma_{\omega_\Lambda}\in\Sigma^{\omega_\Lambda}} \left(\prod_{\eta\in\mathcal{H}(\omega_\Lambda)} \exp\left[-J\left(1-\delta_{\sigma_{\omega_\Lambda}}(\eta)\right)\right] \right), \tag{2.4}$$

the parameter $J := \ln(1-p)^{-1}$ satisfies $0 \le J \le \infty$ and for a hyperedge $\eta \in \mathcal{H}(\omega_\Lambda)$,

$$\delta_{\sigma_{\omega_\Lambda}}(\eta) = \begin{cases} 1 & \text{if } \sigma_{\omega_\Lambda}(x) = \sigma_{\omega_\Lambda}(y), \text{ for all pairs } \{x,y\} \in \eta, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

The random cluster model, on the other hand, has realisations in the set $\mathcal{E}$ given by $\mathcal{E} = \{0,1\}^{\mathcal{H}(\omega_\Lambda)}$. Each such realisation $v \in \mathcal{E}$ is a vector of 0's and 1's called a hyperedge configuration: each hyperedge $\eta \in \mathcal{H}(\omega_\Lambda)$ is either open or closed. For $v \in \mathcal{E}$, let

$$E_v := \{\eta \in \mathcal{H}(\omega_\Lambda) : v(\eta) = 1\}$$

represent the set of open hyperedges. To represent a realisation of a random cluster measure, we only use E_v which, with ω_Λ forms a hypersubgraph (ω_Λ, E_v) of G. However, the open hyperedge set E_v is uniquely determined by $v \in \mathcal{E}$ which is chosen randomly according to the probability mass function $\mu_p^{(q)}$ on $\mathcal{E}$ given by

$$\mu_p^{(q)}(v) = \frac{1}{Z_2} p^{|E_v|}(1-p)^{|\mathcal{H}(\omega_\Lambda)\setminus E_v|} q^{K(E_v)}, \tag{2.6}$$

where $K(E_v)$ is the number of connected components of the hypergraph (ω_Λ, E_v) and Z_2 is the normalising constant given by

$$Z_2 := \sum_{v \in \mathcal{E}} \left(p^{|E_v|}(1-p)^{|\mathcal{H}(\omega_\Lambda)\setminus E_v|} q^{K(E_v)} \right).$$

Notice that by setting $q = 1$ we arrive at the standard bond percolation model where each hyperedge is open (resp. closed) with probability p (resp. $1-p$). Before we give some important properties of the random cluster measure in this discrete hyperedge setting, we generalise to the case of non-constant p. By this we mean that a hyperedge $\eta \in \mathcal{H}(\omega_\Lambda)$ is open with probability $p(\eta)$, which depends on the geometry of the hyperedge. A simple example is when $p(\eta)$ is a function of $|x-y|$ where $\eta = \{x, y\}$. Thus, Equation 2.6 can be rewritten as

$$\mu_p^{(q)}(v) = \frac{1}{Z_3} \left(\prod_{\eta \in \mathcal{H}(\omega_\Lambda)} p(\eta)^{v(\eta)}(1-p(\eta))^{1-v(\eta)} \right) q^{K(E_v)}, \tag{2.7}$$

where the normalising constant Z_3 is given by

$$Z_3 := \sum_{v \in \mathcal{E}} \left(\left(\prod_{\eta \in \mathcal{H}(\omega_\Lambda)} p(\eta)^{v(\eta)}(1-p(\eta))^{1-v(\eta)} \right) q^{K(E_v)} \right).$$

The measure $\mu_p^{(q)}$ satisfies the Fortuin-Kasteleyn-Ginibre (FKG) inequality, see [FKG71], if and only if $q \geq 1$. Before we specify the FKG inequality, we give some preliminary definitions. A function $f : \mathcal{E} \to \mathbf{R}$ is said to be increasing if $f(E_1) \leq f(E_2)$ whenever $E_1 \subseteq E_2$. Similarly, a function $f : \mathcal{E} \to \mathbf{R}$ is called decreasing if $f(E_1) \geq f(E_2)$ whenever $E_1 \subseteq E_2$. An event A is called increasing if $\mathbf{1}_A$ is an increasing function.

Lemma 2.2. (FKG inequality – as a generalisation of [G94])
Fix $q \geq 1$. Then if f and g are both increasing functions on $\mathcal{E}$, then we have

$$\mathbb{E}_{\mu_p^{(q)}}(fg) \geq \mathbb{E}_{\mu_p^{(q)}}(f)\mathbb{E}_{\mu_p^{(q)}}(g). \tag{2.8}$$

In particular, if A and B are both increasing events, then

$$\mu_p^{(q)}(A \cap B) \geq \mu_p^{(q)}(A)\mu_p^{(q)}(B). \tag{2.9}$$

We now give a generalisation of a well-known comparison inequality (see [G94]) for random-cluster measures in the discrete setting. We generalise to our hypergraph framework and to the case of non-constant p. The result relies on the FKG inequality in Lemma 2.2. For any two probability measures μ_1, μ_2 on $\mathcal{E}$, we say μ_1 stochastically dominates μ_2, and write $\mu_1 \succcurlyeq \mu_2$, if $\mathbb{E}_{\mu_1}(f) \geq \mathbb{E}_{\mu_2}(f)$ for all increasing functions $f : \mathcal{E} \to \mathbf{R}$.

Proposition 2.3. (Comparison inequality)
Let $q \geq 1$ and suppose p and $\tilde{p}$ are two probability measures on $\mathcal{H}(\omega_\Lambda)$ such that

$$\frac{p(\eta)}{q^{|\eta|-1}(1-p(\eta))} \geq \frac{\tilde{p}(\eta)}{(1-\tilde{p}(\eta))}, \tag{2.10}$$

for all $\eta \in \mathcal{H}(\omega_\Lambda)$. Then, $\mu_p^{(q)} \succcurlyeq \mu_{\tilde{p}}^{(1)}$.

Proof. The proof in a slight adaptation of the proof in [G94] to the case of non-constant p and to our hypergraph framework. By the assumption that $q \geq 1$, the measure $\mu_p^{(q)}$ satisfies the FKG inequality. See that, for $v \in \mathcal{E}$

$$\mu_{\tilde{p}}^{(1)}(v) = \frac{g(v)\left(\prod_{\eta \in \mathcal{H}(\omega_\Lambda)}(1-\tilde{p}(\eta))\left(\frac{p(\eta)}{1-p(\eta)}\right)^{v(\eta)}\right)q^{K(E_v)}}{\sum_{v' \in \mathcal{E}} g(v')\left(\prod_{\eta \in \mathcal{H}(\omega_\Lambda)}(1-\tilde{p}(\eta))\left(\frac{p(\eta)}{1-p(\eta)}\right)^{v'(\eta)}\right)q^{K(E_{v'})}} \tag{2.11}$$

where g satisfies

$$g(v) = \frac{1}{q^{K(E_v)+(|\eta|-1)|E_v|}} \prod_{\eta \in E_v} \left(\frac{\tilde{p}(\eta)}{(1-\tilde{p}(\eta))} \Big/ \frac{p(\eta)}{q^{|\eta|-1}(1-p(\eta))} \right).$$

However, by multiplying Equation (2.11) by $\frac{\prod_{\eta\in\mathcal{H}(\omega_\Lambda)}(1-p(\eta))}{\prod_{\eta\in\mathcal{H}(\omega_\Lambda)}(1-p(\eta))} = 1$, we are left with

$$\mu_{\tilde{p}}^{(1)}(v) = \frac{g(v)\mu_p^{(q)}(v)}{\sum_{v'\in\mathcal{E}} g(v')\mu_p^{(q)}(v')}.$$

Since $K(E_v) + (|\eta| - 1)|E_v|$ is an increasing function of E_v and

$$\frac{p(\eta)}{q^{|\eta|-1}(1-p(\eta))} \geq \frac{\tilde{p}(\eta)}{(1-\tilde{p}(\eta))},$$

it follows that g is a decreasing function of E_v. Therefore, if f is increasing, then, by the FKG inequality,

$$\mathbb{E}_{\mu_{\tilde{p}}^{(1)}}(f) = \frac{\mathbb{E}_{\mu_p^{(q)}}(fg)}{\mathbb{E}_{\mu_p^{(q)}}(g)} \leq \mathbb{E}_{\mu_p^{(q)}}(f).$$

□

Note that we do not require p and $\tilde{p}$ to be constant: they can vary as long as equation (2.10) is satisfied. This stochastic domination result is frequently used to prove the existence of a phase transition for the infinite volume discrete random cluster models with varying values of p. Indeed, whilst proving a phase transition in continuum random cluster models, we also make use of this discrete result.

2.4 Gibbs measures

Instead of working with a fixed hypergraph as in Section 2.3, we will now look to sample point configurations from $\overline{\Omega}$. In order to do this, we introduce the concept of Gibbs measures on a marked hypergraph structure $\mathcal{H}$. In later Chapters we will look for phase transitions of Delaunay Potts models using probabilistic techniques, however, before this can be done, we must first show the existence of at least one Gibbs measure. We outline how this is done in the general hypergraph structure setting. Only once we can show the existence of a Gibbs measure for a particular model, can we use similar strategies to the Fortuin-Kasteleyn representation of the Potts model [FK72] to show non-uniqueness of Gibbs measures. This non-uniqueness, as we shall see, is equivalent to the existence of a phase transition which is a principle goal of this work.

Having established a definition for a hypergraph structure in Section 2.2, we look to introduce geometric interactions between particles. Such interactions will give rise to configurations $\overline{\omega} \in \overline{\Omega}$ that have a much higher energy than others. A Gibbs measure will sample these high energy configurations less frequently, i.e. low energy configurations will be favoured by the Gibbs measure. To measure the energy of a configuration, we introduce hyperedge potentials.

Definition 2.4. *A hyperedge potential is a measurable function, $\overline{\Phi} : \overline{\Omega}_f \times \overline{\Omega} \to \mathbf{R} \cup \{\infty\}$, from a marked hypergraph structure, $\mathcal{H} \subset \overline{\Omega}_f \times \overline{\Omega}$, to the extended reals.*

Definition 2.5. *A marked hypergraph structure $\mathcal{H}$ and a hyperedge potential $\overline{\Phi}$ are called translation invariant if for all $(\overline{\eta}, \overline{\omega}) \in \mathcal{H}$ and for all $x \in \mathbf{R}^2$,*

$$(\vartheta_x(\overline{\eta}), \vartheta_x(\overline{\omega})) \in \mathcal{H} \quad \text{and} \quad \overline{\Phi}(\vartheta_x(\overline{\eta}), \vartheta_x(\overline{\omega})) = \overline{\Phi}(\overline{\eta}, \overline{\omega}).$$

Hypergraph potentials allow us to control the random hypergraph by giving more or less weight to certain configurations. Let $\overline{\Phi} : \overline{\Omega}_f \times \overline{\Omega} \to \mathbf{R} \cup \{\infty\}$ be one such hyperedge potential attributing a real valued energy to each hyperedge in a hypergraph structure. By summing over the whole configuration, we arrive at the Hamiltonian energy with respect to $\overline{\Phi}$ for a given configuration $\overline{\omega} \in \overline{\Omega}$:

$$H^{\overline{\Phi}}(\overline{\omega}) := \sum_{\overline{\eta} \in \mathcal{H}(\overline{\omega})} \overline{\Phi}(\overline{\eta}, \overline{\omega}). \tag{2.12}$$

This is the formal Hamiltonian: it is an infinite sum. In order to work with finite sums, we restrict our Hamiltonian to a bounded region $\Lambda \in \mathcal{B}(\mathbf{R}^2)$. Consider the subset

$$\mathcal{H}_\Lambda(\overline{\zeta}) := \{\overline{\eta} \in \mathcal{H}(\overline{\zeta}) : \overline{\eta}_\Lambda \neq \emptyset\} \tag{2.13}$$

and take $\overline{\xi} \in \overline{\Omega}_{\Lambda^c}$, a fixed configuration outside of Λ, to be the prescribed boundary condition. The Hamiltonian in Λ with marked boundary condition $\overline{\xi}$ is given by the formula

$$H^{\overline{\Phi}}_{\Lambda|\overline{\xi}}(\overline{\omega}) = \sum_{\overline{\eta} \in \mathcal{H}_\Lambda(\overline{\omega} \cup \overline{\xi})} \overline{\Phi}(\overline{\eta}, \overline{\omega} \cup \overline{\xi}), \quad \text{for } \overline{\omega} \in \overline{\Omega}_\Lambda, \tag{2.14}$$

provided the sum is well defined. The Hamiltonian describes the model, and in particular, the energy excess of the marked configuration $\overline{\omega} \cup \overline{\xi}$ over the boundary configuration, $\overline{\xi}$. Notice that it depends only on the marked hypergraph structure $\mathcal{H}$ and the hyperedge potential $\overline{\Phi}$. If $\overline{\Phi}$ only depends on the single hyperedges, and not on their neighbourhood, then for any two distinct boundary configurations $\overline{\xi}, \overline{\zeta} \in \overline{\Omega}_{\Lambda^c}$,

$$H^{\overline{\Phi}}_{\Lambda|\overline{\xi}}(\overline{\omega}) = H^{\overline{\Phi}}_{\Lambda|\overline{\zeta}}(\overline{\omega}) \ \text{ provided that } \ \mathcal{H}_\Lambda(\overline{\omega} \cup \overline{\xi}) = \mathcal{H}_\Lambda(\overline{\omega} \cup \overline{\zeta}).$$

This is the case for all of the hyperedge potentials that we will consider in this thesis – it is called the local horizon property. We use the Hamiltonian and a reference measure to define a probability measure for the model. Recall our Poisson point process reference measure $\overline{\Pi}^z$ on $\overline{\Omega}$ with intensity measure $z\nu \otimes \lambda$ and define $\overline{\Pi}^z_\Lambda := \overline{\Pi}^z \circ \mathrm{pr}_\Lambda^{-1}$ to be the reference measure on the local restricted configuration space $\overline{\Omega}_\Lambda$. Together with the Hamiltonian, we define the partition function

$$Z_{\Lambda|\bar{\xi}} := \int_{\overline{\Omega}_\Lambda} \exp\left[-H^{\overline{\Phi}}_{\Lambda|\bar{\xi}}(\overline{\omega})\right] \overline{\Pi}_\Lambda(d\overline{\omega}), \tag{2.15}$$

and the negative part of $H^{\overline{\Phi}}_{\Lambda|\bar{\xi}}$:

$$H^{\overline{\Phi}_-}_{\Lambda|\bar{\xi}}(\overline{\omega}) := \sum_{\overline{\eta} \in \mathcal{H}_\Lambda(\overline{\omega} \cup \bar{\xi})} \overline{\Phi}_-(\overline{\eta}, \overline{\omega} \cup \bar{\xi}),$$

where $\overline{\Phi}_-$ is simply the negative part of $\overline{\Phi}$. Classically, there should be a factor A in the exponent of (2.15) to represent the inverse temperature of the system. However, we will assume this information is encompassed in $\overline{\Phi}$.

Definition 2.6. *A marked configuration $\bar{\xi} \in \overline{\Omega}_{\Lambda^c}$ is called admissible for a bounded region $\Lambda \in \mathcal{B}(\mathbf{R}^2)$ and an activity $z > 0$, if*

$$H^{\overline{\Phi}_-}_{\Lambda|\bar{\xi}}(\overline{\omega}) < \infty$$

for $\overline{\Pi}^z_\Lambda-$ almost all $\overline{\omega} \in \overline{\Omega}_\Lambda$ and if $0 < Z_{\Lambda|\bar{\xi}} < \infty$. Denote by $\overline{\Omega}^*_{\Lambda,z}$ the space of marked configurations whose restrictions to $\Lambda^c \times \Sigma$ are admissible configurations for Λ and z.

For $\Lambda \in \mathcal{B}(\mathbf{R}^2)$, we define the Gibbs distribution for $(\mathcal{H}, \overline{\Phi}, z)$ in the region Λ with admissible boundary condition $\bar{\xi}$ as follows:

$$Q_{\Lambda|\bar{\xi}}(d\overline{\omega}) = \frac{1}{Z_{\Lambda|\bar{\xi}}} \exp\left[-H^{\overline{\Phi}}_{\Lambda|\bar{\xi}}(\overline{\omega})\right] \overline{\Pi}^z_\Lambda(d\overline{\omega}). \tag{2.16}$$

Definition 2.7. *Given a marked hypergraph structure $\mathcal{H}$, a hyperedge potential $\overline{\Phi}$ and an activity $z > 0$, a probability measure P on $(\overline{\Omega}, \overline{\mathcal{F}})$ is called a Gibbs measure for $\mathcal{H}$, $\overline{\Phi}$ and z if $P(\overline{\Omega}^*_{\Lambda,z}) = 1$ and*

$$\int f dP = \int_{\overline{\Omega}^*_{\Lambda,z}} Z^{-1}_{\Lambda|\bar{\xi}_{\Lambda^c}} \int_{\overline{\Omega}_\Lambda} f(\overline{\omega} \cup \bar{\xi}_{\Lambda^c}) \exp\left[-H^{\overline{\Phi}}_{\Lambda|\bar{\xi}_{\Lambda^c}}(\overline{\omega})\right] \overline{\Pi}^z_\Lambda(d\overline{\omega}) P(d\bar{\xi}), \tag{2.17}$$

for all $\Lambda \in \mathcal{B}(\mathbf{R}^2)$ and every bounded measurable function $f : \overline{\Omega} \to \mathbf{R}$.

The equations of Definition 2.7 are known as the the DLR equations (after Dobrushin, Lanford and Ruelle), see for example [Geo99].

Let $\mathcal{A}^*$ be the universal completion of a σ-algebra $\mathcal{A}$. Precisely, $\mathcal{A}^*$ is the σ-algebra of all sets that belong to the ν-completion of $\mathcal{A}$ for all probability measures ν on $\mathcal{A}$. The Hamiltonian in (2.14) and the partition function in (2.15) are not measurable with respect to the σ-algebras introduced in Section 2.1, but only with respect to their universal completion. More precisely,

1. The function $(\overline{\omega}, \overline{\xi}) \to H^{\overline{\Phi}}_{\Lambda|\overline{\xi}}(\overline{\omega})$ is measurable with respect to $(\overline{\mathcal{F}}'_{\Lambda} \otimes \overline{\mathcal{F}}_{\Lambda^c})^*$.
2. The function $\overline{\xi} \to Z_{\Lambda|\overline{\xi}}$ is measurable with respect to $\overline{\mathcal{F}}^*_{\Lambda^c}$.
3. The set $\overline{\Omega}^*_{\Lambda,z}$ belongs to $\overline{\mathcal{F}}^*_{\Lambda^c}$.
4. $Q_{\Lambda|\overline{\xi}}(d\overline{\omega})$ is a probability kernel from $(\overline{\Omega}^*_{\Lambda,z}, \overline{\mathcal{F}}^*_{\Lambda^c}|_{\overline{\Omega}^*_{\Lambda,z}}) \to (\overline{\Omega}, \overline{\mathcal{F}})$.

According to the definition of the reference measure $\overline{\Pi}^z_{\Lambda}$, the marks are chosen independently of the spatial positions, henceforth, the above measurability statements can be obtained from Claims A.2 and A.3 of [DDG10] via consideration of $\mathbf{R}^2 \times \Sigma$ instead of $\mathbf{R}^2$ as the state space of the point process.

2.4.1 Existence of Gibbs measures

The existence of a Gibbs measure $\mathcal{P}$ on $(\Omega, \mathcal{F})$ with activity $z > 0$, marked hypergraph structure $\mathcal{H}$ and hyperedge potential $\overline{\Phi}$, relies on the following conditions.

Definition 2.8. *We say the couple $(\overline{\Phi}, \mathcal{H})$ satisfy the range condition **(R)** if there exists $\ell_R, n_R \in \mathbf{N}$, such that, for all $(\overline{\eta}, \overline{\omega}) \in \mathcal{H}$, there exist a bounded region $\Delta \in \mathcal{B}(\mathbf{R}^2)$ such that; $(\overline{\eta}, \overline{\omega}') \in \mathcal{H}$ and $\overline{\Phi}(\overline{\eta}, \overline{\omega}') = \overline{\Phi}(\overline{\eta}, \overline{\omega})$ whenever $\overline{\omega}'_{\Delta} = \overline{\omega}_{\Delta}$; and for every $x, y \in \Delta$, there exist ℓ open balls $B_1, \ldots, B_\ell$ (with $\ell \leq \ell_R$) such that $\cup_{i=1}^{\ell} \overline{B}_i$ is connected and contains x and y, and $N_{B_i}(\omega) \leq n_R$ for each i. Here, $\overline{B}_i$ denotes the closure of the open ball $B_i \subset \mathbf{R}^2$.*

Let $\Lambda \in \mathcal{B}(\mathbf{R}^2)$, $r > 0$ and $\Lambda \oplus r := \cup_{x\in\Lambda} B(x, r)$, where $B(x, r)$ denotes the open ball of radius r centred at x. A configuration $\overline{\xi} \in \overline{\Omega}$ is said to confine the range of $\overline{\Phi}$ from Λ if there exists a boundary layer $\partial\Lambda(\overline{\xi}) = (\Lambda \oplus r)\backslash\Lambda$ for some $r = r_{\Lambda,\overline{\xi}}$, where $r_{\Lambda,\overline{\xi}}$ is chosen as small as possible, such that $\mathcal{H}_{\Lambda}(\overline{\omega} \cup \overline{\xi}_{\Lambda^c}) = \mathcal{H}_{\Lambda}(\overline{\omega} \cup \overline{\xi}'_{\Lambda^c})$ whenever $\overline{\xi}' = \overline{\xi}$ on $\partial\Lambda(\overline{\xi})$. We define the set $\overline{\Omega}^{\Lambda}_{\text{cr}}$ as follows:

$$\overline{\Omega}^{\Lambda}_{\text{cr}} := \left\{ \overline{\xi} \in \overline{\Omega} : \overline{\xi} \text{ confines the range of } \overline{\Phi} \text{ from } \Lambda \right\}.$$

Remark 2.9. *In order to show phase transition in Section 2.7, we will need an adaptation of Proposition 3.1 in [DDG10] to the marked case. If a hyperedge potential $\overline{\Phi}$ and a hypergraph structure $\mathcal{H}$ satisfy the range **(R)** condition, then for each $\Lambda \in \mathcal{B}(\mathbf{R}^2)$, there exists a set $\widehat{\Omega}_{\mathrm{cr}}^{\Lambda} \in \overline{\mathcal{F}}_{\Lambda^c}$ such that $\widehat{\Omega}_{\mathrm{cr}}^{\Lambda} \subset \overline{\Omega}_{\mathrm{cr}}^{\Lambda}$ and $P(\widehat{\Omega}_{\mathrm{cr}}^{\Lambda}) = 1$ for all translation invariant probability measures P on $(\overline{\Omega}, \overline{\mathcal{F}})$ with $P(\{\emptyset\}) = 0$. The proof of this statement is an adaptation of Proposition 5.4 in [DDG10] to the marked case. It is straightforward as the mark distribution, with respect to our reference measure $\overline{\Pi}^{z}$ does not depend on the spatial location of the particles and henceforth we omit it and just give the statement.*

We now define stability which will ensure that the partition functions are finite.

Definition 2.10. *We say the couple $(\overline{\Phi}, \mathcal{H})$ satisfy the stability condition **(S)** if there exists a constant $c_S \geq 0$ such that*

$$H_{\Lambda|\overline{\xi}}(\overline{\omega}) \geq -c_S \left(N_{\Lambda}(\omega) + N_{\partial\Lambda(\overline{\xi})}(\xi) \right)$$

for all bounded $\Lambda \in \mathbf{R}^2$, $\overline{\omega} \in \overline{\Omega}_{\Lambda}$ and $\overline{\xi} \in \overline{\Omega}_{\mathrm{cr}}^{\Lambda}$.

To avoid the meaningless case when $\overline{\Phi} \equiv \infty$, we need a final condition to control the Hamiltonian from above. Let $M \in \mathbf{R}^{2\times 2}$ be an invertible 2×2 matrix, and define a partition of $\mathbf{R}^2$ as

$$\bigcup_{k \in \mathbf{Z}^2} \nabla(k)$$

where each

$$\nabla(k) := \{Mx \in \mathbf{R}^2 : x - k \in [-1/2, 1/2]^2\}. \tag{2.18}$$

is a rhombus. Furthermore, let Γ^s be the set of all configurations that consist of a single point with mark $s \in \Sigma$ whose position lies in some Borel set $D \subset \nabla(0)$ and define the set of all configurations whose restriction to a cell $\nabla(k)$, when shifted back to $\nabla(0)$, belongs to Γ^s as

$$\widehat{\Gamma}^s = \{\overline{\omega} \in \overline{\Omega} : \vartheta_{Mk}(\overline{\omega}_{\nabla(k)}) \in \Gamma^s \text{ for all } k \in \mathbf{Z}^2\}. \tag{2.19}$$

A marked configuration of $\overline{\omega} \in \widehat{\Gamma}^s$ is called pseudo-periodic. Note that when the mark of the particles is obvious, we drop the superscript and write $\widehat{\Gamma}$ instead of $\widehat{\Gamma}^s$. The required partial upper bound for the Hamiltonians is then achieved via the following property.

Definition 2.11. *Let $\overline{\Phi}^{+}$ be the positive part of $\overline{\Phi}$. We say the couple $(\overline{\Phi}, \mathcal{H})$ and the activity $z > 0$ satisfy the upper regularity condition **(U)** if M and D can be chosen so that the following hold:*

1. *Uniform confinement:* $\widehat{\Gamma}^s \subset \overline{\Omega}^\Lambda_{\mathrm{cr}}$ *for all bounded* $\Lambda \subset \mathbf{R}^2$*, and*

$$r_{\Gamma^s} := \sup_\Lambda \sup_{\overline{\omega}\in\widehat{\Gamma}^s} r_{\Lambda,\overline{\omega}} < \infty.$$

2. *Uniform summability:*

$$c_D^+ := \sup_{\overline{\omega}\in\widehat{\Gamma}^s} \sum_{\substack{\overline{\eta}\in\mathcal{H}(\overline{\omega}):\\ \rho(\overline{\eta})\cap\nabla(0)\neq\emptyset}} \frac{\overline{\Phi}^+(\overline{\eta},\overline{\omega})}{|\overline{\eta}|} < \infty. \tag{2.20}$$

3. *Strong non-rigidity:*

$$z|D| > e^{c_D}$$

where c_D *is defined as in (2.20), but with* $\overline{\Phi}$ *instead of* $\overline{\Phi}^+$*.*

If the ***(R)***, ***(S)*** and ***(U)*** conditions are satisfied, the existence of a Gibbs measure is given by the following theorem:

Theorem 2.12. *For every translation invariant marked hypergraph structure* $\mathcal{H} \subset \overline{\Omega}_f \times \overline{\Omega}$*, hyperedge potential* $\overline{\Phi}$ *and activity* $z > 0$ *satisfying the* ***(R)****,* ***(S)*** *and* ***(U)*** *conditions, there exists at least one translation invariant Gibbs measure* $\mathcal{P}$ *on* $(\overline{\Omega}, \overline{\mathcal{F}})$ *with reference measure* $\overline{\Pi}^z$*.*

The proof is an immediate adaptation of the proof of Theorem 3.2 and Corollary 3.4 in [DDG10] to the marked case following Remark 3.7 in [DDG10]. Further details of that adaptation can be found in [No13], Theorem 2.1. The proof relies on an entropy bound and a careful control of the interaction range. The entropy bound is used to obtain tightness of certain Gibbs distributions with pseudo-periodic boundary conditions.

2.5 Geometric Continuum Potts Model

We now go on to define Potts models on random structures in the continuum. The extension to the continuum was first introduced as the Widom-Rowlinson model, a continuum analogue of the Ising model, in [WR70]. Much later, [GH96] generalised this further and defined continuum Potts models. We present these ideas here, but generalise further to our hypergraph structure framework and to geometric interactions. To do this, we define a subclass of hyperedge potentials. Let $\mathcal{H}$ be a marked hypergraph structure. Then for

$(\overline{\eta}, \overline{\omega}) \in \mathcal{H}$ with $\overline{\eta} = (\eta, \sigma_\eta)$ and $\overline{\omega} = (\omega, \sigma_\omega)$, set

$$\overline{\Phi}(\overline{\eta}, \overline{\omega}) = \psi(\eta, \omega) + \varphi(\eta, \omega)(1 - \delta_{\sigma_\eta}(\eta)) \tag{2.21}$$

where

$$\delta_{\sigma_\eta}(\eta) = \begin{cases} 1 & \text{if } \sigma_\eta(x) = \sigma_\eta(y) \text{ for all pairs } \{x, y\} \in \eta, \\ 0 & \text{otherwise,} \end{cases} \tag{2.22}$$

and $\psi, \varphi : \Omega_f \times \Omega \to \mathbf{R} \cup \{\infty\}$. The marks of the particles contribute to $\Phi(\overline{\eta}, \overline{\omega})$ only through the δ_{σ_η} term. We call ψ the background interaction and φ the mark or type interaction because $(1 - \delta_{\sigma_\eta})$ is only non-zero when hyperedges contain particles with different marks. Let $q \geq 2$, then, given a finite box $\Lambda \in \mathcal{B}(\mathbf{R}^2)$, and an admissible boundary configuration $\overline{\xi} = (\xi, \sigma_\xi)$, the finite volume Gibbs distribution $Q_{\Lambda|\overline{\xi}}$ is given by

$$Q_{\Lambda|\overline{\xi}}(d\overline{\omega}) = \frac{1}{Z_{\Lambda|\overline{\xi}}} \exp\left[-H^{\overline{\Phi}}_{\Lambda|\overline{\xi}}(\overline{\omega})\right] \overline{\Pi}^{zq}_{\Lambda}(d\overline{\omega}) \tag{2.23}$$

where for $\overline{\omega} = (\omega, \sigma_\omega) \in \overline{\Omega}_\Lambda$,

$$H^{\overline{\Phi}}_{\Lambda|\overline{\xi}}(\overline{\omega}) = H^{\psi}_{\Lambda|\overline{\xi}}(\overline{\omega}) + H^{\varphi}_{\Lambda|\overline{\xi}}(\overline{\omega}),$$

$$H^{\psi}_{\Lambda|\overline{\xi}}(\overline{\omega}) = \sum_{\eta \in \mathcal{H}_\Lambda(\omega \cup \xi)} \psi(\eta, \omega \cup \xi),$$

and

$$H^{\varphi}_{\Lambda|\overline{\xi}}(\overline{\omega}) = \sum_{\substack{\overline{\eta} = (\eta, \sigma_\eta) \\ \in \mathcal{H}_\Lambda(\overline{\omega} \cup \overline{\xi})}} \varphi(\eta, \omega \cup \xi)(1 - \delta_{\sigma_\eta}(\eta)).$$

From (2.23), for fixed $\omega \in \Omega_\Lambda$, notice that the conditional distribution of σ_ω under the condition $\rho(\overline{\omega}) = \omega$ with respect to $Q_{\Lambda|\overline{\xi}}$ is just the discrete Potts model, given in (2.3), on ω with parameter J a function of φ. This is the justification given by [GH96] to call their model a continuum Potts model. As we are working in a hypergraph structure framework, we call the model defined by (2.23) the geometric continuum Potts model. A Gibbs measure on $(\overline{\Omega}, \overline{\mathcal{F}})$, if one exists for this model, is called a geometric continuum Potts measure with activity z and hyperedge potentials $\overline{\psi}$ and $\overline{\varphi}$ on a marked hypergraph structure $\mathcal{H}$.

We will use Theorem 2.12 to show the existence of geometric continuum Potts measures for a number of models with different geometric interactions. A phase transition is said to occur if more than one geometric continuum Potts measure exists. We will show this non-uniqueness for a number of different models. Our main tool is the geometric continuum random cluster model which is introduced in Section 2.6.

2.6 Geometric Continuum Random Cluster Model

Fix $k \in \mathbf{N}$. For the remainder of this section, let $\mathcal{H}$ be a marked hypergraph structure with $|\eta| = k$, for all $\eta \in \mathcal{H}(\omega)$. Our aim is to introduce a hyperedge process on the random marked hypergraph structure $\mathcal{H}$ via a joint construction with the related geometric continuum Potts model with parameters $\mathcal{H}$, ψ, φ and z as outlined in Section 2.5. This is akin to the joint construction of the discrete Potts model and the Fortuin-Kasteleyn representation of [FK72]. Like in Section 2.3, the idea behind the random cluster representation is to introduce the concept of open and closed hyperedges between the particle positions. This will enable us to set up percolation problems. Our outline of the continuum random cluster representation is adapted from [GH96]: we give the main results here for convenience and to show they fit within our hypergraph structure framework. Let $\Lambda \in \mathcal{B}(\mathbf{R}^2)$ and $\xi \in \Omega_{\Lambda^c}$. The construction consists of three main steps. First, we use a finite volume, Gibbs distribution $P^{zq}_{\Lambda|\xi}$ in Λ with boundary condition ξ, marked hypergraph structure $\mathcal{H}$, hyperedge potential ψ and activity $zq > 0$ to sample particles without marks. Recall, from (2.2), the definition of $\mathcal{H}(\omega)$ for a marked hypergraph structure and an unmarked configuration, $\omega \in \Omega_\Lambda$. Then,

$$P^{zq}_{\Lambda|\xi}(d\omega) = \frac{1}{Z_{\Lambda|\xi}} \exp\left[- \sum_{\eta \in \mathcal{H}_\Lambda(\omega \cup \xi)} \psi(\eta, \omega \cup \xi) \right] \Pi^{zq}_\Lambda(d\omega), \tag{2.24}$$

where

$$Z_{\Lambda|\xi} := \int_{\Omega_\Lambda} \exp\left[- \sum_{\eta \in \mathcal{H}_\Lambda(\omega \cup \xi)} \psi(\eta, \omega \cup \xi) \right] \Pi^{zq}_\Lambda(d\omega). \tag{2.25}$$

This gives us the distribution of particle positions in a box Λ given a configuration of particle positions ξ, outside Λ. Note that $\xi \in \Omega_{\Lambda^c}$ must be chosen such that

$$\sum_{\eta \in \mathcal{H}_\Lambda(\omega \cup \xi)} \psi_-(\eta, \omega \cup \xi) < \infty,$$

for Π^{zq}_Λ – almost all $\omega \in \Omega_\Lambda$, where ψ_- is the negative part of ψ, and also, such that $0 < Z_{\Lambda|\xi} < \infty$. Such a boundary configuration is admissible in the sense of Definition 2.6, although, in the unmarked regime, and with ψ in place of $\overline{\Phi}$. We will view $P^{zq}_{\Lambda|\xi}$ as a probability measure on Ω supported on the set $\Omega_{\Lambda|\xi} = \{\omega \in \Omega : \omega \cap \Lambda^c = \xi\}$. Secondly, given a fixed set of positions, $\omega \in \Omega_{\Lambda|\xi}$, we mark each particle in ω independently with a mark in $\Sigma = \{1, \ldots, q\}$. This gives us a marked configuration $\overline{\omega} \in \overline{\Omega}$. We denote by $\lambda_{\omega,\Lambda}$ the distribution of the random mark vector $\sigma_\omega \in \Sigma^\omega$, where

$$\sigma_\omega = (\sigma_\omega(x) : x \in \omega).$$

The random variables $(\sigma_\omega(x))_{x\in\omega_\Lambda}$ are independent and uniformly distributed on $\Sigma = \{1, \ldots, q\}$, whereas $\sigma_\omega(x) = 1$ for $x \in \omega_{\Lambda^c} = \xi$.

Finally, we introduce the hyperedge process by declaring each $\eta \in \mathcal{H}(\omega)$ open or closed. Let $\mathcal{E}$ be the space of locally finite hyperedge configurations. More precisely, denote the space of all hyperedge configurations in the plane:

$$E_{\mathbf{R}^2} := \left\{\eta = \{x_1, \ldots x_k\} \subset \mathbf{R}^2 : x_1 \neq \ldots \neq x_k\right\}$$

and let

$$\mathcal{E} := \{E \subset E_{\mathbf{R}^2} : E \text{ is locally finite}\}\,. \tag{2.26}$$

Then, given $\omega \in \Omega_{\Lambda|\xi}$, let $\mu_{\omega,\Lambda}$ denote the distribution of the random hyperedge configuration

$$\{\eta \in \mathcal{H}(\omega) : v(\eta) = 1\} \in \mathcal{E} \tag{2.27}$$

where $(v(\eta))_{\eta\in\mathcal{H}(\omega)}$ are independent Bernoulli random variables with probability

$$Prob(v(\eta) = 1) = p_\Lambda(\eta) := \begin{cases} 1 - e^{-\varphi(\eta,\omega)} & \text{if } \eta \in \mathcal{H}_\Lambda(\omega), \\ 1 & \text{otherwise .} \end{cases} \tag{2.28}$$

Note that $\mu_{\omega,\Lambda}$ is nothing more than a point process on the space of all hyperedge configurations in the plane, and can be seen as a p_Λ thinning of the complete hyperedge set $\mathcal{H}(\omega)$. We call a hyperedge η open if $v(\eta) = 1$ and closed if $v(\eta) = 0$. Ensuring all hyperedges outside of Λ are open with probability one, as in (2.28), is called the wired boundary condition. The hyperedge probability (2.28) is the only place that φ enters the construction of the geometric continuum random cluster measure.

Combining the above, we have the probability measure $\mathbf{P}^{zq}_{\Lambda|\xi}$ on $\overline{\Omega} \times \mathcal{E}$ which is given by

$$\mathbf{P}^{zq}_{\Lambda|\xi} := \int P^{zq}_{\Lambda|\xi}(d\omega)\lambda_{\omega,\Lambda} \otimes \mu_{\omega,\Lambda}. \tag{2.29}$$

We address measurability in the following Lemma, and show that $\omega \to \lambda_{\omega,\Lambda}$ and $\omega \to \mu_{\omega,\Lambda}$ are probability kernels.

Lemma 2.13. *The maps $\omega \to \lambda_{\omega,\Lambda}$ and $\omega \to \mu_{\omega,\Lambda}$ from Ω to $\overline{\Omega}$, respectively $\mathcal{E}$, are probability kernels.*

Proof. Due to the fundamental properties of randomisation of point processes – see Lemma 2.4(a) of [GH96] – $\lambda_{\omega,\Lambda}$ depends measurably on ω, and hence, the first result follows. For $\mu_{\omega,\Lambda}$, let $\mathcal{L}_{\omega,\Lambda}$ denote the Laplace transform of $\mu_{\omega,\Lambda}$. That is, for $f : E_{\mathbf{R}^2} \to [0,\infty[$ measurable, we have

$$\begin{aligned}\mathcal{L}_{\omega,\Lambda}(f) &= \int \exp\Big[-\sum_{\eta\in E} f(\eta)\Big]\mu_{\omega,\Lambda}(dE)\\ &= \prod_{\eta\in\mathcal{H}(\omega)} \Big[p_\Lambda(\eta)e^{-f(\eta)} + 1 - p_\Lambda(\eta)\Big]\\ &= \exp\Big[-\sum_{\eta\in\mathcal{H}(\omega)} \tilde{f}(\eta)\Big],\end{aligned}$$

where

$$\tilde{f}(\eta) = \log\Big[e^{-f(\eta)} + \mathbf{1}_{\{\eta\in\mathcal{H}_\Lambda(\omega)\}}e^{-\varphi(\eta,\omega)}(1-e^{-f(\eta)})\Big].$$

Therefore, since $\tilde{f}$ is measurable, the function $\omega \to \mathcal{L}_{\omega,\Lambda}(f)$ is also measurable. This implies that $\omega \to \mu_{\omega,\Lambda}$ depends measurably on ω. $\square$

Let $A \subset \overline{\Omega} \times \mathcal{E}$ be the event that no two vertices with different types belong to the same open hyperedge. Formally, we have

$$A := \left\{ (\overline{\omega}, E) \in \overline{\Omega} \times \mathcal{E} : \sum_{\eta\in E}(1 - \delta_{\sigma_\omega}(\eta)) = 0 \right\}. \tag{2.30}$$

Let σ^1_ξ denote the event that all points of ξ are assigned a mark of 1. Then, from (2.29), we see that

$$\mathbf{P}^{zq}_{\Lambda|\xi}(A) \geq P^{zq}_{\Lambda|\xi}(\{\xi\})\lambda_{\omega,\Lambda}(\sigma^1_\xi)\mu_{\omega,\Lambda}(\mathcal{H}(\xi)) = P^{zq}_{\Lambda|\xi}(\{\xi\}) = Z^{-1}_{\Lambda|\xi}e^{-zq|\Lambda|} > 0,$$

hence, we can define the following conditional measure, which we call the random-cluster representation measure:

$$\mathbf{P}_A = \mathbf{P}^{zq}_{\Lambda|\xi}(\ \cdot \ |A).$$

If we ignore the open hyperedges of the random-cluster representation, i.e. we only consider $\overline{\omega}$ which gives us the particle positions and their types, then we obtain the continuum Potts model with geometric interactions.

Proposition 2.14. *Let pr denote the projection from $\overline{\Omega} \times \mathcal{E}$ onto $\overline{\Omega}$. Then $\mathbf{P}_A \circ pr^{-1} = Q_{\Lambda|\overline{\xi}}$*

Proof. For $\overline{\omega} \in \overline{\Omega}_{\Lambda|\overline{\xi}}$, let $A_{\overline{\omega}}$ be the $\overline{\omega}-$section of A in (2.30). That is

$$A_{\overline{\omega}} := \{E \in \mathcal{E} : (\overline{\omega}, E) \in A\}.$$

Then, by equations (2.23) and (2.30), it follows that for $\omega = \rho(\overline{\omega})$,

$$\begin{aligned}
\mu_{\omega,\Lambda}(A_{\overline{\omega}}) &= \prod_{\eta \in \mathcal{H}(\omega)} \Big(1 - p_\Lambda(\eta)\big(1 - \delta_{\sigma_\omega}(\eta)\big)\Big) \\
&= \exp\Big[-H^{\varphi}_{\Lambda|\overline{\xi}}(\overline{\omega})\Big]
\end{aligned}$$

Hence, for any bounded measurable function f on $\overline{\Omega}$, we have

$$\begin{aligned}
\int f \circ \mathrm{pr} \, d\mathbf{P}_A &= c_1 \int_A f \circ \mathrm{pr} \, d\mathbf{P}^{zq}_{\Lambda|\xi} \\
&= c_1 \int P^{zq}_{\Lambda|\xi}(d\omega) \int \lambda_{\omega,\Lambda}(d\sigma_\omega) f(\overline{\omega}) \mu_{\omega,\Lambda}(A_{\overline{\omega}}) \\
&= c_2 \int \Pi^{zq}_{\Lambda}(d\omega) \int \lambda_{\omega,\Lambda}(d\sigma_\omega) f(\overline{\omega} \cup \overline{\xi}) \exp\Big[-H^{\psi}_{\Lambda|\xi}(\overline{\omega}) - H^{\varphi}_{\Lambda|\xi}(\overline{\omega})\Big] \\
&= c_2 \int \overline{\Pi}^{zq}_{\Lambda}(d\overline{\omega}) f(\overline{\omega} \cup \overline{\xi}) \exp\Big[-H^{\psi}_{\Lambda|\xi}(\overline{\omega}) - H^{\varphi}_{\Lambda|\xi}(\overline{\omega})\Big] \qquad (2.31) \\
&= c_3 \int f dQ_{\Lambda|\overline{\xi}},
\end{aligned}$$

where c_1, c_2, c_3 are relevant constants. Equation (2.31) is down to the fact that $\int \Pi^z_\Lambda(d\omega)\lambda_{\omega,\Lambda}$ is precisely the Poisson point process $\overline{\Pi}^z$ on $\overline{\Omega}$ with intensity measure $z\nu \otimes \lambda_{\omega,\Lambda}$. Realising that $\mathbf{P}_A$ and $Q_{\Lambda|\overline{\xi}}$ are both probability measures, we have $c_3 = 1$. The Proposition follows. □

On the other hand, if we ignore the type or mark of the particles, and only consider the particle positions together with the configuration of open hyperedges, then we will show in Proposition 2.15 that we are left with the continuum random cluster distribution with geometric interactions, which is defined by

$$C_{\Lambda|\xi}(d\omega, dE) := \frac{1}{Z_{2,\Lambda|\xi}} q^{K(\omega,E)} P^z_{\Lambda|\xi}(d\omega)\mu_{\omega,\Lambda}(dE) \tag{2.32}$$

where $K(\omega, E)$, with $E \subset \mathcal{H}(\omega)$, is the number of connected components of the hypergraph (ω, E), including the single connected component that intersects Λ^c, if $\xi \neq \emptyset$. Note that $P^z_{\Lambda|\xi}$ is the same as in (2.24) but with activity z rather than zq. The normalisation constant $Z_{2,\Lambda|\xi}$ is given by

$$Z_{2,\Lambda|\xi} := \int_{\Omega_\Lambda} \int_{\mathcal{E}} q^{K(\omega,E)} \mu_{\omega,\Lambda}(dE) P^z_{\Lambda|\xi}(d\omega), \tag{2.33}$$

where the subscript 2 is just to differentiate from the normalisation constant in (2.25). Notice that the continuum random cluster distribution is completely mark independent and is defined by using the wired boundary condition.

Proposition 2.15. *Let $\rho : \overline{\Omega} \times \mathcal{E} \to \Omega \times \mathcal{E}$ be defined by $\rho(\overline{\omega}, E) = (\omega, E)$. Then*

$$\mathbf{P}_A \circ \rho^{-1} = C_{\Lambda|\xi}.$$

Proof. For $(\omega, E) \in \Omega \times \mathcal{E}$ with $\omega \in \Omega_{\Lambda|\xi}$ and $\mathcal{H}(\omega) \backslash \mathcal{H}_\Lambda(\omega) \subseteq E \subseteq \mathcal{H}(\omega)$, let $A_{(\omega,E)}$ be the set of mark vectors $\sigma_\omega \in \Sigma^\omega$ such that marks are constant across connected components of (ω, E), i.e.

$$A_{(\omega,E)} := \{\sigma_\omega \in \Sigma^\omega : (\overline{\omega}, E) \in A \text{ for } \overline{\omega} = (\omega, \sigma_\omega)\}. \tag{2.34}$$

Under $\lambda_{\omega,\Lambda}$ there are exactly $q^{|\omega\cap\Lambda|}$ distinct mark vectors $\sigma_\omega \in \Sigma^\omega$, each of which have equal probability. For $\sigma_\omega \in A_{(\omega,E)}$, we require $\sigma_\omega(x) = \sigma_\omega(y)$ for all x and y in the same connected component of (ω, E). In particular, $\sigma_\omega(x) = 1$ for x in the unique connected component intersecting with Λ^c. This gives $q^{K_\Lambda(\omega,E)}$ distinct possible markings. Therefore,

$$\lambda_{\omega,\Lambda}(A_{(\omega,E)}) = \frac{q^{K_\Lambda(\omega,E)}}{q^{|\omega\cap\Lambda|}}, \tag{2.35}$$

where $K_\Lambda(\omega, E)$ is the number of connected components of the hypergraph (ω, E) that lie wholly inside Λ. Let $\mathcal{E}_\Lambda := \{E \in \mathcal{E} : \eta \cap \Lambda \neq \emptyset \text{ for all } \eta \in E\}$. Then, for every measurable function f on $\Omega_\Lambda \times \mathcal{E}_\Lambda$,

$$\begin{aligned}\int f \circ \rho^{-1} d\mathbf{P}_A &= c_1 \int_A f \circ \rho^{-1} \mathbf{P}^z_{\Lambda|\xi} \\ &= c_1 \int P^{zq}_{\Lambda|\xi}(d\omega) \int \mu_{\omega,\Lambda}(dE) f(\omega, E) \lambda_{\omega,\Lambda}(A_{(\omega,E)}) \\ &= c_2 \int P^{z}_{\Lambda|\xi}(d\omega) \int \mu_{\omega,\Lambda}(dE) f(\omega, E) q^{K_\Lambda(\omega,E)} \end{aligned} \tag{2.36}$$

with suitable constants $c_1, c_2 > 0$. Equation (2.36) is due to the fact that $P^{zq}_{\Lambda|\xi}$ is absolutely continuous with respect to $P^{z}_{\Lambda|\xi}$, with density proportional to $\omega \to q^{|\omega \cap \Lambda|}$. In the case when $\xi = \emptyset$, the result follows because $K(\omega, E) = K_\Lambda(\omega, E)$. In the case when $\xi \neq \emptyset$, $K(\omega, E) - K_\Lambda(\omega, E) = 1$, so the result follows by letting $c_3 = c_2 q^{-1}$. ☐

For any boxes $\Delta, \Lambda \in \mathcal{B}(\mathbf{R}^2)$, with $\Delta \subset \Lambda$ and any marked particle configuration $\overline{\omega} = (\omega, \sigma_\omega) \in \overline{\Omega}$, let

$$N_{\Delta,s}(\overline{\omega}) := |\{x \in \omega_\Delta : \sigma_\omega(x) = s\}|, \text{ for } s = 1, \dots, q,$$

be the random variable for the number of particles in $\Delta \times \Sigma$ with mark s. Also, let

$$\begin{aligned} N_{\Delta \leftrightarrow \Lambda^c}(\omega, E) := |\{x \in \omega_\Delta : x \text{ belongs to a connected component of} \\ (\omega, E) \text{ that intersects } \Lambda^c\}|. \end{aligned}$$

Lemma 2.16. *For all $\Delta, \Lambda \in \mathcal{B}(\mathbf{R}^2)$, with $\Delta \subset \Lambda$, the functions $K_\Lambda : \Omega \times \mathcal{E} \to \mathbf{N}$ and $N_{\Delta \leftrightarrow \Lambda^c} : \Omega \times \mathcal{E} \to \mathbf{N}$ are measurable.*

Proof. Let

$$B := \{(x, y, \omega, E) \in \mathbf{R}^2 \times \mathbf{R}^2 \times \Omega \times \mathcal{E} : x, y \in \omega, x \neq y \text{ and } x \leftrightarrow y\}$$

where $x \leftrightarrow y$ denotes that x and y are connected in the graph $(\omega, E \cap \mathcal{H}(\omega))$. Also, let

$$\begin{aligned} B_1 := \{(x, y, \omega, E) \in \mathbf{R}^2 \times \mathbf{R}^2 \times \Omega \times \mathcal{E} : \\ x, y \in \omega \text{ and } \exists \eta \in E : \{x, y\} \subseteq \eta\} \end{aligned}$$

and, for $n \geq 1$

$$B_{n+1} := \{(x, y, \omega, E) \in \mathbf{R}^2 \times \mathbf{R}^2 \times \Omega \times \mathcal{E} : \sum_{v \in \omega} \mathbf{1}_{B_1}(x, v, \omega, E) \mathbf{1}_{B_n}(v, y, \omega, E) > 0\}.$$

Therefore,

$$B = \bigcup_{n \geq 1} B_n.$$

Since B_n is measurable for any n, so is B. Now, for $k \geq 1$, $K_\Lambda(\omega, E) \geq k$ if and only if there exists k distinct points $x_1 \neq \cdots \neq x_k$ in ω_Λ such that, for each pair x_i, x_j, with $1 \leq i < j \leq k$, $(x_i, x_j, \omega, E) \in B^c$. More precisely, $K_\Lambda(\omega, E) \geq k$ if and only if

$$\sum_{x_1, \ldots, x_k \in \omega_\Lambda} \prod_{1 \leq i < j \leq k} \mathbf{1}_{\{x_i \neq x_j\}} \mathbf{1}_{B^c}(x_i, x_j, \omega, E) > 0,$$

which gives us the measurability of K_Λ. By noticing that

$$N_{\Delta \leftrightarrow \Lambda^c}(\omega, E) = \sum_{x \in \omega_\Delta} g(x, \omega, E)$$

with $g(x, \omega, E) = 1$ if $\sum_{y \in \omega_{\Lambda^c}} \mathbf{1}_B(x, y, \omega, E) > 0$ and $g(x, \omega, E) = 0$ if not, we also see that $N_{\Delta \leftrightarrow \Lambda^c}$ is measurable. $\square$

The final proposition of this section states a relationship between $N_{\Delta,1}(\overline{\omega})$ in the geometric continuum Potts model and the percolation property of the geometric continuum random cluster model. It is this proposition that forms an integral part of the proof of phase transitions in continuum Potts models with various geometric interactions. Notice that our choice of wired boundary condition with all particles with mark 1 was arbitrary. The same applies to each of the marks $s \in \Sigma$ and each random variable $N_{\Delta,s}$.

Proposition 2.17. *For all $\Delta, \Lambda \in \mathcal{B}(\mathbf{R}^2)$, with $\Delta \subset \Lambda$,*

$$\int (qN_{\Delta,1} - N_\Delta) dQ_{\Lambda|\overline{\xi}} = (q-1) \int N_{\Delta \leftrightarrow \Lambda^c} dC_{\Lambda|\xi}. \tag{2.37}$$

Proof. By Proposition 2.14 and (2.34), the left hand side of (2.37) is equal to

$$\mathbf{P}^{zq}_{\Lambda|\xi}(A)^{-1} \int P^{zq}_{\Lambda|\xi}(d\omega) \int \mu_{\omega,\Lambda}(dE) \sum_{x \in \omega_\Delta} \int_{A_{(\omega,E)}} \lambda_{\omega,\Lambda}(d\sigma_\omega)(q\mathbf{1}_{\{\sigma_\omega(x)=1\}} - 1).$$

Now suppose that x is connected to Λ^c in the hypergraph (ω, E), and hence $\sigma_\omega(x) = 1$ for all $\sigma_\omega \in A_{(\omega,E)}$. It follows that

$$\int_{A_{(\omega,E)}} \lambda_{\omega,\Lambda}(d\sigma_\omega)(q\mathbf{1}_{\{\sigma_\omega(x)=1\}} - 1) = (q-1)\lambda_{\omega,\Lambda}(A_{(\omega,E)}). \tag{2.38}$$

On the other hand, if x is not connected to Λ^c in the hypergraph (ω, E), then the mark $\sigma_\omega(x)$ is independent of $A_{(\omega,E)}$, under the measure $\lambda_{\omega,\Lambda}$, and takes values in $\{1, \ldots, q\}$

with equal probability. This implies that

$$\int_{A_{(\omega,E)}} \lambda_{\omega,\Lambda}(d\sigma_\omega)(q\mathbf{1}_{\{\sigma_\omega(x)=1\}} - 1) = 0. \tag{2.39}$$

Therefore, by (2.38), (2.39) and Proposition 2.15, the result follows. □

2.7 Percolation in the Geometric Continuum Random Cluster Model

In the next chapters, we investigate several different geometric hyperedge potentials for background and type interactions. For each of these models, we will establish the existence of percolation in the geometric continuum random cluster model $C_{\Lambda|\xi}$ when $q \geq 1$ is an arbitrary parameter, z is sufficiently large and for other appropriately chosen parameters, for the pseudo-periodic boundary condition $\overline{\xi} \in \widehat{\Gamma}_{\Lambda^c}$ where $\widehat{\Gamma}$ was defined in (2.19) and M and Γ are chosen such that **(U)** is satisfied. More precisely, let

$$\Lambda_n := \bigcup_{k\in\{-n,\ldots,n\}^2} \nabla(k).$$

for large $n \in \mathbf{N}$, where $\nabla(k)$ is given in (2.18). We show that for any $\Delta \in \mathcal{B}(\mathbf{R}^2)$, there exists $c > 0$, such that for all $\Lambda_n \supset \Delta$, and for all $\overline{\xi} = (\xi, \sigma_\xi) \in \widehat{\Gamma}_{\Lambda^c}$

$$\int C_{\Lambda_n|\xi}(d\omega, dE) N_{\Delta\leftrightarrow\Lambda_n^c}(\omega, E) \geq c. \tag{2.40}$$

This is the main result upon which the proof of the phase transition in all our different models is based on, and is analogous to the statement of Proposition 3.1 in [GH96], but adapted to the hypergraph structure. The proof of (2.40) is deferred until later, but supposing it is satisfied, we can relate it to the notion of a phase transition in the geometric continuum Potts model. We give an outline of the main arguments here.

Assuming that the couple $(\Phi, \mathcal{H})$ satisfy **(R)**, **(S)** and **(U)**, a Gibbs measure is constructed as a limit of Gibbs distributions in boxes Λ_n. Let $\overline{\xi} = (\xi, \sigma_\xi) \in \widehat{\Gamma}_{\Lambda_n^c}$ such that $\sigma_\xi(x) = 1$ for all $x \in \xi$ be a pseudo-periodic boundary condition with mark 1. The upper regularity **(U)** and stability **(S)** conditions then show that $\overline{\xi}$ is admissible for Λ_n and z. Therefore,

$$Q_n := Q_{\Lambda_n|\overline{\xi}} \circ \mathrm{pr}_{\Lambda_n}^{-1}$$

is a Gibbs distribution in Λ_n with boundary condition $\overline{\xi}$ and activity zq, projected to Λ_n. Recall the matrix $M \in \mathbf{R}^{2\times 2}$ from (2.18) and let P_n be the probability measure on $(\overline{\Omega}, \overline{\mathcal{F}})$ such that the marked configurations in the disjoint boxes $\Lambda_n + (2n+1)Mk$, $k \in \mathbf{Z}^2$, are independent with respect to P_n and have distribution Q_n. Rather than obtaining a full asymptotic translation invariance now, we first confine ourselves to the skewed lattice translations, in recognition of our cell structure. So we define the spatial average of P_n,

$$\widehat{P}_n = \frac{1}{|\Lambda_n|} \sum_{i \in L(n)} P_n \circ \vartheta_i^{-1},$$

where $L(n) = \Lambda_n \cap M\mathbf{Z}^2$. To obtain a limit for the sequence $(\widehat{P}_n)_{n\geq 1}$, a suitable topology must be specified. A measurable function $f : \overline{\Omega} \to \mathbf{R}$ is called local and tame if

$$f(\overline{\omega}) = f(\overline{\omega}_\Lambda) \quad \text{and} \quad |f(\overline{\omega})| \leq aN_\Lambda(\omega) + b$$

for all $\overline{\omega} = (\omega, \sigma_\omega) \in \overline{\Omega}$, some $\Lambda \in \mathcal{B}(\mathbf{R}^2)$ and suitable constants $a, b \geq 0$, where $N_\Lambda(\omega)$ is the counting variable defined in (2.1). The set of all local and tame functions is denoted by $\mathcal{T}$. The $\mathcal{T}$–topology, on the set of all translation invariant probability measures on $(\overline{\Omega}, \overline{\mathcal{F}})$, is defined as the smallest topology for which the mappings $P \mapsto \int f dP$ are continuous, where $f \in \mathcal{T}$. It can be shown that there exists a subsequence of $(\widehat{P}_n)_{n\geq 1}$ which converges in the $\mathcal{T}$–topology. The limit of this subsequence $\widehat{P}$, once found, cannot be shown to be concentrated on admissible configurations, see [DDG10], and so it is not known to be Gibbs. However, since $\widehat{P}$ is non-degenerate, it follows by the Range condition **(R)** and Remark 2.9 that

$$P := \widehat{P}(\cdot | \{\emptyset\}^c)$$

is a Gibbs measure and hence, a Delaunay continuum Potts measure for $\mathcal{H}, \overline{\Phi}$ and z, and invariant under the skewed lattice translations. To obtain full translation invariance under $(\vartheta_x)_{x\in\mathbf{R}^2}$, the spatial average of P,

$$P^{(1)} := \frac{1}{|\nabla(0)|} \int_{\nabla(0)} P \circ \vartheta_x^{-1} dx,$$

is taken. The superscript identifies the choice for the marks of the particles in the boundary when constructing $\widehat{P}_n$. An application of (2.40) and Proposition 2.37 shows that

$$\int (qN_{\Delta,1} - N_\Delta) d\widehat{P}_n = (q-1) \int N_{\Delta \leftrightarrow \Lambda^c} dC_{\Lambda_n|\xi} \tag{2.41}$$

$$\geq (q-1)c. \tag{2.42}$$

Since $N_{\Delta,1}$ and N_Δ are in $\mathcal{T}$, the same inequality holds after replacing $\widehat{P}_n$ by $P^{(1)}$. Since

$P^{(1)}$ is symmetric under changes to the marks $2, \dots, q$, this means that

$$\int (qN_{\Delta,1} - N_\Delta) dP^{(1)} > \int (qN_{\Delta,2} - N_\Delta) dP^{(1)} = \cdots = \int (qN_{\Delta,q} - N_\Delta) dP^{(1)}.$$

For $t \in \{2, \dots, q\}$, let $P^{(t)}$ be the translation invariant, Delaunay continuum Potts measure which is obtained from $P^{(1)}$ by switching the marks 1 and t. It is then evident, that there exists q distinct translation invariant, Delaunay continuum Potts measures – a phase transition.

The only missing step so far, is the proof of (2.40), i.e. the existence of percolation in the Delaunay continuum random cluster models. This depends heavily on the specific model in question, and is the dedicated work of this thesis – it forms the bulk of Chapters 3 and 4. Although we use different techniques for different classes of model, we present here a theme that runs through each: a relationship involving the existence of percolation for geometric continuum random cluster distributions $C_{\Lambda|\xi}$ and the existence of percolation in certain site percolation models. Let $M_{\Lambda|\xi}$ be the distribution of particle positions given by the marginal distribution $C_{\Lambda|\xi}(\cdot, \mathcal{E})$. We can then write $C_{\Lambda|\xi}$ as follows:

$$C_{\Lambda|\xi}(d\omega, dE) = M_{\Lambda|\xi}(d\omega)\mu^{(q)}_{\omega,\Lambda}(dE), \tag{2.43}$$

with

$$\mu^{(q)}_{\omega,\Lambda}(dE) = \frac{q^{K(\omega,E)}\mu_{\omega,\Lambda}(dE)}{\int q^{K(\omega,E)}\mu_{\omega,\Lambda}(dE)}. \tag{2.44}$$

Let $\tilde{\mu}_\omega$ be an alternative distribution of open hyperedge configurations where each hyperedge $\eta \in \mathcal{H}(\omega)$ is declared open with probability

$$p_\Lambda(\eta) = p\mathbf{1}_{\mathcal{H}^*(\omega)}(\eta), \tag{2.45}$$

and closed otherwise, where $\mathcal{H}^*(\omega)$ is some subset of $\mathcal{H}(\omega)$ and $p \in [0,1]$. Instead of declaring hyperedges as open or closed, we may wish, instead, to declare the mark of each particle. This leads us to the definition of a continuum site percolation model, $\tilde{C}^{\text{site}}_{\Lambda|\xi}$, with the same distribution of particle positions as our geometric continuum random cluster model. We write

$$\tilde{C}^{\text{site}}_{\Lambda|\xi}(d\overline{\omega}) = M_{\Lambda|\xi}(d\omega)\tilde{\lambda}_\omega(d\overline{\omega}), \tag{2.46}$$

where $\tilde{\lambda}_\omega$ denotes the distribution of the random vector $\sigma_\omega = (\sigma_\omega(x) : x \in \omega)$ with elements in Σ, where $(\sigma_\omega(x))_{x\in\omega}$ are independent random variables with probability

$$Prob(\sigma_\omega(x) = 1) = p\mathbf{1}_{\omega^*}(x). \tag{2.47}$$

Using the same arguments as in Lemma 2.13, we see that the map $\omega \to \tilde{\lambda}_\omega$ is a probability kernel from Ω to $\overline{\Omega}$. Note that the value of p is the same as in (2.45) and $\omega^* \subset \omega$ is the set of points of ω that build the hyperedges in $\mathcal{H}^*(\omega)$. Finally, define the event that there is a path that intersects both Δ and Λ^c and is made up of points of mark 1 connected by edges in the reduced hypergraph $\mathcal{H}^*(\omega)$. That is:

$$\begin{aligned}\{\Delta \leftrightarrow \Lambda^c\} := \Big\{\overline{\omega} \in \Omega \times \{1\} : \exists x_1, \dots, x_n \in \omega \text{ with } x_1 \in \Delta, x_n \in \Lambda^c \\ \text{and for } i = 1, \dots, n-1,\ \exists \eta \in \mathcal{H}^*(\omega) : \{x_i, x_{i+1}\} \subseteq \eta\Big\}.\end{aligned} \tag{2.48}$$

Proposition 2.18. *Let $\tilde{\mu}_\omega$ and $\tilde{\lambda}_\omega$ be as in (2.45) and (2.47). Then, if $\mu^{(q)}_{\omega,\Lambda} \succcurlyeq \tilde{\mu}_\omega$, we have for all $\Delta, \Lambda \in \mathcal{B}(\mathbf{R}^2)$, with $\Delta \subset \Lambda$*

$$\int N_{\Delta\leftrightarrow\Lambda^c} dC_{\Lambda|\xi} \geq \tilde{C}^{\text{site}}_{\Lambda|\xi}(\{\Delta \leftrightarrow \Lambda^c\}).$$

Proof.

$$\begin{aligned}\int C_{\Lambda|\xi}(d\omega, dE) N_{\Delta\leftrightarrow\Lambda^c}(\omega, E) &= \int M_{\Lambda|\xi}(d\omega) \int \mu^{(q)}_{\omega,\Lambda}(dE) N_{\Delta\leftrightarrow\Lambda^c}(\omega, E) \\ &\geq \int M_{\Lambda|\xi}(d\omega) \int \tilde{\mu}_\omega(dE) N_{\Delta\leftrightarrow\Lambda^c}(\omega, E) \\ &\geq \int M_{\Lambda|\xi}(d\omega) \int \tilde{\mu}_\omega(dE) \mathbf{1}_{\{N_{\Delta\leftrightarrow\Lambda^c} \geq 1\}}(\omega, E) \\ &\geq \int M_{\Lambda|\xi}(d\omega) \int \tilde{\lambda}_\omega(d\sigma_\omega) \mathbf{1}_{\{\Delta\leftrightarrow\Lambda^c\}}(\omega, \sigma_\omega) \\ &= \tilde{C}^{\text{site}}_{\Lambda|\xi}(\{\Delta \leftrightarrow \Lambda^c\}),\end{aligned}$$

where the first inequality is a direct application of the assumption in the statement of the Proposition (since $N_{\Delta\leftrightarrow\Lambda^c}$ is an increasing event), and the third inequality is due to the fact that site percolation implies hyperedge percolation. □

We have shown that to prove the existence of percolation in a geometric continuum random cluster model, it is sufficient to show percolation in a suitable continuum site model. This is essentially the work of the next two chapters as we investigate several different geometric hyperedge potentials for background and type interactions.

2.8 Voronoi tessellations and Delaunay triangulations

Having outlined general hypergraph structures and marked hypergraph structures in Section 2.2, and continued in this generic way up until now, we turn towards our main object of study: the Delaunay hypergraph structures. It is these so called 'nearest neighbour' hypergraph structures that underlie the models we present in Chapters 3 and 4, so we introduce them now. One way to define a nearest neighbour hypergraph structure is to look at Voronoi tessellations. A Voronoi tessellation is a decomposition of a metric space (in our case $\mathbf{R}^2$), into a discrete set of objects which we attribute the label Voronoi cells. Given $\omega \in \Omega$, each point $x \in \omega$ lies inside its own Voronoi cell $\mathrm{Vor}_\omega(x)$: the set of all points in $\mathbf{R}^2$ that are closer to x than any other point $y \in \omega$. That is

$$\mathrm{Vor}_\omega(x) := \left\{ z \in \mathbf{R}^2 : |x - z| \leq |x' - z| \text{ for all } x' \in \omega \right\}. \tag{2.49}$$

Often easier to work with than Voronoi tessellations, are Delaunay triangulations. The Delaunay triangulation of a given spatial configuration $\omega \in \Omega$ corresponds to the dual graph of the Voronoi tessellation. It is straightforward to construct the Delaunay triangulation from the Voronoi tessellation – just join, with an edge, any two points of ω whose Voronoi cells share a 1-dimensional face. The set Del of Delaunay hyperedges is given by

$$\begin{aligned} \mathrm{Del} := \Big\{ & (\overline{\eta}, \overline{\omega}) \in \overline{\Omega}_f \times \overline{\Omega} : \overline{\eta} \subset \overline{\omega} \text{ and } \exists \text{ an open ball } B(\overline{\eta}, \overline{\omega}) \subset \mathbf{R}^2 \text{ with} \\ & \overline{\omega} \cap (\partial B(\overline{\eta}, \overline{\omega}) \times \Sigma) = \overline{\eta} \text{ and } \overline{\omega} \cap (B(\overline{\eta}, \overline{\omega}) \times \Sigma) = \emptyset \Big\}. \end{aligned} \tag{2.50}$$

Clearly Del is a marked hypergraph structure, indeed, $\mathrm{Del}(\overline{\omega})$ is the set of hyperedges on $\overline{\omega}$. These include all singletons, edges and triangles of the Delaunay triangulation. Other, less general marked Delaunay hypergraph structures include the subsets

$$\mathrm{Del}_k = \{(\overline{\eta}, \overline{\omega}) : \overline{\eta} \in \mathrm{Del}(\overline{\omega}), |\rho(\overline{\eta})| = k\}, \text{ for } k \in \{1, 2, 3\},$$

which we call the singletons, edges and triangles of Del respectively. For $\overline{\omega} = (\omega, \sigma_\omega) \in \overline{\Omega}$, the set of marked hyperedges of $\overline{\omega}$ is

$$\mathrm{Del}_k(\overline{\omega}) = \{\overline{\eta} : (\overline{\eta}, \overline{\omega}) \in \mathrm{Del}_k\}.$$

We will also make use of $\mathrm{Del}(\omega)$ and $\mathrm{Del}_k(\omega)$: the sets of unmarked hyperedges of ω, as defined in (2.2). It is possible that for some $\overline{\eta} \in \mathrm{Del}(\overline{\omega})$, $|\overline{\eta}| > 3$ i.e. $\overline{\eta}$ is neither a singleton, edge, nor triangle. This happens when $\overline{\eta}$ consists of four or more points whose positions lie on a circle in $\mathbf{R}^2$ with no points of $\rho(\overline{\omega})$ inside. In fact, for this not to happen, we must consider configurations in *general position* as in [Mø94]. More precisely, this means that no four points lie on the boundary of a circle and every half-plane contains at least one point. Fortunately, this occurs with probability one for our Poisson reference measures Π^z and $\overline{\Pi}^z$.

In Chapters 3 and 4, we will make use of the following two classes of Delaunay hyperedge potentials. The first consists of pair interactions of the form

$$\varphi(\eta, \omega) = \phi(|x - y|) \text{ for } \eta = \{x, y\} \in \mathrm{Del}_2(\omega),$$

which take the 'length' of an edge as argument. The second class consists of triplet interactions of the form

$$\varphi(\tau, \omega) = \phi(\beta(\tau)) \text{ for } \tau \in \mathrm{Del}_3(\omega),$$

where $\beta(\tau)$ denotes the smallest interior angle of a triangle τ.

Chapter 3

Delaunay Potts models with infinite range

3.1 Introduction

In [BBD03], the authors prove the existence of a phase transition for the nearest-neighbour continuum Potts model with finite range type interaction on the Delaunay graph. We extend to the case of infinite range, giving a presentation of different results, also seen in [AE15], by the same author of the thesis. We work in the hypergraph structure framework presented in Chapter 2.

3.2 Hardcore background interaction

The models that we consider in this Chapter are those with a hardcore background interaction ψ between pairs of particles, or edges, in Del_2 or triples of particles, or triangles, in Del_3 . For fixed $\delta_0 > 0$, we give an infinite energy to hyperedges that contain particles of distance less than δ_0 to each other. This guarantees, with probability one, that particles cannot get too close. We show, in Lemma 3.1, that these interactions are equivalent to the classical case of a hardcore repulsion between all pairs of particles that form a hyperedge in the complete hypergraph CG. In particular, let the background interaction satisfy

$$\psi(\eta,\omega) \equiv \begin{cases} \psi(|x_1 - x_2|) & \text{for} \quad \eta = \{x_1, x_2\} \in \mathrm{CG}_2(\omega), \\ \psi(|x_1 - x_2|) & \text{for} \quad \eta = \{x_1, x_2\} \in \mathrm{Del}_2(\omega), \\ \psi\left(\min_{1\le i<j\le 3} |x_i - x_j|\right) & \text{for} \quad \eta = \{x_1, x_2, x_3\} \in \mathrm{Del}_3(\omega), \end{cases} \tag{3.1}$$

where $\mathrm{CG}_2(\omega)$ is the set of all pairs of particles in ω and $\psi : \mathbf{R} \to \mathbf{R} \cup \{\infty\}$ is defined as

$$\psi(r) := \begin{cases} +\infty & \text{if } r < \delta_0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Lemma 3.1. *Let $\omega \in \Omega$. Then,*

$$\exp\left[-\sum_{\eta \in \mathrm{CG}_2(\omega)} \psi(\eta,\omega)\right] = \exp\left[-\sum_{\eta \in \mathrm{Del}_2(\omega)} \psi(\eta,\omega)\right] = \exp\left[-\sum_{\tau \in \mathrm{Del}_3(\omega)} \psi(\tau,\omega)\right]. \tag{3.3}$$

Proof. The second equality in (3.3) comes directly from (3.2) upon realising that every triangle $\tau \in \mathrm{Del}_3$ is made up of three edges $\eta \in \mathrm{Del}_2$, but equally, each edge $\eta \in \mathrm{Del}_2$ is the subset of a triangle $\tau \in \mathrm{Del}_3$. The first equality needs some more work. We begin by defining the sets of hyperedges in $\mathrm{Del}_2(\omega)$ and $\mathrm{CG}_2(\omega)$ that contribute an infinite energy to the sums in (3.3). Let

$$\begin{aligned} \mathrm{HC} :&= \{\eta \in \mathrm{Del}_2(\omega) : \psi(\eta,\omega) = \infty\} \\ &= \{\eta = \{x_1, x_2\} \in \mathrm{Del}_2(\omega) : |x_1 - x_2| < \delta_0\}, \end{aligned}$$

and

$$\begin{aligned} \mathrm{HC}_* :&= \{\eta \in \mathrm{CG}_2(\omega) : \psi(\eta,\omega) = \infty\} \\ &= \{\{x_1, x_2\} \in \omega : x_1 \neq x_2 \text{ and } |x_1 - x_2| < \delta_0\}. \end{aligned}$$

If we can show that $\mathrm{HC}_* = \emptyset \Leftrightarrow \mathrm{HC} = \emptyset$, the Lemma will follow. It is trivial to see that $\mathrm{HC} \subset \mathrm{HC}_*$ because each hyperedge $\eta \in \mathrm{Del}_2(\omega)$ comprises exactly two disjoint particles $x_1, x_2 \in \omega$. Therefore, $\mathrm{HC}_* = \emptyset \Rightarrow \mathrm{HC} = \emptyset$. The other direction is a little more tricky. We make use of the set $\mathrm{Vor}_\omega(x) \subset \mathbf{R}^2$ defined in (2.49), and also its boundary, denoted by $\partial\mathrm{Vor}_\omega(x)$. Suppose $\{x_1, x_2\} \in \mathrm{HC}_*$ and let $\overline{x_1 x_2} \subset \mathbf{R}^2$ be the line segment between x_1 and x_2 given by

$$\overline{x_1 x_2} := \{z \in \mathbf{R}^2 : z = x_1 + (x_2 - x_1)t \text{ for some } t \in [0,1]\}. \tag{3.4}$$

Take y to be the midpoint of $\overline{x_1 x_2}$ so that $|x_1 - y| = |x_2 - y| < \delta_0/2$. Obviously, if $\{x_1, x_2\} \in \mathrm{Del}_2(\omega)$, we are done, so assume $\{x_1, x_2\} \notin \mathrm{Del}_2(\omega)$. This implies that $y \notin \mathrm{Vor}_\omega(x_1)$ and $y \notin \mathrm{Vor}_\omega(x_2)$. Let y' be the furthest point away from x_1 on $\overline{x_1 x_2}$ such that $y' \in \mathrm{Vor}_\omega(x_1)$:

$$y' = x_1 + (x_2 - x_1)t_{\max}$$

where

$$t_{\max} = \sup_{t\in[0,1]} \{t : x_1 + (x_2 - x_1)t \in \mathrm{Vor}_\omega(x_1)\}.$$

By the convexity of $\mathrm{Vor}_\omega(x_1)$, we have $|x_1 - y'| < |x_1 - y| < \delta_0/2$ and $y' \in \partial\mathrm{Vor}_\omega(x_1)$. Therefore, there exists some $x_3 \in \omega\backslash x_1$ such that $y' \in \partial\mathrm{Vor}_\omega(x_3)$ also. In particular, $\mathrm{Vor}_\omega(x_1)$ and $\mathrm{Vor}_\omega(x_3)$ share a 1-dimensional face, $\{x_1, x_3\} \in \mathrm{Del}_2(\omega)$ and

$$|x_1 - x_3| \le |x_1 - y'| + |x_3 - y'| < \delta_0/2 + \delta_0/2 = \delta_0.$$

Hence, $\eta = \{x_1, x_3\} \in \mathrm{HC}$ and therefore $\mathrm{HC} = \emptyset \Rightarrow \mathrm{HC}_* = \emptyset$. □

3.3 Type interaction

We also use a type interaction φ on hyperedges of $\mathrm{Del}_m(\omega)$, for $m \in \{2, 3\}$ that do not share a common mark between particles. This type interaction will depend on a parameter $A > 1$ which acts as the inverse temperature for type interactions. For large A the type interaction is strong. All of the type interactions in this thesis will be positive and will satisfy the local horizon property (2.13), i.e. the hyperedge potential will only depend on the hyperedge and not of the neighbourhood of the hyperedge. For a marked configuration $\overline{\omega} = (\omega, \sigma_\omega) \in \overline{\Omega}$, the Hamiltonian in a box $\Lambda \in \mathbf{B}(\mathbf{R}^2)$, with boundary condition $\xi = (\xi, \sigma_\xi) \in \overline{\Omega}_{\Lambda^c}$, is given by

$$H^{\overline{\Phi}}_{\Lambda|\overline{\xi}}(\overline{\omega}) = \sum_{\substack{\eta\in\mathrm{Del}_m(\omega\cup\xi)\\ \eta_\Lambda\neq\emptyset}} \psi(\eta, \omega \cup \xi) + \sum_{\substack{(\eta.\sigma_\eta)\in\mathrm{Del}_m(\overline{\omega}\cup\overline{\xi})\\ \eta_\Lambda\neq\emptyset}} \varphi(\eta, \omega \cup \xi)(1 - \delta_{\sigma_\eta}(\eta)) \tag{3.5}$$

where δ_{σ_ω} is the indicator defined in (2.22).

3.4 Existence

In order to show the existence of a Gibbs measure for the Delaunay Potts model with hard-core background interaction and type interaction as described above, we use the results of [DDG10] that we adapted in Section 2.4.1, and generalise to our new setting of twin hyperedge interactions ψ and φ. We will see that we do not need to explicitly define φ at this point – for existence of Gibbs measure, we only require that it is positive and satisfies the local horizon property.

Proposition 3.2. *For all $z > 0$, $A > 0$ and $\delta_0 \geq 0$, there exists at least one Gibbs measure for the Hamiltonian given in (3.5). Such a Gibbs measure is called a Delaunay Potts measure.*

Proof. For $(\overline{\eta}, \overline{\omega}) \in \mathrm{Del}_m$ with $\overline{\eta} = (\eta, \sigma_\eta)$ and $\overline{\omega} = (\omega, \sigma_\omega)$, we will apply Theorem 2.12 to

$$\overline{\Phi}(\overline{\eta}, \overline{\omega}) := \psi(\eta, \omega) + \varphi(\eta, \omega)(1 - \delta_{\sigma_\eta}(\eta)).$$

Therefore, if we can show that the couple $(\overline{\Phi}, \mathrm{Del}_m)$ satisfies the **(R)**, **(S)** and **(U)** conditions, we are done. By the definition of Del given in (2.50), there exists an open ball $B(\overline{\eta}, \overline{\omega}) \subset \mathbf{R}^2$ with $\overline{\omega} \cap (\partial B(\overline{\eta}, \overline{\omega}) \times \Sigma) = \overline{\eta}$ and $\overline{\omega} \cap (B(\overline{\eta}, \overline{\omega}) \times \Sigma) = \emptyset$. By setting $\Delta = B(\overline{\eta}, \overline{\omega})$, the range condition **(R)** follows from our local horizon assumption (2.13). Stability **(S)** follows because $\overline{\Phi} \geq 0$. For the upper regularity condition **(U)**, we start off by defining the matrix M, from 2.18, so that $\nabla(k)$ take the form of squares of side length L:

$$M = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix},$$

with L to be determined later. Furthermore, for any $\nabla \in \mathcal{B}(\mathbf{R}^2)$ and any real $r > 0$, let

$$\nabla \ominus r := \{x \in \nabla : B(x, r) \subset \nabla\} \tag{3.6}$$

and define Γ^s as the set of all configurations that consist of a single point of $\mathbf{R}^2 \times s$, whose projection under ρ lies in the Borel set $D := \nabla(0) \ominus \delta_0$, where $\nabla(0)$ is defined in (2.18). In particular, let

$$\Gamma^s := \left\{\overline{\omega} \in \overline{\Omega}_{\nabla(0)} : \overline{\omega} = \{\overline{x}\} \text{ for some } \overline{x} \in D \times s\right\}. \tag{3.7}$$

This gives our first constraint on L: it must be larger than $2\delta_0$. The translations $\vartheta_x \in \Theta$ do not alter the marks of a marked configuration, therefore, we obtain a set of pseudo-periodic marked configurations $\widehat{\Gamma}^s$, defined in (2.19), with each $\overline{\omega} \in \widehat{\Gamma}^s$ consisting of only particles of a single mark. The particle positions are also distance at least δ_0 from their neighbours, so using Lemma 3.1, we conclude that $c_D = c_D^+ = 0$. Uniform confinement and uniform summability both follow whilst strong non-rigidity is satisfied if

$$z|\nabla(0) \ominus \delta_0| > e^{c_D},$$

or equivalently, if

$$L > z^{-1/2} + \delta_0.$$

Therefore, we choose L to satisfy

$$L > \max\left\{2\delta_0, \delta_0 + z^{-1/2}\right\}. \tag{3.8}$$

□

Remark 3.3. *Note that Proposition 3.2 holds for the particular case that $\delta_0 = 0$. This is equivalent to there being no background interaction at all, or to $\psi \equiv 0$.*

3.5 Non-uniqueness

In order to show a phase transition for such a system, we first show percolation with respect to the continuum random cluster model associated to the Hamiltonian in (3.5) and then use the theory from Section 2.7. Before embarking on a proof of percolation for this conveniently named Delaunay random cluster distribution $C_{\Lambda|\xi}$, we give a brief outline of the techniques and main ideas that we use. We aim to show that percolation occurs in $C_{\Lambda|\xi}$, for constants z and A large enough, and for suitable pseudo periodic boundary condition ξ. We show this by finding a stochastically smaller measure $\tilde{C}_{\Lambda|\xi}$ and a corresponding site percolation model $\tilde{C}^{\text{site}}_{\Lambda|\xi}$. We then show the existence of site percolation under $\tilde{C}^{\text{site}}_{\Lambda|\xi}$ using a coarse graining technique and by bounding the expected change in $K(\omega, E)$ as a point x_0 is added to a configuration $\omega \in \Omega$. Finally, we use Proposition 2.18 to show percolation in $C_{\Lambda|\xi}$. The phase transition result of Theorem 3.13 then follows.

3.5.1 Coarse graining

We use a coarse graining technique to compare the Delaunay random cluster model with site percolation on $\mathbf{Z}^2$. We set up the coarse graining procedure in the following. Let $\Lambda \in \mathcal{B}(\mathbf{R}^2)$ be a rectangle made up of a finite union of square boxes, $\Delta_{k,l}$, each of side length $9L$, for some $L > 0$. That is, for some $I \subset \mathbf{Z}^2$

$$\Lambda = \bigcup_{(k,l)\in I} \Delta_{k,l}, \tag{3.9}$$

We also split each of these smaller boxes into 81 tiny square boxes (or cells) of side length L:

$$\Delta_{k,l} = \bigcup_{i,j=0}^{8} \Delta_{k,l}^{i,j}. \tag{3.10}$$

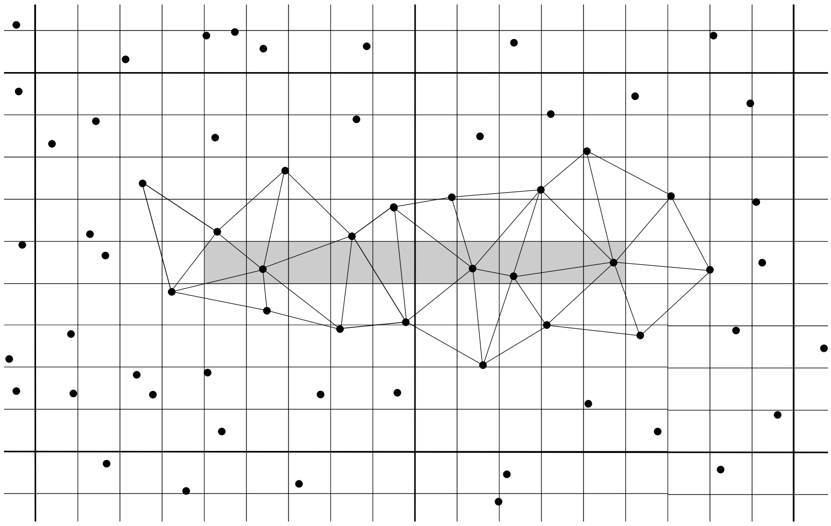


Figure 3.1: The shaded areas show the central band $CB^L_{k:k+1,l}$ of two boxes $\Delta_{k,l}$ and $\Delta_{k+1,l}$ in an L-splitting of Λ. The set of hyperedges $H^L_{k:k+1,l}(\omega)$ is also shown.

This is called the L-splitting of Λ. Given an L-splitting of Λ, we define the central band of $\Delta_{k,l} \cup \Delta_{k+1,l}$ to be

$$CB^L_{k:k+1,l} := \left(\bigcup_{i=0}^{4} \Delta^{4+i,4}_{k,l} \right) \cup \left(\bigcup_{i=0}^{4} \Delta^{i,4}_{k+1,l} \right). \tag{3.11}$$

Given $\omega \in \Omega$, let $H^L_{k:k+1,l}(\omega)$ be the subset of triangles of $\mathrm{Del}_3(\omega)$ whose circumscribing circle has a non-empty intersection with $CB^L_{k:k+1,l}$, see Figure 3.1. More precisely,

$$H^L_{k:k+1,l}(\omega) := \left\{ \tau \in \mathrm{Del}_3(\omega) : B(\tau, \omega) \cap CB^L_{k:k+1,l} \neq \emptyset \right\}. \tag{3.12}$$

Indeed, for boxes $\Delta_{k,l}$ and $\Delta_{k,l+1}$ we consider a possible vertical connection by defining $CB^L_{k,l:l+1}$ and $H^L_{k,l:l+1}(\omega)$ analogously. In the following, we limit ourselves to the horizontal case, however, due to rotational symmetry, all definitions and results hold for the vertical case too. Define the event $F_{k,l}$ that all small boxes $\Delta^{i,j}_{k,l} \subset \Delta_{k,l}$ contain at least one point of ω. Precisely:

$$F_{k,l} := \bigcap_{i,j=0}^{8} (|\omega \cap \Delta^{i,j}_{k,l}| \geq 1). \tag{3.13}$$

In order to use a coarse graining technique to compare to site percolation in $\mathbf{Z}^2$, we require a method to ensure adjacent boxes are connected with high probability. The following definition will allow for this. Recall that the role of the inverse temperature A is encapsulated inside the potential φ.

Definition 3.4. *The Delaunay random cluster distribution $C_{\Lambda|\xi}$ with ψ and φ as above is called coarse-grain ready **(CGR)** if there exists constants $U_1 > 0$ and $U_2 \geq 2\delta_0$ such that for all $\omega \in F_{k,l} \cap F_{k+1,l}$,*

$$\varphi(\eta, \omega) \geq \varphi(U_1) > 0$$

for all $\eta \subseteq \tau \in H^L_{k:k+1,l}(\omega)$ and $L \in [2\delta_0, U_2]$.

The condition of U_2 ensures that we can choose a splitting of Λ with boxes of side length at least $2\delta_0$. This becomes important in Lemma 3.9. Different potentials φ will give different values of U_2 and hence, varying ranges of acceptable box length L.

3.5.2 Papangelou conditional intensity

Definition 3.5. *Let ν on $(\Omega, \mathcal{F})$ be absolutely continuous with respect to the Poisson point process Π^z and denote by $f : \Omega \to [0, \infty)$ the Radon-Nikodym derivative given by $d\nu/d\Pi^z$. The Papangelou conditional intensity for ν with respect to Π^z is then defined by*

$$\frac{f(\omega \cup \{x\})}{f(\omega)},$$

for $\omega \in \Omega$ and $x \in \mathbf{R}^2 \backslash \omega$.

Roughly speaking, the Papangelou conditional intensity, see [DVJ08], can be seen as the conditional intensity for finding a point at x, given the configuration ω. Let $m \in \{2, 3\}$ and recall from (2.32) the continuum random-cluster distribution, which is colour independent and given by

$$C_{\Lambda|\xi}(d\omega, dE) := \frac{1}{Z_{2,\Lambda|\xi}} q^{K(\omega,E)} P^z_{\Lambda|\xi}(d\omega)\mu_{\omega,\Lambda}(dE) \tag{3.14}$$

where $K(\omega, E)$, with $E \subset \mathrm{Del}_m(\omega)$, is the number of connected components of the graph (ω, E) and $Z_{2,\Lambda|\xi}$ is the normalising constant defined in (2.33). Let $M_{\Lambda|\xi}$ be the distribution of particle positions given by the marginal distribution $C_{\Lambda|\xi}(\cdot, \mathcal{E})$ and recall from (2.43) that

$$C_{\Lambda|\xi}(d\omega, dE) = M_{\Lambda|\xi}(d\omega)\mu^{(q)}_{\omega,\Lambda}(dE), \tag{3.15}$$

with

$$\mu^{(q)}_{\omega,\Lambda}(dE) = \frac{q^{K(\omega,E)}\mu_{\omega,\Lambda}(dE)}{\int q^{K(\omega,E)}\mu_{\omega,\Lambda}(dE)}.$$

We define h_Λ to be the Radon-Nikodym derivative of $M_{\Lambda|\xi}$ with respect to $P^z_{\Lambda|\xi}$. That is:

$$h_\Lambda(\omega) = Z^{-1}_{2,\Lambda|\xi} \int q^{K(\omega,E)} \mu_{\omega,\Lambda}(dE). \tag{3.16}$$

The Papangelou conditional intensity for $M_{\Lambda|\xi}$ with respect to $P^z_{\Lambda|\xi}$ is then given by

$$\frac{h_\Lambda(\omega \cup \{x\})}{h_\Lambda(\omega)}, \tag{3.17}$$

for $\omega \in \Omega$ and $x \in \mathbf{R}^2 \backslash \omega$. The second thing we need in order to use a coarse graining technique to compare the Delaunay random cluster model with site percolation on $\mathbf{Z}^2$, is to exhibit some control over the distribution of particle positions $M_{\Lambda|\xi}$. For this control, we require a lower bound on the Papangelou conditional intensity (3.17). In particular, we state the following condition.

Definition 3.6. *The Delaunay random cluster distribution $C_{\Lambda|\xi}$, with ψ and φ as above, has bounded Papangelou conditional intensity **(BPI)** if there exists $\alpha > 0$ (depending on φ and ψ) such that, for any finite box $\Lambda \in \mathcal{B}(\mathbf{R}^2)$, $M_{\Lambda|\xi}$ – almost all $\omega \in \Omega_{\Lambda|\xi}$ and a point x_0, with $x_0 \in \Lambda \backslash \omega$,*

$$\frac{h_\Lambda(\omega \cup \{x_0\})}{h_\Lambda(\omega)} \geq q^{-\alpha}. \tag{3.18}$$

3.5.3 Percolation

Proposition 3.7. *Fix $q \in [1, \infty)$. Let ψ be a hardcore background interaction satisfying (3.1) and (3.2) and let φ be a type interaction on hyperedges of Del_m dependent on parameter A. Suppose the Delaunay random cluster distribution $C_{\Lambda|\xi}$ corresponding to ψ and φ is coarse-grain ready **(CGR)**, has a bounded Papangelou conditional intensity **(BPI)** and that z and A large enough. Then, for all $\Delta \in \mathcal{B}(\mathbf{R}^2)$, there exists $c > 0$, such that for all $\Lambda \supset \Delta$, and for all $\overline{\xi} = (\xi, \sigma_\xi) \in \widehat{\Gamma}_{\Lambda^c}$,*

$$\int C_{\Lambda|\xi}(d\omega, dE) N_{\Delta \leftrightarrow \Lambda^c}(\omega, E) \geq c. \tag{3.19}$$

Remark 3.8. *Note that, despite the need for $q \in \mathbf{N}$ in the Delaunay random cluster representation of the Delaunay Potts model, Proposition 3.7 holds for all real $q > 1$, and hence for many more random cluster models than those with corresponding Delaunay Potts models.*

For the proof, we use the coarse graining framework described in Section 3.5.1 to compare $\tilde{C}^{\text{site}}_{\Lambda|\xi}$ from (2.46) to site percolation on $\mathbf{Z}^2$. In particular, we show that the conditional probability (with respect to $\tilde{C}^{\text{site}}_{\Lambda|\xi}$) that a box $\Delta_{k,l}$ is 'nice' – given the configuration outside $\Delta_{k,l}$ is admissible – is larger than $p_c^{\text{site}}(\mathbf{Z}^2)$: the critical probability for site percolation on the integer lattice. Of course, this guarantees an infinite chain of 'nice' boxes. If we define 'nice' boxes in such a way that an infinite chain of them implies an infinite path of edges in the Delaunay graph passing only through points of mark 1, then we have percolation for $\tilde{C}^{\text{site}}_{\Lambda|\xi}$. This is made rigorous in Lemmas 3.9 and 3.10 below. Then, once we have shown the stochastic domination of $\mu^{(q)}_{\omega,\Lambda}$ over $\tilde{\mu}_\omega$ in Lemma 3.11, Proposition 3.7 follows from Proposition 2.18.

An important component of the discretization method is to estimate the conditional probability that at least one point of the configuration lies inside $\Delta_{k,l}$, given the configuration outside of $\Delta_{k,l}$. To begin, let $\Delta \subset \Lambda$ and write

$$M_{\Lambda|\xi}(A,B) = \int_B \int_A f(\omega', \omega'', \xi) \Pi^z_\Delta(d\omega') \Pi^z_{\Lambda\backslash\Delta}(d\omega'')$$

for $A \in \mathcal{F}_\Delta$, $B \in \mathcal{F}_{\Lambda\backslash\Delta}$ and $\xi \in \Omega_{\Lambda^c}$ where f is given by

$$f(\omega', \omega'', \xi) = Z^{-1}_{\Lambda|\xi} h_\Lambda(\omega' \cup \omega'' \cup \xi) \exp[-H^\psi_{\Lambda|\xi}(\omega' \cup \omega'')] \tag{3.20}$$

and $Z_{\Lambda|\xi}$ is the normalising constant from (2.25), but with $q = 1$. The marginal distribution of the point configuration inside $\Lambda\backslash\Delta$ is then given by

$$M^{\Lambda\backslash\Delta}_{\Lambda|\xi}(d\omega'') = f_{\Lambda\backslash\Delta}(\omega'') \Pi^z_{\Lambda\backslash\Delta}(d\omega''),$$

where the density $f_{\Lambda\backslash\Delta}(\omega'') = \int_{\Omega_\Delta} f(\omega', \omega'', \xi) \Pi^z_\Delta(d\omega')$. Now, let

$$g(\omega'|\omega'') := \begin{cases} f(\omega', \omega'', \xi)/f_{\Lambda\backslash\Delta}(\omega'') & \text{for all } \omega'' \text{ such that } f_{\Lambda\backslash\Delta}(\omega'') \neq 0, \\ \text{an arbitrary density } f_0(\omega') & \text{for all } \omega'' \text{ such that } f_{\Lambda\backslash\Delta}(\omega'') = 0 \end{cases} \tag{3.21}$$

and define

$$M_{\Lambda|\xi}(A|\omega_{\Lambda\backslash\Delta} = \omega'') = \int_A g(\omega'|\omega'') \Pi^z_\Delta(d\omega') \text{ for all } A \in \mathcal{F}_\Delta.$$

Call $g(\omega'|\omega'')$ the conditional density (relative to $M_{\Lambda|\xi}$) of the configuration inside Δ given that the configuration in $\Lambda\backslash\Delta$ is equal to ω''. We check that $M_{\Lambda|\xi}(A|\omega_{\Lambda\backslash\Delta} = \omega'')$ is a regular conditional distribution. Indeed,

$$\begin{aligned}
\int_B \int_A g(\omega'|\omega'')\Pi^z_\Delta(d\omega')M^{\Lambda\setminus\Delta}_{\Lambda|\xi}(d\omega'') &= \int_B \left[\int_A g(\omega'|\omega'')\Pi^z_\Delta(d\omega')\right] f_{\Lambda\setminus\Delta}(\omega'')\Pi^z_{\Lambda\setminus\Delta}(d\omega'') \\
&= \int_{B\cap S} \left[\int_A \frac{f(\omega',\omega'',\xi)}{f_{\Lambda\setminus\Delta}(\omega'')}\Pi^z_\Delta(d\omega')\right] f_{\Lambda\setminus\Delta}(\omega'')\Pi^z_{\Lambda\setminus\Delta}(d\omega'') \\
&= \int_{B\cap S}\int_A f(\omega',\omega'',\xi)\Pi^z_\Delta(d\omega')\Pi^z_{\Lambda\setminus\Delta}(d\omega'') \\
&= M_{\Lambda|\xi}(A,B) \\
&= \int_B M_{\Lambda|\xi}(A|\omega_{\Lambda\setminus\Delta}=\omega'')M^{\Lambda\setminus\Delta}_{\Lambda|\xi}(d\omega'').
\end{aligned}$$

Thus, $\int_A g(\omega'|\omega'')\Pi^z_\Delta(d\omega') = M_{\Lambda|\xi}(A|\omega_{\Lambda\setminus\Delta}=\omega'')$ almost surely with respect to $M^{\Lambda\setminus\Delta}_{\Lambda|\xi}$ and so works as a version. We note that

$$\int_A g(\omega'|\omega'')\Pi^z_\Delta(d\omega') = \mathbf{1}_S(\omega'')\left(\frac{\int_A f(\omega',\omega'',\xi)\Pi^z_\Delta(d\omega')}{f_{\Lambda\setminus\Delta}(\omega'')}\right) + \mathbf{1}_{S^c}(\omega'')\int_A f_0(\omega')\Pi^z_\Delta(d\omega') \tag{3.22}$$

is measurable on $(\Omega_{\Lambda\setminus\Delta},\mathcal{F}_{\Lambda\setminus\Delta})$ for any fixed $A\in\mathcal{F}_\Delta$, and acts as a probability distribution for any fixed $\omega''\in\Omega_{\Lambda\setminus\Delta}$. Thus, $\int_A g(\omega'|\omega'')\Pi^z_\Delta(d\omega')$ defines completely a regular conditional probability distribution. For brevity, we will write $M_{\Lambda,\Delta|\xi'}(\cdot)$ instead of $M_{\Lambda|\xi}(\cdot|\omega_{\Lambda\setminus\Delta}=\omega'')$ where $\xi'=\omega''\cup\xi\in\Omega_{\Delta^c}$. It follows, together with (3.20) and (3.22), that for any admissible boundary configuration $\xi'\in\Omega_{\Delta^c}$,

$$\begin{aligned}
M_{\Lambda,\Delta|\xi'}(A) &= Z^{-1}_{\Lambda,\Delta|\xi'}\int_A h_\Lambda(\omega'\cup\xi')\exp[-H^\psi_{\Lambda|\xi}(\omega'\cup\omega'')]\Pi^z_\Delta(d\omega') \\
&= Z^{-1}_{\Lambda,\Delta|\xi'}\int_A h_\Lambda(\omega'\cup\xi')\exp[-H^\psi_{\Delta|\xi'}(\omega')]\Pi^z_\Delta(d\omega'),
\end{aligned} \tag{3.23}$$

where the second equality is due to the fact that ξ' is admissible and ψ is a hard core potential. The normalisation constant is given by

$$Z_{\Lambda,\Delta|\xi'} = Z_{\Lambda|\xi'}f_{\Lambda\setminus\Delta}(\omega'').$$

Fix $\epsilon = \frac{1-p_c^{\text{site}}(\mathbf{Z}^2)}{2}$ and $L\in[2\delta_0,U_2]$ where U_2 is as in Definition 3.4. Then, we have the following lower bound on the probability that a small box, $\Delta^{i,j}_{k,l}$ in the L-splitting of Λ, contains at least one point of ω.

Lemma 3.9. *Let $z > z_0 := \frac{81q^\alpha}{\epsilon U_2^2}$. Then, for any finite union Λ of boxes $\Delta\in\mathcal{B}(\mathbf{R}^2)$, and* *for any pseudo-periodic boundary condition $\xi \in \widehat{\Gamma}_{\Lambda^c}$,*

$$M_{\Lambda,\nabla|\xi'}(|\omega \cap \nabla| \geq 1) > 1 - \frac{\epsilon}{81}$$

for all cells $\nabla = \Delta^{i,j}_{k,l}$ of the L-splitting of Λ and for any admissible boundary condition $\xi' \in \Omega_{\nabla^c}$, with $\xi' \backslash \Lambda = \xi$.

Proof. We make use of the fact that

$$\int f(\omega)\Pi^z_\nabla(d\omega) = e^{-z|\nabla|}\sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{\nabla^n} f(\{x_1,\dots x_n\})dx_1\dots dx_n, \tag{3.24}$$

for any bounded measurable function $f : \Omega_\nabla \to [0,\infty)$. It follows from (3.23), that

$$\begin{aligned}
\frac{M_{\Lambda,\nabla|\xi'}(|\omega \cap \nabla| = 1)}{M_{\Lambda,\nabla|\xi'}(|\omega \cap \nabla| = 0)} &= \frac{e^{-z|\nabla|}z\int_\nabla \exp\left[-H^\psi_{\nabla|\xi'}(\{x\})\right] h_\Lambda(\omega \cup \{x\})dx}{e^{-z|\nabla|}h_\Lambda(\omega)} \\
&= z\int_\nabla \exp\left[-H^\psi_{\nabla|\xi'}(\{x\})\right]\frac{h_\Lambda(\omega \cup \{x\})}{h_\Lambda(\omega)}dx \\
&\geq q^{-\alpha}z\int_\nabla \exp\left[-H^\psi_{\nabla|\xi'}(\{x\})\right]dx
\end{aligned}$$

using the **(BPI)** condition. Let $\nabla_0 \subset \nabla$ be such that $\nabla \ominus \delta_0 = \nabla_0$. Then, by splitting the integral, we obtain

$$\begin{aligned}
\frac{M_{\Lambda,\nabla|\xi'}(|\omega \cap \nabla| = 1)}{M_{\Lambda,\nabla|\xi'}(|\omega \cap \nabla| = 0)} \geq q^{-\alpha}z\Bigg(&\int_{\nabla_0} \exp\left[-H^\psi_{\nabla|\xi'}(\{x\})\right]dx \\
&+ \underbrace{\int_{\nabla\backslash\nabla_0} \exp\left[-H^\psi_{\nabla|\xi'}(\{x\})\right]dx}_{\geq 0}\Bigg)
\end{aligned} \tag{3.25}$$

$$\geq q^{-\alpha}z\int_{\nabla_0}\exp\Big[-\sum_{\{x,y\}\in\mathrm{Del}_2(\xi'\cup\{x\})}\psi(|x-y|)\Big]dx \geq q^{-\alpha}z|\nabla_0|,$$

where the integrand in the penultimate term is equal to 1 since $|x-y| > \delta_0$ for all $y \in \xi'$. It is apparent that $|\nabla_0| > 0$ because of our assumption that $\delta_0 < \frac{L}{2}$ and that ∇ has side length L. Thus,

$$M_{\Lambda,\nabla|\xi'}(|\omega \cap \nabla| = 0) \leq \frac{q^\alpha}{z|\nabla_0|} < \frac{\epsilon}{81}$$

for $z > z_0$. $\square$

We now introduce site percolation which we will use in our comparison argument.

Recall the hyperedge drawing mechanism $\mu_{\omega,\Lambda}$ described in (2.28). Given $\omega \in \Omega$, let $\tilde{\mu}_\omega$ denote an alternative distribution of the random hyperedge configurations

$$\{\eta \in \mathcal{H}(\omega) : v(\eta) = 1\}$$

where $(v(\eta))_{\eta\in\mathcal{H}(\omega)}$ are independent Bernoulli random variables with probability

$$\text{Prob}(v(\eta) = 1) = \tilde{p}(\eta) := \underbrace{\frac{1 - \exp\left(-\varphi(U_1)\right)}{1 + (q^{|\eta|-1} - 1)\exp\left(-\varphi(U_1)\right)}}_{=:\tilde{p}} \mathbf{1}_{\text{Del}^*_m(\omega)}(\eta), \tag{3.26}$$

where, for $m \in \{2, 3\}$,

$$\text{Del}^*_m(\omega) := \Big\{\eta \in \text{Del}_m(\omega) : \varphi(\eta, \omega) \geq \varphi(U_1)\Big\}, \tag{3.27}$$

and U_1 is as in Definition 3.4. Note that unlike p_Λ of (2.28), $\tilde{p}$ has no dependence on the box Λ. Note also that $\tilde{p}$ is increasing in A, although to reduce excessive notation, we don't explicitly write this. Recall the definition of the continuum site percolation model from (2.46):

$$\tilde{C}^{\text{site}}_{\Lambda|\xi}(d\overline{\omega}) = M_{\Lambda|\xi}(d\omega)\tilde{\lambda}_\omega(d\overline{\omega}),$$

where $\tilde{\lambda}_\omega$ denotes the distribution of the random vector $\sigma_\omega = (\sigma_\omega(x) : x \in \omega)$ with elements in Σ, where $(\sigma_\omega(x))_{x\in\omega}$ are independent Bernoulli random variables satisfying

$$\text{Prob}(\sigma_\omega(x) = 1) = \tilde{p}\mathbf{1}_{\text{Del}^*_1(\omega)}(x) \tag{3.28}$$

and

$$\text{Prob}(\sigma_\omega(x) \neq 1) = 1 - \tilde{p}\mathbf{1}_{\text{Del}^*_1(\omega)}(x)$$

where $\tilde{p}$ is given in (3.26) and $\text{Del}^*_1(\omega)$ is the set of points that build the hyperedges of $\text{Del}^*_m(\omega)$.

Lemma 3.10. *Let $z > z_0$ and $A > A_0$ where z_0 is given in Lemma 3.9 and*

$$A_0 := U_1^{-1}\left(\log\left[\frac{1 + (q^{|\eta|-1} - 1)(1-\epsilon)^{\pi(\frac{\delta_0}{9U_2+2\delta_0})^2}}{1 - (1-\epsilon)^{\pi(\frac{\delta_0}{9U_2+2\delta_0})^2}}\right]\right).$$

Then, there exists $c > 0$ such that

$$\tilde{C}^{\text{site}}_{\Lambda|\xi}(\{\Delta \leftrightarrow \Lambda^c\} \geq c > 0$$

for any box $\Delta \in \mathcal{B}(\mathbf{R}^2)$, any finite union Λ of boxes, and for any pseudo-periodic boundary *condition* $\xi \in \widehat{\Gamma}_{\Lambda^c}$.

Proof. Let $\nabla = \Delta_{k,l}^{i,j}$. By Lemma 3.9, we take z large enough such that, for all configurations $\xi' \in \Omega_{\nabla^c}$, with $\xi' \backslash \Lambda = \xi$,

$$M_{\Lambda,\nabla|\xi'}(|\omega \cap \nabla| = 0) \leq \frac{\epsilon}{81}. \tag{3.29}$$

This implies that, for all configurations $\xi'' \in \Omega_{\Delta^c}$ with $\xi'' \backslash \Lambda = \xi$,

$$\begin{aligned} M_{\Lambda,\Delta|\xi''}(|\omega \cap \nabla| = 0) &= \int_{\Omega_{\Delta \backslash \nabla}} M_{\Lambda,\nabla|\xi'' \cup \zeta}(|\omega \cap \nabla| = 0) M_{\Lambda,\Delta|\xi''}^{\Delta \backslash \nabla}(d\zeta) \\ &\leq \epsilon/81 \int_{\Omega_{\Delta \backslash \nabla}} M_{\Lambda,\Delta|\xi''}^{\Delta \backslash \nabla}(d\zeta) \\ &= \epsilon/81, \end{aligned} \tag{3.30}$$

where the inequality is due to the fact that $\xi'' \cup \zeta \in \Omega_{\nabla^c}$, and hence satisfies the conditions for (3.29). Recall from (3.13) the event $F_{k,l}$ that all small boxes $\Delta_{k,l}^{i,j} \subset \Delta_{k,l}$ contain at least one point of ω and notice that

$$M_{\Lambda,\Delta|\xi''}(F_{k,l}) \geq 1 - \sum_{i,j=0}^{8} M_{\Lambda,\Delta|\xi''}(|\omega \cap \Delta_{k,l}^{i,j}| = 0) > 1 - \epsilon.$$

Let $C_{k,l} \in \overline{\mathcal{F}}$ be an event such that each small box $\Delta_{k,l}^{i,j} \subset \Delta_{k,l}$ contains at least one point and all points in $\Delta_{k,l} \cap \mathrm{Del}_1^*(\omega)$ are of mark 1:

$$C_{k,l} = \{\overline{\omega} = (\omega, \sigma_\omega) \in \overline{\Omega} : \omega \in F_{k,l} \text{ and } \sigma_\omega(x) = 1 \text{ for all } x \in \Delta_{k,l} \cap \mathrm{Del}_1^*(\omega)\}.$$

Recall from (3.28), that under $\tilde{C}^{\text{site}}_{\Delta_{k,l}|\xi''}$, a point $\overline{x} \in \overline{\omega}$, where $\rho(\overline{x}) \in \Delta_{k,l} \cap \mathrm{Del}_1^*(\omega)$, has mark 1 with probability $\tilde{p}$. Also note that the maximum number of particles in $\Delta_{k,l} \cap \mathrm{Del}_1^*(\omega)$ is no larger than the maximum number of particles in $\omega_{\Delta_{k,l}}$. Hence, by the hardcore background interaction,

$$|\omega_{\Delta_{k,l}} \cap \mathrm{Del}_m^*(\omega)| \leq \frac{(9L + 2\delta_0)^2}{\pi \delta_0^2} =: M.$$

Therefore

$$\begin{aligned} \tilde{C}^{\text{site}}_{\Delta_{k,l}|\xi''}(C_{k,l}) &\geq \int M_{\Lambda,\Delta_{k,l}|\xi''}(d\omega) \mathbf{1}_{\{F_{k,l}\}}(\omega) \tilde{p}^M \\ &\geq (1-\epsilon)\tilde{p}^M. \end{aligned} \tag{3.31}$$

However, by taking $A > A_0$, and since $L \leq U_2$, we obtain that

$$\tilde{p} = \frac{1 - \exp[-\varphi(U_1)]}{1 + (q^{|\eta|-1} - 1)\exp[-\varphi(U_1)]} \geq (1-\epsilon)^{\frac{\pi\delta_0^2}{(9L+2\delta_0)^2}} = (1-\epsilon)^{1/M},$$

and hence $\tilde{p}^M \geq (1-\epsilon)$. Combining this with (3.31), it follows that for all $A > A_0$,

$$\tilde{C}^{\text{site}}_{\Delta_{k,l}|\xi''}(C_{k,l}) \geq (1-\epsilon)^2 > 1 - 2\epsilon = p_c^{\text{site}}(\mathbf{Z}^2). \tag{3.32}$$

Then, by standard percolation results, there exists, with positive probability independent of Λ, a chain of boxes $\Delta_{i,j}$ from $\Delta_{k,l} \subset \Lambda$ to Λ^c such that $C_{i,j}$ occurs for each. It remains to check this implies $\{\Delta \leftrightarrow \Lambda^c\}$. Recall the definitions for the central band $CB^L_{k:k+1,l}$ and the subset of hyperedges $H^L_{k:k+1,l}(\omega)$ given by (3.11) and (3.12) respectively and let

$$H_m(\omega) = \{\eta \in \text{Del}_m(\omega) : \exists \tau \in H^L_{k:k+1,l}(\omega) : \eta \subseteq \tau\}$$

be the subset of hyperedges of $\text{Del}_m(\omega)$ that build the triangles of $H^L_{k:k+1,l}(\omega)$. Suppose $\overline{\omega} = (\omega, \sigma_\omega) \in C_{k,l} \cap C_{k+1,l}$. Then, since $C_{\Lambda|\xi}$ satisfies **(CGR)**, it follows that $\varphi(\eta, \omega) \geq \varphi(U_1) > 0$ for all $\eta \in H_m(\omega)$. Therefore $H_m(\omega) \subset \text{Del}^*_m(\omega)$ and hence, $\sigma_\omega(x) = 1$ for all $x \in H_m(\omega)$. Therefore, we can connect $\Delta^{4,4}_{k,l}$ to $\Delta^{4,4}_{k+1,l}$ in the graph Del^*_m inside $\Delta_{k,l} \cup \Delta_{k+1,l}$, through points of mark 1. Hence, using (3.32), we have $\tilde{C}^{\text{site}}_{\Lambda|\xi}(\{\Delta \leftrightarrow \Lambda^c\}) \geq c > 0$.

$\square$

Finally, we show that $\tilde{\mu}_\omega$ is stochastically smaller than $\mu^{(q)}_{\omega,\Lambda}$ using the following Lemma. This completes the proof of Proposition 3.7.

Lemma 3.11. *For all $q \geq 1$ and $\omega \in \Omega$, we have $\mu^{(q)}_{\omega,\Lambda} \succcurlyeq \tilde{\mu}_\omega$.*

Proof. Fix $\eta \in \text{Del}_m(\omega)$. Then, considering (3.27), it follows that

$$\begin{aligned}
\frac{p(\eta)}{q^{|\eta|-1}(1-p(\eta))} &= \frac{1 - \exp[-\varphi(\eta,\omega)]}{q^{|\eta|-1}\exp[-\varphi(\eta,\omega)]} \\
&\geq \frac{1 - \exp[-U_1(A)\mathbf{1}_{\text{Del}^*_m(\omega)}(\eta)]}{q^{|\eta|-1}\exp[-U_1(A)\mathbf{1}_{\text{Del}^*_m(\omega)}(\eta)]}
\end{aligned}$$

$$= \frac{1 - \exp[-U_1(A)\mathbf{1}_{\text{Del}^*_m(\omega)}(\eta)]}{1 + (q^{|\eta|-1} - 1)\exp[-U_1(A)\mathbf{1}_{\text{Del}^*_m(\omega)}(\eta)]} \Bigg/ \frac{q^{|\eta|-1}\exp[-U_1(A)\mathbf{1}_{\text{Del}^*_m(\omega)}(\eta)]}{1 + (q^{|\eta|-1} - 1)\exp[-U_1(A)\mathbf{1}_{\text{Del}^*_m(\omega)}(\eta)]}$$

$$= \frac{\tilde{p}(\eta)}{1 - \tilde{p}(\eta)}.$$

The result then follows from Proposition 2.3. $\square$

3.6 Model 1 – Delaunay Potts model with restricted triangles

The first model to consider is one with a hardcore background interaction ψ between triplets of particles, or triangles, in Del_3 as described in (3.1) and (3.2). We also use a type interaction φ between triplets of particles, or triangles, in Del_3 that gives a positive energy to triangles without a common mark between its particles. This interaction will depend of the smallest angle inside a triangle. In particular, fix $\beta_0 \in (0, \pi/3]$ and let the type interaction satisfy

$$\varphi(\tau, \omega) \equiv \varphi(\beta(\tau)) \quad \text{for} \quad \tau \in \mathrm{Del}_3(\omega),$$

where $\beta(\tau)$ denotes the smallest interior angle of a triangle τ and $\varphi : [0, \pi/3] \to \mathbf{R} \cup \{\infty\}$ is defined as

$$\varphi(\theta) := \begin{cases} 0 & \text{if } \theta < \beta_0, \\ A & \text{otherwise,} \end{cases} \tag{3.33}$$

where A assumes the role of inverse temperature and controls the level of the type interaction. This is an infinite range type interaction as it can act between triplets of particles far away from one another, providing they form a hyperedge in Del_3 and do not share a common mark. For a marked configuration $\overline{\omega} = (\omega, \sigma_\omega) \in \overline{\Omega}_\Lambda$ and an admissible boundary configuration $\overline{\xi} = (\xi, \sigma_\xi)$, the Hamiltonian is given by

$$H^{\overline{\Phi}}_{\Lambda|\overline{\xi}}(\overline{\omega}) = \sum_{\substack{\tau \in \mathrm{Del}_3(\omega \cup \xi) \\ \tau_\Lambda \neq \emptyset}} \psi(\tau, \omega \cup \xi) + \sum_{\substack{(\tau, \sigma_\tau) \in \mathrm{Del}_3(\overline{\omega} \cup \overline{\xi}) \\ \tau_\Lambda \neq \emptyset}} \varphi(\tau, \omega \cup \xi)(1 - \delta_{\sigma_\tau}(\tau)) \tag{3.34}$$

where δ_{σ_τ} is the indicator defined in (2.22). Since ψ is a hardcore background interaction and φ is non-negative and satisfies the local horizon property, the existence of at least one Delaunay Potts measure for the Hamiltonian (3.34) follows from Proposition 3.2.

Remark 3.12. *In this model, we have specified φ to be a step function. In later models, we look at other functions for φ, including those that depend more smoothly on $\theta \in [0, \pi/3]$.*

Theorem 3.13. *For all $\delta_0 > 0$ and $\beta_0 \in (0, \frac{1}{4\sqrt{2}}]$, there exists $z_0 = z_0(\beta_0, \delta_0)$ and $A_0 = A_0(\beta_0, \delta_0)$ such that for $z > z_0$ and $A > A_0$, there exists at least q different Delaunay Potts measures for the Hamiltonian 3.34.*

To prove Theorem 3.13, we show that the Delaunay random cluster distribution $C_{\Lambda|\xi}$ corresponding to ψ and φ is coarse-grain ready **(CGR)** and has a bounded Papangelou conditional intensity **(BPI)**. Then we apply Proposition 3.7.

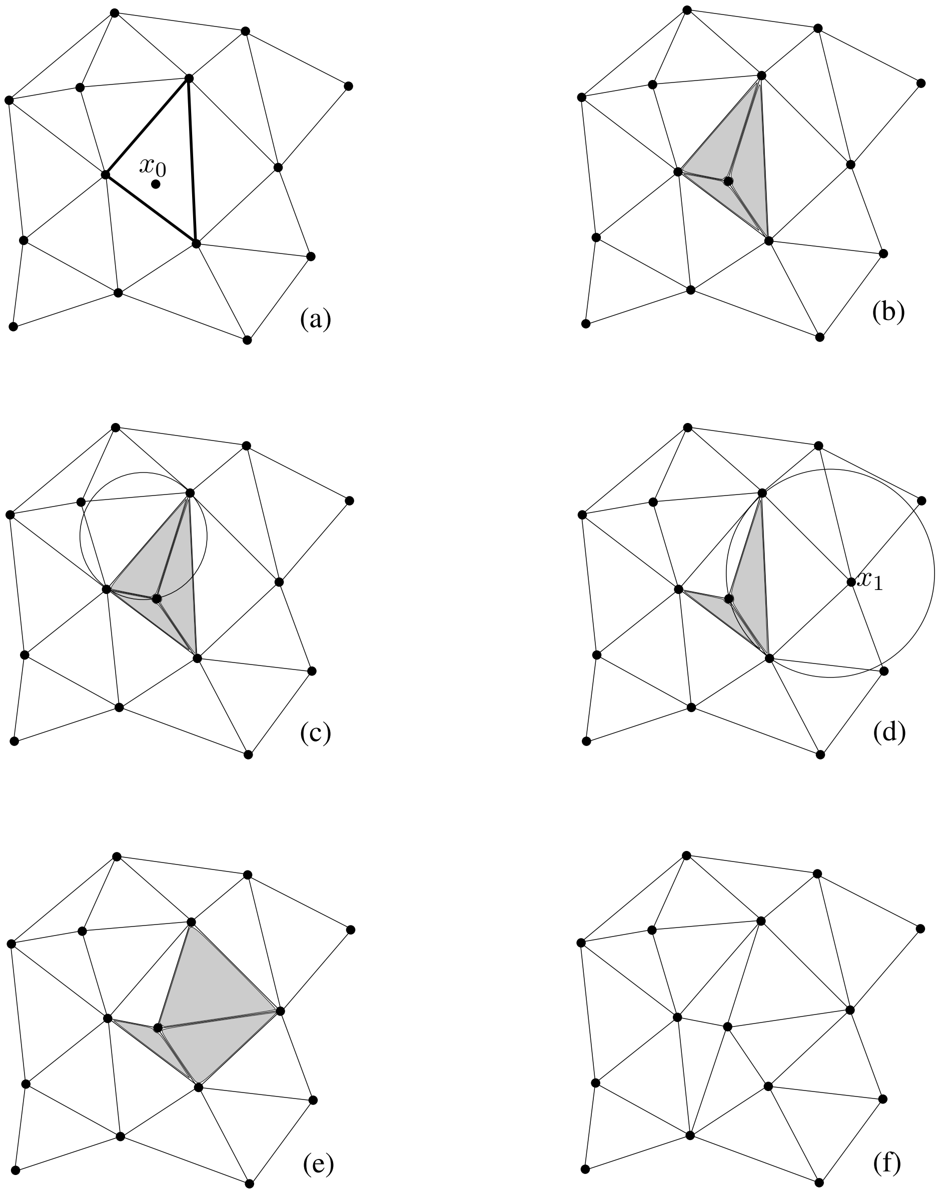


Figure 3.2: Inserting a point x_0 into a configuration ω.

In order to show **(BPI)**, we must first investigate the geometry of the Delaunay triangulation $\mathrm{Del}(\omega)$, and in particular, what happens to it when we augment ω with a new point $x_0 \notin \omega$. Some hyperedges may be destroyed, some are created, and some remain. This process is well described in [Li94], but we give an account here for completeness. Figure 3.2 illustrates the differences in the structure of the Delaunay triangulations $\mathrm{Del}(\omega)$ and $\mathrm{Del}(\omega \cup \{x_0\})$. We first locate the triangle of $\mathrm{Del}(\omega)$ in which x_0 is positioned (a). We then create three new edges that join x_0 to each of the three vertices of the triangle (b).

This creates three new triangles, and destroys one. At this point, we need to verify that the new triangles each satisfy the Delaunay condition (2.50). That is, their circumscribing balls contain no points of ω. If the condition is satisfied, the triangle remains (c). If, on the other hand, the condition is not satisfied, and there exists some point x_1 of ω inside the circumscribing ball (d), we remove the edge not connected to x_0, and replace it by an edge connecting x_0 and x_1 (e). This results in the creation of two new triangles. Each of these new triangles must be checked as above and the process continues. Once all triangles satisfy the Delaunay condition, we arrive at the Delaunay triangulation $\mathrm{Del}(\omega \cup \{x_0\})$ (f).

Before we can show **(BPI)**, we must introduce some notation for our particular case: the hypergraph structure Del_3. Whilst most hyperedges $\tau \in \mathrm{Del}_3(\omega)$ remain intact in $\mathrm{Del}_3(\omega \cup \{x_0\})$, some hyperedges are created, and some are destroyed. Let

$$E^{\text{ext}}_{x_0|\omega} := \mathrm{Del}_3(\omega) \cap \mathrm{Del}_3(\omega \cup \{x_0\}), \tag{3.35}$$

$$E^{+}_{x_0|\omega} := \mathrm{Del}_3(\omega \cup \{x_0\}) \backslash \mathrm{Del}_3(\omega) = \mathrm{Del}_3(\omega \cup \{x_0\}) \backslash E^{\text{ext}}_{x_0|\omega}, \tag{3.36}$$

$$E^{-}_{x_0|\omega} := \mathrm{Del}_3(\omega) \backslash \mathrm{Del}_3(\omega \cup \{x_0\}) = \mathrm{Del}_3(\omega) \backslash E^{\text{ext}}_{x_0|\omega}, \tag{3.37}$$

be the exterior, created, and destroyed hyperedge sets respectively, see Figure 3.3. Note that any created hyperedges must contain x_0, and hence,

$$E^{+}_{x_0|\omega} = \{\tau \in \mathrm{Del}_3(\omega \cup \{x_0\}) : \tau \cap x_0 = x_0\}.$$

We also define $\mu^{-}_{x_0|\omega}$, $\mu^{+}_{x_0|\omega}$ and $\mu^{\text{ext}}_{x_0|\omega}$ to be the edge drawing mechanisms on $E^{-}_{x_0|\omega}$, $E^{+}_{x_0|\omega}$ and $E^{\text{ext}}_{x_0|\omega}$ respectively, which are derived from the edge drawing mechanism $\mu_{\omega,\Lambda}$, as defined in (2.28). In fact

$$\mu^{+}_{x_0|\omega} \otimes \mu^{\text{ext}}_{x_0|\omega} = \mu_{\omega \cup \{x_0\},\Lambda}, \tag{3.38}$$

$$\mu^{-}_{x_0|\omega} \otimes \mu^{\text{ext}}_{x_0|\omega} = \mu_{\omega,\Lambda}. \tag{3.39}$$

Let $E \in \mathcal{E}$ be the resulting subset after a p_Λ thinning of the hyperedge set $\mathrm{Del}_3(\omega)$ for a finite box $\Lambda \in \mathcal{B}(\mathbf{R}^2)$. Throughout this thesis, we are interested in the number of connected components $K(\omega, E)$ of the hypergraph (ω, E). More precisely, we are interested in the change to K when we add or remove points or hyperedges from (ω, E). The addition of a single point $x_0 \in \Lambda$ to ω without also the introduction of hyperedges to E will always

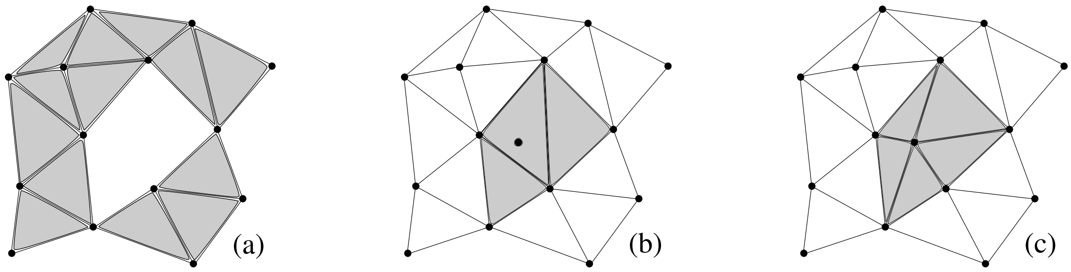


Figure 3.3: The hyperedge sets $E^{\text{ext}}_{x_0|\omega}$ (a), $E^{-}_{x_0|\omega}$ (b) and $E^{+}_{x_0|\omega}$ (c).

increase the number of connected components by one. On the other hand, the augmentation of a single hyperedge $\tau \in \text{Del}_3(\omega)$ to E can result in the connection of a maximum of three different connected components, leaving one. Therefore, for $\omega \in \Omega$, $E \subset \text{Del}_3(\omega)$, $x_0 \in \Lambda \backslash \omega$ and $\tau \in \text{Del}_3(\omega) \backslash E$, we conclude that

$$K(\omega \cup \{x_0\}, E) - K(\omega, E) = 1 \tag{3.40}$$

and

$$-2 \leq K(\omega, E \cup \tau) - K(\omega, E) \leq 0. \tag{3.41}$$

We are now in a position to prove that the Delaunay random cluster distribution for (3.34) has a bounded Papangelou conditonal intensity.

Lemma 3.14. (BPI – Model 1). *For every finite box $\Lambda \subset \mathbf{R}^2$, $M_{\Lambda|\xi}$ – almost all $\omega \in \Omega_{\Lambda|\xi}$ and a point x_0, with $x_0 \in \Lambda \backslash \omega$,*

$$\frac{h_\Lambda(\omega \cup \{x_0\})}{h_\Lambda(\omega)} \geq q^{-\frac{4\pi}{\beta_0}}.$$

Proof. By adding the point x_0 to ω we change the structure of the Delaunay triangulation, as described above. Some hyperedges are destroyed, some are created, and some are not changed. Recall the space of locally finite hyperedge sets $\mathcal{E}$ defined in Equation (2.26) and let

$$\mathcal{E}^* := \{E \subset \mathcal{E} : \beta_0 \leq \beta(\tau) \leq \pi/3 \text{ for all } \tau \in E \text{ with } \tau \cap \Lambda \neq \emptyset\},$$

be the set of 'nice' hyperedge configurations where each hyperedge $\tau \in E$ that intersects Λ

has smallest interior angle at least β_0. We then have

$$\frac{h_\Lambda(\omega\cup\{x_0\})}{h_\Lambda(\omega)} = \frac{\int q^{K(\omega\cup\{x_0\},E)}\mu_{\omega\cup\{x_0\},\Lambda}(dE)}{\int q^{K(\omega,E)}\mu_{\omega,\Lambda}(dE)}$$

$$= \frac{\int_{\mathcal{E}^*} q^{K(\omega\cup\{x_0\},E)}\mu_{\omega\cup\{x_0\},\Lambda}(dE) + \int_{\mathcal{E}\setminus\mathcal{E}^*} q^{K(\omega\cup\{x_0\},E)}\mu_{\omega\cup\{x_0\},\Lambda}(dE)}{\int_{\mathcal{E}^*} q^{K(\omega,E)}\mu_{\omega,\Lambda}(dE) + \int_{\mathcal{E}\setminus\mathcal{E}^*} q^{K(\omega,E)}\mu_{\omega,\Lambda}(dE)}.$$

The second term on both the numerator and denominator vanish because of φ. In particular, $p_\Lambda(\tau) = 1-e^{-\varphi(\tau,\omega)}$ for triangles having a non-empty intersection with Λ and if $\beta(\tau) < \beta_0$, then $p_\Lambda(\tau) = 0$. It follows that,

$$\frac{h_\Lambda(\omega\cup\{x_0\})}{h_\Lambda(\omega)} = \frac{\int_{\mathcal{E}^*} q^{K(\omega\cup\{x_0\},E)}\mu_{\omega\cup\{x_0\},\Lambda}(dE)}{\int_{\mathcal{E}^*} q^{K(\omega,E)}\mu_{\omega,\Lambda}(dE)}$$

$$= \frac{\int_{\mathcal{E}^*\cap E^{\text{ext}}_{x_0|\omega}} \int_{\mathcal{E}^*\cap E^{+}_{x_0|\omega}} q^{K(\omega\cup\{x_0\},E_1\cup E_2)}\mu^{+}_{x_0|\omega}(dE_2)\mu^{\text{ext}}_{x_0|\omega}(dE_1)}{\int_{\mathcal{E}^*\cap E^{\text{ext}}_{x_0|\omega}} \int_{\mathcal{E}^*\cap E^{-}_{x_0|\omega}} q^{K(\omega,E_3\cup E_4)}\mu^{-}_{x_0|\omega}(dE_4)\mu^{\text{ext}}_{x_0|\omega}(dE_3)}$$

$$= \frac{\int_{\mathcal{E}^*\cap E^{\text{ext}}_{x_0|\omega}} \int_{\mathcal{E}^*\cap E^{+}_{x_0|\omega}} q^{K(\omega\cup\{x_0\},E_1\cup E_2)-K(\omega,E_1)}\mu^{+}_{x_0|\omega}(dE_2)q^{K(\omega,E_1)}\mu^{\text{ext}}_{x_0|\omega}(dE_1)}{\int_{\mathcal{E}^*\cap E^{\text{ext}}_{x_0|\omega}} \int_{\mathcal{E}^*\cap E^{-}_{x_0|\omega}} q^{K(\omega,E_3\cup E_4)-K(\omega,E_3)}\mu^{-}_{x_0|\omega}(dE_4)q^{K(\omega,E_3)}\mu^{\text{ext}}_{x_0|\omega}(dE_3)}. \tag{3.42}$$

However, since $E_1 \cup E_2 \in \mathcal{E}^*$, either $\beta(\tau) \geq \beta_0$ for all $\tau \in E_2$ or $E_2 = \emptyset$. It follows, since $E_2 \subset E^{+}_{x_0|\omega}$, that the maximum number of hyperedges in E_2 is $\frac{2\pi}{\beta_0}$. Therefore, by Equations (3.40) and (3.41), we conclude that

$$K(\omega\cup\{x\}, E_1\cup E_2) - K(\omega, E_1) \geq -\frac{4\pi}{\beta_0},$$

and

$$K(\omega, E_3\cup E_4) - K(\omega, E_3) \leq 0.$$

Combining with Equation (3.42), we obtain

$$\frac{h_\Lambda(\omega\cup\{x\})}{h_\Lambda(\omega)} \geq q^{-\frac{4\pi}{\beta_0}}.$$

$\square$

Lemma 3.15. (CGR – Model 1)
There exists a constant $U_1 > 0$ such that for all $\omega \in F_{k,l} \cap F_{k+1,l}$ and for all $\tau \in H^L_{k:k+1,l}(\omega)$

$$\varphi(\tau, \omega) \geq \varphi(U_1),$$

where L satisfies

$$0 < \delta_0 < \frac{L}{2} \quad \text{and} \quad L < \frac{\delta_0}{\sqrt{8}\beta_0} =: U_2. \tag{3.43}$$

Proof. Suppose $L \in [2\delta_0, U_2]$ (the interval is non-empty due to our restriction on β_0, namely $\beta_0 \in (0, \frac{1}{4\sqrt{2}}]$). Then, given $\omega \in F_{k,l} \cap F_{k+1,l}$, recall the central band of $\Delta_{k,l} \cup \Delta_{k+1,l}$,

$$CB^L_{k:k+1,l} = \left(\bigcup_{i=0}^{4} \Delta^{4+i,4}_{k,l} \right) \cup \left(\bigcup_{i=0}^{4} \Delta^{i,4}_{k+1,l} \right).$$

and the subset of hyperedges of $\mathrm{Del}_3(\omega)$ that have non-empty intersection with $CB^L_{k:k+1,l}$

$$H^L_{k:k+1,l}(\omega) = \left\{ \tau \in \mathrm{Del}_3(\omega) : \tau \cap CB^L_{k:k+1,l} \neq \emptyset \right\}.$$

Since all of the little squares $\Delta^{i,j}_{k,l}, i, j = 0, \ldots, 8$ contain at least one point, we have that every open ball of radius at least $\sqrt{2}L$ and centre $y \in CB_{k:k+1,l}$ has a non-empty intersection with ω. Therefore, for each $\tau \in H^L_{k:k+1,l}(\omega)$, the circumscribing ball $B(\tau)$ has radius less than $\sqrt{2}L$.

Let $\tau = \{a, b, c\}$ be such a triangle. Without loss of generality, let $\beta(\tau)$ be the angle $\widehat{acb}$ and let ℓ be the arc length of the arc on $\partial B(\tau)$ between a and b. Let x be the centre of $B(\tau)$. It follows that $\widehat{axb} = 2\beta(\tau)$ and $\ell = 2r\beta(\tau)$ where r is the radius of $B(\tau)$. By the hardcore condition $\ell > |a - b| \geq \delta_0$. Combining these statements with the fact that $r < \sqrt{2}L$ gives

$$\beta(\tau) > \frac{\delta_0}{\sqrt{8}L}.$$

Therefore, because φ is an increasing function of θ, we obtain that, for all $\tau \in H^L_{k:k+1,l}(\omega)$,

$$\varphi(\tau, \omega) = \varphi(\beta(\tau)) \geq \varphi\left(\frac{\delta_0}{\sqrt{8}L} \right) \geq \varphi(\beta_0).$$

By setting, $U_1 = \beta_0$, the result follows. □

Remark 3.16. *The upper bound constraint of β_0 is due to our proof method: we are not claiming that there is no phase transition for $\beta_0 \in (\frac{1}{4\sqrt{2}}, \pi/3]$. Indeed in the following* *model, such a harsh restriction does not apply.*

3.7 Model 1b – Relaxing of the type interaction

Rather than having a hard type interaction as in (3.33), we now look at a smooth potential on $\beta(\tau)$, the smallest angle in a triangle τ. We take an increasing function of $\beta(\tau)$ which will make interactions stronger between triplets of points that do not share a common colour if they form a triangle closer to an equilateral triangle. The interpretation in the Delaunay random cluster model is that triangles with larger minimum angle are more likely to exist. The hardcore background interaction stays the same.

Let ψ be as in (3.1), and let the type interaction satisfy

$$\varphi(\tau, \omega) \equiv \varphi(\beta(\tau)) \quad \text{for} \quad \tau \in \mathrm{Del}_3(\omega),$$

where $\beta(\tau)$ denotes the smallest interior angle of the triangle τ, and $\varphi : [0, \pi/3] \to \mathbf{R} \cup \{\infty\}$ is defined as

$$\varphi(\theta) := \log(1 + A\theta^3) \tag{3.44}$$

Again, A takes the role of the inverse temperature. Equation (3.44) is another infinite range type interaction as it can act between triplets of particles far away from one another, providing they form a hyperedge in Del_3 and do not share a common mark. Note that we chose the exponent of θ to be the lowest sufficient to satisfy a technical constraint of our proof of Lemma 3.20. For a marked configuration $\overline{\omega} = (\omega, \sigma_\omega) \in \overline{\Omega}$, the Hamiltonian is given by

$$H^{\overline{\Phi}}_{\Lambda|\overline{\xi}}(\overline{\omega}) = \sum_{\substack{\tau \in \mathrm{Del}_3(\omega \cup \xi) \\ \tau_\Lambda \neq \emptyset}} \psi(\tau, \omega \cup \xi) + \sum_{\substack{(\tau, \sigma_\tau) \in \mathrm{Del}_3(\overline{\omega} \cup \overline{\xi}) \\ \tau_\Lambda \neq \emptyset}} \varphi(\tau, \omega \cup \xi)(1 - \delta_{\sigma_\tau}(\tau)). \tag{3.45}$$

Since ψ is a hardcore background interaction and φ is non-negative and satisfies the local horizon property, the existence of at least one Delaunay Potts measure follows from Proposition 3.2.

Theorem 3.17. *For all $\delta_0 > 0$, there exists $A_0 = A_0(\delta_0)$ and $z_0 = z_0(\delta_0, A_0)$ such that for all $A > A_0$ and $z > z_0$, there exists at least q different Delaunay Potts measures for the Hamiltonian (3.45).*

Again it is sufficient to show that $C_{\Lambda|\xi}$ satisfies the **(CGR)** and **(BPI)** conditions. The first is straightforward.

Lemma 3.18. (CGR – Model 1b)
There exists a constant $U_1 > 0$ such that for all $\omega \in F_{k,l} \cap F_{k+1,l}$ and for all $\tau \in H^L_{k:k+1,l}(\omega)$,

$$\varphi(\tau, \omega) \geq \varphi(U_1),$$

where $L \in [2\delta_0, U_2]$ for some $U_2 > 2\delta_0$.

Proof. Fix $U_2 = 4\delta_0$. From the proof of Lemma 3.15, we have that $\beta(\tau) > \frac{\delta_0}{\sqrt{8}L}$ for all $\tau \in H^L_{k:k+1,l}(\omega)$, for $\omega \in F_{k,l} \cap F_{k+1,l}$. Since φ is an increasing function of θ, choose $U_1 = \frac{\delta_0}{\sqrt{8}U_2} = \frac{1}{4\sqrt{8}}$. □

Remark 3.19. *Note that the choice of U_2 in the proof of Lemma 3.18 is arbitrary, so long as it is strictly greater than $2\delta_0$. This allows us to split Λ into very large boxes, which gives a better estimate for z_0 and a worse estimate for A_0. To find a better estimate for A_0 in Theorem 3.17 we must choose a smaller box length L.*

Showing the **(BPI)** condition is a little harder. Given a configuration $\omega \in \Omega$, we want to investigate the effect of adding a point $x_0 \in \mathbf{R}^2$ to ω on the number of connected components in the the continuum random cluster model. In particular, we would like to bound below the following:

$$K(\omega \cup \{x_0\}, E_1 \cup E) - K(\omega, E_1),$$

where $E_1 \subset E^{\text{ext}}_{x_0|\omega}$ and $E \subset E^+_{x_0|\omega}$. Given $E \subset E^+_{x_0|\omega}$, let $|E|$ denote its cardinality. A single application of Equation (3.40) and repeated applications of Equation (3.41) show us that

$$K(\omega \cup \{x_0\}, E_1 \cup E) - K(\omega, E_1) \geq -2|E|. \tag{3.46}$$

In the case of the Delaunay random cluster representation from Model 1, $|E|$ was easy to bound above because we had a strict interaction prohibiting open hyperedges $\tau \in \text{Del}_3(\omega)$ with an interior angle less that β_0. However, in the current model, we have no hard restriction on the smallest interior angles. Open hyperedges $\tau \in \text{Del}_3(\omega)$ with very small interior angles are permitted, if not likely, therefore, we cannot find an upper bound for $|E|$. Instead, we look to bound the expected value.

Lemma 3.20. *Let $\Lambda \in \mathcal{B}(\mathbf{R}^2)$. Then $\int |E| \mu^+_{x_0|\omega}(dE) \leq 2\pi \left(1 + \frac{A\pi}{3}^2\right)$ for all $\omega \in \Omega$ and $x_0 \in \Lambda \backslash \omega$.*

Before we prove Lemma 3.20, we show how it can be used in order to satisfy the **(BPI)** condition.

Lemma 3.21. (BPI – Model 1b)
For all $\Lambda \in \mathcal{B}(\mathbf{R}^2)$, for all configurations $\omega \in \Omega$ and for all points $x_0 \in \Lambda \backslash \omega$,

$$\frac{h_\Lambda(\omega \cup \{x_0\})}{h_\Lambda(\omega)} \geq q^{-4\pi(1+\frac{A\pi^2}{3})}.$$

Proof. Recalling the definition of $h_\Lambda(\omega)$ from (3.16), we write the integrals in terms of $\mu^-_{x_0|\omega}$, $\mu^+_{x_0|\omega}$ and $\mu^{\text{ext}}_{x_0|\omega}$.

$$\begin{aligned}\frac{h_\Lambda(\omega \cup \{x_0\})}{h_\Lambda(\omega)} &= \frac{\int q^{K(\omega\cup\{x_0\},E)}\mu_{\omega\cup\{x_0\},\Lambda}(dE)}{\int q^{K(\omega,E)}\mu_{\omega,\Lambda}(dE)} \\ &= \frac{\int q^{K(\omega,E_1)}\left(\int q^{K(\omega\cup\{x_0\},E_1\cup E_2)-K(\omega,E_1)}\mu^+_{x_0|\omega}(dE_2)\right)\mu^{\text{ext}}_{x_0|\omega}(dE_1)}{\int q^{K(\omega,E_3)}\left(\int q^{K(\omega,E_3\cup E_4)-K(\omega,E_3)}\mu^-_{x_0|\omega}(dE_4)\right)\mu^{\text{ext}}_{x_0|\omega}(dE_3)}.\end{aligned} \tag{3.47}$$

However,

$$\begin{aligned}\int q^{K(\omega\cup\{x_0\},E_1\cup E_2)-K(\omega,E_1)}\mu^+_{x_0|\omega}(dE_2) &\geq \int q^{-2|E_2|}\mu^+_{x_0|\omega}(dE_2) \\ &\geq q^{-2\int|E_2|\mu^+_{x_0|\omega}(dE_2)} \\ &\geq q^{-4\pi\left(1+\frac{A\pi}{3}^2\right)}.\end{aligned} \tag{3.48}$$

The first inequality comes from Equation 3.46. The second inequality is due to Jensen's inequality and the third to Lemma 3.20. We also know, from (3.41), that

$$K(\omega, E_3 \cup E_4) - K(\omega, E_3) \leq 0, \tag{3.49}$$

because adding hyperedges can only reduce the number of clusters. Combining (3.47), (3.48) and (3.49), we obtain

$$\begin{aligned}\frac{h_\Lambda(\omega \cup \{x_0\})}{h_\Lambda(\omega)} &\geq \frac{\int q^{K(\omega,E_1)}\mu^{\text{ext}}_{x_0|\omega}(dE_1)q^{-4\pi\left(1+\frac{A\pi}{3}^2\right)}}{\int q^{K(\omega,E_3)}\mu^{\text{ext}}_{x_0|\omega}(dE_3)q^0} \\ &= q^{-4\pi\left(1+\frac{A\pi}{3}^2\right)}.\end{aligned}$$

□

Proof of Lemma 3.20. The number of open hyperedges, $|E|$, that have a non-empty intersection with $\{x_0\}$ is obviously dominated by the total number of hyperedges of $E^+_{x_0|\omega}$. We also know, by (2.28), that the probability that a hyperedge is open with respect to $\mu^+_{x_0|\omega}$ is p_Λ. Therefore, the expectation of $|E|$ with respect to $\mu^+_{x_0|\omega}$ satisfies

$$\int |E|\mu^+_{x_0|\omega}(dE) = \sum_{\tau \in E^+_{x_0|\omega}} p_\Lambda(\tau) = \sum_{\tau \in E^+_{x_0|\omega}} \left[1 - e^{-\varphi(\tau,\omega)}\right]. \tag{3.50}$$

The problem we have is that the number of hyperedges in $E^+_{x_0|\omega}$ has no upper bound. Therefore, we partition $E^+_{x_0|\omega}$ into a sequence of disjoint sets $(H_i)_{i\geq 1}$ where

$$H_1 := \left\{\tau \in E^+_{x_0|\omega} : \beta(\tau) \geq 1\right\}$$

and

$$H_i := \left\{\tau \in E^+_{x_0|\omega} : \frac{1}{i} \leq \beta(\tau) < \frac{1}{i-1}\right\}$$

for $i \geq 2$, $i \in \mathbf{N}$. For $i \in \mathbf{N}$, the cardinality of the set H_i is no larger than $2\pi i$. Due to the fact that φ is an increasing function of θ, it follows that $p_\Lambda(\tau) \leq 1 - \exp[-\varphi(\frac{1}{i-1})]$ for all $\tau \in H_i$ providing $i \geq 2$. Hence, by (3.50), we obtain

$$\begin{aligned}\int |E|\mu^+_{x_0|\omega}(dE) &= \sum_{i=1}^{\infty}\sum_{\tau \in H_i} \left[1 - e^{-\varphi(\tau,\omega)}\right] \leq 2\pi + \sum_{i=2}^{\infty} 2\pi i \left[1 - e^{-\varphi\left(\frac{1}{i-1}\right)}\right] \\ &= 2\pi + \sum_{i=2}^{\infty} 2\pi i \frac{A}{(i-1)^3+1} \leq 2\pi \left[1 + A\sum_{i=2}^{\infty}\frac{i}{(i-1)^3}\right].\end{aligned}$$

To bound the infinite sum, we use the fact that

$$\sum_{i=2}^{\infty}\frac{i}{(i-1)^3} \leq \sum_{i=2}^{\infty}\frac{2}{(i-1)^2} = 2\sum_{i=1}^{\infty}\frac{1}{i^2} = \frac{\pi^2}{3}. \tag{3.51}$$

Therefore,

$$\int |E|\mu^+_{x_0|\omega}(dE) \leq 2\pi\left(1 + \frac{A\pi^2}{3}\right). \tag{3.52}$$

□

3.8 Model 2 - Delaunay Potts model with infinite range

In [BBD03], the existence of a phase transition is shown for the nearest-neighbour continuum Potts model with finite range on the Delaunay graph. We are going to extend this study to the case of infinite range type interactions. The hardcore background interaction ψ

remains as in (3.1) and (3.2). However, we propose a type interaction φ with infinite range between pairs of particles, or edges, in Del_2 that gives a positive energy to edges with no common mark between particles. This type interaction will depend on the length of the hyperedge. In particular,

$$\varphi(\eta, \omega) \equiv \varphi(|x-y|) \quad \text{for} \quad \eta = \{x, y\} \in \mathrm{Del}_2(\omega),$$

where $|x-y|$ is the Euclidean distance between x and y in $\mathbf{R}^2$, and $\varphi : [0, \infty] \to \mathbf{R} \cup \{\infty\}$ is defined as

$$\varphi(\ell) := \log\left(1 + A\left(\frac{\delta_0}{\ell}\right)^3\right)$$

where $A > 0$ is an inverse temperature parameter for the type interaction. This is an infinite range type interaction as it can act between pairs of particles far away from one another, providing they form a hyperedge in Del_2 and do not share a common mark. Define the hyperedge Hamiltonian of a marked particle configuration $\overline{\omega} = (\omega, \sigma_\omega) \in \overline{\Omega}_\Lambda$, with an admissible boundary configuration $\overline{\xi} = (\xi, \sigma_\xi)$, by

$$H^{\overline{\Phi}}_{\Lambda|\overline{\xi}}(\overline{\omega}) = \sum_{\substack{\eta \in \mathrm{Del}_2(\omega \cup \xi) \\ \eta \cap \Lambda \neq \emptyset}} \psi(\eta, \omega \cup \xi) + \sum_{\substack{(\eta, \sigma_\eta) \in \mathrm{Del}_2(\overline{\omega} \cup \overline{\xi}) \\ \eta \cap \Lambda \neq \emptyset}} \varphi(\eta, \omega \cup \xi)(1 - \delta_{\sigma_\eta}(\eta)) \tag{3.53}$$

where δ_{σ_η} is the indicator defined in (2.22). We now turn to the existence and uniqueness questions. Again, since ψ is a hardcore background interaction and φ is non-negative and satisfies the local horizon property, the existence of at least one Delaunay Potts measure for the Hamiltonian (3.53) follows from Proposition 3.2.

Theorem 3.22. *For all $\delta_0 > 0$, there exists $A_0 = A_0(\delta_0)$ and $z_0 = z_0(A, \delta_0)$ such that for $A > A_0$ and $z > z_0$, there exists at least q different Delaunay Potts measures for the Hamiltonian (3.53).*

Lemma 3.23. (CGR – Model 2)
There exists a constant $U_1 > 0$ such that for all $\omega \in F_{k,l} \cap F_{k+1,l}$ and for all $\eta \in H^L_{k:k+1,l}(\omega)$,

$$\varphi(\eta, \omega) \geq \varphi(U_1),$$

where $L \in [2\delta_0, U_2]$ for some $U_2 > 2\delta_0$.

Proof. Fix $U_2 > 2\delta_0$ and suppose $L \in [2\delta_0, U_2]$. Then, given $\omega \in F_{k,l} \cap F_{k+1,l}$, recall the central band of $\Delta_{k,l} \cup \Delta_{k+1,l}$

$$CB^L_{k:k+1,l} = \left(\bigcup_{i=0}^{4} \Delta^{4+i,4}_{k,l} \right) \cup \left(\bigcup_{i=0}^{4} \Delta^{i,4}_{k+1,l} \right).$$

and the subset of hyperedges of $\mathrm{Del}_2(\omega)$ that have non-empty intersection with $CB^L_{k:k+1,l}$

$$H^L_{k:k+1,l}(\omega) = \left\{ \eta \in \mathrm{Del}_2(\omega) : \eta \cap CB^L_{k:k+1,l} \neq \emptyset \right\}.$$

Let $\eta = \{x_1, x_2\} \in H^L_{k:k+1,l}(\omega)$. It follows that there exists $x_3 \in \omega$ such that $\tau = \{x_1, x_2, x_3\} \in \mathrm{Del}_3(\omega)$ and $\tau \cap CB^L_{k:k+1,l} \neq \emptyset$. Since all of the little squares $\Delta^{i,j}_{k,l}$ for $i, j = 0, \ldots, 8$ contain at least one point, we have that every open ball of radius at least $\sqrt{2}L$ that has a non-empty intersection with $CB_{k:k+1,l}$, also has a non-empty intersection with ω. Therefore, the circumscribing ball $B(\tau, \omega)$ has radius less than $\sqrt{2}L$ and hence, $|x_1 - x_2| < 2\sqrt{2}L$. Since φ is a decreasing function of ℓ, choose $U_1 = 2\sqrt{2}U_2$ and the result follows.

□

As we are now working with the hypergraph structure Del_2 as opposed to Del_3 in Models 1 and 1b, we switch our focus from triangles to edges and therefore need to realign our notation before we can show **(BPI)**. In particular, we define the sets of exterior, created and destroyed hyperedges of $\mathrm{Del}_2(\omega)$ and $\mathrm{Del}_2(\omega \cup \{x_0\})$. Let

$$E^{\mathrm{ext}}_{x_0|\omega} := \mathrm{Del}_2(\omega) \cap \mathrm{Del}_2(\omega \cup \{x_0\}), \tag{3.54}$$

$$E^{+}_{x_0|\omega} := \mathrm{Del}_2(\omega \cup \{x_0\}) \backslash \mathrm{Del}_2(\omega) = \Big\{ \eta \in \mathrm{Del}_2(\omega \cup \{x_0\}) : \eta \cap x_0 = x_0 \Big\}, \tag{3.55}$$

and

$$E^{-}_{x_0|\omega} := \mathrm{Del}_2(\omega) \backslash \mathrm{Del}_2(\omega \cup \{x_0\}) = \mathrm{Del}_3(\omega) \backslash E^{\mathrm{ext}}_{x_0|\omega}, \tag{3.56}$$

be the exterior, created, and destroyed hyperedges respectively, see Figure 3.4, and define $\mu^-_{x_0|\omega}$, $\mu^+_{x_0|\omega}$ and $\mu^{\mathrm{ext}}_{x_0|\omega}$ to be the edge drawing mechanisms on $E^-_{x_0|\omega}$, $E^+_{x_0|\omega}$ and $E^{\mathrm{ext}}_{x_0|\omega}$ respectively.

We show **(BPI)** with a similar method as to the one used for Model 1b. Namely, for a p_Λ thinning of $E^+_{x_0|\omega}$, E, we look for an upper bound for the cardinality of E. The augmentation of a single hyperedge $\eta \in \mathrm{Del}_2(\omega)$ to E can result in the connection of a

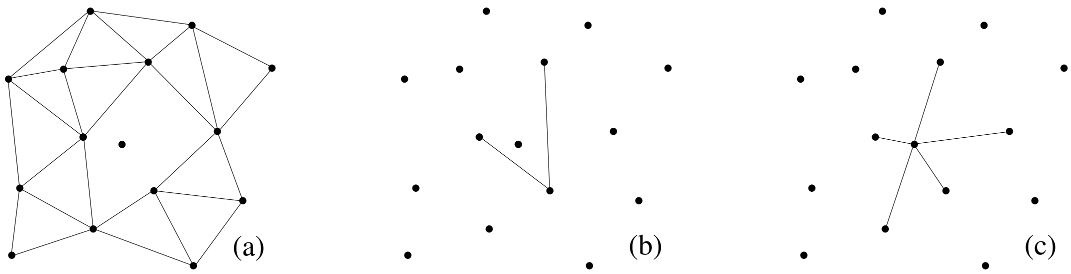


Figure 3.4: The hyperedge sets $E^{\text{ext}}_{x_0|\omega}$ (a), $E^{-}_{x_0|\omega}$ (b) and $E^{+}_{x_0|\omega}$ (c).

maximum of two different connected components, leaving one. Therefore, for $\omega \in \Omega$, $E \subset \text{Del}_2(\omega)$ and $\eta \in \text{Del}_2(\omega)\backslash E$,

$$-1 \leq K(\omega, E \cup \eta) - K(\omega, E) \leq 0. \tag{3.57}$$

A single application of Equation (3.40) and repeated applications of Equation (3.57) show us that

$$K(\omega \cup \{x_0\}, E_1 \cup E) - K(\omega, E_1) \geq -|E|, \tag{3.58}$$

where $E_1 \subset E^{\text{ext}}_{x_0|\omega}$ and $E \subset E^{+}_{x_0|\omega}$. The right hand side of (3.58) differs from that of (3.46) by a factor 2 due to the fact that we are using edges and not triangles. For the finite range nearest neighbour model of [BBD03], it is easy to obtain an upper bound for $|E|$. However, in the current regime, we have neither a restriction on the number of neighbours, like in Model 1 and 1b, nor a restriction on the range of the interaction, as in [BBD03], therefore, there is no upper bound for $|E|$. Again, we look to bound its expected value with respect to $\mu^{+}_{x_0|\omega}$. This is shown in Lemma 3.24 – the proof requires that particles not be too close together – we conveniently make use of the hardcore background interaction.

Lemma 3.24. *Let $\Lambda \in \mathcal{B}(\mathbf{R}^2)$. Then $\int |E|\mu^{+}_{x_0|\omega}(dE) \leq 4\left[9 + \frac{A\pi^2}{3}\right]$, for $M_{\Lambda|\xi}$ – almost all $\omega \in \Omega$ and $x_0 \in \Lambda\backslash\omega$.*

By taking $\alpha = 4\left[9 + \frac{A\pi^2}{3}\right]$ and by using (3.58) instead of (3.46), it can then be shown, by the proof of Lemma 3.21, that

$$\frac{h_\Lambda(\omega \cup \{x_0\})}{h_\Lambda(\omega)} \geq q^{-4\left[9+\frac{A\pi^2}{6}\right]}, \tag{3.59}$$

almost surely with respect to $M_{\Lambda|\xi}$, and hence that $C_{\Lambda|\xi}$ satisfies the **(BPI)** condition.

Proof. Given $\omega \cup \{x_0\} \in \Omega$, such that $x_0 \notin \omega$, let $B(x_0, r) \subset \mathbf{R}^2$ be the ball of radius r centred at x_0. The background interaction ψ ensures that with probability one, under $M_{\Lambda|\xi}$, no two particles of ω form a hyperedge of length less than δ_0 in $\mathrm{Del}_2(\omega)$. Moreover, with probability one, any two particles are at least distance δ_0 apart – see Lemma 3.1. Therefore, almost surely with respect to $M_{\Lambda|\xi}$, we obtain the following upper bound for the number of points $y \in \omega$ with $|y - x_0| < r$:

$$|B(x_0, r) \cap \omega| \leq \frac{\pi(r + \delta_0)^2}{\pi(\delta_0/2)^2} = 4\left(\frac{r + \delta_0}{\delta_0}\right)^2. \tag{3.60}$$

This is just the maximum number of balls of radius δ_0 that you can pack together in $\mathbf{R}^2$, whose centres lie within Euclidean distance r of x_0. Therefore, using (3.60) for values $r = 2\delta_0, 4\delta_0 \ldots$, we see that

$$|B(x_0, 2\delta_0) \cap \omega| \leq 4\left(\frac{3\delta_0}{\delta_0}\right)^2 = 36 =: b_1, \tag{3.61}$$

$$|B(x_0, 4\delta_0) \cap \omega| \leq 4\left(\frac{5\delta_0}{\delta_0}\right)^2 = 100 =: b_2, \tag{3.62}$$

$$\vdots$$

$$|B(x_0, 2n\delta_0) \cap \omega| \leq 4\left(\frac{(2n+1)\delta_0}{\delta_0}\right)^2 = 4(2n+1)^2 =: b_n. \tag{3.63}$$

The number of hyperedges of $E^+_{x_0|\omega}$ that have length less than r is obviously bounded above by $|B(x_0, r) \cap \omega|$, however, as r increases, this bound grows quadratically. On the other hand, according to φ, hyperedges with large length are less likely to remain in a p_Λ–thinning of $E^+_{x_0|\omega}$ than their counterparts with small length. In the following, we try to play these two facts against each other. We start by partitioning the plane into annuli of increasing radius centred at x_0 such that

$$\mathbf{R}^2 = \bigcup_{n=0}^{\infty} AN_n,$$

where

$$AN_0 := B(x_0, \delta_0),$$
$$AN_n := B(x_0, 2n\delta_0) \backslash AN_{n-1} \quad \text{for } n \in \mathbf{N}.$$

We then let E be a p_Λ thinning of the hyperedge set $E^+_{x_0|\omega}$ for a finite box $\Lambda \in \mathcal{B}(\mathbf{R}^2)$. Let $\eta = \{x_0, x\} \in E^+_{x_0|\omega}$. Since x_0 lies in Λ, we recall from (2.28) that the probability of η appearing in E is given by $p_\Lambda(\eta) = 1 - \exp\{-\varphi(|x_0 - x|)\}$. For convenience we let p_Λ take a real valued argument and write

$$p_\Lambda(\ell) := 1 - \exp\{-\varphi(\ell)\}.$$

Then for $n \in \mathbf{N}$, define

$$E^n := \{\eta = \{x_0, x\} \in E : |x_0 - x| < 2n\delta_0\},$$

the subset of hyperedges of E that have length less than $2n\delta_0$. From (3.61), we see there can be at most 36 particles of ω within distance $2\delta_0$ of x_0. Therefore,

$$\int_{\mathcal{E}(\omega,x_0)} |E^1| \mu^+_{x_0|\omega}(dE) \leq 36.$$

Similarly, for $n = 2$, there are $|B(x_0, 4\delta_0) \cap \omega|$ particles to consider. As above, at most 36 of these are of distance less than $2\delta_0$ from x_0 and have probability at most 1 of sharing a hyperedge with x_0 in E. The remaining particles lie in the annulus AN_2 and due to the fact that p_Λ is a decreasing function of distance, they have at most probability $p_\Lambda(2\delta_0)$ of sharing a hyperedge with x_0 in E. Therefore,

$$\int_{\mathcal{E}(\omega,x_0)} |E^2| \mu^+_{x_0|\omega}(dE) \leq 36 + (b_2 - b_1) p_\Lambda(2\delta_0),$$

and in general

$$\begin{aligned}
\int_{\mathcal{E}(\omega,x_0)} |E^n| \mu^+_{x_0|\omega}(dE) &\leq 36 + \sum_{i=1}^{n-1} (b_{i+1} - b_i) p_\Lambda(2i\delta_0) \\
&= 4 \left[9 + \sum_{i=1}^{n-1} \Big((2i+1)^2 - (2i-1)^2 \Big) p_\Lambda(2i\delta_0) \right] \\
&= 4 \left[9 + \sum_{i=1}^{n-1} 8i p_\Lambda(2i\delta_0) \right].
\end{aligned}$$

We have defined a non-negative sequence of random variables $\{|E^n|\}_{n\in\mathbf{N}}$ such that $|E^n| \nearrow |E|$ as $n \to \infty$ for all $E \in \mathcal{E}(\omega)$. By the monotone convergence theorem, we find

$$\begin{aligned}
\int_{\mathcal{E}(\omega,x_0)} |E| \mu^+_{x_0|\omega}(dE) &= \int_{\mathcal{E}(\omega,x_0)} \lim_{n\to\infty} (|E^n|)\, \mu^+_{x_0|\omega}(dE) \\
&= \lim_{n\to\infty} \left(\int_{\mathcal{E}(\omega,x_0)} |E^n| \mu^+_{x_0|\omega}(dE) \right) \\
&\leq 4 \left[9 + \lim_{n\to\infty} \sum_{i=1}^{n-1} 8i p_\Lambda(2i\delta_0) \right]. \qquad (3.64)
\end{aligned}$$

Recall that

$$\varphi(\ell) := \log\left(1 + A\left(\frac{\delta_0}{\ell}\right)^3\right),$$

and therefore,

$$p_\Lambda(\ell\delta_0) = 1 - e^{-\varphi(\ell\delta_0)} = 1 - e^{\log(1-\frac{A}{\ell^3+A})} = \frac{A}{\ell^3 + A}. \tag{3.65}$$

Then by combining (3.65) with (3.64), we obtain

$$\begin{aligned}\int |E|\mu^+_{x_0|\omega}(dE) &\leq 4\left[9 + \sum_{n=1}^{\infty} \frac{8nA}{(2n)^3 + A}\right] \leq 4\left[9 + \sum_{n=1}^{\infty} \frac{8nA}{(2n)^3}\right] \\ &= 4\left[9 + A\sum_{n=1}^{\infty} \frac{1}{n^2}\right] = 4\left[9 + \frac{A\pi^2}{6}\right].\end{aligned}$$

which gives the statement of the Lemma. □

Having shown that the Delaunay random cluster distribution $C_{\Lambda|\xi}$ satisfies both the **(CGR)** and **(BPI)** conditions, Theorem 3.22 follows from Proposition 3.7.

3.9 Lower bounds for activity and temperature parameters in the phase transition regime

Table 3.9 presents our estimates for z_0 and A_0 for each of the models in this Chapter. As discussed in Remark 3.19, the choice of box length L in the splitting of Λ can be used to optimise either z_0 or A_0. However, L must be chosen in the interval $[2\delta_0, U_2]$. We choose $L = 4\delta_0$ in each model.

	z_0	A_0
Model 1	$\frac{81q^{\frac{4\pi}{\beta_0}}}{\epsilon\delta_0^2\left(\frac{1}{\sqrt{2}\beta_0}-1\right)^2}$	$\log\left[\frac{1+(q^2-1)a}{1-a}\right]$
Model 1b	$\frac{81q^{4\pi(1+A\pi^2/3)}}{4\epsilon\delta_0^2}$	$32^{3/2}\left(1+\frac{q^2a}{1-a}\right)$
Model 2	$\frac{81q^{4\left(9+\frac{A\pi^2}{6}\right)}}{4\epsilon\delta_0^2}$	$32\sqrt{2}\delta_0^2\left(1+\frac{qa}{1-a}\right)$

Table 3.1: Activity and inverse temperature estimates. For brevity, we denote by a the constant $(1-\epsilon)^{\frac{\pi}{1444}}$.

Chapter 4

Soft Widom-Rowlinson Model on Delaunay Graph

4.1 Introduction

Chapters 2 and 3 concerned the extension of [BBD03] to the case of type interactions with hyperedge potentials with infinite range. Our arguments relied on the hardcore background interactions. We now look at a class of models that do not have any background interaction at all. The motivation comes from the Widom-Rowlinson model [WR70] – a marked point process with particles of two types: the mark space takes the form $\Sigma = \{-1, +1\}$. There is no background interaction, only a hardcore exclusion between particles of different type. We present the model here in the hypergraph structure framework of our Chapter 2. Although we present it in two dimensions to fall in line with the work of this thesis, the Widom-Rowlinson model is actually well defined on $\mathbf{R}^d$ for all $d \geq 2$. The underlying hypergraph structure is simply the complete hypergraph CG. Given a marked configuration $\overline{\omega} \in \overline{\Omega}$, recall that $\mathrm{CG}_2(\overline{\omega})$ is the set of hyperedges built from a pair of particles in $\overline{\omega}$. That is

$$\mathrm{CG}_2(\overline{\omega}) := \{\overline{\eta} \subset \overline{\omega} : |\overline{\eta}| = 2\}\,.$$

In the particular case of the Widom-Rowlinson model, for $\overline{\omega} = (\omega, \sigma_\omega) \in \overline{\Omega}$, interactions correspond to the Hamiltonian

$$H^{\varphi}(\overline{\omega}) = \sum_{(\eta,\sigma_\eta) \in \mathrm{CG}_2(\overline{\omega})} \varphi(\eta, \omega)(1 - \delta_{\sigma_\eta}(\eta)), \tag{4.1}$$

where δ_{σ_η} is the indicator defined in (2.22), and φ, for some fixed $r > 0$ and $\eta = \{x, y\}$, takes the form

$$\varphi(\eta, \omega \cup \xi) = \begin{cases} \infty & \text{if } |x - y| \le r, \\ 0 & \text{otherwise.} \end{cases}$$

There exists constants $0 < z_0 \le z_0' < \infty$, such that for activity $z < z_0$ the Widom-Rowlinson model has a unique Gibbs measure, whereas, for activity $z > z_0'$, there are multiple Gibbs measures. This shows the existence of a phase transition. The problem of non-uniqueness of Gibbs measures for large activity z was first solved by Ruelle using a Peierls-type argument. It was later also shown using a continuum random-cluster representation where the Widom-Rowlinson model is obtained by independently assigning to each connected component in the continuum random-cluster representation a mark of either -1 or $+1$ with probability 1/2 for each. This more modern stochastic geometric method was due to [CCK94]. There are a variety of different generalisations of the Widom-Rowlinson model including the case when particles of different type are just discouraged to get too close, rather than forbidden. Specifically, the type interaction takes the form φ

$$\varphi(\eta, \omega \cup \xi) = \begin{cases} A & \text{if } |x - y| \le r, \\ 0 & \text{otherwise.} \end{cases}$$

for some constant $A > 0$ representing the inverse temperature. Otherwise, the Hamiltonian is the same as in (4.1). For this generalisation, the problem of non-uniqueness of Gibbs measures, for a required high activity or low temperature, was solved by [LL72]. The result was later repeated using a continuum random cluster representation in [GH96]. These Widom-Rowlinson models each admit an analogue with $q \ge 3$ types of particle. These analogues each exhibit a phase transition which can be shown using a suitable continuum random cluster representation.

We propose an extension of [CCK94] and [LL72] by only considering interactions on the Delaunay hypergraph structure Del rather than the complete hypergraph structure CG. We use a soft exclusion between pairs of particles of different type that form a hyperedge together in Del_2. This geometric flavour is the key difference between our model and the Widom-Rowlinson model on the complete hypergraph structure, where geometric interactions are absent. However, just as in the Widom-Rowlinson model, there is no background interaction in our model, so two particles of the same type can suffuse: they can get infinitesimally close without penalty. We show non-uniqueness of this proposed model for large activity and low temperature. In this sense, it is an extension of the study in Chapters 2 and 3, and in particular a substantial generalisation of the model analysed in [BBD03]. The proof techniques and methods in [BBD03] and Chapter 3 of this thesis use the background hardcore interaction to ensure an upper bound on the number of particles in a box $\Lambda \in \mathcal{B}(\mathbf{R}^2)$. Here, we develop a novel approach without using any background potential. Our methods require a smooth soft type interaction on edges of the Delaunay graph. The potential will decay with an increase in edge length and disappear for lengths greater than $R > 0$, i.e. the interaction has a finite range. On the other hand, as distance tends to 0, the potential tends to infinity – smoothly. The main task is to uniformly bound, from above and below, the quotient of densities defined as a Papangelou conditional intensity in (3.17). Once this is achieved, we use coarse-graining techniques to compare our model to a mixed site-bond percolation model on the integer lattice.

Define the background potential as $\psi \equiv 0$. Let the type interaction φ between pairs of particles, or edges, in Del_2 depend of the length of the hyperedge, and satisfy

$$\varphi(\eta, \omega) \equiv \varphi(|x-y|) \quad \text{for} \quad \eta = \{x, y\} \in \mathrm{Del}_2(\omega),$$

where $|x-y|$ is the Euclidean distance between x and y in $\mathbf{R}^2$, and $\varphi : (0, \infty] \to \mathbf{R}$ is defined as

$$\varphi(\ell) = \log\left(\frac{\ell^4 + A}{\ell^4}\right) \mathbf{1}\{\ell \leq R\}. \tag{4.2}$$

The parameter $R > 0$ is the finite range of the interaction and A assumes the role of inverse temperature and controls the level of the type interaction. Define the hyperedge Hamiltonian of a marked particle configuration $\overline{\omega} = (\omega, \sigma_\omega) \in \overline{\Omega}_\Lambda$, with an admissible boundary configuration $\overline{\xi} = (\xi, \sigma_\xi)$, by

$$H^{\overline{\Phi}}_{\Lambda|\overline{\xi}}(\overline{\omega}) = \sum_{\substack{(\eta,\sigma_\eta) \in \mathrm{Del}_2(\overline{\omega} \cup \overline{\xi}) \\ \eta \cap \Lambda \neq \emptyset}} \varphi(\eta, \omega \cup \xi)(1 - \delta_{\sigma_\eta}(\eta)), \tag{4.3}$$

where δ_{σ_η} is the indicator defined in (2.22). It follows from Proposition 3.2 and in particular Remark 3.3 that at least one such Gibbs measure exists.

Theorem 4.1. *There exists $z_0(R, q) > 0$ and $A_0(z, R, q) > 0$, such that, for all $z > z_0(R, q)$ and $A > A_0(z, R, q)$, there exists at least q different Delaunay Potts measures for the Hamiltonian (4.3).*

The proof structure has similarities with those in Chapter 3, but the techniques are vastly different. For instance, we again find a suitable edge drawing mechanism $\tilde{\mu}_\omega$, show percolation for $\tilde{C}^{\text{site}}_{\Lambda|\xi}$ and relate it the non-uniqueness of Delaunay Potts measures through the application of Proposition 2.18. We also again use a discretization argument to show percolation in $\tilde{C}^{\text{site}}_{\Lambda|\xi}$, however, instead of the coarse grain structure of Section 3.5.1, we use an alternative regime and compare to mixed site-bond percolation on the square lattice, rather than standard site percolation as before.

4.2 Mixed site-bond percolation

Given a graph $G = (V, E)$, let $\mathbf{P}_p$ be the probability measure on configurations of open and closed sites of G. Each site of G is open with probability p and closed with probability $1 - p$. Similarly, let $\tilde{\mathbf{P}}_p$ be the probability measure on configurations of open and closed edges of G. Each edge of G is open with probability p and closed with probability $1 - p$. For $x_0 \in V$ and a subset of vertices $X \subset V$, let

$$\sigma(p, x_0, X, G) := \mathbf{P}_p(\exists \text{ a path } v_0, e_1, \ldots, e_n, v_n \text{ with } v_0 = x_0, v_n \in X \\ \text{and all its vertices open } | x_0 \text{ is open }),$$

and

$$\beta(p, x_0, X, G) := \tilde{\mathbf{P}}_p(\exists \text{ a path } v_0, e_1, \ldots, e_n, v_n \text{ with } v_0 = x_0, v_n \in X \\ \text{and all its edges open }).$$

Then the following inequality holds for $0 \leq p \leq 1$, and is often used to show that site percolation implies bond percolation – see [Ke82].

$$\sigma(p, x_0, X, G) \leq \beta(p, x_0, X, G). \tag{4.4}$$

In mixed percolation, both edges and vertices may be open or closed, possibly with different probabilities. Each edge or bond is open independently of anything else with probability p' and each site is open independently of anything else with probability p. The edges and sites that are not open, along with the edges to or from these sites, are closed. We now look for paths of open sites and open edges. For $x_0 \in V$ and a subset of vertices $X \subset V$, let

$$\gamma(p, p', x_0, X, G) := \mathbf{P}_{p,p'}(\exists \text{ a path } v_0, e_1, \ldots, e_n, v_n \text{ with } v_0 = x_0, v_n \in X \\ \text{and all its vertices open, and all its edges open }).$$

Let G' be the reduced graph where each edge and site of G is removed independently with probability $1 - p'$ and $1 - p$ respectively. By taking expectations of inequality

(4.4), on G', with respect to $\mathbf{P}_\delta$ and $\tilde{\mathbf{P}}_\delta$, we arrive at the mixed percolation result of Hammersley: a generalisation of the work of McDiarmid [Ha80]. That is, for $0 \le \delta, p, p' \le 1$:

$$\gamma(\delta p, p', x_0, X, G) \le \gamma(p, \delta p', x_0, X, G). \tag{4.5}$$

By setting $\delta = p$ and $p' = 1$ in (4.5), and noticing that $\gamma(p^2, 1, x_0, X, G) = \sigma(p^2, x_0, X, G)$ we arrive at

$$\sigma(p^2, x_0, X, G) \le \gamma(p, p, x_0, X, G), \tag{4.6}$$

and hence

$$\theta_{\text{mixed}}(p, p) \ge \theta_{\text{site}}(p^2), \tag{4.7}$$

where $\theta_{\text{mixed}}(p, p')$ is the mixed percolation probability with parameters p and p', and $\theta_{\text{site}}(p)$ is the site percolation probability with parameter p. Inequality (4.7) will be used in a new coarse graining scheme for continuum random cluster models without a hardcore background interaction.

4.3 Coarse graining

We endeavour to use a coarse graining technique to compare the site percolation model $\tilde{C}^{\text{site}}_{\Lambda|\xi}$ with mixed site-bond percolation on $\mathbf{Z}^2$. Let $\Lambda \in \mathcal{B}(\mathbf{R}^2)$ be a box made up of a finite union of smaller boxes, $\Delta_{k,l}$, each of side length $8L$, for some $L > 0$:

$$\Lambda = \bigcup_{k,l=0}^{N} \Delta_{k,l}. \tag{4.8}$$

We also split each of these smaller boxes into 64 tiny boxes of side length L:

$$\Delta_{k,l} = \bigcup_{i,j=0}^{7} \Delta_{k,l}^{i,j}. \tag{4.9}$$

This is called the L-splitting of Λ. Given an L-splitting of Λ, define

$$\Delta^-_{k,l} = \bigcup_{i,j=2}^{5} \Delta_{k,l}^{i,j}$$

to be the union of the 16 smaller boxes towards the centre of $\Delta_{k,l}$. These will act as the sites in the mixed site-bond percolation model on $\mathbf{Z}^2$. Finally we define the link boxes between

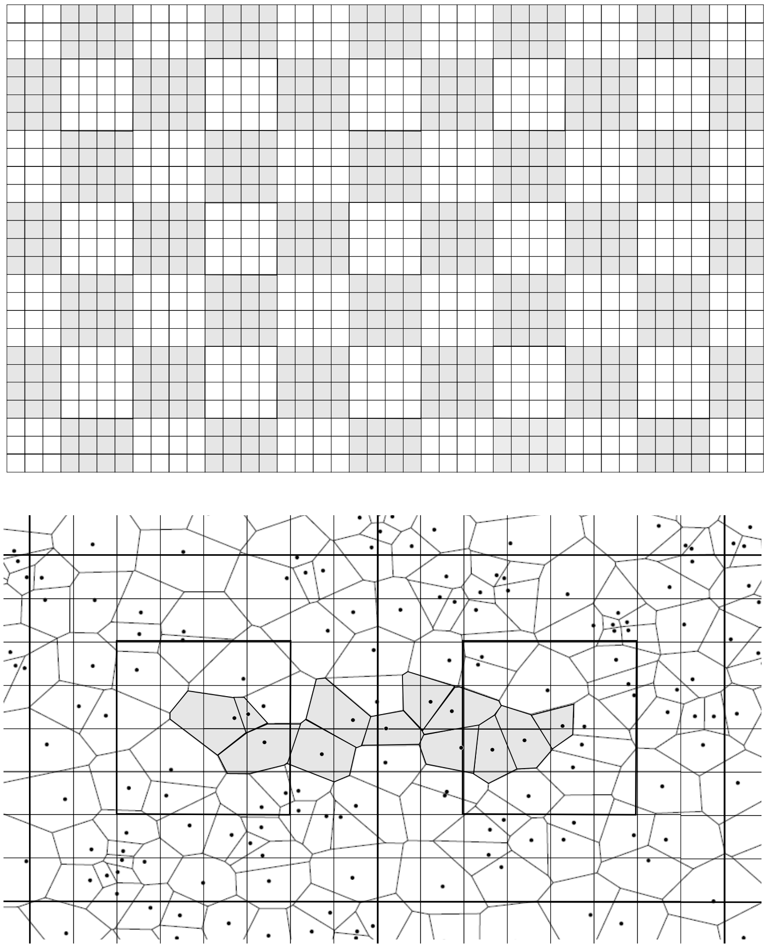


Figure 4.1: (a). Part of an L-splitting of Λ. The shaded boxes are the link boxes. (b) The shaded area is the union of the Voronoi cells with centre $x \in H^L_{k:k+1,l}(\omega)$.

$\Delta^-{}_{k,l}$ and $\Delta^-{}_{k+1,l}$ as

$$\Delta^{k:k+1,l}_{\text{link}} = \left(\bigcup_{j=0}^{3} \Delta^{6,j+2}_{k,l} \right) \cup \left(\bigcup_{j=0}^{3} \Delta^{7,j+2}_{k,l} \right) \cup \left(\bigcup_{j=0}^{3} \Delta^{0,j+2}_{k+1,l} \right) \cup \left(\bigcup_{j=0}^{3} \Delta^{1,j+2}_{k+1,l} \right) \quad (4.10)$$

which act as the bonds in the mixed site-bond percolation model on $\mathbf{Z}^2$, see Figure 4.1 (a). This completes the coarse grain procedure. When we establish percolation in the mixed site-bond model on $\mathbf{Z}^2$, i.e. the existence of an infinite chain of open sites and bonds, we would like to relate it to the existence of an infinite connected component of hyperedges in Del_2, built only from points of mark 1, in the continuum site percolation model $\tilde{C}^{\text{site}}_{\Lambda|\xi}$. To do this, we define $CB^L_{k:k+1,l}$ to be the straight line segment between the centres of the boxes

$\Delta_{k,l}$ and $\Delta_{k+1,l}$ and let

$$H^L_{k:k+1,l}(\omega) := \left\{x \in \omega : \mathrm{Vor}_\omega(x) \cap CB^L_{k:k+1,l} \neq \emptyset\right\}. \tag{4.11}$$

be the subset of points of a configuration, whose Voronoi cells intersect the line segment $CB^L_{k:k+1,l}$. See Figure 4.1 (b).

4.4 Percolation

Without the hardcore background interaction to lean on, our previous strategies of discretization seem not to work at all. Instead, we look much deeper into the underlying geometry and properties of the Delaunay tessellations. Throughout this chapter, we work exclusively on Del_2, therefore we often refer to hyperedges as just 'edges'. For brevity, given a configuration $\omega \in \Omega$, we write

$$\eta_{xy} := \{x, y\} \quad \text{for} \quad \{x, y\} \in \mathrm{Del}_2(\omega). \tag{4.12}$$

Recall what happens to a Delaunay graph when we add a point x_0 to a configuration $\omega \in \Omega$, and in particular, recall the sets of exterior, created and destroyed edges denoted by $E^{\mathrm{ext}}_{x_0|\omega}$, $E^+_{x_0|\omega}$ and $E^-_{x_0|\omega}$ respectively and defined in (3.54 – 3.56). To add to these, we also define the neighbourhood of x_0.

Definition 4.2. *Given a graph G, a polygon $P \subset G$ is a sequence of vertices starting and ending at the same vertex, with each two consecutive vertices in the sequence adjacent to each other in the graph. No repetitions of vertices are allowed, other than the starting and ending vertex.*

Definition 4.3. *Let $\partial_{x_0|\omega} := (V_{x_0|\omega}, E^{\mathrm{nbd}}_{x_0|\omega})$ where $V_{x_0|\omega}$ is the set of points that share an edge with x_0 in $\mathrm{Del}_2(\omega \cup \{x_0\})$ and $E^{\mathrm{nbd}}_{x_0|\omega}$ is the set of edges in $\mathrm{Del}_2(\omega \cup \{x_0\})$ that have both endpoints in $V_{x_0|\omega}$. More precisely,*

$$V_{x_0|\omega} := \left\{x \in \omega : \eta_{xx_0} \in E^+_{x_0|\omega}\right\}$$

and

$$E^{\mathrm{nbd}}_{x_0|\omega} := \left\{\eta_{xy} \in E^{\mathrm{ext}}_{x_0|\omega} : x, y \in V_{x_0|\omega}\right\}.$$

The graph $\partial_{x_0|\omega} := (V_{x_0|\omega}, E^{\mathrm{nbd}}_{x_0|\omega})$ splits the plane into two regions. The region containing x_0 we call the neighbourhood of x_0, and $\partial_{x_0|\omega}$ is called the boundary of the neighbourhood of x_0.

Let $B := B(x_0, R)$ be the ball of radius R centred at x_0 and let $V_B \subset V_{x_0|\omega}$ be the restriction of $V_{x_0|\omega}$ to B. Recall, from Equation 2.44, that

$$\mu^{(q)}_{\omega,\Lambda}(dE) = \frac{q^{K(\omega,E)}\mu_{\omega,\Lambda}(dE)}{\int q^{K(\omega,E)}\mu_{\omega,\Lambda}(dE)},$$

where $\mu_{\omega,\Lambda}$ is the distribution of edge configurations, associated with the edge probability $p_\Lambda(\eta_{xy}) = 1 - e^{-\varphi(|x-y|)}$. Also recall that μ^-_ω, μ^+_ω and μ^{ext}_ω are the edge drawing mechanisms on $E^-_{x_0|\omega}$, $E^+_{x_0|\omega}$ and $E^{\text{ext}}_{x_0|\omega}$ respectively and define

$$\mu^{(q)}_{\text{ext},\omega}(dE) := \frac{q^{K(\omega,E)}\mu^{\text{ext}}_\omega(dE)}{\int q^{K(\omega,E)}\mu^{\text{ext}}_\omega(dE)}. \tag{4.13}$$

The main task in this section is to find an upper bound, independent of $\omega \in \Omega$, for the expected number of connected components of (ω, E) that intersect V_B, where E is sampled from $\mu^{(q)}_{\text{ext},\omega}$. This will enable us to bound the Papangelou conditional intensity of $M_{\Lambda|\xi}$ (3.17), both from above *and* below, see Lemma 4.5 and Lemma 4.7 below. This in turn allows us to exhibit control over the distribution of particle positions – a necessity for our discretization approach, see Lemma 4.6 and Lemma 4.8.

Proposition 4.4. *Given a graph (ω, E), let $N^{\text{cc}}_{V_B}(\omega, E)$ denote the number of connected components that intersect V_B. There exists $0 < \alpha < \infty$ such that*

$$\int N^{\text{cc}}_{V_B}(\omega, E)\mu^{(q)}_{\text{ext},\omega}(dE) \leq \alpha,$$

for all $\omega \in \Omega$.

Proposition 4.4 is the key step in proving percolation and hence the non-uniqueness of Gibbs measures. The proof is substantial and constitutes a significant part of this thesis. For this reason we defer it to Section 4.5.

Lemma 4.5. *For every finite box $\Lambda \subset \mathbf{R}^2$, $M_{\Lambda|\xi}$ – almost all $\omega \in \Omega_{\Lambda|\xi}$ and a point x_0, with $x_0 \in \Lambda\backslash\omega$,*

$$\frac{h_\Lambda(\omega \cup \{x_0\})}{h_\Lambda(\omega)} \geq q^{-\alpha}, \tag{4.14}$$

where α is given in Proposition 4.4.

Proof. It follows by similar arguments to the proof of Lemma 3.14, that

$$\frac{h_\Lambda(\omega \cup \{x_0\})}{h_\Lambda(\omega)} \geq \frac{\int q^{K(\omega,E_2)} \int q^{K(\omega\cup\{x_0\},E_1\cup E_2)-K(\omega,E_2)} \mu_\omega^+(dE_1)\mu_\omega^{\text{ext}}(dE_2)}{\int q^{K(\omega,E_4)}\mu_\omega^{\text{ext}}(dE_4)}.$$

Therefore, by using the measure in (4.13), we have that

$$\frac{h_\Lambda(\omega \cup \{x_0\})}{h_\Lambda(\omega)} \geq \iint q^{K(\omega\cup\{x_0\},E_1\cup E_2)-K(\omega,E_2)} \mu_\omega^+(dE_1)\mu_{\text{ext},\omega}^{(q)}(dE_2).$$

As the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = q^x$ is convex, we can apply Jensen's inequality to obtain

$$\frac{h_\Lambda(\omega \cup \{x_0\})}{h_\Lambda(\omega)} \geq q^{\iint K(\omega\cup\{x_0\},E_1\cup E_2)-K(\omega,E_2)\mu_\omega^+(dE_1)\mu_{\text{ext},\omega}^{(q)}(dE_2)}. \tag{4.15}$$

Notice that new edges, made by the insertion of x_0 to the configuration ω can only be open with respect to μ_ω^+ if they have length less than R. Therefore,

$$K(\omega \cup \{x_0\}, E_1 \cup E_2) - K(\omega, E_2) \geq -N_{V_B}^{\text{cc}}(\omega, E_2) \tag{4.16}$$

and by Proposition 4.4,

$$\int N_{V_B}^{cc}(\omega, E_2)\mu_{\text{ext},\omega}^{(q)}(dE_2) \leq \alpha < \infty. \tag{4.17}$$

Together, (4.15), (4.16) and (4.17) show that

$$\frac{h_\Lambda(\omega \cup \{x_0\})}{h_\Lambda(\omega)} \geq q^{-\alpha}.$$

□

Having a uniform lower bound for $\frac{h_\Lambda(\omega\cup\{x_0\})}{h_\Lambda(\omega)}$ allows us to exhibit some control over the distribution $M_{\Lambda,\nabla|\xi'}(\cdot)$ defined in (3.23). In particular, fix

$$\epsilon = \frac{1 - \left(p_c^{site}(\mathbf{Z}^2)\right)^{1/2}}{4}, \tag{4.18}$$

and choose L to satisfy

$$z^{-1/2} < L \leq \frac{R}{4\sqrt{2}}, \tag{4.19}$$

then, we obtain the following lower bound on the probability that a small box, $\Delta_{k,l}^{i,j}$ in the

L-splitting of Λ, contains at least one point of ω.

Lemma 4.6. *Let $\xi \in \Omega_{\Lambda^c}$. Then for all cells $\nabla = \Delta_{k,l}^{i,j}$ of an L-splitting of Λ and for any $\xi' \in \Omega_{\nabla^c}$, such that $\xi' \backslash \Lambda = \xi$, we have*

$$M_{\Lambda,\nabla|\xi'}(|\omega \cap \nabla| \geq 1) > 1 - \frac{\epsilon}{64}$$

for all $z > z_0 := \frac{2048 q^\alpha}{\epsilon R^2}$.

Proof. We want to show that $M_{\Lambda,\nabla|\xi'}(|\omega \cap \nabla| \geq 1) > 1 - \frac{\epsilon}{64}$ holds for large enough z. Since we have $\psi \equiv 0$, this is straightforward.

$$\frac{M_{\Lambda,\nabla|\xi'}(|\omega \cap \nabla| = 1)}{M_{\Lambda,\nabla|\xi'}(|\omega \cap \nabla| = 0)} = z \int_\nabla \frac{h_\Lambda(\omega \cup \{x\})}{h_\Lambda(\omega)} dx \geq q^{-\alpha} z |\nabla|,$$

where the inequality is a direct application of Lemma 4.5. It follows that

$$M_{\Lambda,\nabla|\xi'}(|\omega \cap \nabla| = 0) \leq q^\alpha (z|\nabla|)^{-1},$$

and hence, for $z > z_0$

$$M_{\Lambda,\nabla|\xi'}(|\omega \cap \nabla| \geq 1) \geq 1 - q^\alpha (z|\nabla|)^{-1} > 1 - q^\alpha \left(\frac{z_0 R^2}{8}\right)^{-1} = 1 - \frac{\epsilon}{64}.$$

$\square$

We have a lower bound for the Papangelou conditional intensity in (4.14) which allows us to bound from below the number of particles in a box ∇ with high probability, for large z. We shall show that the number of particles in a box also has an upper bound, dependent on z, with high probability. Having such an upper bound is important as it allows us to quantify the probability that *all* particles in a box have mark 1. In Chapter 3, and indeed the work of [BBD03], an upper bound for the number of particles in a box followed immediately as a consequence of the hardcore background potential. However, without such a background potential, we turn to finding an upper bound for the Papangelou conditional intensity of $M_{\Lambda|\xi}$, uniform over suitable events.

Let $\Delta_{k,l}$ be an element of an L-splitting of Λ and define $F_{k,l}^{\text{ext}}$ to be the event that each of the smaller boxes $\Delta_{k,l}^{i,j} \subset \Delta_{k,l}$ that are not contained in $\Delta_{k,l}^-$, contain at least one point of ω. More precisely, it is a specific set of "well-behaved" configurations given by:

$$F^{\text{ext}}_{k,l} := \bigcap_{\substack{i,j\in\{0,\ldots,7\}:\\ \Delta^{i,j}_{k,l}\subset\Delta\setminus\Delta^-}} \left\{\omega\in\Omega : |\omega\cap\Delta^{i,j}_{k,l}|\geq 1\right\}. \tag{4.20}$$

Lemma 4.7. *Given a finite box $\Lambda\subset\mathbf{R}^2$, an element of the L-splitting $\Delta_{k,l}\subset\Lambda$, a configuration $\omega\in F^{ext}_{k,l}$ and a point $x_0\in{\Delta^-}_{k,l}\setminus\omega$,*

$$\frac{h_\Lambda(\omega\cup\{x_0\})}{h_\Lambda(\omega)}\leq q^{\alpha+1},$$

where α is given in Proposition 4.4.

Proof. Suppose $\omega\in F^{\text{ext}}_{k,l}$ and $x_0\in{\Delta^-}_{k,l}\setminus\omega$. Then, by similar techniques to that used in Lemma 4.5, we see that

$$\begin{aligned}
\frac{h_\Lambda(\omega\cup\{x_0\})}{h_\Lambda(\omega)} &= \frac{\int q^{K(\omega,E_2)}\int q^{K(\omega\cup\{x_0\},E_1\cup E_2)-K(\omega,E_2)}\mu^+_\omega(dE_1)\mu^{\text{ext}}_\omega(dE_2)}{\int q^{K(\omega,E_4)}\int q^{K(\omega,E_3\cup E_4)-K(\omega,E_4)}\mu^-_\omega(dE_3)\mu^{\text{ext}}_\omega(dE_4)}\\
&\leq \frac{q\int q^{K(\omega,E_2)}\mu^{\text{ext}}_\omega(dE_2)}{\int q^{K(\omega,E_4)}\int q^{K(\omega,E_3\cup E_4)-K(\omega,E_4)}\mu^-_\omega(dE_3)\mu^{\text{ext}}_\omega(dE_4)} \qquad (4.21)\\
&= q\left(\iint q^{K(\omega,E_3\cup E_4)-K(\omega,E_4)}\mu^-_\omega(dE_3)\mu^{(q)}_{\text{ext},\omega}(dE_4)\right)^{-1}
\end{aligned}$$

where (4.21) uses the inequality

$$K(\omega\cup\{x_0\},E_1\cup E_2)-K(\omega,E_2)\leq 1, \tag{4.22}$$

which was established in (3.40) and (3.57). We can then apply Jensen's inequality to the integral in the denominator to obtain the upper bound

$$\frac{h_\Lambda(\omega\cup\{x_0\})}{h_\Lambda(\omega)}\leq q\left(q^{\iint K(\omega,E_3\cup E_4)-K(\omega,E_4)\mu^-_\omega(dE_3)\mu^{(q)}_{\text{ext},\omega}(dE_4)}\right)^{-1}. \tag{4.23}$$

Recall that $V_{x_0|\omega}=\{x\in\omega:\eta_{xx_0}\in E^+_{x_0|\omega}\}$ and notice that for all $\omega\in F^{\text{ext}}_{k,l}$, $x_0\in{\Delta^-}_{k,l}\setminus\omega$, and $x\in V_{x_0|\omega}$, we have that $|x-x_0|<4\sqrt{2}L$. This basically ensures that the neighbourhood of x_0 is contained in the ball of radius $4\sqrt{2}L$ and centre x_0:

$$V_{x_0|\omega}\subset\mathcal{B}_{4\sqrt{2}L}(x_0)\subset B.$$

Therefore, since $x,y\in V_{x_0|\omega}$ for all $\eta_{xy}\in E^-_{x_0}(\omega)$, it follows that adding edges in $E^-_{x_0}(\omega)$ can only fuse together two connected components (reducing the number of connected components by one) if they each intersect B. Hence,

$$K(\omega, E_3 \cup E_4) - K(\omega, E_4) \geq -N_{V_B}^{\mathrm{cc}}(\omega, E_4). \tag{4.24}$$

Combining (4.23) and (4.24), together with Proposition 4.4, we conclude that

$$\frac{h_\Lambda(\omega \cup \{x_0\})}{h_\Lambda(\omega)} \leq q^{\alpha+1},$$

for all $\omega \in F_{k,l}^{\mathrm{ext}}$ and for all $x_0 \in {\Delta^-}_{k,l} \backslash \omega$.

□

In the following, we fix $\Delta_{k,l} \subset \Lambda$ and drop the subscript k and l from our notation. For example, $F_{k,l}^{\mathrm{ext}}$ will be written as F^{ext} and ${\Delta^-}_{k,l}$ will be written as Δ^-. Having an upper bound for $\frac{h_\Lambda(\omega \cup \{x_0\})}{h_\Lambda(\omega)}$ allows us to exhibit some control over the conditional probability measure $M_{\Lambda,\Delta^-|\xi'}(\cdot|F^{\mathrm{ext}})$. In particular, we can find a lower bound for the conditional probability, given $\omega \in F^{\mathrm{ext}}$, that Δ^- contains no more than $m(z) \in \mathbf{N}$ points of $\omega \in \Omega$.

Lemma 4.8. *Fix $z > 0$. Let $\Delta = \Delta_{k,l}$ be an element of the L-splitting of Λ. Then, for any $\xi' \in \Omega_{(\Delta^-)^c}$, admissible for Δ^- and z, with $\xi' \backslash \Lambda = \xi$, and $\xi' \in F^{ext}$, we have*

$$M_{\Lambda,\Delta^-|\xi'}(|\omega \cap \Delta^-| \leq m(z)) > 1 - \epsilon,$$

where $m(z) := 2\epsilon^{-1} q^{\alpha+1} |\Delta^-| z$ and ϵ is given in (4.18).

Proof. First of all, we define the random variable N to be the number of points of a configuration in Δ^-, i.e. $N = |\omega \cap \Delta^-|$. Explicitly, we can write $M_{\Lambda,\Delta^-|\xi'}(d\omega)$ as

$$M_{\Lambda,\Delta^-|\xi'}(d\omega) = Z_{\Lambda,\Delta^-|\xi'}^{-1} h_\Lambda(\omega \cup \xi') \Pi_{\Delta^-}^z(d\omega), \tag{4.25}$$

as given in (3.23). We also use the fact that

$$\int f(\omega) \Pi_{\Delta^-}^z(d\omega) = e^{-z|\Delta^-|} \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{(\Delta^-)^n} f(\{x_1, \ldots x_n\}) dx_1 \ldots dx_n, \tag{4.26}$$

Combining Equations (4.25) and (4.26), and setting $Z' := e^{-z|\Delta^-|} Z_{\Lambda,\Delta^-|\xi'}^{-1}$ for brevity, we see that

$$
\begin{aligned}
M_{\Lambda,\Delta^-|\xi'}(N=n+1) &= \frac{z^{n+1}}{(n+1)!} Z' \int_{(\Delta^-)^{n+1}} h_\Lambda(\{x_1,\ldots,x_{n+1}\} \cup \xi') dx_1 \ldots dx_{n+1} \\
&= \frac{z^n}{n!} \left(\frac{z}{n+1}\right) Z' \int_{(\Delta^-)^n} \int_{\Delta^-} h_\Lambda(\omega \cup \xi' \cup \{x\}) dx d\omega \\
&= \frac{z^n}{n!} \left(\frac{z}{n+1}\right) Z' \int_{(\Delta^-)^n} h_\Lambda(\omega \cup \xi') g_{\Delta^-|\xi'}(\omega) d\omega,
\end{aligned}
$$

where

$$
g_{\Delta^-|\xi'}(\omega) = \int_{\Delta^-} \frac{h_\Lambda(\omega \cup \xi' \cup \{x\})}{h_\Lambda(\omega \cup \xi')} dx.
$$

Again, using Equation (4.25), it follows that

$$
\begin{aligned}
M_{\Lambda,\Delta^-|\xi'}(N=n+1) &= \left(\frac{z}{n+1}\right) \int_{\Omega_{\Delta^-}} M_{\Lambda,\Delta^-|\xi'}(d\omega, \mathbf{1}_{\{N(\omega)=n\}}) g_{\Delta^-|\xi'}(\omega) \\
&= \left(\frac{z}{n+1}\right) M_{\Lambda,\Delta^-|\xi'}(N=n) \int_{\Omega_{\Delta^-}} M_{\Lambda,\Delta^-|\xi'}(d\omega | \mathbf{1}_{\{N(\omega)=n\}}) g_{\Delta^-|\xi'}(\omega). \quad (4.27)
\end{aligned}
$$

Notice that, since $\omega \cup \xi' \in F^{\text{ext}}$, we can use Lemma 4.7 to bound the function $g_{\Delta^-|\xi'}(\omega)$ above by $q^{\alpha+1}|\Delta^-|$. Therefore, following (4.27), we obtain

$$
\begin{aligned}
\frac{M_{\Lambda,\Delta^-|\xi'}(N=n+1)}{M_{\Lambda,\Delta^-|\xi'}(N=n)} &= \frac{z}{n+1} \int_{\Omega_{\Delta^-}} M_{\Lambda,\Delta^-|\xi'}(d\omega | N=n) g_{\Delta^-|\xi'}(\omega) \\
&\leq \frac{z}{n+1} q^{\alpha+1} |\Delta^-| \underbrace{\int_{\Omega_{\Delta^-}} M_{\Lambda,\Delta^-|\xi'}(d\omega | N=n)}_{=1} \\
&= \frac{q^{\alpha+1}|\Delta^-| z}{n+1}. \quad (4.28)
\end{aligned}
$$

This gives us a relationship between the probabilities, with respect to $M_{\Lambda,\Delta^-|\xi'}(\cdot)$, that there are n and $n+1$ particles in $|\omega \cap \Delta^-|$ respectively. However, for all $n > m(z)$ we would like to extend this relationship to compare the probabilities, that there are n and $\lfloor m(z) \rfloor$ particles in $|\omega \cap \Delta^-|$ respectively, where $\lfloor m(z) \rfloor$ denotes the highest integer not larger than

$m(z)$. To do this, we apply (4.28) multiple $(n - \lfloor m(z) \rfloor)$ times.

$$
\begin{aligned}
M_{\Lambda,\Delta^-|\xi'}(N = n) &\leq \frac{q^{\alpha+1}|\Delta^-|z}{n} M_{\Lambda,\Delta^-|\xi'}(N = n-1) \\
&\leq \frac{q^{\alpha+1}|\Delta^-|z}{n} \cdot \frac{q^{\alpha+1}|\Delta^-|z}{n-1} \cdot \cdots \cdot \frac{q^{\alpha+1}|\Delta^-|z}{\lfloor m(z) \rfloor + 1} \underbrace{M_{\Lambda,\Delta^-|\xi'}(N = \lfloor m(z) \rfloor)}_{\leq 1} \\
&\leq \frac{\left(q^{\alpha+1}|\Delta^-|z\right)^{n-\lfloor m(z) \rfloor}}{n!} \lfloor m(z) \rfloor !.
\end{aligned}
$$

Therefore, using a combinatorial argument, it follows that

$$
\begin{aligned}
M_{\Lambda,\Delta^-|\xi'}(N > m(z)) &\leq \sum_{n=\lfloor m(z) \rfloor+1}^{\infty} \frac{\left(q^{\alpha+1}|\Delta^-|z\right)^{n-\lfloor m(z) \rfloor}}{n!} \lfloor m(z) \rfloor ! && (4.29) \\
&\leq \sum_{n=\lfloor m(z) \rfloor+1}^{\infty} \left(\frac{q^{\alpha+1}|\Delta^-|z}{\lfloor m(z) \rfloor}\right)^{n-\lfloor m(z) \rfloor}. && (4.30)
\end{aligned}
$$

By recalling that $m(z) = 2\epsilon^{-1}q^{\alpha+1}|\Delta^-|z$, this gives us

$$
M_{\Lambda,\Delta^-|\xi'}(N > m(z)) \leq \sum_{n=\lfloor m(z) \rfloor+1}^{\infty} \left(\frac{\epsilon}{2}\right)^{n-\lfloor m(z) \rfloor} = \sum_{n=1}^{\infty} \left(\frac{\epsilon}{2}\right)^{n}. \tag{4.31}
$$

Since $\epsilon < 1/2$, the right hand side of (4.31) is less than ϵ. By taking complements, we have established that

$$
M_{\Lambda,\Delta^-|\xi'}(N \leq m(z)) \geq 1 - \epsilon.
$$

$\square$

Before we reintroduce the site percolation measure, we define an alternative edge drawing mechanism to $\mu_{\omega,\Lambda}$ described in (2.28). Given $\omega \in \Omega$, let $\tilde{\mu}_\omega$ denote the distribution of the random hyperedge configurations

$$
\{\eta \in \mathcal{H}(\omega) : v(\eta) = 1\}
$$

where $(v(\eta))_{\eta \in \mathcal{H}(\omega)}$ are independent Bernoulli random variables with probability

$$
\text{Prob}(v(\eta) = 1) = \tilde{p}(\eta) := \underbrace{\frac{1}{\frac{q}{A}64L^4 + 1}}_{=:\tilde{p}} \mathbf{1}_{\mathrm{Del}_2^{**}(\omega)}(\eta) \tag{4.32}
$$

where

$$\mathrm{Del}_2^{**}(\omega) := \left\{ \eta = \{x, y\} \in \mathrm{Del}_2(\omega) : |x - y| < \sqrt{8}L \right\}.$$

The probability $\tilde{p}$, is constant on the graph $\mathrm{Del}_2^{**}(\omega)$ which allows us to use the fact that site percolation implies bond percolation in Proposition 2.18. Hence, we define a site percolation measure, $\tilde{C}^{\text{site}}_{\Lambda|\xi_{\Lambda^c}}$, and show that site percolation occurs for large enough A and z. Recall the definition of the continuum site percolation model

$$\tilde{C}^{\text{site}}_{\Lambda|\xi}(d\overline{\omega}) = M_{\Lambda|\xi}(d\omega)\tilde{\lambda}_\omega(d\overline{\omega}),$$

where $\tilde{\lambda}_\omega$ denotes the distribution of the random vector $\sigma_\omega = (\sigma_\omega(x) : x \in \omega)$ with elements in Σ, where $(\sigma_\omega(x))_{x\in\omega}$ are independent Bernoulli random variables satisfying

$$\mathrm{Prob}(\sigma_\omega(x) = 1) = \tilde{p}\mathbf{1}_{\mathrm{Del}_1^{**}(\omega)}(x) \tag{4.33}$$

and

$$\mathrm{Prob}(\sigma_\omega(x) \neq 1) = 1 - \tilde{p}\mathbf{1}_{\mathrm{Del}_1^{**}(\omega)}(x)$$

where $\tilde{p}$ is given in (4.32) and $\mathrm{Del}_1^{**}(\omega)$ is the set of points that build the hyperedges of $\mathrm{Del}_2^{**}(\omega)$. Using analogous arguments to the proof of Lemma 3.11, it follows that

$$\tilde{\mu}^{(q)}_\omega \succcurlyeq \tilde{\mu}_\omega. \tag{4.34}$$

Therefore, by Proposition 2.18, the following Lemma completes the proof of Theorem 4.1.

Lemma 4.9. *Let $z > z_0$ and $A > A_0(z)$ where z_0 is given in Lemma 4.6 and*

$$A_0(z) := \frac{qR^4}{(1 - 2\epsilon)^{-1/m(z)} - 1}$$

where $m(z)$ is given in Lemma 4.8. Then, there exists $c > 0$ such that

$$\tilde{C}^{\text{site}}_{\Lambda|\xi}(\{\Delta \leftrightarrow \Lambda^c\} \geq c > 0$$

for any box $\Delta \in \mathcal{B}(\mathbf{R}^2)$, any finite union Λ of boxes, and for any pseudo-periodic boundary condition $\xi \in \widehat{\Gamma}_{\Lambda^c}$.

Proof. We start by bounding the probability, with respect to $M_{\Lambda,\Delta|\xi''}$, that all small boxes $\nabla = \Delta^{i,j}_{k,l}$ of $\Delta\backslash\Delta^-$ contain at least one particle – precisely the event $F^{\text{ext}}_{k,l}$ defined in (4.20). Using the result of Lemma 4.6 and a similar argument to that in the proof of Lemma 3.10, we can choose z large enough such that, for all configurations $\xi'' \in \Omega_{\Delta^c}$ with $\xi''\backslash\Lambda = \xi$,

$$M_{\Lambda,\Delta|\xi''}(F^{\text{ext}}_{k,l}) \geq 1 - \sum_{\Delta^{i,j}_{k,l} \subset \Delta \setminus \Delta^-} M_{\Lambda,\Delta|\xi''}(|\omega \cap \Delta^{i,j}_{k,l}| = 0) > 1 - \frac{48\epsilon}{64} = 1 - \frac{3\epsilon}{4}. \quad (4.35)$$

Next, we define the event $G_{k,l} := \{\omega \in \Omega : |\omega \cap \Delta^-| \leq m(z)\}$ that there are at most $m(z)$ points of ω in the subset $\Delta^- \subset \Delta$ and also the event

$$F^-_{k,l} := \bigcap_{\substack{i,j \in \{0,\ldots,7\}: \\ \Delta^{i,j}_{k,l} \subset \Delta^-}} \left\{\omega \in \Omega : |\omega \cap \Delta^{i,j}_{k,l}| \geq 1\right\}.$$

that all small boxes $\nabla = \Delta^{i,j}_{k,l}$ of Δ^- contain at least one particle. Then we have that, for all configurations $\xi'' \in \Omega_{\Delta^c}$ with $\xi'' \setminus \Lambda = \xi$,

$$M_{\Lambda,\Delta|\xi''}(F^-_{k,l}, F^{\text{ext}}_{k,l}, G_{k,l}) = \int_{\Omega_\Delta} \mathbf{1}_{F^-_{k,l}}(\omega)\mathbf{1}_{F^{\text{ext}}_{k,l}}(\omega)\mathbf{1}_{G_{k,l}}(\omega) M_{\Lambda,\Delta|\xi''}(d\omega)$$

$$= \int_{\Omega_{\Delta\setminus\Delta^-}} \mathbf{1}_{F^{\text{ext}}_{k,l}}(\zeta) \left[\int_{\Omega_{\Delta^-}} \mathbf{1}_{F^-_{k,l}}(\omega)\mathbf{1}_{G_{k,l}}(\omega) M_{\Lambda,\Delta|\xi''}(d\omega|\omega\setminus\Delta^- = \zeta\setminus\Delta^-)\right] M_{\Lambda,\Delta|\xi''}(d\zeta)$$

$$= \int_{\Omega_{\Delta\setminus\Delta^-}} \mathbf{1}_{F^{\text{ext}}_{k,l}}(\zeta) M_{\Lambda,\Delta^-|\xi'}(F^-_{k,l}, G_{k,l}) M_{\Lambda,\Delta|\xi''}(d\zeta), \quad (4.36)$$

where $\xi' := \zeta\setminus\Delta^-$ and where we used Equation (3.23). However, by Lemma 4.8, and through our choice of z in (4.35), it follows that

$$\begin{aligned} M_{\Lambda,\Delta^-|\xi'}(F^-_{k,l}, G_{k,l}) &\geq 1 - M_{\Lambda,\Delta^-|\xi'}((F^-_{k,l})^c) - M_{\Lambda,\Delta^-|\xi'}((G_{k,l})^c) \\ &> 1 - \frac{16\epsilon}{64} - \epsilon \\ &= 1 - \frac{5\epsilon}{4} \end{aligned} \quad (4.37)$$

and hence, combining with (4.35) and (4.36), we conclude that

$$\begin{aligned} M_{\Lambda,\Delta|\xi''}(F^-_{k,l}, F^{\text{ext}}_{k,l}, G_{k,l}) &> \left(1 - \epsilon - \frac{5\epsilon}{4}\right) \int_{\Omega_{\Delta\setminus\Delta^-}} \mathbf{1}_{F^{\text{ext}}_{k,l}}(\zeta) M_{\Lambda,\Delta|\xi''}(d\zeta) \\ &> \left(1 - \frac{5\epsilon}{4}\right)\left(1 - \frac{3\epsilon}{4}\right) \\ &> 1 - 2\epsilon. \end{aligned} \quad (4.38)$$

The next step is to condition the marks of the particles. Let $C_{k,l} \in \overline{\mathcal{F}}$ be the event that each small box $\Delta_{k,l}^{i,j} \subset \Delta$ contains at least one point, $\Delta^- = \Delta_{k,l}^-$ contains no more than $m(z)$ points and all points in $\Delta^- \cap \mathrm{Del}_1^{**}$ have mark 1:

$$C_{k,l} = \Big\{ \overline{\omega} = (\omega, \sigma_\omega) \in \overline{\Omega} : \omega \in F_{k,l}^- \cap F_{k,l}^{\text{ext}} \cap G_{k,l} \text{ and} \\ \sigma_\omega(x) = 1 \text{ for all } x \in \Delta^- \cap \mathrm{Del}_1^{**}(\omega) \Big\}. \quad (4.39)$$

Recall from (4.33), that a point $\overline{x} \in \overline{\omega}$, where $\rho(\overline{x}) \in \Delta^- \cap \mathrm{Del}_1^{**}(\omega)$, has mark 1 with probability

$$\tilde{p} = \frac{1}{\frac{64qL^4}{A} + 1},$$

with respect to $\tilde{C}^{\text{site}}_{\Delta_{k,l}|\xi''}$. It follows that

$$\begin{aligned} \tilde{C}^{\text{site}}_{\Delta_{k,l}|\xi''}(C_{k,l}) &\geq \int M_{\Lambda,\Delta_{k,l}|\xi''}(d\omega) \mathbf{1}_{F_{k,l}^-}(\omega) \mathbf{1}_{F_{k,l}^{\text{ext}}}(\omega) \mathbf{1}_{G_{k,l}}(\omega) \tilde{p}^{|\mathrm{Del}_1^{**}(\omega) \cap \Delta^-|} \\ &\geq \tilde{p}^{\lfloor m(z) \rfloor} M_{\Lambda,\Delta_{k,l}|\xi''}(F_{k,l}^-, F_{k,l}^{\text{ext}}, G_{k,l}) \end{aligned} \quad (4.40)$$

However, by taking $A > A_0$, and since $L \leq R/\sqrt{8}$, we obtain that

$$\tilde{p} = \frac{1}{\frac{64qL^4}{A} + 1} \geq (1 - 2\epsilon)^{1/\lfloor m(z) \rfloor},$$

and hence $\tilde{p}^{\lfloor m(z) \rfloor} \geq (1 - 2\epsilon)$. Combining this with (4.38) and (4.40), we conclude, for all $A > A_0(z)$, that

$$\tilde{C}^{\text{site}}_{\Delta_{k,l}|\xi''}(C_{k,l}) \geq (1 - 2\epsilon)^2 > 1 - 4\epsilon > \left(p_c^{\text{site}}(\mathbf{Z}^2)\right)^{1/2}. \quad (4.41)$$

If $\omega \in C_{k,l}$, we say that $\Delta_{k,l}$ is a 'good' cell. Two neighbouring cells $\Delta_{k,l}$ and $\Delta_{k+1,l}$ are said to be 'linked' if the box $\Delta_{\text{link}} := \Delta_{\text{link}}^{k:k+1,l}$ defined in (4.10) has an intersection with $\mathrm{Del}_1^{**}(\omega)$ that contains only points of mark 1. More precisely, the event that $\Delta_{k,l}$ and $\Delta_{k+1,l}$ are linked, is

$$L_{k,k+1}^l := \Big\{ \overline{\omega} \in \overline{\Omega} : \sigma_\omega(x) = 1 \text{ for all } x \in \Delta_{\text{link}}^{k:k+1,l} \cap \mathrm{Del}_1^{**}(\omega) \Big\}.$$

Also define

$$F_{\text{link}} := \left(F_{k,l}^- \cap F_{k,l}^{\text{ext}} \right) \cap \left(F_{k+1,l}^- \cap F_{k+1,l}^{\text{ext}} \right)$$

and

$$G_{\text{link}} := \{\omega \in \Omega : |\omega \cap \Delta_{\text{link}}| \leq m(z)\},$$

and let $\xi' \in \Omega_{\Delta^c_{\text{link}}}$ be a boundary configuration outside Δ_{link} such that $\xi' \backslash \Lambda = \xi$. The conditional probability that $\Delta_{k,l}$ and $\Delta_{k+1,l}$ are linked, given they are both 'good' cells, is then given by

$$\begin{aligned} \tilde{C}^{\text{site}}_{\Delta_{\text{link}}|\xi'}(L^l_{k,k+1}|C_{k,l} \cap C_{k+1,l}) &\geq \int M_{\Lambda,\Delta_{\text{link}}|\xi'}(d\omega|F_{\text{link}})\mathbf{1}_{\{G_{\text{link}}\}}(\omega)\tilde{p}^{\lfloor m(z)\rfloor} \\ &\geq (1-\epsilon)(1-2\epsilon) \\ &\geq 1-4\epsilon \\ &\geq \left(p_c^{\text{site}}(\mathbf{Z}^2)\right)^{1/2}, \end{aligned} \tag{4.42}$$

where the second inequality is another application of Lemma 4.8, but with Δ_{link} in place of Δ^-. Then, by (4.41), (4.42) and the results of McDiarmid and Hammersley, in particular, (4.7), mixed site-bond percolation occurs. There exists a chain of good boxes joined by links from $\Delta_{k,l} \subset \Lambda$ to Λ^c.

It remains to check this implies $\{\Delta \leftrightarrow \Lambda^c\}$. For this, we recall the set $H^L_{k:k+1,l}(\omega)$ from (4.11). Using the argument of the proof of Lemma 3.23, we know that all hyperedges $\eta = \{x,y\} \in \text{Del}_2(\omega)$ that have a non-empty intersection with $H^L_{k:k+1,l}(\omega)$ satisfy $|x-y| < \sqrt{8}L$. This implies that $H^L_{k:k+1,l}(\omega) \subset \text{Del}_1^{**}(\omega)$. Let $x, y \in \omega$ be such that $\text{Vor}_\omega(x)$ and $\text{Vor}_\omega(y)$ contain the centres of the boxes $\Delta_{k,l}$ and $\Delta_{k+1,l}$ respectively. Since $H^L_{k:k+1,l}(\omega) \subset \text{Del}_1^{**}(\omega)$, we can to connect x and y in the graph $\text{Del}_2^{**}(\omega)$ inside $\Delta^-{}_{k,l} \cup \Delta_{\text{link}} \cup \Delta^-{}_{k+1,l}$. Hence, by (4.41) and (4.42), we have

$$\tilde{C}^{\text{site}}_{\Lambda|\xi_{\Lambda^c}}(\{\Delta \leftrightarrow \Lambda^c\}) > c > 0.$$

□

Theorem 4.1 follows as a consequence of Proposition 2.18 and Proposition 2.17.

4.5 Proof of Proposition 4.4

4.5.1 Sketch

The proof of Proposition 4.4 is rather long, so we first outline the strategy. We want to bound the number of connected components that intersect V_B in the reduced graph of $E^{\text{ext}}_{x_0|\omega}$ under $\mu^{(q)}_{\text{ext},\omega}$. Note that if we had an upper bound on $|V_B|$, this would be trivial. Unfortunately, we do not. However, we do know that for large $|V_B|$, a large number of particles must be close together and thus, are more likely to be connected to each other due to the soft exclusion between particles of different type. The graph $\partial_B := (V_B, E_B)$, with edge set

$$E_B := \left\{ \eta_{xy} \in E^{\text{ext}}_{x_0|V_B} : x, y \in V_B \right\} \tag{4.43}$$

is connected, but does not necessarily contain a polygon, like $\partial_{x_0|\omega}$. Here, $E^{\text{ext}}_{x_0|V_B}$ is simply $E^{\text{ext}}_{x_0|\omega}$ evaluated for $\omega = V_B$. The case where x_0 lies in the opposite half plane to all $x \in V_B$ is one such example of ∂_B not forming a loop. We call ∂_B the contraction of the boundary $\partial_{x_0|\omega}$ to B, see Figure 4.3.

The crux of the proof is to find an upper bound for the number of edges in the edge set E_B that have length greater than some fixed real number. This will allow us to construct a sequence of edge subsets of E_B, defined by edge length, to balance the unbounded number of particles against the increased probability that they are connected. The smaller the edge lengths, the greater the possible number of edges in the subset, but also the greater the probability that they are open. It turns out that such an upper bound can be found in the scenario where there are no defects in the geometry of V_B. These defects which we give the logical name 'kinks' are defined below in Definition 4.11. An upper bound cannot be found if the geometry of V_B contains kinks, so we devise a plan to discount them.

Notice that it is not necessary that $E_B \subset E^{\text{ext}}_{x_0|\omega}$. This creates a problem when we consider an edge drawing mechanism on $E^{\text{ext}}_{x_0|\omega}$. To overcome this, we introduce an edge drawing mechanism on E_B and build a structure that will allow us to compare events between the two probability spaces. This technique relies heavily on some geometric properties of the Delaunay tessellation. Having outlined a sketch of the strategy, we begin to define some important notation.

4.5.2 Notation

We begin by introducing a polar coordinate system in $\mathbf{R}^2$. Let x_0 be the pole, and let L be the polar axis. For $z \in \mathbf{R}^2$, denote $\hat{z}$ to be the angular coordinate of z taken counter clockwise from the polar axis L. Given two points $x, y \in \mathbf{R}^2$, $\overleftrightarrow{xy}$ will denote the unique straight line that intersects x and y in the plane, $\overleftarrow{xy}$ will denote the half line that stops at y and $\overline{xy}$ will denote the line segment between x and y only. Given two straight lines $\ell_1, \ell_2 \subset \mathbf{R}^2$ that intersect at a point $z \in \mathbf{R}^2$, $\angle(\ell_1, \ell_2)$ will denote the angle between them. More precisely, it is the angle needed in order to rotate ℓ_1 onto ℓ_2 with z as the centre of rotation. Notice that it is certainly not true that $\angle(\ell_1, \ell_2) = \angle(\ell_2, \ell_1)$, however, it holds that

$\angle(\ell_1, \ell_2) + \angle(\ell_2, \ell_1) = \pi$. When we consider a triangle in the plane with vertices x, y, z, for example, we sometimes refer to the interior angle at y as $\widehat{xyz}$. In this case, as we specify the interior angle, $\widehat{xyz} = \widehat{zyx}$.

Definition 4.10. *Given a set of points $V = \{x_i \in \mathbf{R}^2 : 1 \le i \le n\}$ with $\hat{x}_1 < \cdots < \hat{x}_n$, the graph*

$$\Gamma = \left(V, \bigcup_{i=1}^{n-1} \eta_{x_i x_{i+1}} \right)$$

is called a spoked chain if $\eta_{x_0 x_i} \in \mathrm{Del}_2(V \cup \{x_0\})$ for all $1 \le i \le n$. The polygon $P(\Gamma, x_0)$, created by adding the point x_0 and edges $\eta_{x_0 x_1}$ and $\eta_{x_n x_0}$ to Γ is called the induced polygon of Γ – see Figure 4.2.

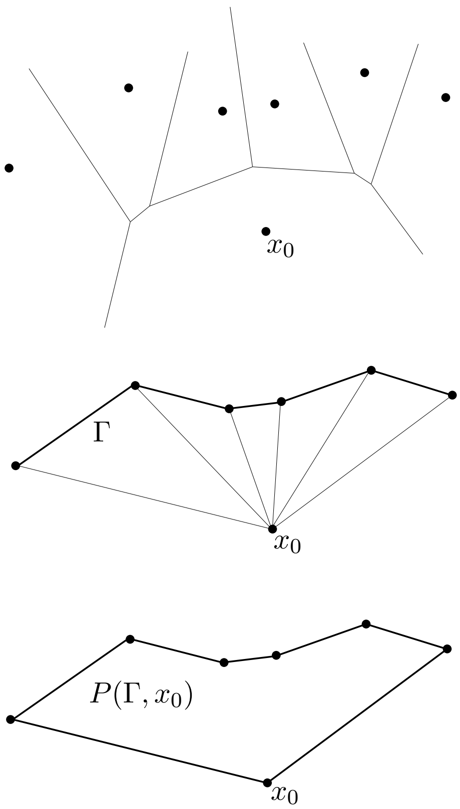


Figure 4.2: From top to bottom we have: 1. A collection of points that neighbour x_0 in the Delaunay/Voronoi tessellation. 2. A spoked chain Γ, shown in bold. 3. The induced polygon $P(\Gamma, x_0)$.

In order to quantify the number of connected components that intersect V_B, we analyse the shape of the contracted boundary, ∂_B. First however, we split B into four quadrants, $Q_i \subset \mathbf{R}^2$ for $i = 1, 2, 3, 4$, where

$$Q_i := \left\{ z \in B : \frac{\pi}{2}(i-1) \leq \hat{z} < \frac{\pi}{2} i \right\}.$$

Instead of bounding the number of connected components that intersect V_B directly, the plan is to bound the number of connected components that intersect $V_B \cap Q_1$ and then multiply this bound by 4. The reasons for doing this are twofold: it not only provides us a framework to define kinks, but also ensures that any two points that we consider will differ in angle by no more than $\pi/2$. This allows us to find a lower bound for the probability that the two points belong to the same connected component.

Definition 4.11. *Let $\Gamma = (V, E)$ be a spoked chain. Suppose $x_i, x_j, x_k \in V$ such that $\hat{x}_i < \hat{x}_j < \hat{x}_k$. We say that x_i, x_j and x_k form a kink in Γ if:*

1. $\widehat{x_i x_j x_k} < \pi/2$
2. $\widehat{x_{i'} x_{j'} x_{k'}} \geq \pi/2$ *for all* $x_{i'}, x_{j'}, x_{k'} \in V$ *with* $\hat{x}_j \leq \hat{x}_{i'} < \hat{x}_{j'} < \hat{x}_{k'} < \hat{x}_k$
3. $\widehat{x_{i'} x_{j'} x_{k'}} \geq \pi/2$ *for all* $x_{i'}, x_{j'}, x_{k'} \in V$ *with* $\hat{x}_j < \hat{x}_{i'} < \hat{x}_{j'} < \hat{x}_{k'} \leq \hat{x}_k$.

Definition 4.12. *Suppose x_i, x_j and x_k form a kink in the spoked chain $\Gamma = (V, E)$. The kink is called intruding if the line segment $\overline{x_i x_k}$ lies outside of the induced Polygon $P(\Gamma, x_0)$ and protruding if it lies inside $P(\Gamma, x_0)$.*

Lemma 4.13. *Let $\Gamma = (V, E)$ be a spoked chain where $V = \cup_{i=1}^n \{x_i\}$ and $\hat{x}_1 < \cdots < \hat{x}_n$. A kink in $\Gamma = (V, E)$ is either intruding or protruding.*

Proof. Suppose not. Then there exists $1 \leq i < j < k \leq n$ such that x_i, x_j and x_k form a kink in Γ and $\overline{x_i x_k}$ lies neither inside nor outside of $P(\Gamma, x_0)$. Let $U \subset \mathbf{R}^2$ be the connected component of $\mathbf{R}^2 \backslash \overleftrightarrow{x_i x_k}$ than does not contain x_j. It follows that Γ crosses $\overline{x_i x_k}$ between $\hat{x}_i$ and $\hat{x}_k$ and hence, there exists $x_{j'} \in V \cap U$ with $\hat{x}_i < \hat{x}_{j'} < \hat{x}_k$. Without loss of generality, let $\hat{x}_j < \hat{x}_{j'} < \hat{x}_k$. Therefore, $\widehat{x_i x_j x_{j'}} < \pi/2$ which contradicts condition 2 of Definition 4.11 for the kink formed by x_i, x_j and x_k. □

Lemma 4.14. *Let $\Gamma = (V, E)$ be a spoked chain with $V = \cup_{i=1}^n \{x_i\}$ and $\hat{x}_1 < \cdots < \hat{x}_n$. If x_i, x_j and x_k form an intruding kink in Γ, then $\angle(\overleftrightarrow{x_i x_{i+1}}, \overleftrightarrow{x_{k-1}, x_k}) < \pi/2$.*

Proof. Suppose there exists a spoked chain $\Gamma = (V, E)$ with $V = \cup_{i=1}^n \{x_i\}$ and $\hat{x}_1 < \cdots < \hat{x}_n$. Let x_i, x_j and x_k form an intruding kink in Γ. Since the kink is intruding, we know that x_l lies in the interior of the triangle $\{x_0, x_i, x_k\}$, for all $i < l < k$. Suppose the statement of the Lemma is false, that is: $\angle(\overleftrightarrow{x_i x_{i+1}}, \overleftrightarrow{x_{k-1}, x_k}) \geq \pi/2$. This forces either

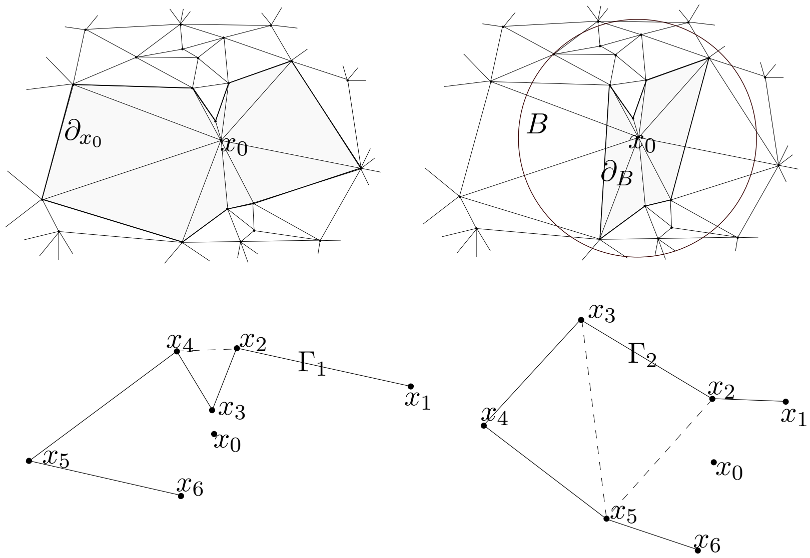


Figure 4.3: From left to right, top to bottom we have: 1. The shaded area is the neighbourhood of x_0, whilst the bold edges form $\partial_{x_0|\omega}$. 2. The contraction of $\partial_{x_0|\omega}$ to ∂_B. 3. The points x_2, x_3 and x_4 form an intruding kink in Γ_1. 4. The points x_2, x_3 and x_5 form a protruding kink in Γ_2.

x_{i+1} or x_{k-1} to be in the interior of the triangle $\{x_i, x_j, x_k\}$. Without loss of generality, suppose, in fact, that x_{i+1} is in the interior of $\{x_i, x_j, x_k\}$. Therefore, $\widehat{x_{i+1}x_jx_k} < \pi/2$ which by Definition 4.11, contradicts that x_i, x_j and x_k form a kink in Γ. $\square$

4.5.3 Intermediary Lemmas

Kinks of intruding and protruding nature may occur in $\partial_B \cap Q_1$, although the number of them is bounded above – see Lemma 4.15 and Lemma 4.16. The aim is to separate $\partial_B \cap Q_1$ into a fixed number of kinkless pieces, each a spoked chain, which are easier to work with. For each of these kinkless pieces, we will find an upper bound for the expected number, with respect to $\mu^{(q)}_{\text{ext},\omega}$, of connected components in a thinning of $E^{\text{ext}}_{x_0|\omega}$ that intersect it.

Lemma 4.15. *The number of intruding kinks in $\partial_B \cap Q_1$ is bounded above by 2.*

Proof. We show that the angle between two intruding kinks in a spoked chain is greater than $\pi/4$. Since $\partial_B \cap Q_1$ lies in the quadrant Q_1, and is a spoked chain by definition, the result will follow. Let $\Gamma = (V, E)$ be a spoked chain and order the elements of V, such that $\hat{x}_1 < \cdots < \hat{x}_n$. Suppose there is an intruding kink in Γ. Therefore, by Definition 4.12 and

Lemma 4.14, there exists $1 \le i < n-1$ and $i+1 < j \le n$, such that

$$\angle(\overleftrightarrow{x_i x_{i+1}}, \overleftrightarrow{x_{j-1} x_j}) < \pi/2 \tag{4.44}$$

and $\overline{x_i x_j}$ lies outside of the induced polygon $P(\Gamma, x_0)$.

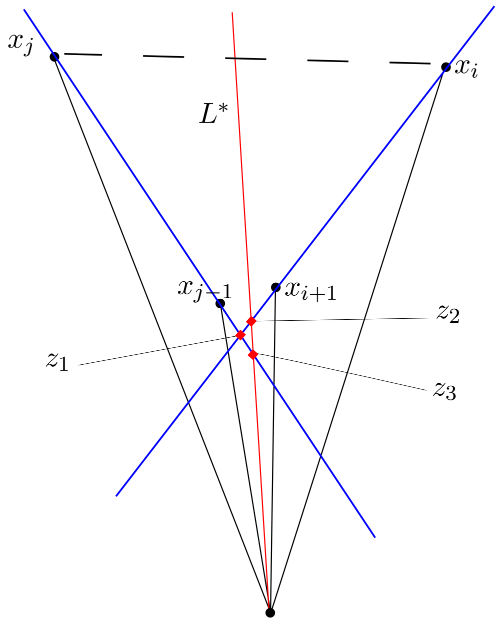


Figure 4.4: Lower bound for angle $x_0 \widehat{x_{j-1}} x_j$

The straight lines $\overleftrightarrow{x_i x_{i+1}}$ and $\overleftrightarrow{x_{j-1} x_j}$ split the plane into four regions. Since the kink is intruding, x_0 must lie in the opposite region to that of the line segment $\overline{x_i x_j}$. Let L^* be the radial line of angle $\frac{\hat{x}_{i+1} - \hat{x}_{j-1}}{2}$. Let $z_1 \in \mathbf{R}^2$ be the point of intersection of $\overleftrightarrow{x_i x_{i+1}}$ and $\overleftrightarrow{x_{j-1} x_j}$ and let z_2, z_3 be the points of intersection of L^* with $\overleftrightarrow{x_i x_{i+1}}$ and $\overleftrightarrow{x_{j-1} x_j}$ respectively – see Figure 4.4. Then,

$$\widehat{x_i z_1 x_j} + \widehat{x_i z_2 x_0} + \widehat{x_0 z_3 x_j} = 2\pi, \tag{4.45}$$

which implies, together with (4.44), that

$$\max\left\{\widehat{x_i z_2 x_0}, \widehat{x_0 z_3 x_j}\right\} \ge \frac{2\pi - \pi/2}{2} = \frac{3\pi}{4}. \tag{4.46}$$

Without loss of generality, let $\widehat{x_0 z_3 x_j} \ge \frac{3\pi}{4}$. Because x_{j-1} lies on the line segment $\overline{z_3 x_j}$, it follows that

$$\widehat{x_0 x_{j-1} x_j} \geq \widehat{x_0 z_3 x_j} \geq \frac{3\pi}{4}. \tag{4.47}$$

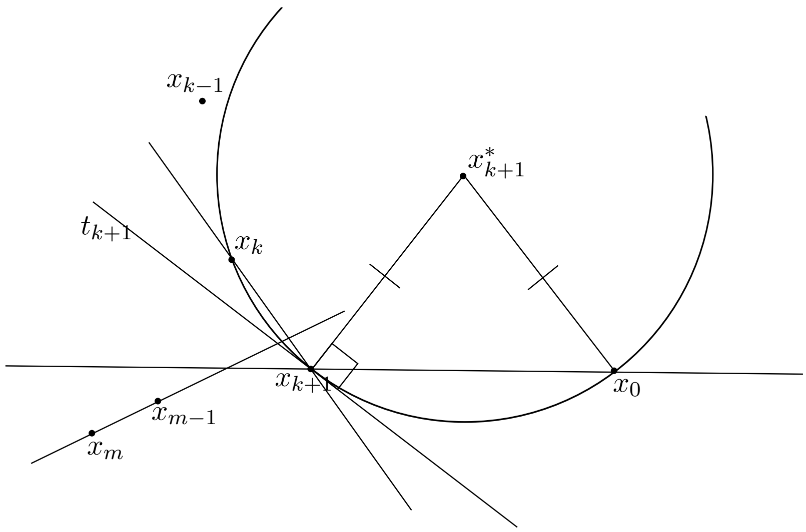


Figure 4.5: The intruding kink formed by x_k, x_l and x_m.

Suppose there is another intruding kink in Γ, formed by x_k, x_l and x_m for $j < k < l < m \leq n$. Then , by Lemma 4.14, we have that

$$\angle(\overleftrightarrow{x_k x_{k+1}}, \overleftrightarrow{x_{m-1} x_m}) < \pi/2.$$

Let t_{k+1} be the tangent to $B(\tau(x_0, x_k, x_{k+1}))$ at x_{k+1}. Then, by noting that

$$|V \cap B(\tau(x_0, x_{j-1}, x_j))| = 0,$$

which is a consequence of the properties of the Delaunay structure (2.50), it follows that

$$\begin{aligned} \angle(t_{k+1}, \overleftrightarrow{x_0 x_{k+1}}) &\leq \angle(\overleftrightarrow{x_k x_{k+1}}, \overleftrightarrow{x_0 x_{k+1}}) && (4.48)\\ &\leq \angle(\overleftrightarrow{x_k x_{k+1}}, \overleftrightarrow{x_{m-1} x_m}) && (4.49)\\ &< \pi/2. && (4.50) \end{aligned}$$

Here, (4.48) is direct from the definition of a tangent and (4.49) is a consequence of the fact that $\hat{x}_{k+1} < \hat{x}_{m-1} < \hat{x}_m$. For $1 \leq r < n$, let x^*_{r+1} denote the centre of the circumscribing circle of $\tau(x_0, x_r, x_{r+1}) \in \text{Del}_3(V \cup \{x_0\})$. Since the triple $\{x_0, x^*_{k+1}, x_{k+1}\}$ form an isosceles triangle, see Figure 4.5, we can conclude that

$$\hat{x}_{k+1} - \hat{x}^*_{k+1} = \pi/2 - \angle(t_{k+1}, \overleftrightarrow{x_0 x_{k+1}}) > 0. \tag{4.51}$$

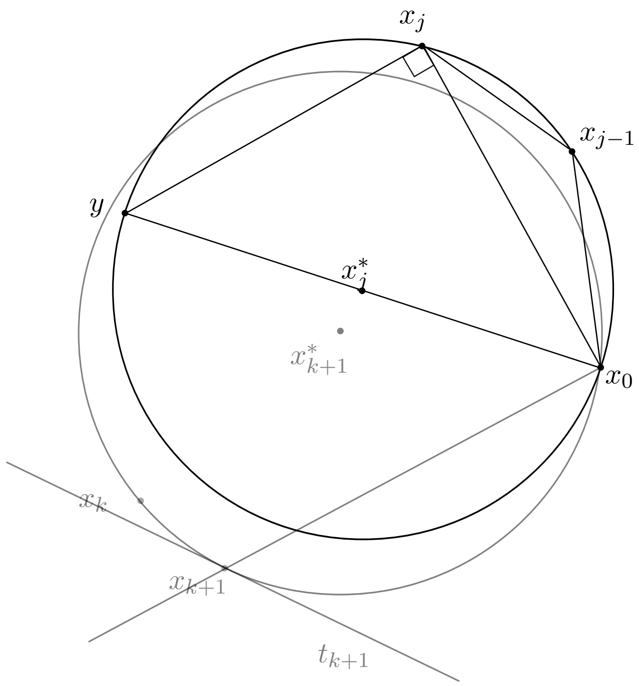


Figure 4.6: Lower bound for angle between kinks of type 2

Let y be the antipodal point to x_0 on the circumscribed ball of $\tau(x_0, x_{j-1}, x_j)$ in $\mathbf{R}^2$. Since, $|x_0 - y|$ is equal to the diameter of the circle, it follows that $\widehat{x_0 x_j y} = \pi/2$, see Figure 4.6. The points x_0, x_{j-1}, x_j and y form a cyclic quadrilateral. Using, (4.47), and the fact that opposite angles of a cyclic quadrilateral add up to π, we see that $\widehat{x_0 y x_j} \leq \pi/4$. Hence, by (4.51)

$$\hat{x}_{k+1} - \hat{x}_j > \hat{x}^*_{k+1} - \hat{x}_j \geq \hat{x}^*_j - \hat{x}_j = \widehat{y x_0 x_j} = \pi - \underbrace{\widehat{x_0 x_j y}}_{=\pi/2} - \underbrace{\widehat{x_0 y x_j}}_{\leq \pi/4} \geq \pi/4, \tag{4.52}$$

where the second inequality is due to a further property of the Delaunay structure that we show in the Appendix A.1. This tells us that the angle between intruding kinks must be greater than $\pi/4$.

□

Lemma 4.16. *There are no protruding kinks in $\partial_B \cap Q_1$.*

Proof. Again, order the elements of V_B such that $\hat{x}_1 < \cdots < \hat{x}_n$. Suppose we have a protruding kink, then, by Definition 4.12, we have for some $1 \leq i < j < k \leq n$

$$\widehat{x_i x_j x_k} < \pi/2. \tag{4.53}$$

The pair $x_i x_k$ does not form an edge of E_B, therefore, by the properties of the Delaunay graph, x_j lies inside $B(\tau(x_0, x_i, x_k))$. The line segment $\overline{x_i x_k}$ is a chord which splits $B(\tau(x_0, x_i, x_k))$ into two regions. Since we have a protruding kink, $\overline{x_i x_k}$ lies inside the induced polygon $P(\partial_B \cap Q_1, x_0)$ and so x_j does not lie in the same region as x_0. Therefore, by (4.53), it follows that $\widehat{x_i x_0 x_k} \geq \pi/2$, and hence, there are no protruding kinks in $\partial_B \cap Q_1$.

□

Before we can prove Proposition 4.4, we need two more results to allow us some control over the edges of $E^{\text{ext}}_{x_0|\omega}$. We first introduce an edge drawing mechanism, where edges are drawn independently of each other, conditionally on a given particle configuration. Let $\tilde{\mu}_{2,\omega}$ be the alternative edge drawing mechanism associated with the edge probability

$$\tilde{p}_2(\eta_{xy}) := \frac{\mathbf{1}\{|x-y| \leq R\}}{\frac{q}{A}\left(|x-y|\right)^4 + 1}, \tag{4.54}$$

and let $\tilde{\mu}^{\text{ext}}_{2,\omega}$ be the corresponding edge drawing mechanism on $E^{\text{ext}}_{x_0|\omega}$ only.

Lemma 4.17. *For all $q \geq 1$ and $\omega \in \Omega$, we have $\mu^{(q)}_{\omega,\Lambda} \succcurlyeq \tilde{\mu}_{2,\omega}$.*

Proof. Fix $\eta = (x, y) \in \text{Del}_2(\omega)$. By Proposition 2.3, it is enough to show that

$$\frac{p(\eta)}{q(1-p(\eta))} \geq \frac{\tilde{p}_2(\eta)}{(1-\tilde{p}_2(\eta))}. \tag{4.55}$$

Recall, from the construction of the continuum random cluster model, that

$$p(\eta) = 1 - e^{-\varphi(|x-y|)} = \frac{\mathbf{1}\{|x-y| \leq R\}}{\frac{1}{A}|x-y|^4 + 1}.$$

If $|x - y| > R$, then $p(\eta) = \tilde{p}_2(\eta) = 0$, and (4.55) is trivial. Suppose $|x - y| < R$, then, in fact, we also have

$$\frac{p(\eta)}{q(1-p(\eta))} = \frac{\tilde{p}_2(\eta)}{(1-\tilde{p}_2(\eta))}.$$

Therefore, (4.55) holds for all $\eta = (x, y) \in \text{Del}(\omega)$ and the result follows. □

Recall, that $V_B \subset V_{x_0|\omega} \subset \omega$. Also recall, from (4.43), that E_B is not necessarily a subset of $E^{\text{ext}}_{x_0|\omega}$. In fact, they belong to different Delaunay tessellations:

$$E_B \subset \text{Del}_2(V_B \cup \{x_0\}) \quad \text{and} \quad E^{\text{ext}}_{x_0|\omega} \subset \text{Del}_2(\omega \cup \{x_0\}).$$

We introduce another edge drawing mechanism, but this time on E_B. Let μ^* denote the distribution of the random hyperedge configurations

$$\{\eta \in E_B : v(\eta) = 1\}$$

where $(v(\eta))_{\eta \in E_B}$ are independent Bernoulli random variables with probability

$$\text{Prob}(v(\eta) = 1) = p^*(\eta) := \frac{\mathbf{1}\{|x-y| \leq \frac{2}{\pi} \wedge R\}\mathbf{1}\{|\hat{x}-\hat{y}| \leq \frac{\pi}{2}\}}{\frac{q}{A}\left(\frac{\pi}{2}|x-y|\right)^4 + 1}, \tag{4.56}$$

for $\eta = \{x, y\} \in E_B$. We then compare the probability that two points are connected with respect to $\tilde{\mu}_{2,\omega}$ and the probability that they are connected with respect to μ^*.

Lemma 4.18. *Fix $\omega \in \Omega$. Let $\eta_{xy} \in E_B$ and let $x \leftrightarrow y$ denote that x and y lie in the same connected component of (ω, E), where E is a $\tilde{p}_2$–thinning of $E^{\text{ext}}_{x_0|\omega}$. Then,*

$$\tilde{\mu}^{\text{ext}}_{2,\omega}(x \leftrightarrow y) \geq p^*(\eta_{xy}). \tag{4.57}$$

Proof. By the definition of p^*, (4.57) follows trivially for $x, y \in \partial_B$ with $|x-y| > \frac{2}{\pi} \wedge R$ or with $|\hat{x}-\hat{y}| > \frac{\pi}{2}$. Therefore, we assume $|x-y| \leq \frac{2}{\pi} \wedge R$ and $|\hat{x}-\hat{y}| \leq \frac{\pi}{2}$.

Case 1: If $\eta_{xy} \in E^{\text{ext}}_{x_0|\omega}$, we have:

$$\tilde{\mu}^{\text{ext}}_{2,\omega}(x \leftrightarrow y) \geq \tilde{p}_2(\eta_{xy}) = \frac{1}{\frac{q}{A}\left(|x-y|\right)^4 + 1} \geq \frac{1}{\frac{q}{A}\left(\frac{\pi}{2}|x-y|\right)^4 + 1} \geq p^*(\eta_{xy}).$$

Case 2: If $\eta_{xy} \notin E^{\text{ext}}_{x_0|\omega}$, the proof is much longer and goes as follows. Since $\eta_{xy} \notin E^{\text{ext}}_{x_0|\omega}$, and $x, y \in \partial_B$ there exists $z \in \omega \backslash \partial_B$, such that $\eta_{zx_0} \in \text{Del}_2(\omega \cup \{x_0\})$. This implies that $z \in V_{x_0|\omega} \backslash V_B$ and $\hat{x} < \hat{z} < \hat{y}$. We now check whether $\eta_{xz}, \eta_{zy} \in E^{\text{ext}}_{x_0|\omega}$. If they are not, we find more points of $V_{x_0|\omega} \backslash V_B$ in a similar way. Continue this process until there are no more points $z \in V_{x_0|\omega} \backslash V_B$ with $\hat{x} < \hat{z} < \hat{y}$. Therefore, there exists $z_1, z_2, \ldots z_n \in V_{x_0|\omega} \backslash V_B$ with $\hat{x} < \hat{z}_1 < \cdots < \hat{z}_n < \hat{y}$ such that

$$\eta_{xz_1}, \eta_{z_1 z_2}, \ldots, \eta_{z_n y} \in E^{\text{ext}}_{x_0|\omega}.$$

The event that each of these hyperedges is open implies the event that x and y belong to the same connected component of open hyperedges, hence

$$\tilde{\mu}^{\text{ext}}_{2,\omega}(x \leftrightarrow y) \geq \tilde{p}_2(\eta_{xz_1})\tilde{p}_2(\eta_{z_1z_2}) \cdots \tilde{p}_2(\eta_{z_ny}). \tag{4.58}$$

For any two points, $x_1, x_2 \in \omega$, with $\hat{x}_1 < \hat{x}_2$, define $C^{x_0}_{x_1x_2}$ to be the arc on the circle $B(\tau(x_1, x_2, x_0))$ between x_1 and x_2 and define $U_{x_1x_2}$ to be the subset of $\mathbf{R}^2$ bounded by $C^{x_0}_{x_1x_2}$ and $\overline{x_1x_2}$, that is, the convex hull of $C^{x_0}_{x_1x_2}$. Let $n = \#\{z \in \partial_{x_0|\omega} : \hat{x} < \hat{z} < \hat{y}\}$. We claim that

$$L(C^{x_0}_{xz_1}) + \cdots + L(C^{x_0}_{z_ny}) \leq L(C^{x_0}_{xy}) \tag{4.59}$$

no matter the value of $n \in \mathbf{N}$, where $L(C)$ denotes the length of the arc C. Before we prove the claim, we show how it implies the Lemma. By our assumption that $|x - y| \leq \frac{2}{\pi} \wedge R$ and $|\hat{x} - \hat{y}| \leq \frac{\pi}{2}$, it follows that $L(C^{x_0}_{xy}) \leq 1$, and by (4.59): $L(C^{x_0}_{xz_1}) + \cdots + L(C^{x_0}_{z_ny}) \leq 1$. Obviously, this shows that

$$|x - z_1| + |z_1 - z_2| + \cdots + |z_{n-1} - z_n| + |z_n - y| \leq 1. \tag{4.60}$$

Consider the following algebraic manipulation. For $a, b \in \mathbf{R}$ with $0 \leq a \leq b \leq 1$, we have

$$\frac{1}{\frac{q}{A}a^4 + 1} \cdot \frac{1}{\frac{q}{A}b^4 + 1} = \frac{1}{\frac{q}{A}\left(\frac{q}{A}a^4b^4 + a^4 + b^4\right) + 1} \geq \frac{1}{\frac{q}{A}\left(a + b\right)^4 + 1}, \tag{4.61}$$

where the inequality follows because $\frac{q}{A} < 1$ and because of the constraints on a and b. Hence, using (4.60), we obtain

$$\begin{aligned}
\tilde{p}_2(\eta_{xz_1}) \cdots \tilde{p}_2(\eta_{z_ny}) &\geq \left(\frac{1}{\frac{q}{A}|x - z_1|^4 + 1}\right) \cdots \left(\frac{1}{\frac{q}{A}|z_n - y|^4 + 1}\right) \\
&\geq \frac{1}{\frac{q}{A}\left(|x - z_1| + |z_1 - z_2| + \cdots + |z_n - y|\right)^4 + 1} \\
&\geq \frac{1}{\frac{q}{A}L(C^{x_0}_{xy})^4 + 1} \\
&\geq \frac{1}{\frac{q}{A}\left(\frac{\pi}{2}|x - y|\right)^4 + 1} = p^*(\eta_{xy}),
\end{aligned} \tag{4.62}$$

where (4.62) is the repeated use of (4.61) with $a = |z_i - z_{i+1}|$ and $b = |z_j - z_{j+1}|$. This, along with (4.58), shows the statement of the Lemma. We finish by verifying the claim (4.59).

Suppose there exists $z \in \partial_{x_0|\omega} \backslash \partial_B$ such that $\eta_{xz}, \eta_{zy} \in E^{\text{ext}}_{x_0|\omega}$. Since $z \notin B$, it must be true that $z \in U_{xy}$. Therefore, by a direct application of Theorem A.2, we have

$$L(C^{x_0}_{xz}) + L(C^{x_0}_{zy}) \le L(C^{x_0}_{xy}).$$

and the claim holds for $n = 1$. Assume the claim holds for $n = k - 1$. Then, let $n = k$. There exists $z_1, \ldots z_k \in \partial_{x_0|\omega} \backslash \partial_B$ such that $\hat{x}_1 < \cdots < \hat{x}_k$ and $\eta_{xz_1}, \ldots, \eta_{z_k y} \in E^{\text{ext}}_{x_0|\omega}$. Let

$$i = \underset{1 \le j \le k}{\operatorname{argmax}} |z_j - \overline{xy}|.$$

It follows that $z_i \in U_{z_{i-1} z_{i+1}}$ where, for convenience, we write $z_0 := x$ and $z_{k+1} := y$. By Theorem A.2 in the Appendices,

$$L(C^{x_0}_{z_{i-1} z_i}) + L(C^{x_0}_{z_i z_{i+1}}) \le L(C^{x_0}_{z_{i-1} z_{i+1}}). \tag{4.63}$$

By making the following notation change:

$$z'_j := \begin{cases} z_j, & \text{for } 1 \le j < i \\ z_{j+1}, & \text{for } i \le j \le k-1, \end{cases}$$

it follows, from (4.63), that

$$L(C^{x_0}_{xz_1}) + \cdots\cdots + L(C^{x_0}_{z_k y}) \le L(C^{x_0}_{xz'_1}) + \cdots + L(C^{x_0}_{z'_{k-1} y}),$$

and hence, by the assumption for $n = k - 1$,

$$L(C^{x_0}_{xz_1}) + \cdots\cdots + L(C^{x_0}_{z_k y}) \le L(C^{x_0}_{xy}).$$

The claim follows by mathematical induction. $\square$

Lemma 4.19. *Let $\delta > 0$ and $\Gamma = (V, E)$ be a spoked chain with $V \subset Q_1$. If Γ does not contain a kink, then the number of edges in E with length greater than 2δ is at most $6\left(\frac{R}{\delta}\right)^2$.*

Proof. Order the elements of $V = \cup_{i=1}^n \{x_i\}$ such that $\hat{x}_1 < \ldots < \hat{x}_n$. For $1 \le i < n$, let $D_i \subset \mathbf{R}^2$ be the disc of radius $\frac{|x_i - x_{i+1}|}{2}$ centred at x_i. Let $S_i \subset \mathbf{R}^2$ be the sector of D_i with interior angle $\pi/2$ and line of symmetry $\overline{x_i x_{i+1}}$. We claim that

$$S_i \cap S_i = \emptyset, \text{ for } i \neq j; \text{ and} \tag{4.64}$$

$$\bigcup_{i=1}^{n-1} S_i \subset Q_1 \oplus \frac{R}{\sqrt{2}}. \tag{4.65}$$

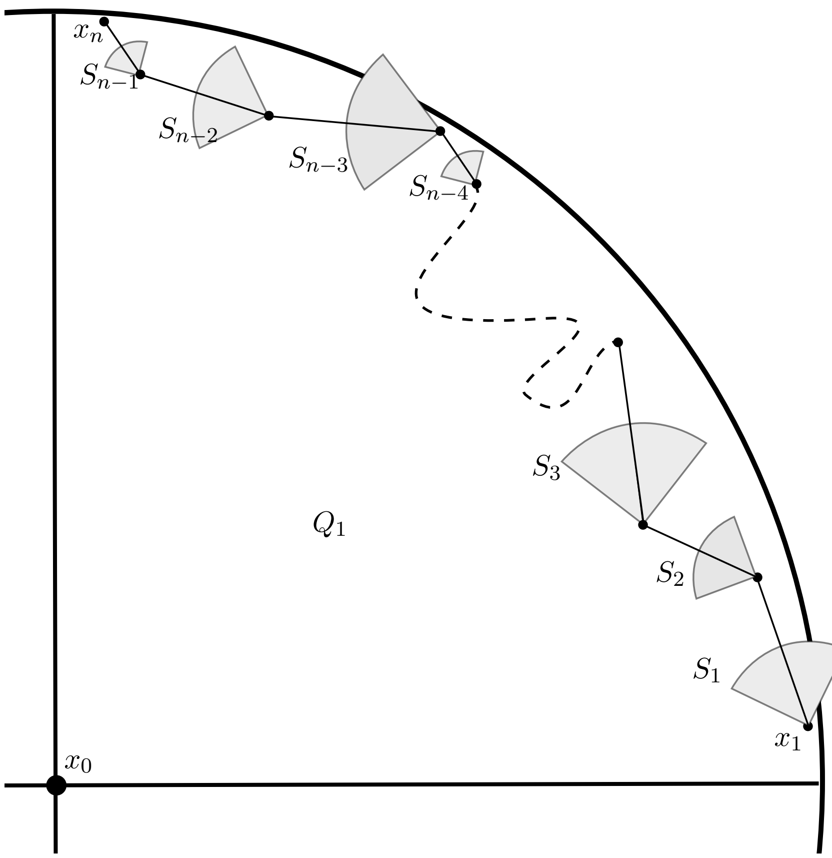


Figure 4.7: The sectors S_i of a spoked chain in Q_1.

Assuming the claim is true, the sum of the areas of the sectors S_i must not exceed the area of $Q_1 \oplus \frac{R}{\sqrt{2}}$ which is less than $\frac{3}{2}\pi R^2$. Each edge $\eta \in E$ of length greater than 2δ contributes a sector of area greater than $\frac{\pi}{4}\delta^2$, therefore, the maximum number of such edges in Γ is simply

$$\frac{\frac{3}{2}\pi R^2}{\frac{\pi}{4}\delta^2} = 6\left(\frac{R}{\delta}\right)^2,$$

which gives the result. All that is left to do, is to prove the claims. Consider $x_i \in V$. Let ℓ_1 be the image of the line $\overleftrightarrow{x_i x_{i+1}}$ under a rotation of angle $\pi/2$, centred at x_{i+1}. There are exactly two connected components of $\mathbf{R}^2 \backslash \ell_1$. Let U denote the one that contains x_i. Now suppose $x_k \in U$ for some $i+1 < k \leq n$. This implies that $x_i \widehat{x_{i+1}} x_k < \pi/2$. Then, by Definition 4.11, this contradicts the fact that Γ does not contain a kink. Therefore, $x_k \in U^c$ for all $i+1 < k \leq n$.

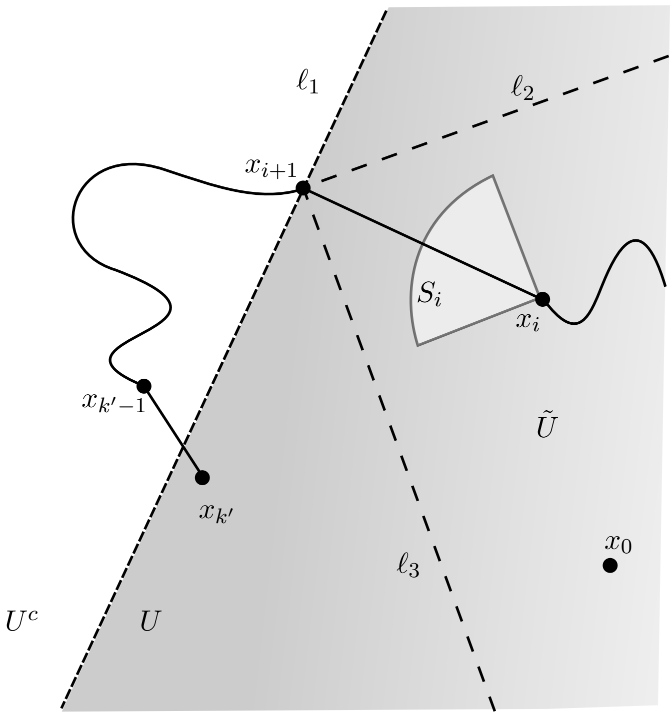


Figure 4.8: The point x_k' is the first time after x_{i+1} that the chain enters U.

Let ℓ_2 and ℓ_3 be the images of the half line $\overleftarrow{x_i x_{i+1}}$ under rotations, centred at x_{i+1}, of angles $\pi/4$ and $-\pi/4$ respectively – see Figure 4.8. Again, there are two connected components of $\mathbf{R}^2 \backslash (\ell_2 \cup \ell_3)$. Let $\tilde{U}$ denote the one that contains x_i. Equation (4.64) follows by noticing that $S_i \subset \tilde{U}$ and $S_k \subset \tilde{U}^c$ for all $i+1 < k \leq n$. Equation (4.65) is easily verified when you consider that $S_i \subset D_i$ for all $1 \leq i < n$ and the maximal radius for D_i is half the maximal edge length, which, considering we are restricted to Q_1, is $\sqrt{2}R$. $\square$

We are now in a position to prove the main result of this section.

4.5.4 Proof

Proof of Proposition 4.4: First of all, we split $B = \mathcal{B}_R(x_0)$ into four quadrants, $Q_i \subset \mathbf{R}^2$ for $i = 1, 2, 3, 4$, where

$$Q_i := \left\{ z \in B : \frac{\pi}{2}(i-1) \leq \hat{z} < \frac{\pi}{2} i \right\}.$$

Consider $\partial_B \cap Q_1$ which contains all vertices and edges of ∂_B that lie wholly in Q_1. By construction, $\partial_B \cap Q_1$ is a spoked chain. It follows from Lemma 4.15 and Lemma 4.16 that there are at most 2 intruding kinks of $\partial_B \cap Q_1$ and zero protruding kinks. For each intruding kink x_i, x_j, x_k, we remove the edge $\eta_{x_j x_{j+1}}$ from $\partial_B \cap Q_1$. Since removing an edge anywhere except from the end of a spoked chain will result in leaving two spoked chains, we are left with at most 3 spoked chains in Q_1. Importantly, none of these will contain an intruding or protruding kink.

Let $\Gamma = (V^\Gamma, E^\Gamma)$ be one such kinkless spoked chain in Q_1. Recall from the statement of Proposition 4.4 that the number of clusters or connected components in a graph (ω, E) that intersect V_B is denoted by $N^{\mathrm{cc}}_{V_B}(\omega, E)$ and define $N^{\mathrm{cc}}_{\Gamma}(\omega, E)$ to be the number of clusters of (ω, E) that intersect V^Γ. We endeavour to bound the expectation of $N^{\mathrm{cc}}_{\Gamma}(\omega, E)$ with respect to $\mu^{(q)}_{\text{ext},\omega,\Lambda}$, the edge drawing mechanism on $E^{\text{ext}}_{x_0}(\omega)$ given in (4.13). Then, to conclude the Proposition, we will use the following:

$$\int N^{\mathrm{cc}}_{V_B}(\omega, E)\mu^{(q)}_{\text{ext},\omega,\Lambda}(dE) \leq 12 \int\int N^{\mathrm{cc}}_{\Gamma}(\omega, E)\mu^{(q)}_{\text{ext},\omega,\Lambda}(dE)$$

where the factor 12 is considering at most three kinkless spoked chains in each of the four quadrants. Order the elements of V^Γ such that $\hat{x}_1 < \cdots < \hat{x}_n$. Let $\{x \leftrightarrow y\}$ denote the event that x and y belong to the same cluster of (ω, E) and notice that

$$\begin{aligned}
\int N^{\mathrm{cc}}_{\Gamma}(\omega, E)\mu^{(q)}_{\text{ext},\omega,\Lambda}(dE) &\leq 1 + \sum_{j=1}^{n-1} \left[1 - \mu^{(q)}_{\text{ext},\omega,\Lambda}\big(\{x_j \leftrightarrow x_{j+1}\}\big)\right] \\
&\leq 1 + \sum_{j=1}^{n-1} \left[1 - \tilde{\mu}_{2,\text{ext},\omega,\Lambda}\big(\{x_j \leftrightarrow x_{j+1}\}\big)\right] && (4.66) \\
&\leq 1 + \sum_{j=1}^{n-1} \left[1 - p^*(\eta_{x_j x_{j+1}})\right] && (4.67) \\
&= 1 + \sum_{\eta \in E^\Gamma} [1 - p^*(\eta)], && (4.68)
\end{aligned}$$

where (4.66) and (4.67) are due to Lemmas 4.17 and 4.18 respectively. We partition the edge set E^Γ of the spoked chain Γ into subsets depending on edge lengths. Let

$$E_1^\Gamma := \left\{ \eta_{xy} \in E^\Gamma : |x-y| > \frac{2}{\pi} \wedge R \right\}$$

and

$$E_i^\Gamma := \left\{ \eta_{xy} \in E^\Gamma : \frac{\frac{2}{\pi} \wedge R}{i} < |x-y| \leq \frac{\frac{2}{\pi} \wedge R}{i-1} \right\}$$

for $i \geq 2$, $i \in \mathbf{N}$. By recalling that

$$p^*(\eta_{xy}) = \frac{\mathbf{1}\{|x-y| \leq \frac{2}{\pi} \wedge R\} \mathbf{1}\{|\hat{x}-\hat{y}| \leq \frac{\pi}{2}\}}{\frac{q}{A}\left(\frac{\pi}{2}|x-y|\right)^4 + 1}.$$

from (4.56), we see that $1 - p^*(\eta) = 1$ for all $\eta \in E_1^\Gamma$. However, since Γ is contained in Q_1, and hence $|\hat{x} - \hat{y}| < \frac{\pi}{2}$, we have

$$1 - p^*(\eta_{xy}) = \frac{1}{\frac{A}{q}\left(\frac{\pi}{2}|x-y|\right)^{-4} + 1},$$

for all $\eta_{xy} \in E_i^\Gamma$, for all $i \geq 2$ and $i \in \mathbf{N}$. Let $r := 1 \wedge \frac{R\pi}{2}$. Then, considering $|x - y| \leq \frac{2r}{\pi(i-1)}$ for all $\eta_{xy} \in E_i^\Gamma$, and noticing that $\cup_{i=1}^\infty E_i^\Gamma = E^\Gamma$ and $E_i^\Gamma \cap E_j^\Gamma = \emptyset$ for $i \neq j$. it follows that

$$\begin{aligned}
\sum_{\eta \in E^\Gamma} \big(1 - p^*(\eta)\big) = \sum_{i=1}^\infty \sum_{\eta \in E_i^\Gamma} \big(1 - p^*(\eta)\big) &\leq \sum_{\eta \in E_1^\Gamma} 1 + \sum_{i=2}^\infty \sum_{\eta \in E_i^\Gamma} \left[\frac{1}{\frac{A}{q}\left(\frac{i-1}{r}\right)^4 + 1} \right] \\
&\leq 6R^2\pi^2 r^{-2} + \sum_{i=2}^\infty \left[6R^2\pi^2 i^2 r^{-2}\right] \left[\frac{1}{\frac{A}{q}\left(\frac{i-1}{r}\right)^4 + 1} \right] \qquad (4.69) \\
&\leq 6R^2\pi^2 r^{-2} \left[1 + \frac{q}{A} \sum_{i=2}^\infty \frac{i^2}{(i-1)^4} \right] \qquad (4.70)
\end{aligned}$$

where (4.69) comes from the application of Lemma 4.19 to each of the sets E_i^Γ. To bound the infinite sum, we use the fact that

$$\sum_{i=2}^\infty \frac{i^2}{(i-1)^4} \leq \sum_{i=2}^\infty \frac{4}{(i-1)^2} = 4\sum_{i=1}^\infty \frac{1}{i^2} = \frac{2}{3}\pi^2. \qquad (4.71)$$

Together, Equations (4.68), (4.70) and (4.71) combine to give

$$\begin{aligned}\int N_\Gamma^{\mathrm{cc}}(\omega, E)\mu^{(q)}_{\mathrm{ext},\omega,\Lambda}(dE) &\leq 1 + 6R^2\pi^2 r^{-2}\left[1 + \frac{2q\pi^2}{3A}\right] \\ &= 1 + 6(4 \vee R^2\pi^2)\left[1 + \frac{2q\pi^2}{3A}\right].\end{aligned}$$

Therefore, setting $\alpha := 12\left(6(4 \vee R^2\pi^2)\left[1 + \frac{2q\pi^2}{3A}\right]\right)$ completes the proof. □

Chapter 5

Russo-Seymour-Welsh Theorem

5.1 Introduction

After studying continuum percolation, we now consider a question which serves as the next logical step in a journey towards a complete understanding of the probabilistic properties of continuum models. Briefly, an open crossing from one domain in $\mathbf{R}^2$ to another exists if there is a connected component of open points and hyperedges that intersects each domain. If a hyperedge is open with probability p, how would the crossing probability behave on larger scales? In particular, what is the probability of crossing a rectangle? This leads to the Russo–Seymour–Welsh Theorem (RSW) (see [BR06d] and references within) which relates the probability of an open horizontal crossing of an $L \times L$ square to that of an open horizontal crossing (the long way) of a $3/2L \times L$ rectangle, see [BS98] and [SW78]. More precisely, for some function $f : (0, 1] \to (0, 1]$ with $f(q) \to 1$ as $q \to 1$, if the probability of an open horizontal crossing of the $L \times L$ square is bounded below by q, then the probability of an open horizontal crossing of a $3/2L \times L$ rectangle, is at least $f(q)$. The function f does not depend on L. The RSW theorem was first proved for Bernoulli bond percolation on the two dimensional square lattice. The model is as follows: each edge is open, independent of each other, with probability p, and closed with probability $1 - p$. Kesten [Ke82] later generalised the lattice RSW theorem in such a way that, given $c < 1$, one just needed to know the probability of crossing a $cL \times L$ rectangle the short way in order to bound the probability of crossing a larger rectangle the long way from below.

The RSW theorem is a fundamental tool that forms the foundation for a wide range of results concerning discrete percolation in the plane. In particular, for the triangular lattice, it was used in the famous proof of Cardy's formula and conformal invariance [Sm01]. However, in the continuum, RSW theorems are too few. This does not mean they are of less importance, rather, more difficult to prove. Indeed, the establishment of results seem promising in the continuum, once an RSW theorem can be found. A small glimpse of this can be seen in [BR06a]: the authors manage to prove a much weaker version of RSW. It turns out to be sufficient for them to show that the critical probability for Voronoi percolation in the plane is $1/2$ – this is a fairly recent and major result. In the Voronoi percolation model alone, where points are distributed with a Poisson point process, and Voronoi cells assigned to be open independently with probability p, a full RSW theorem would possibly lead to such results as conformal invariance and Cardy's formula, see [Ey11]. There are also many open problems in Gibbsian models, where particles have both type dependent and type independent interactions. Beffara and Duminil-Copin [BD11] showed a box crossing estimate for the two dimensional random cluster model on the lattice, and, having established the existence of a phase transition in the Delaunay Potts models discussed in Chapters 3 and 4, it is the next logical step to look for an RSW estimate, which would, in this area, be the very first such result.

For bond percolation on the square lattice and indeed site percolation on the triangular lattice, the proof of RSW relies on planar duality. For other discrete models – those without the planar duality property – other techniques are used, however, underlying them all is the use of independence. In the continuum setting, this independence of events is usually lost, and so, a proof of RSW seems to be more difficult. Alexander [Al96] shows one way around this problem with the continuum when he proves a version of the RSW theorem for the Boolean model. The Boolean model, sometimes referred to as the Poisson blob model, is driven by a Poisson point process, X. Each point of X has an independent identically distributed random radius attached to it. These radii give rise to closed balls around the points of X. As well as being independent of each other, the radii are also independent of X. Points in the plane are then said to be open if they lie in one of the closed balls, or closed if not. We denote the open region by C and note that C and C^c are both made up of connected components. We say there exists an open (respectively closed) path between two points x and y in the plane, if x and y belong to the same connected component of C (respectively C^c). There is an open crossing between two domains $\nabla_1, \nabla_2 \in \mathcal{B}(\mathbf{R}^2)$ if there exists $x \in \nabla_1$ and $y \in \nabla_2$ with an open path between x and y. Roy [Ro90] shows an RSW theorem for the Boolean model where the radii are bounded, however, he did not transfer the property that $f(q) \to 1$ as $q \to 1$ across from the discrete model. Alexander proves the RSW theorem for the non random fixed radius case of the Boolean model. More precisely, let X be a Poisson point process in $\mathbf{R}^2$ with intensity λ. For $A \subset \mathbf{R}^2$ and $r \geq 0$ let $A \oplus r := \cup_{x \in A} B(x, r)$. Obviously, $C = X \oplus r$ is the open region.

Theorem 5.1. *[Al96] Let X be a Poisson point process in the plane. Let λ be the intensity*

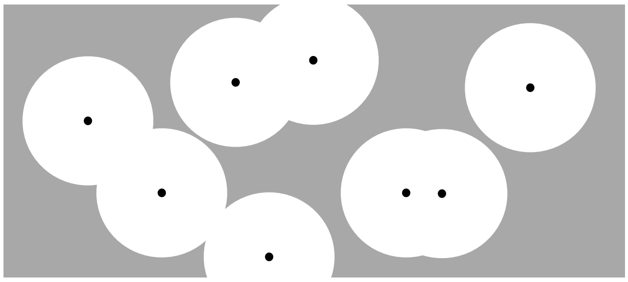


Figure 5.1: Fixed radius continuum percolation model

of X. Suppose $r > 0$, then, for some constant $K(\lambda r) > 0$, with $K(\cdot)$ nondecreasing,

$$\mathbf{P}(H^+([0, 3L/2 - 15r/2] \times [0, L])) \geq K(\lambda r)\mathbf{P}(H^+([0, L-2r]^2))^4\mathbf{P}(H^+([0, L+5r]^2))^2$$

where $H^+([a, b] \times [c, d])$ is the event of a horizontal open crossing of a $(b - a) \times (d - c)$ rectangle.

Alexander uses the fixed radius of the discs, r, to ascertain that events in regions of the plane separated by a distance $4r$ are independent. This is the key aspect which enables RSW despite the dependence in the model. His proof is then largely similar to that of RSW on the lattice – he uses a 'canonical low crossing' to act in the same way as the 'lowest occupied crossing' of [Ke82]. That is, given a lowest crossing, or path $\gamma \subset \mathbf{R}^2$ of a rectangle $R \subset \mathbf{R}^2$, the lowest canonical crossing is simply the shifted path $\vartheta_{(0,4r)}(\gamma)$. Using symmetry and independence, it can then be shown that, with non-negligible probability, this canonical crossing can be combined with another open path, to form an open crossing of R, ableit, with a gap of width $4r$. The technical difficulty is crossing this gap of width $4r$ and connecting the two paths together.

5.2 Problems with Voronoi percolation

The Voronoi percolation model is most easily described as the union of two Poisson point processes: one with intensity λp that gives open points and another with intensity $\lambda(1 - p)$ that gives closed points. Voronoi cells of open points are themselves open and Voronoi cells of closed points are closed. We assign $\mathbb{P}_p$ to be the associated probability measure. For $p \geq 1/2$, as shown in [BR06a], there exists an infinite cluster of open Voronoi cells with probability one. We fix $p = 1/2$ in the following discussion and set $\mathbb{P} := \mathbb{P}_{1/2}$. Unfortunately, the techniques of [Al96] do not carry over to the Voronoi percolation case. Alexander showed that events in well separated (by distance $4r$) regions are independent in the non random fixed radius boolean model. However, in the Voronoi tessellations, there is no upper bound on the diameter of individual Voronoi cells. Large cells are rare, but do occur. Given $\Lambda \in \mathcal{B}(\mathbf{R}^2)$, let $A_\Lambda := \{\omega \in \Omega : diam(\text{Vor}(x)) < R$ for every $x \in \omega \cap \Lambda\}$, for some fixed $R > 0$, then

$$\mathbb{P}(A) \to 0$$

as the size of Λ tends to infinity. In short, large Voronoi cells do exist. Recall from Section 2.1 the σ-algebra $\mathcal{F}_\Lambda$ of all events that happen in Λ only. Then, no matter how separated two regions $\Lambda_1, \Lambda_2 \in \mathcal{B}(\mathbf{R}^2)$, the events in $\mathcal{F}_{\Lambda_1}$ are not independent of those in $\mathcal{F}_{\Lambda_2}$. This is in contrast to the Boolean model with fixed or even bounded radii. We say the Boolean model exhibits spatial independence in the sense of the following definition.

Definition 5.2. *A point process X is said to be spatially independent if there exists $r > 0$ such that given Λ_1, $\Lambda_2 \subset \mathbf{R}^2$ with $d(\Lambda_1, \Lambda_2) > r$, any two events A and B are independent if $A \in \mathcal{F}_{\Lambda_1}$ and $B \in \mathcal{F}_{\Lambda_2}$.*

Even though independence (and even spatial independence) fails in their case, Beffara and Duminil-Copin [BD11] prove RSW for the discrete random cluster model that we described in (2.6), but on the square lattice, $\mathbf{Z}^2$. They invoke self duality and use stochastic domination of different boundary conditions.

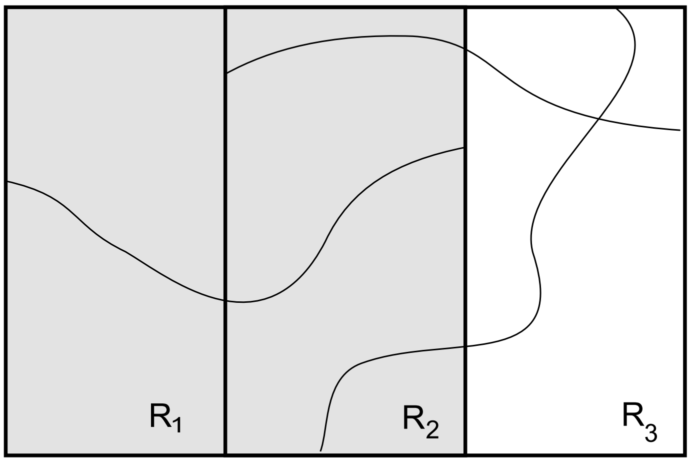


Figure 5.2: Event A

The basics of their proof are as follows. Take two squares S_1 and S_2 say, so that they overlap as in Figure 5.2. This results in three regions of equal width: $R_1 = S_1 \backslash S_2, R_2 = S_1 \cap S_2$ and $R_3 = S_2 \backslash S_1$. The probability of a horizontal open crossing of a square, in the discrete random cluster model on the lattice, is known to be $1/2$ for critical p. Let A be the event that there is a horizontal open crossing of both S_1 and S_2, and that there is a vertical open crossing of S_2 that starts at the bottom edge of R_2. Since open crossings are increasing events, it follows, by the FKG inquality presented in Lemma 2.2, that $\mu_p^q(A) \geq 1/16$.

Suppose now that event A holds. Therefore, we have a ‘highest’ horizontal open crossing of S_1 and a ‘right-most’ vertical open crossing of S_2. Denote these by Γ_1 and Γ_2 respectively. The areas above Γ_1 in S_1 and to the right of Γ_2 in S_2 are known. Omitting some technical details, and supposing our crossings do not combine to form a horizontal open crossing of $S_1 \cup S_2$, we take the reflections of Γ_1 and Γ_2 in the shared vertical boundary of R_2 and R_3. Denote these reflections as $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$. $\Gamma_1, \Gamma_2, \overline{\Gamma}_1$ and $\overline{\Gamma}_2$ enclose an unknown area, which we call Δ, see Figure 5.3. Let B be the event that there is an open crossing from Γ_1 to Γ_2 in Δ. Note that Γ_1 and Γ_2 are both open. Also note that Δ is symmetrical. The authors then see that $\mu_p^q(B|A) \geq \mu_p^q(B|A, C)$ where C is the event that $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ are closed. But $\mu_p^q(B|A, C)$ is shown to be $1/2$ by a simple use of symmetry and duality.

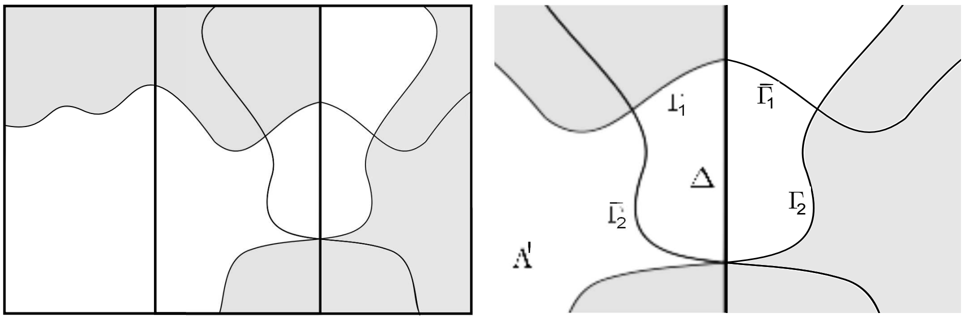


Figure 5.3: The shaded areas have been discovered as we are conditioning on event A.

Lets try and construct a similar approach for Poisson Voronoi percolation and see where it breaks down. Since the FKG inequality holds for Poisson Voronoi percolation, and, again, the probability of a horizontal open crossing of a square is known to be $1/2$ for critical p, see [BR06b], the event A is bounded below by $1/16$. Suppose again that A holds and that γ_1 and γ_2 are our ‘highest’ horizontal crossing of S_1 and a ‘right-most’ vertical crossing of S_2. Again, the areas above γ_1 and to the right of γ_2 are known in the sense that we know whether each point belongs to an open Voronoi cell or a closed Voronoi cell. The curve γ_1 is an interface graph of S_1 with open Voronoi cells ‘below’ and closed Voronoi cells ‘above’. The positions of the Poisson points whose Voronoi cells are adjacent to and below γ_1 are known. These cells are called ’half-known’ as they are yet to be completed: we do not know the positions of the Poisson points below. Therefore, given A, we know part of the Voronoi tessellation, but do not know other parts. This is known as having a partial Voronoi tessellation.

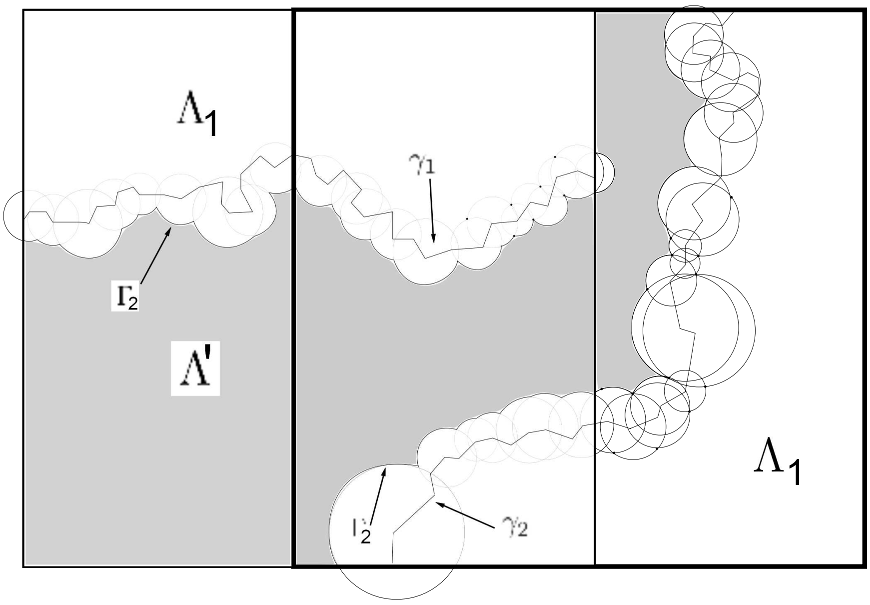


Figure 5.4: Construction of Λ', as shown by the shaded area.

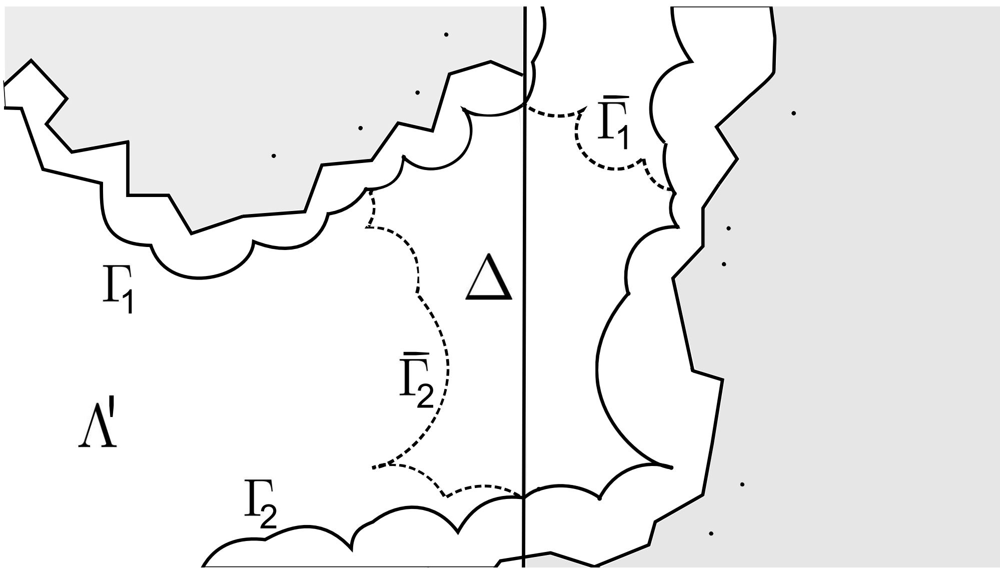


Figure 5.5: Construction of Δ

We want to discover the positions of particles in Δ. The only constraint is that the structure of the tessellation that is already known, i.e. in the areas above γ_1 and to the right of γ_2, must be preserved. Almost surely, each vertex of our partial Voronoi tessellation has three equidistant closest Poisson points: each vertex has a triple of associated Poisson points. Therefore, for each vertex v, there is a unique disc circumscribed by its associated triple of Poisson points, and centred at v. Let the union of these discs be denoted Λ_1 and define $\Lambda' := (S_1 \cup S_2) \backslash \Lambda_1$, see Figure 5.4. In summary, given the event A, we have a known configuration $\omega_{\Lambda_1} \in \Omega_{\Lambda_1}$ of Poisson points. By taking a second Poisson point process restricted to Λ' we can 'complete' the Voronoi tessellation. The structure of the tessellation above γ_1 and to the right of γ_2 will be preserved.

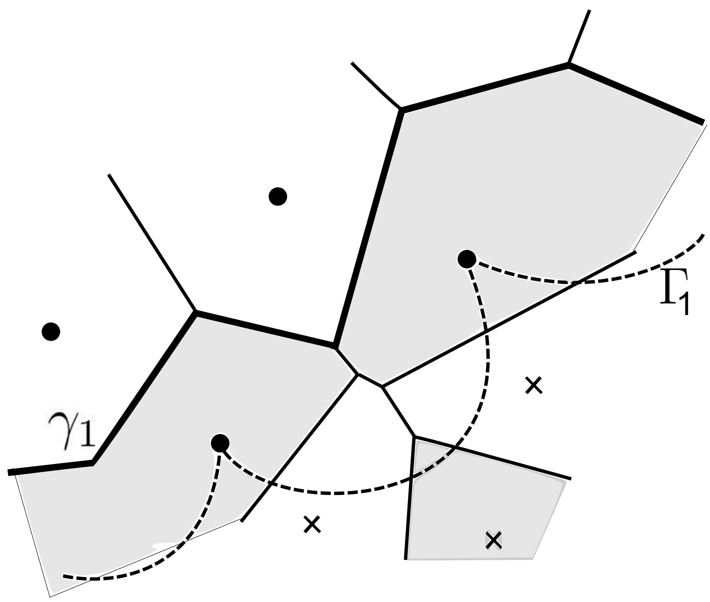


Figure 5.6: This is where the proof fails. You can see a piece of an open crossing from Γ_1 to Γ_2 inside Δ. However, it does not reach γ_1. If we bound the probability of crossing the gap between γ_1 and Γ_1, we would be done. However, this is a Poisson point process, and so the width of the gap is not bounded above. There could be many very very thin Voronoi cells between Γ_1 and γ_1. We say Poisson Voronoi percolation cannot "cross a gap" of unknown width.

The interface between Λ_1 and Λ' is denoted by Γ_1 and Γ_2 as shown in Figures 5.4 and 5.5. We take the reflection of Γ_1 and Γ_2 in the shared vertical boundary of R_2 and R_3. Denote these reflections as $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$. Γ_1, Γ_2, $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ enclose an area, which we call Δ. The next step is to bound the probability of having an open crossing from Γ_1 to Γ_2 inside Δ away from zero. This is slightly more technical than before, but using a suitable decreasing event at the boundary, C, it can be shown, using the FKG inequality, that

$$\mathbb{P}(B|A) \geq \mathbb{P}(B|A, C) \geq 1/2.$$

However, there is a problem. We still may not have a horizontal crossing of $S_1 \cup S_2$. See Figure 5.6 for an example.

Remark 5.3. *Shortly after the submission of this thesis, an RSW result was shown for Voronoi percolation in [Tas14]. The author uses the FKG inequality, rotational symmetry, and quasi-independence (similar to Definition 5.2) properties, along with renormalisation techniques in their proof.*

5.3 Gibbsian models with geometric interactions

The existence of small and large Voronoi cells caused the problems in the Poisson Voronoi percolation model. We now look at other continuum models with a Voronoi/Delaunay structure where such defects do not occur. One model that restricts Voronoi tessellations so that cells are neither too large nor too small is the double hardcore model of [Der08]. Let the mark space Σ contain only two marks: open and closed. We say that $\overline{\omega} = (\omega, \sigma_\omega) \in \overline{\Omega}$ satisfies the double hardcore condition (DHC) if, for $R > r > 0$,

$$B(x, r) \subset \mathrm{Vor}_\omega(x) \subset B(x, R) \text{ for all } \mathbf{x} = (x, s) \in \overline{\omega}.$$

The hyperedge potential acts on all hyperedges $x \in \mathrm{Del}_1(\omega)$ but does not satisfy the local horizon property (2.13). It is defined as

$$\psi(x, \omega) \equiv \psi(\mathrm{Vor}_\omega(x)) := \begin{cases} +\infty & \text{if (DHC) not satisifed} \\ 0 & \text{if (DHC) satisifed.} \end{cases} \tag{5.1}$$

A Gibbs measure $\mathcal{P}_{\mathrm{DHC}}$ for such a hyperedge potential is shown to exist in [Der08], but also by an adaptation of the proof of Proposition 3.2 in this thesis. In this model, the Voronoi cells are open and closed independently of each other and, crucially, independent of the particle positions.

Conjecture 5.4. *Fix $z = 1$ and $\rho > 1$. Denote $R_{\rho,L} \subset \mathbf{R}^2$ to be a $\rho L \times L$ rectangle. Then*

$$\mathcal{P}_{DHC}(H^+(R_{1,L})) > \epsilon > 0 \implies \mathcal{P}_{DHC}(H^+(R_{\rho,L})) > g(\rho, \epsilon) > 0$$

where $H^+(R_{\rho,L})$ is the event of a horizontal open crossing of $R_{\rho,L}$ and $g(\rho, \epsilon)$ does not *depend on L.*

By complementing the background interaction (5.1) with a type interaction, we move into the familiar area of continuum random cluster models with a random Delaunay/Voronoi structure. In this case, whether or not a Voronoi cell is open depends heavily on the particle positions. What is particularly interesting is that the problems outlined in Section 5.2 concerning small and large cells, and their detrimental effect on our efforts to prove RSW, are very similar to the problems faced in Chapters 3 and 4. In particular, throughout the proofs of continuum percolation for the various Delaunay random cluster models, the recurring elements were to bound below the probability that a small box ∇ contains at least one particle (no large Voronoi cells), and to bound below the probability that a box Δ contained no more than $M > 0$ particles (average size of Voronoi cells in Δ is not too small). We therefore conjecture that the RSW theorem holds for the soft Widom-Rowlinson model on the Delaunay graph studied in Chapter 4.

Conjecture 5.5. *Fix $z > z_0$, $A > A_0$ and $\rho > 1$. Denote $R_{\rho,L} \subset \mathbf{R}^2$ to be a $\rho L \times L$ rectangle. Then*

$$\mathcal{P}^{z,A,R}(H^+(R_{1,L})) > \epsilon > 0 \implies \mathcal{P}^{z,A,R}(H^+(R_{\rho,L})) > g(\rho,\epsilon) > 0$$

where $H^+(R_{\rho,L})$ is the event of a horizontal open crossing of $R_{\rho,L}$, $g(\rho,\epsilon)$ does not depend on L and $\mathcal{P}^{z,A,R}$ is the Gibbs measure corresponding to the Hamiltonian (4.3).

In summary, there are many ways to prove the RSW theorem, but all depend on the spatial independence property, defined in Definition 5.2, in a critical way. There is one exception to this, however, as we saw in Section 5.2 where the authors of [BD11] take advantage of planar duality and symmetry arguments. The case of Poisson Voronoi percolation falls down in the same way in both scenarios – the size of the Voronoi cells is unbounded. We conjecture that an RSW estimate can be shown in the case where geometric interactions occur between the particles to discourage large (and small) Voronoi cells. This by itself however, does not give us spatial independence, in fact, more dependence has been built into the system. However, without large Voronoi cells, an adaptation of the proof method of [BD11] seems more possible. The size of any gap, as shown in Figure 5.6, is bounded above, and without small Voronoi cells, the number of Voronoi cells that can fill said gap is also bounded above, enabling us to "cross the gap". There are other things to consider though. Do symmetry arguments still work in this setting? Non-hereditary processes like the double sided hardcore one described above would cause this to fail for instance. This brings us to the FKG inequality – a key ingredient for stochastic domination arguments used in many of the proofs we have discussed. A point process analogue was given in [GK97], however, as we have seen, using geometric interactions erases the clear distinction between attraction and repulsion. Will a weaker result be sufficient? Finally, due to the existence of a Gibbs measure, and therefore a consistency relation, it is supposed that dependence decays with distance. This decay of correlations, if shown to be strong enough, could enable us to separate events and evaluate their intersection. These are all interesting questions and ones which must be answered in order to find a proof of RSW in the Delaunay/Voronoi continuum setting.

Chapter 6

Conclusions and Outlook

In conclusion, in this thesis, we have shown the existence of percolation, for high activity and low temperature, in two classes of Delaunay random cluster models. Those with a hardcore background interaction, and those without any background interaction at all. Following a joint construction of these Delaunay random cluster models with corresponding Delaunay Potts models, we can interpret these percolation results with respect to the latter. They imply that multiple distinct Gibbs measures (Delaunay Potts measures) exist for large enough activity and low enough temperature for each of our models, and providing uniqueness of a Gibbs measures at the opposite end of the phase space, show the existence of a phase transition. This extension of the continuum random cluster representation to the Delaunay hypergraph structure is formulated in Chapter 2 for hyperedge potentials that only depend on the hyperedge and not the neighbourhood of the hyperedge. Follow on work might include constructing a similar continuum random cluster representation for the case when hyperedge potentials do depend on the neighbourhood of a hyperedge, and then finding a suitable model where percolation exists – it is thought that dependent percolation may be needed. One example of such a model has a hyperedge potential that acts on single particles, but is a function of the number of neighbours to that particle in the Delaunay graph. Clearly this depends on the neighbourhood of the hyperedge (single particle) and not just the hyperedge itself.

In Chapter 3, we study a class of Delaunay random cluster models with hardcore background interactions: a generalisation and extension of the work of [BBD03] to the case of infinite range type interactions both on triangles and edges of the Delaunay graph. The first model we consider has a type interaction that discourages triangles with large smallest interior angle if all three particles that build the triangle do not have the same mark. We show non-uniqueness of Gibbs measures for large activity z and low temperature

A^{-1}. Our estimates for these activity and temperature thresholds are similar to those in [BBD03], although the former depends on the angle threshold β_0 between discouraged and non-discouraged triangles in our case, rather than the finite range R of the interaction as in [BBD03]. Our estimates show that if we decrease the angle threshold, we must increase the activity to retain percolation. This is counter intuitive behaviour because, in the random cluster representation, we are increasing the hyperedge drawing probability so one would expect a more connected hypergraph. However, the positions of the particles also depend on our type interaction. For this model, and indeed all those we consider in this thesis, the required activity and inverse temperature are shown to increase with the cardinality of the mark space. This on the other hand, is intuitive: a wider selection of marks will make it more difficult for one of them to percolate. Our second model is just a relaxation of the first, using a smooth type interaction that gives increasingly more penalty for triangles with a larger smallest interior angle. Our proof of percolation only relies on the hardcore background interaction to control the maximum number of particles in a box. A possible extension of this model would be to lose the background hardcore interaction altogether. This would result in the case of an infinite range type interaction with no background interaction. To do this, we would simply need to find a bound for the expected number of destroyed hyperedges when a particle is added to an existing configuration. We could then follow the methods of Chapter 4 to control the maximum number of particles in a box. The final model in Chapter 3 is a direct extension of [BBD03] to the case of infinite range type interaction on the edges of the Delaunay graph. The novelty is to bound the expected number of created hyperedges without using a finite range assumption. Our estimates for the activity and temperature thresholds are weaker, as expected. Comparing with [BBD03], the temperature parameter threshold is identical, however, our activity parameter threshold, although now independent of the range of interaction R, depends heavily on the temperature. In fact, we require z to be proportional to q to the power $4A\pi/3$. By taking advantage of the self similarity property of the Delaunay triangulation, it is thought that this model can be generalised to the case where $\varphi \equiv A$, although our 'annulus' proof technique would not work.

In Chapter 4, in the case of Delaunay random cluster models without any background interaction at all, we present an extension of [LL72] and [CCK94] to the Delaunay structure where we use a smooth decreasing type interaction that explodes in the neighbourhood of the origin to ensure percolation. The problem of non-uniqueness of Gibbs measures for large activity z was first solved for the Widom-Rowlinson model by Ruelle using a Peierls-type argument [R71]. It was later also shown using a more modern stochastic geometric method in [CCK94]: the authors used a continuum random-cluster representation. The work of [LL72] provided a solution to the problem for the more general case of the Widom-Rowlinson model when particles of different type are just discouraged to get too close, rather than forbidden: non-uniqueness of Gibbs measures required high activity or low temperature. Again, much later, a continuum random-cluster representation was used to make the same result – this time by [GH96]. The Widom-Rowlinson and its generalisations discussed here do not incorporate a background interaction: only a repulsion between particles of different type in the form of a pair interaction on the edges of the complete graph. We consider a much more restricted version of these models, where the inter-type interaction only occurs on edges (hyperedges) of the Delaunay graph. This is the key feature of the present study and is the first time non-uniqueness of a Gibbs measure has been shown for a Delaunay Potts model without a background interaction. Our result, as usual, is for large activity and low temperature, which each depend on R, the finite range of our type interaction.

One might presume that an increase in the range of the type interaction would facilitate percolation. And whilst this is certainly true in the classical systems of non-geometric interactions on the hyperedges of the complete graph, where it represents an increase to the repulsion, it is not so clear cut in our hypergraph structure framework where the lines between repulsion and attraction are blurred due to the non-additive nature of the geometric interactions. In fact, although our estimates show that a larger activity is needed to maintain a phase transition for type interactions with either a large range, or very small range, the temperature does not need to be as low in the latter case. However, our proof relies on a bound α for the change in the expected number of connected components when you augment a configuration with a new particle. The bound we found is quadratic in R, but this is due to our particular proof methods. Due to the self-similarity property of the Delaunay hypergraphs, it is believed that a bound independent of R can be obtained. This would grant significantly tighter estimates for the required activity and temperature for percolation. In particular, in the case of a constant type interaction on edges of the Delaunay graph, percolation would exist for all activity $z > 0$ and for temperature small enough – very interesting yet intuitive behaviour. To see this, we follow our proof method. Choose any acticity $z > 0$. To ensure, for this activity, that the small boxes in an L'-partition of Λ contain at least one particle with high probability, we must choose L large enough. This choice of L, due to our bound for α being independent of the length of hyperedges, is inversely proportional to the square root of z. To ensure percolation, we must finally choose A small enough such that all particles in an $8L \times 8L$ box have mark 1 with large probability. This only depends on the number of particles in the box and will therefore be constant due to our choices of z and L.

As a final remark, all of the estimates in this thesis depend on our choice of coarse graining structure – in particular the square lattice. By using an alternative skewed lattice of rhombi with interior angle $\pi/3$, we can improve our estimates by a constant factor. However, a comparison with another continuum point process instead of a discretization approach is not thought possible. These thoughts are due to the non-additivity of the geometric hyperedge interactions which do not allow for stochastic comparison of point processes in the form of the analogue Holley–Preston inequality presented in [GK97].

Appendix A

Geometrical Lemmas

Lemma A.1. *Let* $G = (V, E)$ *be a spoked chain where* $V = \cup_{i=1}^{n} x_i$ *and* $\hat{x}_1 < \cdots < \hat{x}_n$*. For* $1 < k \leq n$*, let* x_k^* *and* x_{k+1}^* *be the centres of the circumscribing circles of the triples* $\{x_0, x_{k-1}, x_k\}$ *and* $\{x_0, x_k, x_{k+1}\}$ *respectively. Then* $\hat{x}_{k+1}^* \geq \hat{x}_k^*$*.*

Proof. The points $\hat{x}_{k+1}^*$ and $\hat{x}_k^*$ both lie on the bisector of the line segment $\overline{x_0 x_k}$. Suppose $\hat{x}_k^* > \hat{x}_k$, then the radius of $B(\{x_0, x_k, x_{k+1}\})$ is greater than the radius of $B(\{x_0, x_{k-1}, x_k\})$ and hence $\hat{x}_{k+1}^* \geq \hat{x}_k^*$. Now suppose $\hat{x}_k^* \leq \hat{x}_k$. If $\hat{x}_{k+1}^* < \hat{x}_k^*$, then x_{k+1} lies in the interior of $B(\{x_0, x_{k-1}, x_k\})$ which contradicts the properties of a Delaunay tessellation. Therefore, $\hat{x}_{k+1}^* \geq \hat{x}_k^*$. $\square$

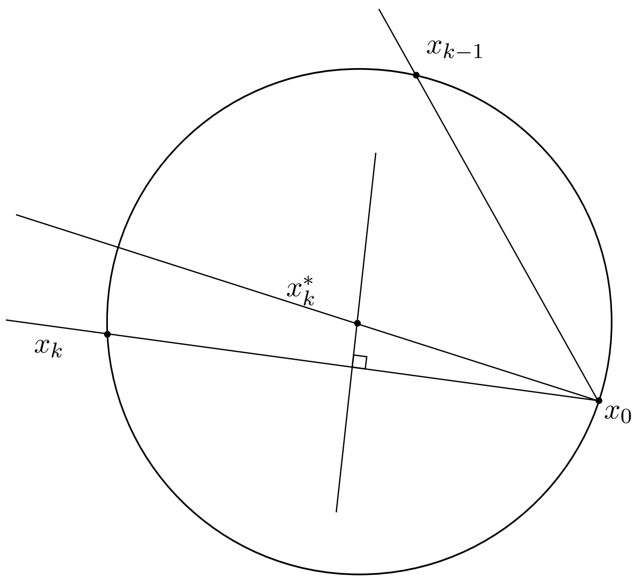


Figure A.1: Linear bisector of the line segment $\overline{x_0 x_k}$

Let $a \in \mathbf{R}^2$ be the pole in a polar coordinate system where $\hat{x}$ denotes the angular coordinate of a point $x \in \mathbf{R}^2$. For $x, y \in \mathbf{R}^2$ with $\hat{x} < \hat{y}$, let $B(\{a, x, y\})$ be the unique circle that intersects a, x and y. There are exactly two arcs of $B(\{a, x, y\})$ with endpoints $\hat{x}$ and $\hat{y}$. Let C^a_{xy} be the one that does not contain a. Given an arc of a circle, C, let $L(C)$ denote its length. For two points $x, y \in \mathbf{R}^2$, recall that $\overleftrightarrow{xy}$ denotes the unique straight line in $\mathbf{R}^2$ that passes through both x and y.

Theorem A.2. *Suppose $a \in \mathbf{R}^2$ is the pole. Let $b, c \in \mathbf{R}^2$, with $0 < \hat{b} < \hat{c} < \pi$. Let $D \subset \mathbf{R}^2$ be the region bounded by $B(\{a, b, c\})$. Let U be the convex hull of C^a_{bc}. Then, for all $z \in U$,*

$$L(C^a_{bz}) + L(C^a_{zc}) \leq L(C^a_{bc}). \tag{A.1}$$

Proof. We start off with some notation. Let $r > 0$ be the radius of the circle $B(\{a, b, c\})$ and let

$$M := |b - c| \ , \ h_1 := |b - z| \ , \ h_2 := |z - c|,$$

$$t := |z - a| \ , \ s_1 := |b - a| \ , \ s_2 := |c - a|,$$

and

$$\theta_1 := \hat{z} - \hat{b} \ , \ \theta_2 := \hat{c} - \hat{z} \ , \ \theta := \theta_1 + \theta_2.$$

Then, $L(C^a_{bz}) = 2\theta_1 \times \text{radius}(B(\{a, b, z\}))$, and $\text{radius}(B(\{a, b, z\})) = h_1/2\sin(\theta_1)$. Therefore, we have the following formulae for $L(C^a_{bz})$, $L(C^a_{zc})$ and $L(C^a_{bc})$:

$$L(C^a_{bz}) = h_1 \frac{\theta_1}{\sin(\theta_1)} \ , \ L(C^a_{zc}) = h_2 \frac{\theta_2}{\sin(\theta_2)} \ , \ L(C^a_{bc}) = M \frac{\theta}{\sin(\theta)}.$$

The strategy of the proof is to first show that $L(C^a_{bz}) + L(C^a_{zc}) = L(C^a_{bc})$ for $z \in C^a_{bc}$ and $L(C^a_{bz}) + L(C^a_{zc}) \leq L(C^a_{bc})$ for $z \in \overline{bc}$. We then define $L(C^a_{bz}) + L(C^a_{zc})$ as a function of θ_1, s_1, t and r, and show that it is convex with respect to t – see Figure A.2 for an illustration of this convexity. Noting that z is uniquely determined by t and θ_1, we conclude the result.

Let $z \in C^a_{bc}$. It follows that $B(\{a, b, c\}) = B(\{a, b, z\}) = B(\{a, z, c\})$. Therefore, $C^a_{bz} \cup C^a_{zc} = C^a_{bc}$. Hence,

$$L(C^a_{bz}) + L(C^a_{zc}) = L(C^a_{bc}). \tag{A.2}$$

Let $z \in \partial U \cap \overline{bc}$. Then, $h_1 + h_2 = M$ and

$$\begin{aligned} L(C^a_{bz}) + L(C^a_{zc}) &= h_1 \frac{\theta_1}{\sin(\theta_1)} + h_2 \frac{\theta_2}{\sin(\theta_2)} \\ &= h_1 \frac{\theta_1}{\sin(\theta_1)} + (M - h_1) \frac{\theta_2}{\sin(\theta_2)} \\ &\leq h_1 \frac{\theta}{\sin(\theta)} + (M - h_1) \frac{\theta}{\sin(\theta)} \\ &= M \frac{\theta}{\sin(\theta)} \\ &= L(C^a_{bc}), \end{aligned} \tag{A.3}$$

where the inequality holds because $\theta \geq \max\{\theta_1, \theta_2\} > 0$ and $g(x) := \frac{x}{\sin(x)}$ is an increasing function on the interval $[0, \pi]$.

Let M and r be fixed constants. To write $L(C^a_{bz}) + L(C^a_{zc})$ as a function of θ_1, s_1 and t, note that by the cosine rule of triangles,

$$h_1^2 = t^2 - 2s_1 t \cos(\theta_1) + s_1^2 \quad \text{and} \quad h_2^2 = t^2 - 2s_2 t \cos(\theta_2) + s_2^2, \tag{A.4}$$

hence,

$$L(C^a_{bz}) = \left(t^2 - 2s_1 t \cos(\theta_1) + s_1^2\right)^{1/2} \frac{\theta_1}{\sin(\theta_1)} =: f_1(\theta_1, s_1, t). \tag{A.5}$$

Furthermore, s_2 is a function of s_1 and θ_1, since

$$M^2 = s_1^2 + s_2^2 - 2s_1 s_2 \cos\left(\sin^{-1}\left(\frac{M}{2r}\right)\right); \tag{A.6}$$

and θ_2 is a function of θ_1:

$$\theta_2 = \theta - \theta_1 = \sin^{-1}\left(\frac{M}{2r}\right) - \theta_1 \tag{A.7}$$

Then, by (A.4), (A.6) and (A.7), we have

$$L(C^a_{zc}) = \left(t^2 - 2s_2(s_1, \theta_1) t \cos(\theta_2) + s_2(s_1, \theta_1)^2\right)^{1/2} \frac{\theta_2(\theta_1)}{\sin(\theta_2(\theta_1))} =: f_2(\theta_1, s_1, t). \tag{A.8}$$

We aim to show that $f(\theta_1, s_1, t) := f_1(\theta_1, s_1, t) + f_2(\theta_1, s_1, t)$ is convex with respect to t.

To do this we start by taking the first and second derivative of f_1 with respect to t:

$$\frac{d}{dt} f_1(\theta_1, s_1, t) = \frac{t - s_1\cos(\theta_1)}{\left(t^2 - 2s_1 t\cos(\theta_1) + s_1^2\right)^{1/2}} \cdot \frac{\theta_1}{\sin(\theta_1)},$$

and

$$\frac{d^2}{dt^2} f_1(\theta_1, s_1, t) = \left[\frac{1}{\left(t^2 - 2s_1 t\cos(\theta_1) + s_1^2\right)^{1/2}} - \frac{2\left(t - s_1\cos(\theta_1)\right)^2}{2\left(t^2 - 2s_1 t\cos(\theta_1) + s_1^2\right)^{3/2}}\right] \cdot \frac{\theta_1}{\sin(\theta_1)}$$

$$= \underbrace{\frac{\theta_1}{\sin(\theta_1)}}_{\geq 0} \cdot \left[\frac{\left(t^2 - 2s_1 t\cos(\theta_1) + s_1^2\right)^{3/2}}{\underbrace{\left(t^2 - 2s_1 t\cos(\theta_1) + s_1^2\right)^2}_{\geq 0}} - \frac{\left(t^2 - 2s_1 t\cos(\theta_1) + s_1^2\right)^{1/2}\left(t - s_1\cos(\theta_1)\right)^2}{\underbrace{\left(t^2 - 2s_1 t\cos(\theta_1) + s_1^2\right)^2}_{\geq 0}}\right].$$

The denominator of the quotients in the square brackets is just h_1^2 which is positive. The function $\frac{x}{\sin(x)}$ is also positive for $0 \leq x \leq \pi$. Therefore, to show that $\frac{d^2}{dt^2} f_1(\theta_1, s_1, t) \geq 0$, we only need to check that the numerator of the quotient in the square brackets, denoted $F(\theta_1, s_1, t)$, is positive. Indeed,

$$\begin{aligned} F(\theta_1, s_1, t) &= \left(t^2 - 2s_1 t\cos(\theta_1) + s_1^2\right)^{3/2} - \left(t^2 - 2s_1 t\cos(\theta_1) + s_1^2\right)^{1/2}\left(t - s_1\cos(\theta_1)\right)^2 \\ &= \left(t^2 - 2s_1 t\cos(\theta_1) + s_1^2\right)^{1/2}\left(t^2 - 2s_1 t\cos(\theta_1) + s_1^2 - \left(t - s_1\cos(\theta_1)\right)^2\right) \\ &= \underbrace{\left(t^2 - 2s_1 t\cos(\theta_1) + s_1^2\right)^{1/2}}_{=h_1 \geq 0} \cdot \underbrace{s_1^2\left(1 - \cos^2(\theta_1)\right)}_{\geq 0} \geq 0. \end{aligned}$$

Therefore, $\frac{d^2}{dt^2} f_1(\theta_1, s_1, t) \geq 0$. Similarly, $\frac{d^2}{dt^2} f_2(\theta_1, s_1, t) \geq 0$. Therefore, $\frac{d^2}{dt^2} f(\theta_1, s_1, t) \geq 0$, which implies convexity with respect to t. Fix $0 \leq \theta_1 \leq \sin^{-1}(\frac{M}{2r})$. There exists $0 < t_{\text{min}}(\theta_1) < t_{\text{max}}(\theta_1) < 2r$ such that $t_{\text{min}}(\theta_1) \leq |z| \leq t_{\text{max}}(\theta_1)$ for all $z \in U$ with $\hat{z} - \hat{b} = \theta_1$. We have shown, in Equations (A.2) and (A.3) that

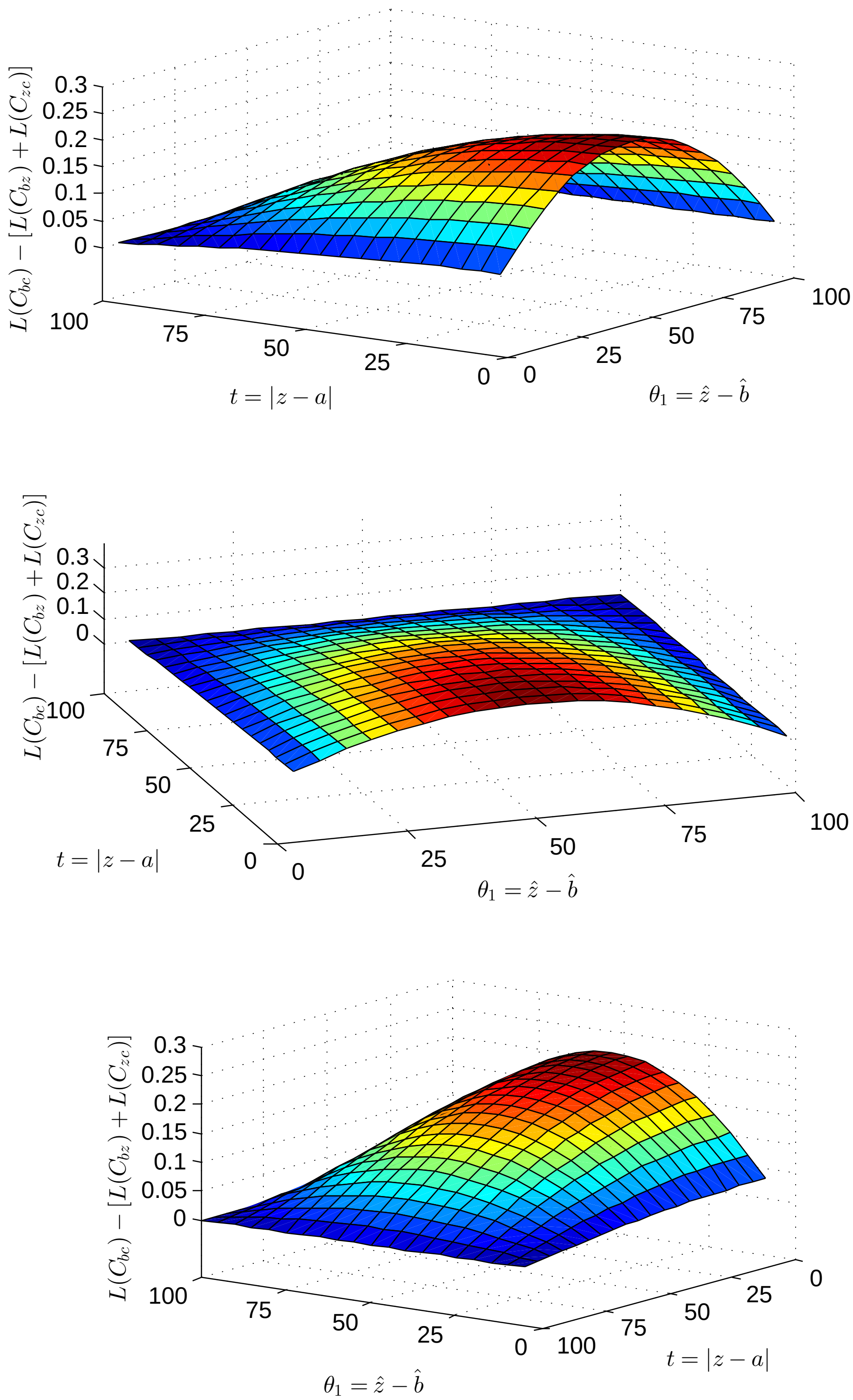
$$f(\theta_1, s_1, t_{\text{min}}(\theta_1)) \leq L(C_{bc}^a) \quad \text{and} \quad f(\theta_1, s_1, t_{\text{max}}(\theta_1)) = L(C_{bc}^a).$$

Therefore, by the convexity of f, for all $t \in [t_{\text{min}}(\theta_1), t_{\text{max}}(\theta_1)]$,

$$\begin{aligned} f(\theta_1, s_1, t) \le \frac{t - t_{\text{min}}(\theta_1)}{t_{\text{max}}(\theta_1) - t_{\text{min}}(\theta_1)} f(\theta_1, s_1, t_{\text{min}}(\theta_1)) \\ + \frac{t_{\text{max}}(\theta_1) - t}{t_{\text{max}}(\theta_1) - t_{\text{min}}(\theta_1)} f(\theta_1, s_1, t_{\text{max}}(\theta_1)) \\ \le f(\theta_1, s_1, t_{\text{max}}(\theta_1)) = L(C^a_{bc}). \end{aligned}$$

Since, θ_1 and s_1 were arbitrary, this completes the proof. □

Figure A.2: The difference between $L(C_{bz}) + L(C_{zc})$ and $L(C_{bc})$ for points $z \in U$. On the plot, t takes values between $t_{\text{min}} = 0$, when $z \in \overline{bc}$ and $t_{\text{max}} = 100$, when $z \in C_{bc}$. Similarly, the values of θ_1 are a percentage of θ, which is the maximum value for θ_1. The following values were used: $r = 10$, $M = 5\sqrt{3}$ and $s_1 = \frac{5\sqrt{3}}{(2\sin(\sin^{-1}(5\sqrt{3}/(20))/2))} \approx 19.5$.



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