RUBBER BANDS, PURSUIT GAMES AND SHY COUPLINGS

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Abstract. In this paper, we consider pursuit-evasion and probabilistic consequences of some geometric notions for bounded and suitably regular domains in Euclidean space that are CAT(κ) for some $\kappa > 0$. These geometric notions are useful for analyzing the related problems of (a) existence/nonexistence of successful evasion strategies for the Man in Lion and Man problems, and (b) existence/nonexistence of shy couplings for reflected Brownian motions. They involve properties of *rubber bands* and the extent to which a loop in the domain in question can be deformed to a point without, in between, increasing its loop length. The existence of a *stable rubber band* will imply the existence of a successful evasion strategy but, if all loops in the domain are *well-contractible*, then no successful evasion strategy will exist and there can be no co-adapted shy coupling. For example, there can be no shy couplings in bounded and suitably regular star-shaped domains and so, in this setting, any two reflected Brownian motions must almost surely make arbitrarily close encounters as $t \to \infty$.

1. Introduction

The motivation for this article is a conjecture about shy couplings, that is, about constructions of pairs of reflected Brownian motions in a bounded Euclidean domain that are contrived so that, for some fixed $\varepsilon > 0$, they never come within distance ε of each other. In Bramson, Burdzy, and Kendall (2012), we showed that strong results about nonexistence of shy couplings could be proved using ideas of pursuit-evasion games and modern metric geometry. In the current paper, we introduce new metric geometry notions (such as"rubber bands" and "well-contractible loops") that can be used to derive general results about pursuit-evasion games and further results about shy coupling. In particular, while Bramson et al. (2012) shows that shy couplings cannot be supported by suitably regular bounded CAT(0) domains, here we show that shy couplings cannot be supported by a substantially larger family of domains including, for example, bounded star-shaped domains with suitably regular boundaries (see Definition 2.5 for the definitions of CAT(0) and CAT(κ) domains). Our results apply to domains $D \subset \mathbb{R}^d$, for $d \geq 2$, but their main interest is

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in $d \geq 3$, since all bounded simply connected domains in $d = 2$ are CAT(0), and hence the results from Bramson et al. (2012) apply in that setting.

We first summarize our results for pursuit-evasion games. In this deterministic setting, there are two players, a Lion and a Man, each of whom is constrained to remain in a given bounded domain D. Both the Lion and the Man are allowed to move within D at up to unit speed. We are interested in the question as to whether, for some strategy of the Lion, the Lion is able to come within distance ε of the Man, irrespective of the strategy of the Man and for any $\varepsilon > 0$. We will say that the Lion *captures* the Man or the Man *evades* the Lion, depending on whether or not such a strategy exists for every pair of initial positions.

The pursuit-evasion problem in a disk is a well-known problem, and includes the question as to whether the Man can avoid the Lion indefinitely (even though the distance between them is allowed to go to 0). See, for example, Isaacs (1965), Littlewood (1986), Nahin (2007). In our current setting, we consider bounded domains $D \subset \mathbb{R}^d$.

For the Lion and Man pursuit-evasion problem, we will determine conditions on the domain D under which the Man can evade the Lion and under which the Lion can capture the Man. Under suitable side conditions, the first scenario holds when D possesses a *stable rubber band*, which is, in essence, a locally distance-minimizing loop. Section 3 is devoted to showing this, with the main result being Theorem 3.7. The second scenario holds when all loops in D are *well-contractible*, which in essence means that the loop can be contracted to a point, with the length of the intermediate loops decreasing at a uniform rate with respect to the homotopy parameter. Section 4 shows that the Lion is able to capture the Man when all loops are well-contractible, with the main result being Theorem 4.6.

The assumption that D is CAT(κ) figures prominently in both arguments, and in the succeeding sections of the paper. Roughly speaking, a domain D satisfies the CAT(κ) condition if suitably small triangles defined using the intrinsic distance in D have angles no greater than angles of triangles with the same side lengths on the surface of the Euclidean sphere of radius $1/\sqrt{\kappa}$ (the formal definition of CAT(κ) domains will be given later in the paper). We will also require some regularity on the boundary of D, which will be given by the uniform exterior sphere and uniform interior cone conditions (see Definitions 2.1-2.2); a domain D satisfying both conditions will be referred to as an ESIC domain. An ESIC domain whose loops are all well-contractible will be referred to as a *CL domain*. Since an ESIC domain is CAT(κ), for some $\kappa \geq 0$ (see Corollary A.5), these two boundary conditions will in fact suffice for many of our results. The definitions of these terms and others that will be employed in the paper are given in Section 2.

The second half of the paper is devoted mostly to shy couplings. A reflected Brownian motion on a domain D is said to admit a *shy coupling* if there exists a coupling of Brownian motions X and Y on D, for some choice of initial points x and y, such that

$$\mathbb{P}\left[\inf\left\{\operatorname{dist}(X_t, Y_t) : 0 \leq t < \infty\right\} > 0 \mid X_0 = x, Y_0 = y\right] \quad > \quad 0\,.$$

(We consider throughout only couplings that are co-adapted, that is, that do not anticipate the future.) An example of a shy coupling is given by Brownian motions X and Y on a circle, where

Y is produced from X by a nontrivial rotation. Except for similar specialized examples, all known results involve the absence of shy couplings, and only a partial theory is known. Benjamini, Burdzy, and Chen (2007), who introduced the notion of shy coupling, showed that no shy couplings exist for reflected Brownian motion in convex bounded planar domains with C^2 boundaries containing no line segments; Kendall (2009) used a direct and somewhat quantitative approach to remove regularity requirements in the convex case. Bramson et al. (2012) showed that no shy couplings exist for bounded ESIC domains that are CAT(0). (Also see Bramson et al. (2012) for further background.)

Section 5 extends the approach taken in Bramson et al. (2012), and shows, in Theorem 5.5, that no shy couplings exist for bounded CL domains. The basic idea behind the argument is to transform the process of coupled Brownian motions, by using the Cameron-Martin-Girsanov transformation and scaling time, to a process where each sample path is approximated by a solution of the Lion and Man problem. In the present context, one can then apply Theorem 4.6 to this Lion and Man problem.

In Section 6, it is shown that there is no analogous application of Theorem 3.7 whereby the existence of a shy coupling follows from the existence of a stable rubber band. In fact, starting with any bounded domain possessing a stable rubber band, it is possible to append another larger domain, which preserves the rubber band, so that the combined domain has no shy couplings.

A number of examples of CL domains and domains with rubber bands are given in Section 7. In particular, in Examples 7.2 -7.4, various examples of CL domains are given, such as restrictions of CAT(0) domains that themselves are not CAT(0), including star-shaped domains. At the end of the section, we conjecture that, off a nowhere dense family of domains (taken with respect to the Gromov-Hausdorff distance), all bounded domains with bounded principal curvatures are either CL or possess a *semi-stable rubber band* (that is, the rubber band is minimal, but not necessarily strictly minimal).

Employing a result from Lytchak (2004), the claim that ESIC domains are CAT(κ), for some $\kappa \geq 0$, is shown in the short appendix.

2. Rubber bands

In this section, we introduce some basic notions for domains in Euclidean space, including: conditions for suitable regularity of the boundary, intrinsic distance and related concepts from metric geometry, and rectifiable loops and their homotopies. Most importantly, we introduce the new notion of *rubber bands*, as well as several associated concepts. The notion of rubber band will play a key rôle in the main results in later sections on pursuit-evasion and on shy coupling of reflected Brownian motion.

Suppose that $D \in \mathbb{R}^d$ is a bounded domain (that is, an open connected set). The *intrinsic distance* $\text{dist}_{\mathbf{I}}(v, z)$ between $v, z \in D$ is the infimum of lengths ℓ_Γ of rectifiable arcs $\Gamma \subset D$ that contain v and z. We will typically wish to restrict our attention to domains for which the notion of intrinsic distance extends to the entire closure $\overline{D}$ without discontinuity at the boundary ∂D. To achieve this, we follow Bramson et al. (2012) in requiring that D satisfy both the uniform exterior sphere condition and the uniform interior cone condition defined below. Here and elsewhere, $\mathcal{B}(z,r)$ denotes the open Euclidean ball of radius r centered at z.

Definition 2.1 (Uniform exterior sphere condition, from Saisho, 1987, §1, Condition (A))**.** *A domain D satisfies a* uniform exterior sphere condition based on radius r *if, for every $z \in \partial D$, the set of "exterior normals" $\mathcal{N}_{z,r} = \{\nu \in \mathbb{R}^d : |\nu| = 1, \mathcal{B}(z + r\nu, r) \cap D = \emptyset\}$ is non-empty, with $\mathcal{N}_{z,r} = \mathcal{N}_{z,s}$ for $0 < s \leq r$.*

Definition 2.2 (Uniform interior cone condition, from Saisho, 1987, §1, Condition (B'))**.** *A domain D satisfies a* uniform interior cone condition based on radius $\delta > 0$ and angle $\alpha \in (0, \pi/2]$ *if, for every $v \in \partial D$, there is at least one unit vector $\mathbf{m}$ such that the cone $C(\mathbf{m}) = \{z : \langle z, \mathbf{m}\rangle > |z| \cos\alpha\}$ satisfies*

$$(w + C(\mathbf{m})) \cap \mathcal{B}(v, \delta) \quad \subseteq \quad D \qquad \textit{for all } w \in D \cap \mathcal{B}(v, \delta)\,.$$

We say that the cone $w + C(\mathbf{m})$ is based on w and angle $\alpha \in (0, \pi/2]$.

It was shown in Bramson et al. (2012, Section 2) that the uniform interior cone condition is equivalent to the better known Lipschitz boundary condition (see Definition A.1).

The uniform exterior sphere and uniform interior cone conditions were employed by Saisho (1987) to define reflecting Brownian motion in D. However, the conditions are also useful in establishing regularity of the intrinsic distance. In particular, if D satisfies both conditions, then the intrinsic distance between two close points in D is comparable to the Euclidean distance (Bramson et al., 2012, Proposition 12), and the intrinsic distance therefore extends to the entire closure $\overline{D}$ without discontinuity at ∂D.

The following two simple examples demonstrate the need for both conditions:

Example 2.3. *Suppose that D is formed from the disc $\mathcal{B}((0,0),1)$ by deleting the line segment from $(0,0)$ to $(1,0)$. Then D satisfies the uniform interior cone condition, although the uniform exterior sphere condition fails on the line segment from $(0,0)$ to $(1,0)$. The intrinsic distance cannot be extended to $\overline{D}$ in a continuous manner.*

Example 2.4. *Suppose that D is formed from the cube $[-1,1]^3$, in 3-space, by deleting the two continuous families of closed balls $\{\overline{\mathcal{B}((1,0,u),1)} : -1/2 \leq u \leq 1/2\}$ and $\{\overline{\mathcal{B}((-1,0,u),1)} : -1/2 \leq u \leq 1/2\}$. Here, D satisfies the uniform exterior sphere condition, although the uniform interior cone condition fails at the open line segment $\{(0,0,u) : -1/2 < u < 1/2\}$. The domain D is connected, with the two points $(0, \pm\varepsilon, 0)$ being distance $\sqrt{1+4\varepsilon^2}$ apart with respect to the intrinsic distance for D. On the other hand, the two points are distance 2ε apart in terms of both the Euclidean metric and the intrinsic distance for $\overline{D}$. Thus, the intrinsic distance cannot be extended to $\overline{D}$ in a continuous manner.*

We therefore typically consider domains that satisfy the uniform exterior sphere and interior cone conditions; we refer to such domains as *ESIC domains* (i.e., uniform Exterior Sphere and Interior Cone domains). (In principle, one might consider generalizing the following results to non-ESIC domains; one then needs to take into account the pathologies illustrated in the two preceding examples.)

The following classic curvature comparison property is central to our arguments. Following Bridson and Haefliger (1999, §II.1, Definition 1.1) we define the CAT(κ) property as follows.

Definition 2.5. *For $\kappa > 0$, the domain D is a CAT(κ) domain if any two distinct points with distance less than $\pi/\sqrt{\kappa}$ are joined by a geodesic and the distance between any two points on the perimeter of any geodesic triangle $\triangle pqr$ of perimeter less than $2\pi/\sqrt{\kappa}$ is no greater than the distance between the corresponding points of the model triangle $\triangle\widetilde{p}\widetilde{q}\widetilde{r}$ with the same side lengths on the 2-dimensional Euclidean sphere of radius $1/\sqrt{\kappa}$. The domain D is a CAT(0) domain if any two distinct points at whatever distance are joined by a geodesic and the distance between any two points on the perimeter of any geodesic triangle $\triangle pqr$ is no greater than the distance between the corresponding points of the model triangle $\triangle\widetilde{p}\widetilde{q}\widetilde{r}$ with the same side lengths in the 2-dimensional Euclidean plane.*

A bounded domain satisfying the uniform exterior sphere and uniform interior cone conditions is CAT(κ), for some $\kappa > 0$. We sketch a proof in in Appendix A. The claim has already been proved in the literature in a slightly weaker form (see Remark A.6). From time to time in the article, we will explicitly recall that ESIC domains satisfy the CAT(κ) property, since our estimates often make use of the curvature parameter κ.

For $\kappa > 0$, the scaling $D \to \sqrt{\kappa}D$ transforms a CAT(κ) domain into a CAT(1) domain. (See, for example, the appendix to Alexander, Bishop, and Ghrist, 2010.) Note that, for $\kappa_1 \leq \kappa_2$, if a domain D is CAT(κ_1), then it is also automatically CAT(κ_2). Where convenient, we will limit our arguments to the cases $\kappa = 0, 1$.

We next introduce some notation for rectifiable loops and the concatenation of curves in $\overline{D}$. Let $\mathcal{S}$ be the circle with radius 1 centered at the origin; it will be convenient to identify $\mathcal{S}$ with $\{e^{2\pi i u}, 0 \leq u < 1\}$. Let $\mathcal{K}$ be the family of all loops K in $\overline{D}$ with finite length, i.e., $K : \mathcal{S} \to \overline{D}$ is a continuous mapping, with $K(\mathcal{S})$ being rectifiable with length $\ell_K < \infty$. We will reparametrize K by its length measured from a base point $K(0)$, i.e., $K = \{K(t) : t \in [0, \ell_K)\}$ such that, for every $s \in [0, \ell_K)$, the length of $\{K(t) : t \in [0, s]\}$ is s. Accordingly, we may view any loop $K \in \mathcal{K}$ as a Lipschitz closed curve with Lipschitz constant 1. The same conventions about parametrization by length will apply to other rectifiable curves that are not necessarily loops. For convenience, we will sometimes abuse notation by writing K instead of $K(\mathcal{S})$, for example, writing $K \subset D$. For $K \in \mathcal{K}$, we define the Euclidean tubular neighbourhood $\mathcal{B}(K, r)$ of K by $\mathcal{B}(K, r) = \{z \in \overline{D} : \operatorname{dist}(z, K) < r\}$. (Recall that $\mathcal{B}(z, r)$ denotes the open Euclidean ball of radius r centered on z.)

The *concatenation* $f * g$ of curves $f : [0, T] \to \mathbb{R}^d$ and $g : [0, S] \to \mathbb{R}^d$, with $f(T) = g(0)$, is the curve $f * g : [0, T + S] \to \mathbb{R}^d$,

$$(f * g)(u) \quad = \quad \begin{cases} f(u) & \text{if } 0 \leq u \leq T\,, \\ g(u - T) & \text{if } T < u \leq T + S\,. \end{cases}$$

We write f^{-1} for the reversed curve $t \mapsto f(T-t)$. If $f(0) = f(T)$, then we write f^{*n} for the n-fold concatenation of f with itself, for $n = 1, 2, \ldots$; in particular, for a loop $K \in \mathcal{K}$ and n a positive integer, the n-fold *concatenation power* $K^{*n} \in \mathcal{K}$ satisfies the conditions $\ell_{K^{*n}} = n\ell_K$ and $K^{*n}(t) = K(t \bmod \ell_K)$ for $t \in [0, n\ell_K)$. If $n = -m$ is negative, then we define $K^{*n} \in \mathcal{K}$ to be the reversal of K^{*m}.

The intrinsic Hausdorff distance between $A, B \subset \mathbb{R}^d$ is defined by

$$d_H(A, B) = \max\left\{\sup_{v\in A}\inf_{z\in B} \operatorname{dist}_{\mathbf{I}}(v, z)\,,\ \sup_{v\in B}\inf_{z\in A} \operatorname{dist}_{\mathbf{I}}(v, z)\right\}.$$

We will use intrinsic Hausdorff distance to measure distance between loops viewed as subsets of the closure $\overline{D}$ of the domain D.

It will be important to identify instances in which loops can be contracted to points, to identify other instances in which loops cannot be contracted at all, and to distinguish between weak contractions as opposed to contractions for which contraction occurs at least at a uniform rate. (We consider only ESIC domains in order to avoid needing to consider the kind of boundary issues illustrated by Examples 2.3, 2.4.)

Definition 2.6. *Suppose $D \subset \mathbb{R}^d$, with $d \geq 2$.*

(a) *A loop $K \in \mathcal{K}$ is a* contractible loop *if there exists a length-monotonic homotopy of K with a point $z \in \overline{D}$, namely, a continuous mapping $H : \mathcal{S} \times [0, 1] \to \overline{D}$ such that*
   (i) *For every $\gamma \in [0, 1)$, there exists $K_\gamma \in \mathcal{K}$ such that $H(e^{2\pi i t}, \gamma) = K_\gamma(t\ell_{K_\gamma})$, for $t \in [0, 1)$.*
   (ii) *$K_0 = K$.*
   (iii) *$H(\mathcal{S}, 1) = K_1 = \{z\}$ for the specified $z \in \overline{D}$.*
   (iv) *The function $\gamma \to \ell_{K_\gamma}$ is non-increasing on $[0, 1]$.*

   We will identify $K(\gamma, t)$ with the family $\{K_\gamma\}_{\gamma\in[0,1)}$ and call it a contraction *of K.*

(b) *A contractible loop $K \in \mathcal{K}$ is* well-contractible, with contractibility constant $c \in (0, \infty)$, *if there exists a length-monotonic homotopy contraction $\{K_\gamma\}_{\gamma\in[0,1)}$ such that, for all $0 \leq \gamma < \eta \leq 1$,*

$$\ell_{K_\gamma} - \ell_{K_\eta} \quad \geq \quad c\, d_H(K_\gamma, K_\eta)\, \ell_{K_\gamma}.$$

   In words, this says that the homotopy can be chosen so that the relative rate of contraction is bounded away from zero when measured using the change in the Hausdorff distance. Note that the contractibility constant c may depend on the point $H(\mathcal{S}, 1)$ to which the loop is contracted.

(c) *A bounded ESIC domain D is a* contractible loop (CL) domain *if there exists a constant $c > 0$ such that, for each $K \in \mathcal{K}$, there exists $z \in \overline{D}$ such that K is well-contractible to z with the contractibility constant c. (We can then also say that the loops in D are* uniformly contractible.*)*

Remark 2.7. *The definitions of contractible loops and well-contractible loops apply to loops in any set $D \subset \mathbb{R}^d$ but we will limit our considerations to ESIC domains D because the behavior of such loops may be strange in non-ESIC domains.*

We introduce the following concepts when the loop length-functional is at a "local minimum".

Definition 2.8. (a) *A loop $K \in \mathcal{K}$ is a* semi-stable rubber band *if, for some $\varepsilon > 0$, the following holds: Suppose that $K_1 \in \mathcal{K}$ and there exists a continuous mapping $H : \mathcal{S} \times [0,1] \to \overline{\mathcal{B}(K,\varepsilon)}$ such that $H(e^{2\pi i t}, 0) = K(t\ell_K)$ for $t \in [0,1)$ and $H(e^{2\pi i t}, 1) = K_1(t\ell_{K_1})$ for $t \in [0,1)$. Then $\ell_{K_1} \geq \ell_K$.*

(b) *A loop $K \in \mathcal{K}$ is a* stable rubber band *if it is semi-stable and if, for some $\varepsilon > 0$ and all $0 < \eta \leq \varepsilon$, there exists $\delta = \delta(\eta,\varepsilon) > 0$ such that the following holds: Suppose that $K_1 \in \mathcal{K}$, $d_H(K, K_1) \geq \eta$ and, for some $n \geq 1$, there exists a continuous mapping $H : \mathcal{S} \times [0,1] \to \overline{\mathcal{B}(K,\varepsilon)}$ such that $H(e^{2\pi i t}, 0) = K(tn\ell_K)$ for $t \in [0,1)$ and $H(e^{2\pi i t}, 1) = K_1(t\ell_{K_1})$ for $t \in [0,1)$. Then $\ell_{K_1} > n\ell_K + \delta$. In words, if a concatenation power K^{*n} of K with $n \neq 0$ can be locally perturbed to a loop K_1, then K_1 must be longer than K^{*n} by at least an amount depending on the intrinsic Hausdorff distance between the two loops.*

As noted above, an ESIC domain D must be CAT(κ), for some $\kappa \geq 0$. We conclude this section with two lemmas that employ the CAT(κ) property, followed by a pair of remarks. The first lemma shows that, in ESIC domains, any two rectifiable loops that are suitably close to each other are also connected by a (not necessarily length-monotonic) local homotopy. We adopt the convention that $\pi/\sqrt{\kappa} = \infty$ if $\kappa = 0$, in order to avoid needing to distinguish between $\kappa = 0$ and $\kappa > 0$.

Lemma 2.9. *Let D be an ESIC domain that is CAT(κ), with $\kappa \geq 0$. Suppose that K_0, K_1 are rectifiable loops such that*

$$\operatorname{dist}_{\mathbf{I}}(K_0(t\ell_{K_0}), K_1(t\ell_{K_1})) \quad \leq \quad \varepsilon$$

for all $0 \leq t \leq 1$, for some $\varepsilon < \pi/\sqrt{\kappa}$. Then K_0 and K_1 are homotopic within $\mathcal{B}(K_0, \varepsilon)$.

Proof. First note that it follows from Definition 2.5 that any geodesic of total length less than $\pi/\sqrt{\kappa}$ is uniquely defined by its end-points, is minimal, and depends continuously on its end-points. (This dependence is uniform in case the total length is bounded away from $\pi/\sqrt{\kappa}$.)

We define the homotopy $H : [0,1]^2 \to \mathcal{B}(K_0, \varepsilon)$ by

$$H(s,t) \quad = \quad \gamma^{(t)}(s\ell_{\gamma^{(t)}}),$$

where $\gamma^{(t)}$ is the unit-speed geodesic from $K_0(t\ell_{K_0})$ to $K_1(t\ell_{K_1})$. The continuity of $H(\cdot,\cdot)$ follows directly from the properties in the first paragraph of the proof. □

We can employ the previous lemma to show that a semi-stable rubber band is locally geodesic.

Lemma 2.10. *If K is a semi-stable rubber band in an ESIC domain D, then it is locally geodesic in the intrinsic distance metric.*

Proof. The loop K is locally geodesic in the intrinsic distance metric if, for some $\varepsilon > 0$ and any $0 \leq s < t < \ell_K$, (i) when $t - s < \varepsilon/2$, then $\{K(v) : s \leq v \leq t\}$ determines a length-minimizing intrinsic geodesic from $K(s)$ to $K(t)$ and (ii) when $(\ell_K - t) + (s - 0) = \ell_K - t + s < \varepsilon/2$, then $\{K(v) : t \leq v < \ell_K\}$ followed by $\{K(v) : 0 \leq v \leq s\}$ determines a length-minimizing intrinsic geodesic from $K(t)$ to $K(s)$.

We will demonstrate case (i); a similar argument holds for case (ii). First note that D must be CAT(κ) for some $\kappa > 0$. Choose $\varepsilon > 0$ as in Definition 2.8(a) so that $\varepsilon < \pi/\sqrt{\kappa}$. Suppose $0 \leq s < t < \ell_K$ and $t - s < \varepsilon/2$. Then $K(v) \in \mathcal{B}(K(s), \varepsilon/2)$ for $s \leq v \leq t$, because K has Lipschitz constant 1. Were $\{K(v) : s \leq v \leq t\}$ not length-minimizing, then it would be possible to replace this section of the loop by a strictly shorter segment, thus producing a new loop K_1 with strictly smaller total length. Moreover, by the triangle inequality, $\text{dist}_{\mathbf{I}}(K(v), K_1(v)) \leq \varepsilon$ for $s \leq v \leq t$. Since $K(v) = K_1(v)$ for $v \notin [s, t]$, we have $\text{dist}_{\mathbf{I}}(K(v), K_1(v)) \leq \varepsilon$ for all v. Hence, by Lemma 2.9, it follows that K and K_1 are homotopic within $\mathcal{B}(K, \varepsilon)$. This contradicts the assertion that K is semi-stable, and therefore implies that the segment $\{K(v) : s \leq v \leq t\}$ must be length-minimizing, and hence is a minimal geodesic. ☐

Remark 2.11. At the intuitive level, a rubber band is almost the same as a non-constant harmonic map from a circle to a closed set in the Euclidean space, or in other words a closed geodesic. However, the theory of harmonic maps does not seem to be relevant to our study. (The literature on harmonic maps is huge. Succinct summaries of the general theory of smooth harmonic maps can be found in Eells and Lemaire (1978, 1988); see also the monograph by Lin and Wang (2008). Non-smooth harmonic maps are discussed in Eells and Fuglede (2001).)

Remark 2.12. Note that the property of K being a stable rubber band, respectively a semi-stable rubber band, in a domain D is *local* to K, in the sense that K remains stable, respectively semi-stable, if the domain D is altered, as long as $D \cap \mathcal{B}(K, \varepsilon)$ is not altered for some $\varepsilon > 0$.

3. Domains with stable rubber bands

In this section, we analyze domains that contain stable rubber bands. In Definition 3.1, we formulate the Lion and Man problem, and specify what it means for the Man to have a successful evasion strategy. Theorem 3.7 is the main result of this section, where we will show that, for ESIC domains containing a stable rubber band, there is always a successful evasion strategy for the Man. The property that any ESIC domain is CAT(κ), for some $\kappa \geq 0$, will be employed repeatedly.

We begin by establishing a mathematical framework for pursuit and evasion. In Definition 3.1, the path of the Man is represented by a continuous curve $y(t)$ and that of the Lion by a continuous curve $x(t)$. Here, $\mathbb{R}_+ = \{t \in \mathbb{R} : t \geq 0\}$ and $\mathcal{C} = C(\mathbb{R}_+, \overline{D})$ is the space of continuous functions on $\mathbb{R}_+$ with values in $\overline{D}$.

Definition 3.1. *Suppose that D is an ESIC domain.*

(i) $\{x(t), t \geq 0\}$ *is an* admissible curve *if it is continuous, locally rectifiable and parametrized so that* $\mathrm{dist}_{\mathbf{I}}(x(s), x(t)) \leq |s-t|$ *for all* $s, t \geq 0$. *Note that this implies* $x'(t)$ *exists for almost all* $t \geq 0$ *and* $x(t) - x(0) = \int_0^t x'(s) ds$ *for every* $t \geq 0$.

(ii) Let Λ *be the family of all quadruples* (x, y, F_x, F_y) *such that* x *and* y *are admissible curves and* x, y, F_x *and* F_y *satisfy the following properties. The functions* $F_x : \mathbb{R}_+ \times \mathcal{C}^2 \to \mathbb{R}^d$ *and* $F_y : \mathbb{R}_+ \times \mathcal{C}^2 \to \mathbb{R}^d$ *are measurable and such that* $x'(t) = F_x(t, x(\cdot), y(\cdot))$ *for all* t *where* $x'(t)$ *exists and, similarly,* $y'(t) = F_y(t, x(\cdot), y(\cdot))$ *for all* t *where* $y'(t)$ *exists. Moreover,* F_x *and* F_y *are non-anticipative in the sense that, if* $x, x^*, y, y^* \in \mathcal{C}$, $x(s) = x^*(s)$ *and* $y(s) = y^*(s)$ *for* $s \leq t$, *then* $F_x(t, x(\cdot), y(\cdot)) = F_x(t, x^*(\cdot), y^*(\cdot))$*; the analogous condition is satisfied by* F_y.

(iii) The Man has a successful evasion strategy *if, for some pair* $x_0, y_0 \in \overline{D}$, *(a) There exists* $(x, y, F_x, F_y) \in \Lambda$, *with* $x(0) = x_0$ *and* $y(0) = y_0$. *(b) Suppose that* F_x *and* x, *with* $x(0) = x_0$, *are such that there exist* y *and* F_y *with* $y(0) = y_0$ *and* $(x, y, F_x, F_y) \in \Lambda$. *Then there exist* y *and* F_y *such that* $y(0) = y_0$, $(x, y, F_x, F_y) \in \Lambda$ *and the evasion condition* $\inf_{t \in [0,\infty)} \mathrm{dist}_{\mathbf{I}}(x(t), y(t)) > 0$ *holds.*

(iv) Conversely, there is no successful evasion strategy for the Man *(or that* the Lion can capture the Man*) if, for each pair* $x_0, y_0 \in \overline{D}$, *with* $x_0 \neq y_0$, *and every* F_y *and* y *with* $y(0) = y_0$, *with at least one tuple* $(x, y, F_x, F_y) \in \Lambda$ *satisfying* $x(0) = x_0$, *there exist* x *and* F_x *with* $(x, y, F_x, F_y) \in \Lambda$ *and with* $x(0) = x_0$, *and satisfying* $\inf_{t \in [0,\infty)} \mathrm{dist}_{\mathbf{I}}(x(t), y(t)) = 0$.

Remark 3.2. Definition 3.1 is stated in the context of ESIC domains that are open subsets of Euclidean spaces. Note however that the concepts of Definition 3.1 still make sense in the more general context of CAT(κ) metric spaces.

Remark 3.3. In contrast to the classical formulation given in Littlewood (1986), we consider an evasion strategy to fail if the Lion is able to approach arbitrarily close to the Man, even if the Lion does not catch the Man in finite time.

Remark 3.4. (i) Assuming that F_x, F_y, x_0 and y_0 are given, the conditions $x'(t) = F_x(t, x(\cdot), y(\cdot))$ and $y'(t) = F_y(t, x(\cdot), y(\cdot))$ specify a system of differential equations, typically with right-hand sides that are discontinuous when viewed as time-varying vector fields. We do not make any claims in general about existence or uniqueness of solutions to this set of equations. It is trivial to see that, for any x_0 and y_0, there exist $(x, y, F_x, F_y) \in \Lambda$ satisfying $x(0) = x_0$ and $y(0) = y_0$. For example, x and y can be constant functions and $F_x \equiv F_y \equiv 0$.

(ii) For curves x and y that represent the Lion and Man, we will tacitly assume that if, for some t, $x(t) = y(t)$, then $x(s) = y(s)$ for all $s \geq t$.

We introduce notation to represent pursuit games in which the Lion has a "fixed path" strategy that does not "take into account" the strategy of the Man. Such strategies provide a useful heuristic to understand the difference of roles for the Lion and the Man in pursuit problems, but will not be directly employed in any of the proofs in the paper.

Definition 3.5. *The set* $\Lambda_0 \subset \Lambda$ *is the collection of all* $(x, y, F_x, F_y) \in \Lambda$ *such that* $F_x(t, x(\cdot), y(\cdot))$ *does not depend on* $y(\cdot)$.

Lemma 3.6. *Suppose that x_0 and y_0 are given. The following conditions are equivalent.*

(i) There exists F_y, with $(x, y, F_x, F_y) \in \Lambda$ for some x, y and F_x, and with $x(0) = x_0$ and $y(0) = y_0$, such that $\inf_{t\in[0,\infty)} \operatorname{dist}_{\mathbf{I}}(x(t), y(t)) > 0$ for every choice of x, y and F_x satisfying $x(0) = x_0$ and $y(0) = y_0$, and $(x, y, F_x, F_y) \in \Lambda$.

(ii) There exists F_y, with $(x, y, F_x, F_y) \in \Lambda$ for some x, y and F_x, and with $x(0) = x_0$ and $y(0) = y_0$, such that $\inf_{t\in[0,\infty)} \operatorname{dist}_{\mathbf{I}}(x(t), y(t)) > 0$ for every choice of x, y and F_x satisfying $x(0) = x_0$ and $y(0) = y_0$, and $(x, y, F_x, F_y) \in \Lambda_0$.

Proof. Since $\Lambda_0 \subset \Lambda$, (i) implies (ii). Suppose that (ii) holds and consider any fixed $(x, y, F_x, F_y) \in \Lambda$ satisfying $x(0) = x_0$ and $y(0) = y_0$. Let $\widehat{F}_x(t, x, y) = \lim_{s\uparrow 0}(x(t + s) - x(t))/s$ if the limit exists and $\widehat{F}_x(t, x, y) = 0$ otherwise. Since $x'(t)$ exists for almost all t, $(x, y, \widehat{F}_x, F_y) \in \Lambda_0$. Since $\liminf_{t\to\infty} \operatorname{dist}_{\mathbf{I}}(x(t), y(t)) > 0$ is true for $(x, y, \widehat{F}_x, F_y)$, it also holds for (x, y, F_x, F_y). This implies (i). □

Thus, the existence of a successful evasion strategy does not depend on whether the Lion is "intelligent". This may seem counterintuitive, so we offer a heuristic explanation. The Lion may choose his strategy randomly and may capture the Man by pure luck. The Man has to protect himself against all strategies, even those chosen randomly.

We note that our heuristic explanation is just that—there is no randomness in the mathematical model discussed in the lemma. Moreover, one should be aware of the following subtle point: Λ_0 does not necessarily rigorously correspond to the intuitive concept of the Lion choosing his strategy without regard to the Man's position, since F_x need not uniquely determine the Lion's path (due to possible bifurcations of $x' = F_x$).

On the other hand a "fixed path" strategy for the Man may fail to successfully evade the Lion if the Lion is intelligent, that is, if the Lion can base his strategy F_x on both x and y. See Remark 3.11 at the end of the section.

We now turn to our main result on pursuit-evasion in this section. Theorem 3.7 states that the existence of a stable rubber band makes it possible for the Man to evade the Lion, as long as the Man starts on the rubber band and the Lion is initially a positive distance away from the Man. Intuitively, this is plausible since the Man simply has to run away from the Lion along the rubber band. The proof involves making this observation precise.

Theorem 3.7. *Suppose that D is an ESIC domain that contains a stable rubber band K. Then there is a successful evasion strategy for the Man whenever the starting positions $x_0, y_0 \in \overline{D}$ are such that $y_0 \in K$ and $x_0 \neq y_0$.*

Proof. We will show there is a $\alpha > 0$, depending on $\operatorname{dist}_{\mathbf{I}}(x(0), y(0)) > 0$, such that, no matter what strategy is adopted by the Lion, the Man can choose a strategy to ensure that $\operatorname{dist}_{\mathbf{I}}(x(t), y(t)) > \alpha$ for all $t \geq 0$.

Suppose that K is a stable rubber band and Definition 2.8(b) is satisfied for some $\varepsilon > 0$ and function $\delta(\eta, \varepsilon)$. It is evident from Definition 2.8(b) that $\delta(\eta, \varepsilon)$ may be chosen to be non-decreasing in ε for $\varepsilon \geq \eta$.

Assume that K is a stable rubber band, $y_0 \in K$ and $x_0 \neq y_0$. Let $\varepsilon > 0$ and $\delta(\eta, \varepsilon)$ be such that Definition 2.8(b) is satisfied. We decrease ε, if necessary, so that $\text{dist}_{\mathbf{I}}(x_0, y_0) \geq \varepsilon/2$.

Since D is ESIC, it is also CAT(κ) for some $\kappa \geq 0$. To ensure the global geometry of D does not interfere, we decrease $\varepsilon > 0$ further, if necessary, so that $\varepsilon < \pi/\sqrt{\kappa}$.

The essence of the argument involves the notion of *hot pursuit* – for a fixed $\varepsilon > 0$, we say that *the Lion x is in ε-hot pursuit of the Man y over the time interval* $[T_0, T_1]$ if

$$\text{dist}_{\mathbf{I}}(x(t), y(t)) \quad \leq \quad \varepsilon \qquad \text{for } T_0 \leq t \leq T_1 \,.$$

We shall show that a Man can always evade a Lion in hot pursuit by running in a judiciously chosen direction along K. On the other hand, the Lion gains nothing by desisting from hot pursuit for a while, since an "up-crossing argument" applied to $\text{dist}_{\mathbf{I}}(x, y)$ shows that the Man can deal with such variations simply by taking rest-periods in the intervals $[s, t]$ satisfying $\text{dist}_{\mathbf{I}}(x(s), y(s)) \geq \varepsilon$ and $\text{dist}_{\mathbf{I}}(x(u), y(u)) \geq \varepsilon/2$ for $u \in [s, t]$.

(i) Consider first the situation in which the Lion x begins at location $x(0)$, at intrinsic distance at least $\varepsilon/6$ from K and at most ε from the Man, who begins at $y(0) \in K$. Without loss of generality, we suppose $y(0) = K(0)$. Choose $\delta = \delta(\varepsilon/6, \varepsilon)$ as required in Definition 2.8(b), and set $\eta = \min\{\delta, \varepsilon/2\}$. The Man has a choice between running "clockwise" $(y(t) = K(t))$ and "counter-clockwise" $(\overline{y}(t) = K(-t))$. We argue that, for at least one of these strategies, the Man can remain at least distance $\eta/3$ from the Lion as long as the Lion continues in ε-hot pursuit.

Arguing by contradiction, suppose that the Lion can use ε-hot pursuit to come within $\eta/3$ of the Man, whichever of the two strategies is adopted by the Man. Let t_c and $\overline{t}_c$ be the two times at which this $\eta/3$-capture occurs. Note that it is possible for either or both of t_c, $\overline{t}_c$ to exceed the length of the loop K; the chase may encircle K several times.

Define non-negative integers n and $\overline{n}$ by

$$\begin{aligned} (n-1)\ell_K \quad &< \quad t_c \quad \leq \quad n\ell_K \,, \\ (\overline{n}-1)\ell_K \quad &< \quad \overline{t}_c \quad \leq \quad \overline{n}\ell_K \,, \end{aligned}$$

and determine rectifiable paths by using the two ε-hot pursuits x and $\overline{x}$ of the Lion:

$$\begin{aligned} \Gamma_1 \quad &= \quad x|_{[0,t_c]} \,, \\ \overline{\Gamma}_1 \quad &= \quad \overline{x}|_{[0,\overline{t}_c]} \,, \\ \Gamma_2 \quad &= \quad \text{minimal geodesic from } x(t_c) \text{ to } K(t_c) \,, \\ \overline{\Gamma}_2 \quad &= \quad \text{minimal geodesic from } \overline{x}(\overline{t}_c) \text{ to } K(-\overline{t}_c) \,, \\ \Gamma_3 \quad &= \quad \text{arc of } K \text{ from } K(t_c) \text{ to } K(n\ell_K) = K(0) \,, \\ \overline{\Gamma}_3 \quad &= \quad \text{arc of } K \text{ from } K(-\overline{t}_c) \text{ to } K(-\overline{n}\ell_K) = K(0) \,. \end{aligned}$$

Here, Γ_3, $\overline{\Gamma}_3$ are defined by continuing the same direction of travel along K as given by y, $\overline{y}$ respectively. The construction of Γ_1, Γ_2, and Γ_3 is illustrated in Figure 1.

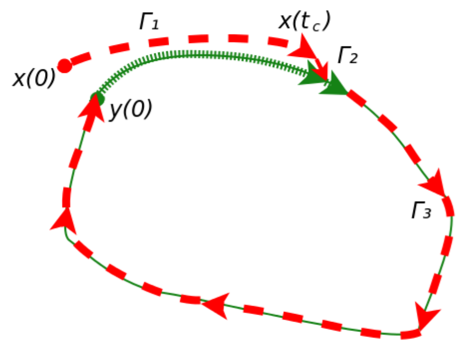

FIGURE 1. One half of the "hot-pursuit" loop K' from part (i).

One can extend the pursuit by the Lion of the Man past time t_c, respectively time $\bar{t}_c$, depending on the strategy adopted by the Man, along the concatenated paths $\Gamma_1 * \Gamma_2 * \Gamma_3$, respectively $\overline{\Gamma}_1 * \overline{\Gamma}_2 * \overline{\Gamma}_3$, with both the Lion and the Man moving at unit speed along the extensions. Since the Lion is within distance $\eta/3 \leq \varepsilon$ of the Man at time t_c, respectively $\bar{t}_c$, it follows that the paths $\Gamma_1 * \Gamma_2 * \Gamma_3$ and $\overline{\Gamma}_1 * \overline{\Gamma}_2 * \overline{\Gamma}_3$ are both ε-hot pursuits of y, $\overline{y}$. We will consider the rectifiable loop running from $x(0) = \overline{x}(0)$ back to itself, given by

$$K' \quad = \quad \Gamma_1 * \Gamma_2 * \Gamma_3 * \overline{\Gamma}_3^{\,-1} * \overline{\Gamma}_2^{\,-1} * \overline{\Gamma}_1^{\,-1} .$$

The ε-hot pursuit property implies that K' lies in $\overline{\mathcal{B}(K,\varepsilon)}$ and that one can construct a mapping H as in Definition 2.8 (b). Moreover $d_H(K, K') \geq \operatorname{dist}_{\mathbf{I}}(x(0), K) \geq \varepsilon/6$. Finally,

$$\ell_{K'} \quad \leq \quad t_c + \eta/3 + (n\ell_K - t_c) + \bar{t}_c + \eta/3 + (\overline{n}\ell_K - \bar{t}_c) \\ = \quad (n + \overline{n})\ell_K + 2\eta/3 \quad \leq \quad (n + \overline{n})\ell_K + \delta\,,$$

which violates the stability of the rubber band K. This contradiction shows that if the Lion starts from distance at least $\varepsilon/6$ from K, and remains in hot pursuit of the Man, then the Man can choose a clockwise or counterclockwise strategy so as to always remain at least distance $\eta/3$ away from the Lion.

(ii) Now consider the situation in which the Lion starts at $x(0)$ that is less than $\varepsilon/6$ from K but greater than or equal to $\varepsilon/2$ and less than ε from the Man's starting point $y(0) = K(0)$. Let z denote the closest point to $x(0)$ on K, and suppose that $z = K(-u)$ for some $u > 0$ (the case $u < 0$ can be dealt with in an analogous way). By the triangle inequality, $u \geq \varepsilon/3$.

Let the Man adopt the strategy $y(t) = K(t)$ and consider the Lion in ε-hot pursuit of the Man. Suppose the Lion comes within distance $\eta/3$ of the Man at time t_h. Define

$$\begin{aligned}\Gamma_1 &= x|_{[0,t_h]}\,,\\ \Gamma_2 &= \text{minimal geodesic from } x(t_h) \text{ to } y(t_h)\,,\\ \Gamma_3 &= y|_{[t_h, m\ell_K - u]}\,,\\ \Gamma_4 &= \text{minimal geodesic from } z = K(-u) \text{ to } x(0)\,,\end{aligned}$$

where m is the integer satisfying

$$(m-1)\ell_K - u \quad < \quad t_h \quad \leq \quad m\ell_K - u\,.$$

Consider the rectifiable loop $K' = \Gamma_1 * \Gamma_2 * \Gamma_3 * \Gamma_4$ based at $x(0)$. The construction of Γ_1, Γ_2, Γ_3, and Γ_4 is illustrated in Figure 2. Arguing as in (i), because of the hot pursuit by the Lion,

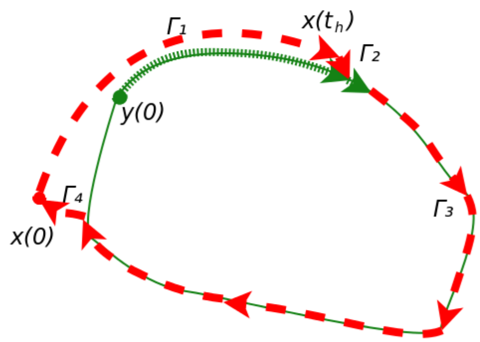

FIGURE 2. The "hot-pursuit" loop from part (ii).

it follows that K' and K are homotopic within $\overline{\mathcal{B}(K,\varepsilon)}$. Once again, we choose $\delta = \delta(\varepsilon/6, \varepsilon)$ as required in Definition 2.8(b), but we now set $\eta = \min\{\delta, \varepsilon/2\}$. A simple computation of lengths shows that K' violates the stable rubber band property of K:

$$\ell_{K'} \quad \leq \quad t_h + \eta/3 + m\ell_K - u - t_h + \varepsilon/6 \quad = \quad m\ell_K - (u - \varepsilon/6 - \eta/3) \quad \leq \quad m\ell_K\,.$$

So, the Lion cannot come within distance $\eta/3$ of the Man if it remains in ε-hot pursuit.

In order to complete the proof, we now spell out the up-crossing argument that was mentioned earlier. Suppose that $\text{dist}_{\mathbf{I}}(x_0, y_0) \geq \varepsilon > 0$. The Man chooses to rest until the Lion is distance $\varepsilon/2$ from him. We need to consider two cases.

If the Lion is within $\varepsilon/6$ of K, then the Man moves in the appropriate direction given by (ii) above. We have seen in (ii) that the Lion cannot come within distance $\eta/3$ of the Man while maintaining ε-hot pursuit. Since neither the Lion nor the Man can travel faster than at unit speed, at least time $\frac{1}{2} \times (\varepsilon - \varepsilon/2) = \varepsilon/4$ must elapse before the Lion and the Man are separated by ε, at which time the Man rests again.

If instead, the Lion is further than $\varepsilon/6$ from K, then the Man moves in the escape direction guaranteed by (i) above. We have seen in (i) that the Lion cannot come within distance $\eta/3$ of the Man while it maintains ε-hot pursuit. Once more, at least time $\frac{1}{2} \times (\varepsilon - \varepsilon/2) = \varepsilon/4$ must elapse before the Lion and the Man are separated by ε, at which time the Man rests again.

Over any finite time interval $[0, T]$, there can be at most $1 + 4T/\varepsilon$ separate periods of ε-hot pursuit and $4T/\varepsilon$ separate periods of rest and, in each of these periods, the Lion and the Man remain separated in intrinsic distance by at least $\min\{\varepsilon/2, \eta/3\} = \eta/3$. Accordingly, this separation holds for all time, and therefore the Man will successfully evade the Lion. □

A crucial part of the above proof was the decomposition of an arbitrary pursuit strategy into alternating periods of ε-hot pursuit and of pursuit at a further distance. There is a related but more stringent notion of *simple pursuit*, which will be employed in the next section. The following definition is adapted from Alexander et al. (2010, Section 6).

Definition 3.8. *We will call (x, y) a* simple pursuit *if x and y are admissible curves, there exists a unique intrinsic geodesic between $x(t)$ and $y(t)$ for all $t \geq 0$, and $x'(t)$ exists for almost all $t \geq 0$, with $|x'(t)| = 1$ for $x(t) \neq y(t)$ and $x'(t)$ pointing towards $y(t)$ along the geodesic between $x(t)$ and $y(t)$. When $x(t_0) = y(t_0)$ for some t_0, we assume that $x(t) = y(t)$ for $t \geq t_0$. We will write Λ_s to denote the family of all simple pursuits (x, y).*

Remark 3.9. Note Definition 3.8 does not assert that, for every $v, z \in \overline{D}$, there exists a unique geodesic between v and z. This may not be true for distant pairs of points.

Remark 3.10. Simple pursuit can be viewed as a greedy solution to the pursuit problem (in the language of algorithm theory); it is therefore described as the "greedy pursuit strategy" in Bramson et al. (2012).

The notion of simple pursuit is easy to illustrate in the presence of stable rubber bands, albeit in a rather elementary way. Recall that a stable rubber band is a local geodesic (Lemma 2.10). As a consequence, if a domain D has a stable rubber band, then there exists a simple pursuit (x, y) in which the Man evades the Lion. Namely, choose any $x_0, y_0 \in K$ with $\text{dist}_{\mathbf{I}}(x_0, y_0) = \varepsilon/2$ and let $y(t)$ move away from $x(t)$ along K at the constant speed 1, starting from y_0. Let $x(t)$ follow $y(t)$ along K at the constant speed 1 as well. The distance between $x(t)$ and $y(t)$ will always be $\varepsilon/2$. Theorem 3.7 establishes a considerably stronger version of this fact, not limited to simple pursuit.

Remark 3.11. For some starting positions of the Lion and the Man, and some (possibly foolish) strategies $y(t)$ of the Man, it is evident that the Man will not evade the Lion under simple pursuit, whatever the geometry of the domain. For example, if $y(0)$ is in the interior of D and the Man adopts the "resting" strategy $y(t) \equiv y_0$, then simple pursuit from any starting point $x(0)$ close enough to $y(0)$ leads to capture of the Man in finite time.

4. Simple pursuit in CL domains

The main result of this section is Theorem 4.6, which shows that, if the Lion adopts an appropriate pursuit strategy, then the Man cannot successfully evade the Lion in a CL domain. We recall that since, by definition, a CL domain D is ESIC, it is also CAT(κ) for some $\kappa > 0$. Also, since the scaling $D \to \sqrt{\kappa} D$ transforms a CAT(κ) domain, $\kappa > 0$, into a CAT(1) domain, it suffices to state our arguments for CL domains that are CAT(1). (The CAT(0) case is covered by the results of Alexander, Bishop, and Ghrist (2006), where the notion of CL domains is not required; also note that CAT(0) domains are automatically CAT(κ), for any $\kappa > 0$.)

Our strategy will be to show that if the Lion and the Man are initially close, specifically, $\operatorname{dist}_{\mathbf{I}}(x(0), y(0)) < \pi$, then there is no successful evasion strategy by the Man if the Lion adopts simple pursuit (in the sense of Definition 3.1). In particular, we will show that, for every admissible curve $y(\,\cdot\,)$, there exists $x(\,\cdot\,)$ such that (x, y) is a simple pursuit and $\lim_{t\to\infty} \operatorname{dist}_{\mathbf{I}}(x(t), y(t)) = 0$. The general case, with arbitrary $x(0)$ and $y(0)$, will follow quickly from this by constructing a chain of points in D from $x(0)$ to $y(0)$, each of which is less than distance π from its neighbors, and applying simple pursuit at each step.

We begin by introducing a number of geometrical results that will be required in order to establish Theorem 4.6. To start with, we require the following general proposition from Alexander et al. (2010, Theorem 20).

Proposition 4.1. *Suppose that $\overline{D}$ is a closed CAT(1) space, $x_0, y_0 \in \overline{D}$, $\operatorname{dist}_{\mathbf{I}}(x_0, y_0) < \pi$ and $\{y(t), t \geq 0\}$ is an admissible curve. Then there exists a unique admissible curve $\{x(t), t \geq 0\}$ such that (x, y) is a simple pursuit.*

We also need the following general geometric observation from Alexander et al. (2010, Proposition 23).

Proposition 4.2. *Let $\overline{D}$ be a compact CAT(1) space. If there is a successful evasion strategy for the Man whenever the Man is initially separated from the Lion by a distance of less than π, then there exists a bilaterally infinite local geodesic in $\overline{D}$.*

The idea of the proof is as follows. Consider a successful evasion by the Man of the simple pursuit strategy provided by Proposition 4.1. The corresponding path will have total curvature that grows sublinearly. A sequence of segments of this evasion path, with lengths tending to ∞, can be used to construct a limit by applying compactness to choose a convergent subsequence. This limit will be a bilaterally infinite path that must have zero total curvature, and hence be a geodesic. (Note that this geodesic may be c1osed!)

A key result in this area of metric geometry is the powerful technique of Reshetnyak majorization, which reduces the essence of many problems to calculations from two-dimensional spherical geometry.

Proposition 4.3 (Reshetnyak majorization)**.** *If the length of a rectifiable closed curve h in a CAT(1) space D is less than 2π, then there is a convex domain C, contained in S^2, that majorizes*

h in the sense that there is a distance non-expanding map from C into D, such that its restriction to the boundary of C is an arc-length preserving map onto the image of h.

For a proof see Reshetnyak (1968); a clear statement can be found in Maneesawarng and Lenbury (2003).

The relevant calculation from two-dimensional spherical geometry is summarized in the following preparatory lemma.

Lemma 4.4. *Let $p\,q\,\widetilde{q}\,\widetilde{p}$ be a geodesic quadrilateral on the unit 2-sphere, such that the interior angles at p and q are obtuse or right-angles, such that $\widetilde{p}$ and $\widetilde{q}$ are on the same side of the great circle passing through p and q, such that $\delta = \operatorname{dist}(p,q) \in (0,\pi)$, and such that the distances $\operatorname{dist}(p,\widetilde{p})$ and $\operatorname{dist}(q,\widetilde{q})$ are both bounded above by some positive $\varepsilon < \delta/2$. If ε is chosen small enough so that*

$$\frac{1-\cos\delta}{\min\{\sin\delta,\sin(\delta-2\varepsilon)\}}\sin\varepsilon \quad < \quad 2, \tag{4.1}$$

then

$$\widetilde{\delta} \quad = \quad \operatorname{dist}(\widetilde{p},\widetilde{q}) \quad \geq \quad \delta - \frac{1-\cos\delta}{\min\{\sin\delta,\sin(\delta-2\varepsilon)\}}\ \sin^2\varepsilon\,. \tag{4.2}$$

In the context of our application of this lemma we will require δ to be small, so that we may take $\min\{\sin\delta,\sin(\delta-2\varepsilon)\} = \sin(\delta-2\varepsilon)$.

Proof. We begin by showing how to reduce the argument to the symmetric case, where $\operatorname{dist}(p,\widetilde{p}) = \operatorname{dist}(q,\widetilde{q}) = \varepsilon$ and the interior angles at p and q are right-angles. First, let $\mathcal{E}$ be the "equatorial" great-circle geodesic that is the perpendicular bisector of the minimal geodesic from p to q. Since the distances $\operatorname{dist}(p,\widetilde{p})$ and $\operatorname{dist}(q,\widetilde{q})$ are bounded by $\varepsilon < \delta/2 < \pi/2$ and the interior angles at p and q are obtuse or right-angles, the points $\widetilde{p}$ and $\widetilde{q}$ lie on the opposite sides of $\mathcal{E}$, and therefore

$$\operatorname{dist}(\widetilde{p},\mathcal{E}) + \operatorname{dist}(\widetilde{q},\mathcal{E}) \quad \leq \quad \operatorname{dist}(\widetilde{p},\widetilde{q})\,.$$

Let $\mathcal{H}$ be the open hemisphere of $S^2\setminus\mathcal{E}$ containing p. Then the function $x \mapsto \operatorname{dist}(x,\mathcal{E})$ of $x \in \mathcal{H}$ is a nonlinear, but strictly increasing function of the vertical height of x above the equatorial plane that is defined by $\mathcal{E}$. Moreover $x \mapsto \operatorname{dist}(x,\mathcal{E})$, restricted to the little circle $\{x : \operatorname{dist}(x,p) = \operatorname{dist}(\widetilde{p},p)\}$, can have just one minimum and just one maximum (since $\operatorname{dist}(p,\mathcal{E}) < \pi/2$). The maximum and minimum must lie on the great-circle geodesic γ defined by p and q, and the vertical height function $x \mapsto \operatorname{dist}(x,\mathcal{E})$ varies strictly monotonically on the two connected components of $\{x : \operatorname{dist}(x,p) = \operatorname{dist}(\widetilde{p},p)\} \setminus \gamma$. All these facts follow immediately from the observation that the little circle $\{x : \operatorname{dist}(x,p) = \operatorname{dist}(\widetilde{p},p)\}$ can be obtained as the intersection of $\mathcal{H}$ with an inclined plane. It follows directly that $\operatorname{dist}(\widetilde{p},\mathcal{E})$ is minimized when the interior angle at p is reduced to a right-angle. Similarly, $\operatorname{dist}(\widetilde{q},\mathcal{E})$ is minimized when the interior angle at q is reduced to a right-angle.

In the case where the interior angle at p (respectively q) is a right angle, we can argue that $\operatorname{dist}(\widetilde{p},\mathcal{E})$ (respectively, $\operatorname{dist}(\widetilde{q},\mathcal{E})$) is minimized when $\operatorname{dist}(\widetilde{p},p)$ (respectively, $\operatorname{dist}(\widetilde{q},q)$) is increased to the maximum allowed value, namely ε. For a similar argument shows that the height function $x \mapsto \operatorname{dist}(x, \mathcal{E})$, when restricted to the great circle through p and perpendicular to pq at p, attains its maximum at $x = p$, and is strictly increasing on the two portions of this geodesic rising from $\mathcal{E}$ to p.

On the other hand, if the interior angles at p and q are right-angles, and $\operatorname{dist}(\widetilde{p}, \mathcal{E}) = \operatorname{dist}(\widetilde{q}, \mathcal{E}) = \varepsilon$, then the geodesic segments realizing $\operatorname{dist}(\widetilde{p}, \mathcal{E})$ and $\operatorname{dist}(\widetilde{q}, \mathcal{E})$ will together form the minimal geodesic from $\widetilde{p}$ to $\widetilde{q}$. This highly symmetric situation can be analyzed using vector geometry. It is immediate from the reduction argument that $\widetilde{\delta} = \operatorname{dist}(\widetilde{p}, \widetilde{q}) < \delta = \operatorname{dist}(p, q)$. So, we can employ Cartesian coordinates such that:

$(1, 0, 0)$ is the point of intersection of $\mathcal{E}$ with the minimal geodesic from p to q;
$p = (\cos(\delta/2), \sin(\delta/2), 0)$ and $q = (\cos(\delta/2), -\sin(\delta/2), 0)$;
$\widetilde{p} = (\cos(\delta/2)\cos\varepsilon, \sin(\delta/2)\cos\varepsilon, \sin\varepsilon)$ and $\widetilde{q} = (\cos(\delta/2)\cos\varepsilon, -\sin(\delta/2)\cos\varepsilon, \sin\varepsilon)$.

Accordingly,

$$\cos\widetilde{\delta} \;=\; \cos\operatorname{dist}(\widetilde{p}, \widetilde{q}) \;=\; \cos^2(\delta/2)\cos^2\varepsilon - \sin^2(\delta/2)\cos^2\varepsilon + \sin^2\varepsilon \;=\; \cos\delta\cos^2\varepsilon + \sin^2\varepsilon\,. \tag{4.3}$$

Set $\eta = \delta - \widetilde{\delta}$ and note that $\widetilde{\delta} \geq \delta - 2\varepsilon$ by the triangle inequality, and so $\eta \leq 2\varepsilon$. Note also that we assumed $\varepsilon < \delta/2 = \frac{1}{2}\operatorname{dist}(p, q)$, and so $\widetilde{\delta} > 0$. Re-arranging (4.3) to read

$$\cos(\delta - \eta) - \cos\delta \quad = \quad (1 - \cos\delta)\sin^2\varepsilon\,, \tag{4.4}$$

and using $2\varepsilon < \delta < \pi$ and $0 < \eta \leq 2\varepsilon$, the left-hand side of (4.4) has partial derivative with respect to η given by

$$\sin(\delta - \eta) \quad \geq \quad \min\{\sin\delta, \sin(\delta - 2\varepsilon)\} \quad > \quad 0\,.$$

Since $\pi > \delta > \delta - 2\varepsilon > 0$, we deduce from (4.1) that

$$(1 - \cos\delta)\sin^2\varepsilon \quad < \quad 2\min\{\sin\delta, \sin(\delta - 2\varepsilon)\}\sin\varepsilon \quad <$$
$$< \quad \min\{\sin\delta, \sin(\delta - 2\varepsilon)\}\cdot 2\varepsilon \quad \leq \quad \int_0^{2\varepsilon} \sin(\delta - \eta)\,\mathrm{d}\,\eta\,.$$

Calculus therefore shows that (4.4) must have a root η in the range $[0, 2\varepsilon)$. Moreover, η must satisfy

$$\eta \cdot \min\{\sin\delta, \sin(\delta - 2\varepsilon)\} \quad \leq \quad (1 - \cos\delta)\sin^2\varepsilon\,,$$

and hence

$$\eta \quad \leq \quad \frac{1 - \cos\delta}{\min\{\sin\delta, \sin(\delta - 2\varepsilon)\}}\sin^2\varepsilon,$$

which implies (4.2) as required. □

Let a *k-times broken geodesic* be a continuous path which is locally geodesic save at k distinct points. Consider now the bilaterally infinite geodesic guaranteed by Proposition 4.2 under a successful evasion strategy. Given any $T > 0$, we can choose a segment of this geodesic of length at least T. By concatenating it with the reverse curve (i.e., the geodesic segment obtained by retracing the path of the original segment), one obtains a closed, twice-broken geodesic of length at least $2T$. The CAT(1) property constrains the constant of contractibility for broken geodesics as follows.

Proposition 4.5. *Let D be an ESIC domain that is CAT(1). Let K be a loop in D that is a k-times broken geodesic. Suppose that K is well-contractible, with contractibility constant c. Then*

$$c \quad \leq \quad \frac{7+11k}{\ell_K}\,. \tag{4.5}$$

Proof. It suffices to show the following: for all sufficiently small $\varepsilon > 0$, any loop $\widetilde{K}$ with

$$\sup_t\{\operatorname{dist}_{\mathbf{I}}(\widetilde{K}(t), K(t))\} \quad < \quad \varepsilon \tag{4.6}$$

must satisfy

$$\ell_K - \ell_{\widetilde{K}} \quad \leq \quad (7+11k)\,\varepsilon + 9\ell_K\varepsilon^2\,. \tag{4.7}$$

Fix some $\varepsilon > 0$ with $\varepsilon < \pi/26$. Partition K into a sequence of $n = \lfloor \ell_K/(7\varepsilon)\rfloor$ segments so that the first $n-1$ of these segments have length exactly 7ε, and so that the n^{th} segment has length in the range $[7\varepsilon, 14\varepsilon)$ and is located so that it contains a "broken point" p of the geodesic that is at distance at least 3ε from each end-point of the segment.

Consider any loop $\widetilde{K}$ satisfying (4.6). For each end point q' of the geodesic segments used to partition K, project q' to the nearest point $\widetilde{q}$ of $\widetilde{K}$, and then project $\widetilde{q}$ back to the nearest point q of K. Using the triangle inequality, it follows that these points q divide K into a sequence of new segments of lengths ℓ_1, ℓ_2, ..., ℓ_n, such that

$$3\varepsilon \quad < \quad \ell_i \quad < \quad 11\varepsilon \qquad \text{for } i = 1, \ldots, n-1\,,$$

and $3\varepsilon < \ell_n < 18\varepsilon$. We associate with each endpoint q of these new segments the point $\widetilde{q}$ on $\widetilde{K}$ that was chosen as above, with the points arranged in corresponding order along $\widetilde{K}$. The construction of q', $\widetilde{q}$ and q is illustrated in Figure 3.

At most k of these segments, including the n^{th} segment, contain broken points of the geodesic in their interior. For a given segment $i = 1, \ldots, n-1$, where the segment is *not* a broken geodesic, this segment, together with the two points on $\widetilde{K}$ corresponding to the segment's end-points, defines a quadrilateral in D. One side of the quadrilateral is a geodesic of length ℓ_i, the two neighboring sides are also geodesics and form obtuse angles or right-angles with the first side (because the points q_i are projections on K of points in $\widetilde{K}$). The fourth side can be replaced by a shorter geodesic of length $\widetilde{\ell}_i$. Using the triangle inequality, the total perimeter length of the quadrilateral corresponding to the segment i, with $i < n$, will therefore be at most $26\varepsilon < \pi$. (Note that the curve formed by the quadrilateral may intersect itself, but will be a rectifiable closed curve.)

Each such quadrilateral is majorized by a convex domain C in S^2, in the sense of Proposition 4.3, with the map from C to the quadrilateral being distance non-expanding and the restriction to the boundary being arc-length preserving. The domain C is therefore a geodesic quadrilateral that

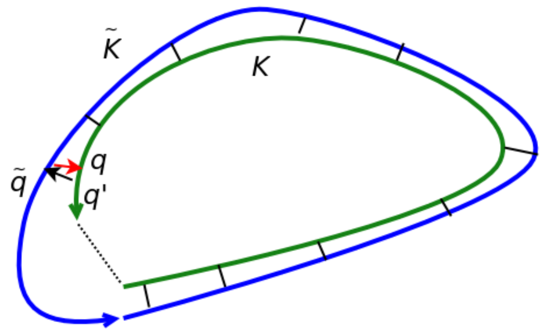

FIGURE 3. A twice-broken geodesic K, together with an approximating loop $\widetilde{K}$. The twice-broken geodesic is divided into a sequence of segments as described in the text.

is *not* self-intersecting; its angles corresponding to the obtuse angles or right-angles of the quadrilateral in D must themselves be obtuse or right-angled. One can also check that the quadrilateral C satisfies the other assumptions in Lemma 4.4. It therefore follows from the lemma that

$$\widetilde{\ell}_i \quad \geq \quad \ell_i - \frac{1-\cos\ell_i}{\min\{\sin\ell_i, \sin(\ell_i - 2\varepsilon)\}} \sin^2\varepsilon\,.$$

Using $3\varepsilon < \ell_i < 11\varepsilon < \pi/2$, it also follows that

$$\widetilde{\ell}_i \quad \geq \quad \ell_i - \frac{1-\cos(11\varepsilon)}{\sin\varepsilon}\sin^2\varepsilon \quad \geq \quad \ell_i - \frac{121\varepsilon^2}{2\sin\varepsilon}\sin^2\varepsilon \quad \geq \quad \ell_i - 61\varepsilon^3\,.$$

This allows us to generate a lower bound on the total length of $\widetilde{K}$. Allowing for the k or fewer segments that contain broken points of the geodesic, this length has lower bound

$$\ell_{\widetilde{K}} \quad \geq \quad \sum_i \ell_i - (18 + 11(k-1))\varepsilon - 61\left(\frac{\ell_K}{7\varepsilon} - k\right)\varepsilon^3 \quad \geq \quad \sum_i \ell_i - (7+11k)\varepsilon - 9\ell_K\varepsilon^2\,.$$

Since $\sum_i \ell_i = \ell_K$, we can rearrange terms to obtain (4.7). Considering arbitrarily small $\varepsilon > 0$, and comparing to the definition of well-contractibility in Definition 2.6(b), we obtain the upper bound on the contractibility constant c given by (4.5). ☐

Propositions 4.2 and 4.5 are the main ingredients in the proof of Theorem 4.6. For the proof, we also note that, by the first variation formula for CAT(1) spaces, the intrinsic distance between the Lion and the Man is non-increasing for simple pursuit (see (Alexander et al., 2010, A.1)).

Theorem 4.6. *Suppose that D is a bounded CL domain. Then there is no successful evasion strategy for the Man if the Lion and Man are initially closer than π and if the Lion conducts a simple pursuit. Moreover, there is a pursuit strategy for the Lion for which there is no successful evasion strategy for the Man, irrespective of the initial positions of the Lion and the Man.*

Proof. The CL domain D is ESIC, and is hence CAT(κ) for some $\kappa > 0$. Rescaling if necessary, we may suppose that D is CAT(1).

We first assume that the Lion and the Man are initially closer than π, and afterwards consider the general case. On account of Proposition 4.2, a successful evasion strategy by the Man in response to the Lion's simple pursuit would result in the construction of a bilaterally infinite local geodesic. By the comment preceding Proposition 4.5, one obtains closed, arbitrarily long twice-broken geodesics in D. But, by Proposition 4.5, a closed twice-broken geodesic of length T will have contractibility constant $c \le \frac{29}{T} \to 0$ as $T \to \infty$, which violates the assumption that D is a CL domain. Consequently, there can be no successful evasion strategy of the Man. In particular, for any $\varepsilon > 0$, the Lion will, at large enough times, remain within this distance of the Man.

In order to extend the above argument to arbitrary initial positions of the Lion and the Man, one can connect these positions by a finite chain of points that are each within distance π of their immediate neighbors. The above argument for simple pursuit by the Lion of the Man can be applied to each pair of neighboring points. Therefore, in each case, the distance between the corresponding pairs of paths will, for large enough times, be within distance ε of one another. The distance between the Lion and the Man, starting from arbitrary initial positions, will therefore eventually be within $n\varepsilon$ of one another, where n is the length of the chain. Since $\varepsilon > 0$ is arbitrary, this completes the proof. ☐

5. Shy couplings

The main result in this section is Theorem 5.5, where we show that CL domains admit no shy couplings. To demonstrate Theorem 5.5, we will relate shy couplings to the deterministic Lion and Man problem, with the Lion adopting the simple pursuit strategy to pursue the Man. We employ a limiting Cameron-Martin-Girsanov transformation to make this comparison, after which we apply the first part of Theorem 4.6 to the corresponding Lion and Man problem.

Let D is a bounded ESIC domain, which is therefore CAT(κ) for some $\kappa > 0$. After rescaling, we can set $\kappa = 1$, and so can assume that D is CAT(1). For any $v, z \in \overline{D}$ with $0 < \text{dist}_{\mathbf{I}}(v, z) < \pi$, there is a unique geodesic between them; we denote by $\chi(v, z)$ the unit tangent vector at v of the geodesic from v to z. (This tangent vector gives the direction of pursuit by the Lion of the Man when the Lion adopts simple pursuit.) Proposition 5.1 states that $\chi(v, z)$ varies continuously in (v, z). It is proved in Bramson et al. (2012, Proposition 12 part (3)).

Proposition 5.1. *Suppose that D is a bounded ESIC domain that is CAT(1) . Then the vector field $\chi(v, z)$ varies continuously in (v, z) on the set $\{(v, z) : \text{dist}_{\mathbf{I}}(v, z) \in (0, \pi)\}$, and hence is uniformly continuous on any compact subset of this region.*

Proposition 5.1 allows us to prove the following useful result about simple pursuit in CAT(1) domains. It states that if, for each pair of initial values $x(0)$ and $y(0)$, the paths x and y eventually become arbitrarily close at certain times, then this occurs uniformly, not depending on $x(0)$ and $y(0)$. Recall that the family Λ_s of simple pursuits is defined in Definition 3.8.

Proposition 5.2. *Suppose that D is a bounded ESIC domain that is CAT(1), and suppose that, for each $(x, y) \in \Lambda_s$ with $\mathrm{dist}_{\mathbf{I}}(x(0), y(0)) \leq \pi/2$, and each $\varepsilon > 0$, there exists some $t < \infty$ such that $\mathrm{dist}_{\mathbf{I}}(x(t), y(t)) \leq \varepsilon$. Then, for each $\varepsilon > 0$, there exists some $t < \infty$ such that, for each $(x, y) \in \Lambda_s$ with $\mathrm{dist}_{\mathbf{I}}(x(0), y(0)) \leq \pi/2$, $\mathrm{dist}_{\mathbf{I}}(x(t), y(t)) \leq \varepsilon$.*

Proof. Assume that, on the contrary, there exists $\varepsilon > 0$ such that, for every integer $n > 0$, there exists $(x_n, y_n) \in \Lambda_s$ such that $\mathrm{dist}_{\mathbf{I}}(x_n(0), y_n(0)) \leq \pi/2$ and $\mathrm{dist}_{\mathbf{I}}(x_n(n), y_n(n)) \geq \varepsilon$. (As observed by Alexander et al., 2010, A.1, the function $t \to \mathrm{dist}_{\mathbf{I}}(x(t), y(t))$ is non-increasing for simple pursuit; we consequently need only consider integer times n.)

Since D is ESIC, we may extend simple pursuit to $\overline{D}$ as well. This allows us to use a variation on the classic Arzela-Ascoli argument. Using the compactness of $\overline{D}$ and passing to a subsequence if necessary, we can assume that $\{x_n(0)\}_{n\geq 1}$ converges to $x_\infty(0)$ and $\{y_n(0)\}_{n\geq 1}$ converges to $y_\infty(0)$; one must have $\mathrm{dist}_{\mathbf{I}}(x_\infty(0), y_\infty(0)) \leq \pi/2$. The functions x_n and y_n are Lipschitz with constant 1 so, for every fixed interval $[0, k]$, there exist subsequences of $\{x_n\}$ and $\{y_n\}$ that converge to admissible functions x_∞ and y_∞ uniformly on $[0, k]$. Using the diagonal method, we can assume that $\{x_n\}$ and $\{y_n\}$ converge to admissible functions x_∞ and y_∞ uniformly on every compact interval. This and $\inf_{0\leq t\leq n} \mathrm{dist}_{\mathbf{I}}(x_n(t), y_n(t)) \geq \varepsilon$ imply that, for every n, $\inf_{0\leq t\leq n} \mathrm{dist}_{\mathbf{I}}(x_\infty(t), y_\infty(t)) \geq \varepsilon$. Hence, $\inf_{0\leq t<\infty} \mathrm{dist}_{\mathbf{I}}(x_\infty(t), y_\infty(t)) \geq \varepsilon$.

It follows from the uniform convergence of x_n to x_∞ and Proposition 5.1 that $\chi(x_n(s), y_n(s)) \to \chi(x_\infty(s), y_\infty(s))$ for every s. Since $x_n(t) - x_n(0) = \int_0^t \chi(x_n(s), y_n(s))ds$ for every $t \geq 0$, by applying bounded convergence as $n \to \infty$, it follows that $x_\infty(t) - x_\infty(0) = \int_0^t \chi(x_\infty(s), y_\infty(s))ds$ for every $t \geq 0$. This shows that $(x_\infty, y_\infty) \in \Lambda_s$. Therefore, by the assumption made in the proposition, $\mathrm{dist}_{\mathbf{I}}(x_\infty(t), y_\infty(t)) \to 0$ as $t \to \infty$. This contradicts our earlier claim and hence completes the proof. □

We next introduce the notion of coupled Brownian motions. As mentioned in the introduction, all probabilistic couplings considered in this paper are assumed to be co-adapted – Brownian motions B and $\widetilde{B}$ are *co-adaptively coupled* if they are defined on the same probability space, are adapted to the same filtration $\{\mathcal{F}_t : t \geq 0\}$ and if, in addition, both have independent increments *with respect to their common filtration*, i.e.,

$$B_{t+s} - B_t \text{ is independent of } \mathcal{F}_t \text{ for all } t, s \geq 0\,, \quad \text{and}$$
$$\widetilde{B}_{t+s} - \widetilde{B}_t \text{ is independent of } \mathcal{F}_t \text{ for all } t, s \geq 0.$$

(The alternative terminology of "jointly immersed" Brownian motions makes explicit use of the theory of co-immersed filtrations of σ-algebras, see Émery, 2005.) Note that $B_{t+s} - B_t$ and $\widetilde{B}_{t+s} - \widetilde{B}_t$ will not in general be independent of each other. Kendall (2009, Lemma 6) gives an explicit proof of the result from the folklore of stochastic calculus that one may represent such a coupling using stochastic integrals, possibly at the cost of augmenting the filtration so as to include a further independent Brownian motion C. Namely, there exist $(d \times d)$-matrix-valued predictable random processes $\mathbb{J}$ and $\mathbb{K}$ such that

$$\widetilde{B} \quad = \quad \int \mathbb{J}^\top \,\mathrm{d}\, B + \int \mathbb{K}^\top \,\mathrm{d}\, C\,, \tag{5.1}$$

with $\mathbb{J}$ and $\mathbb{K}$ satisfying

$$\mathbb{J}^\top \mathbb{J} + \mathbb{K}^\top \mathbb{K} = I_d \tag{5.2}$$

at all times, where I_d is the $(d \times d)$ identity matrix. (Informally one may view $\mathbb{J}$ as the matrix of infinitesimal covariances between the Brownian differentials $\mathrm{d}\, B$ and $\mathrm{d}\, \widetilde{B}$.)

In the context of stochastic calculus, a pair of processes X and $\widetilde{X}$ is said to form a co-adapted coupling if they can be defined by strong solutions of stochastic differential equations driven by B, $\widetilde{B}$ respectively. (There is of course a wider theory of co-adapted coupling applying to general Markov chains and other random processes.) We will employ the stochastic differential equation obtained from the Skorokhod transformation for reflected Brownian motion in an ESIC domain D. Saisho (1987) has shown for ESIC domains that, given a driving Brownian motion B, there exists a unique solution pair $(X, \int \nu_X \,\mathrm{d}\, L^X)$ satisfying

$$\begin{aligned} &\mathrm{d}\, X \quad = \quad \mathrm{d}\, B - \nu_X \,\mathrm{d}\, L^X\,, \\ &L^X \text{ is non-decreasing and increases only when } X \in \partial D\,, \\ &\nu_X \in \mathcal{N}_{X,r}\,. \end{aligned} \tag{5.3}$$

Here, L^X may be viewed as the local time of the reflected Brownian motion X on the boundary ∂D, while ν_X is a unit vector defined only when $X \in \partial D$. (In the case of smooth boundary, ν_X may be taken to be the unit outward-pointing normal vector at $X \in \partial D$; in the more general case with uniform exterior sphere and interior cone conditions, the definitions of L^X and ν_X will be interdependent, but all choices lead to the same process X.) Note that the solutions of (5.3) are pathwise unique, and the process X is strong Markov.

Consider a co-adapted coupling of reflecting Brownian motions X and Y in the bounded ESIC domain $D \subset \mathbb{R}^d$. We can use (5.1) to represent this coupling as

$$\mathrm{d}\, X \quad = \quad \mathrm{d}\, B - \nu_X \,\mathrm{d}\, L^X\,, \tag{5.4}$$

$$\mathrm{d}\, Y \quad = \quad \left(\mathbb{J}^\top \,\mathrm{d}\, B + \mathbb{K}^\top \,\mathrm{d}\, A\right) - \nu_Y \,\mathrm{d}\, L^Y\,, \tag{5.5}$$

where A and B are independent d-dimensional Brownian motions, and $\mathbb{J}$, $\mathbb{K}$ are predictable $(d \times d)$-matrix processes such that (5.2) is satisfied. Here L^X and L^Y may be viewed informally as the local times of X and Y that have accumulated on the boundary. We interpret the Brownian particle X as the Brownian Lion or pursuer, and the other Brownian particle Y as the Brownian Man or evader. It will be convenient for the following work to suppose that the coupling given in (5.4)-(5.5) holds only up to the time $T^* = \inf\{t \geq 0 : X(t) = Y(t)\}$ (the time of "capture"); we define the coupling for all times $t \geq T^*$ by $Y(t) = X(t)$, with X satisfying (5.4) after time T^*.

The main result of this section, Theorem 5.5, is that a bounded CL domain cannot support a shy coupling. Most of the work is carried out in Proposition 5.3, which is then applied in the proof of the theorem. We state both the proposition and the theorem first, and then give their proofs. These results are related to those in Bramson et al. (2012), although the proofs differ in significant details. Theorem 1 of Bramson et al. (2012) only holds for ESIC domains that are CAT(0), whereas Theorem 5.5 here covers the more general CL domains. The latter family includes, for example, star-shaped ESIC domains, and more general ESIC domains that are mentioned in Example 7.4, neither of which need be CAT(0).

Proposition 5.3. *Suppose that the CL domain D is bounded in the Euclidean metric. For any $\varepsilon > 0$, there exists a $t > 0$ such that, for any X and Y satisfying (5.4)-(5.5) with $X(0)$, $Y(0) \in \overline{D}$,*

$$\mathbb{P}\left[\inf_{0\leq s\leq t} \operatorname{dist}_{\mathbf{I}}(X(s), Y(s)) \leq \varepsilon\right] \quad > \quad 0\,. \tag{5.6}$$

Remark 5.4. Our proof of Proposition 5.3 actually yields the following stronger result. For any $\varepsilon > 0$ and all $0 < t_1 < t_2 < \infty$,

$$\mathbb{P}\left[\operatorname{dist}_{\mathbf{I}}(X(s), Y(s)) \leq \varepsilon \text{ whenever } t_1 \leq s \leq t_2\right] \quad > \quad 0\,.$$

The version (5.6) suffices for Theorem 5.5 and is needed for Theorem 6.1.

Theorem 5.5. *Suppose that the CL domain D is bounded. Then, there is no shy co-adapted coupling for reflected Brownian motion in D.*

Proof of Proposition 5.3. Since D is CL and is hence ESIC, it can be scaled so that it is CAT(1). We first demonstrate (5.6) when $\operatorname{dist}_{\mathbf{I}}(X(0), Y(0)) \leq \pi/2$, which will provide motivation for the general case.

The first step is to alter the stochastic dynamics of the coupled Brownian motions X and Y given in (5.4)-(5.5) by adding a large drift. The new equations are given in (5.7)-(5.8); the drift there for the X^n component is given by n times the unit tangent vector field χ introduced before Proposition 5.1, and the drift of Y^n is given by adding the corresponding large drift governed by the product of the coupling matrix $\mathbb{J}^\top$ with χ. Setting $T^{*,n} = \inf\{t \geq 0 : X^n(t) = Y^n(t)\}$, for $t < T^{*,n}$, one has

$$X^n(t) \quad = \quad X(0) + B(t) + \int_0^t n\chi(X^n(s), Y^n(s))\,\mathrm{d}\,s - \int_0^t \nu_{X^n(s)}\,\mathrm{d}\,L_s^{X^n}\,, \tag{5.7}$$

$$\begin{aligned} Y^n(t) \quad &= \quad Y(0) + \int_0^t \left(\mathbb{J}_s^\top\,\mathrm{d}\,B(s) + \mathbb{K}_s^\top\,\mathrm{d}\,A(s)\right) \\ &\qquad + \int_0^t n\mathbb{J}_s^\top \chi(X^n(s), Y^n(s))\,\mathrm{d}\,s - \int_0^t \nu_{Y^n(s)}\,\mathrm{d}\,L_s^{Y^n}\,. \end{aligned} \tag{5.8}$$

As after (5.4)-(5.5), for $t > T^{*,n}$, we set $Y^n(t) = X^n(t)$ and let $X^n(t)$ evolve as the ordinary reflected Brownian motion after $T^{*,n}$. (Note that $\chi(v, z)$ is not defined for $v = z$.) We also set

$T^n = \inf\{t \geq 0 : \operatorname{dist}_{\mathbf{I}}(X^n(t), Y^n(t)) \geq 3\pi/4\}$. Since $\chi(v,z)$ is not necessarily uniquely defined if $\operatorname{dist}_{\mathbf{I}}(v,z) \geq \pi$, we will need to analyze the stopped processes $X^n(t \wedge T^n)$ and $Y^n(t \wedge T^n)$.

By the Cameron-Martin-Girsanov theorem, the distributions of the solutions of (5.4)-(5.5) and (5.7)-(5.8) are mutually absolutely continuous on every interval $[0, T^n \wedge T^{*,n} \wedge k]$, for $k < \infty$. On the other hand, as we will show, after rescaling time and taking n to be very large, the paths of $(X^n(\cdot), Y^n(\cdot))$ can be viewed as being uniformly close to those for the corresponding Lion and Man problem. Since D is assumed to be a CL domain, this will allow us to apply Theorem 4.6 to establish (5.6).

We rescale time by making the substitutions $X^n(t) = \widetilde{X}^n(nt)$, $Y^n(t) = \widetilde{Y}^n(nt)$, $B(t) = \widetilde{B}^n(nt)/\sqrt{n}$, $A(t) = \widetilde{A}^n(nt)/\sqrt{n}$, $\mathbb{J}(t) = \widetilde{\mathbb{J}}^{(n)}(nt)$, $\mathbb{K}(t) = \widetilde{\mathbb{K}}^{(n)}(nt)$. Then (5.7)-(5.8) take the form

$$\widetilde{X}^n(t) = X(0) + \frac{1}{\sqrt{n}}\widetilde{B}^n(t) + \int_0^t \chi(\widetilde{X}^n(s), \widetilde{Y}^n(s))\,\mathrm{d}\,s - \int_0^t \nu_{\widetilde{X}^n(s)}\,\mathrm{d}\,L_s^{\widetilde{X}^n}\,, \tag{5.9}$$

$$\begin{aligned}\widetilde{Y}^n(t) = {} & Y(0) + \frac{1}{\sqrt{n}}\int_0^t \Big((\widetilde{\mathbb{J}}_s^{(n)})^\top\,\mathrm{d}\,\widetilde{B}^n(s) + (\widetilde{\mathbb{K}}_s^{(n)})^\top\,\mathrm{d}\,\widetilde{A}^n(s)\Big) \\ & + \int_0^t (\widetilde{\mathbb{J}}_s^{(n)})^\top \chi(\widetilde{X}^n(s), \widetilde{Y}^n(s))\,\mathrm{d}\,s - \int_0^t \nu_{\widetilde{Y}^n(s)}\,\mathrm{d}\,L^{\widetilde{Y}_s^n}\,.\end{aligned} \tag{5.10}$$

As before, for $t > \widetilde{T}^{*,n} = \inf\{t \geq 0 : \widetilde{X}^n(t) = \widetilde{Y}^n(t)\}$, we set $\widetilde{Y}^n(t) = \widetilde{X}^n(t)$ and let $\widetilde{X}^n(t)$ evolve as ordinary reflected Brownian motion after $\widetilde{T}^{*,n}$. Note that $\widetilde{B}^n$ and $\widetilde{A}^n$ are standard Brownian motions. Corresponding to the previous definition of T^n, we define stopping times $\widetilde{T}^n = \inf\{t \geq 0 : \operatorname{dist}_{\mathbf{I}}(\widetilde{X}^n(t), \widetilde{Y}^n(t)) \geq 3\pi/4\}$ and analyze the stopped processes $\widetilde{X}^n(t \wedge \widetilde{T}^n)$ and $\widetilde{Y}^n(t \wedge \widetilde{T}^n)$.

Now, consider the analog of (5.9)-(5.10), but without boundary:

$$\widetilde{U}^n(t) = \frac{1}{\sqrt{n}}\widetilde{B}^n(t) + \int_0^t \chi(\widetilde{X}^n(s), \widetilde{Y}^n(s))\,\mathrm{d}\,s\,, \tag{5.11}$$

$$\widetilde{V}^n(t) = \frac{1}{\sqrt{n}}\int_0^t \Big((\widetilde{\mathbb{J}}_s^{(n)})^\top\,\mathrm{d}\,\widetilde{B}^n(s) + (\widetilde{\mathbb{K}}_s^{(n)})^\top\,\mathrm{d}\,\widetilde{A}^n(s)\Big) + \int_0^t (\widetilde{\mathbb{J}}_s^{(n)})^\top \chi(\widetilde{X}^n(s), \widetilde{Y}^n(s))\,\mathrm{d}\,s\,. \tag{5.12}$$

The criterion of Stroock and Varadhan (1979, §1.4) establishes tightness of the sextuplet

(5.13)

$$\begin{aligned}\mathbf{H}^n(t) = \Big(&\widetilde{U}^n(t), \frac{1}{\sqrt{n}}\widetilde{B}^n(t), \int_0^t \chi(\widetilde{X}^n(s), \widetilde{Y}^n(s))\,\mathrm{d}\,s\,, \\ &\widetilde{V}^n(t), \frac{1}{\sqrt{n}}\int_0^t \Big((\widetilde{\mathbb{J}}_s^{(n)})^\top\,\mathrm{d}\,\widetilde{B}^n(s) + (\widetilde{\mathbb{K}}_s^{(n)})^\top\,\mathrm{d}\,\widetilde{A}^n(s)\Big), \int_0^t (\widetilde{\mathbb{J}}_s^{(n)})^\top \chi(\widetilde{X}^n(s), \widetilde{Y}^n(s))\,\mathrm{d}\,s\Big)\,,\end{aligned}$$

since the diffusion coefficients and the drifts are bounded by 1. So, there exists an appropriate subsequence of $\mathbf{H}^n$ that converges weakly (in the uniform metric) to a limiting process $\mathbf{H}^\infty$. In a harmless abuse of notation, we re-index, denoting this subsequence by $\{\mathbf{H}^n : n \geq 1\}$. In particular, $\widetilde{U}^n(t)$ and $\widetilde{V}^n(t)$ converge weakly so, by Saisho (1987, Thm. 4.1), $(\widetilde{X}^n, \widetilde{Y}^n)$ converges weakly to a limiting continuous process $(\widetilde{X}^\infty, \widetilde{Y}^\infty)$ along the same subsequence. It follows that the octuplet

(5.14)
$$\mathbf{K}^n(t) = \Big(\widetilde{X}^n(t), \widetilde{Y}^n(t), \widetilde{U}^n(t), \frac{1}{\sqrt{n}}\widetilde{B}^n(t), \int_0^t \chi(\widetilde{X}^n(s), \widetilde{Y}^n(s))\,\mathrm{d}\,s\,,$$
$$\widetilde{V}^n(t), \frac{1}{\sqrt{n}}\int_0^t \Big((\widetilde{\mathbb{J}}_s^{(n)})^\top\,\mathrm{d}\,\widetilde{B}^n(s) + (\widetilde{\mathbb{K}}_s^{(n)})^\top\,\mathrm{d}\,\widetilde{A}^n(s)\Big)\,, \int_0^t (\widetilde{\mathbb{J}}_s^{(n)})^\top \chi(\widetilde{X}^n(s), \widetilde{Y}^n(s))\,\mathrm{d}\,s\Big)$$

is tight, and therefore converges weakly along a further subsequence. Once again, we commit a harmless abuse of notation and re-index, denoting the weakly converging subsequence by $\{\mathbf{K}^n : n \geq 1\}$. We now employ the Skorokhod representation of weak convergence to construct the sequence of $\mathbf{K}^n$ on the same probability space so that it converges almost surely, uniformly on compact intervals.

The fourth and seventh components of $\mathbf{K}^n$ are Brownian motions run at rate $\frac{1}{n}$, so they each converge to the zero process as $n \to \infty$. The fifth and eighth components of $\mathbf{K}^n$ are both Lip(1); their limits are therefore also Lip(1). These observations and (5.11)-(5.12) imply that the limits $\widetilde{V}^\infty$ and $\widetilde{U}^\infty$ of $\widetilde{V}^n$ and $\widetilde{U}^n$ are also Lip(1).

Let $\widetilde{T}^\infty = \liminf_{n\to\infty} \widetilde{T}^n$ and $\widetilde{T}^* = \inf\{t \geq 0 : \widetilde{X}^\infty(t) = \widetilde{Y}^\infty(t)\}$, and note that $\widetilde{T}^* \leq \liminf_{n\to\infty} \widetilde{T}^{*,n}$. We will argue that

$$\widetilde{U}^\infty(t) = \int_0^t \chi(\widetilde{X}^\infty(s), \widetilde{Y}^\infty(s))\,\mathrm{d}\,s \qquad \text{for } t < \widetilde{T}^\infty \wedge \widetilde{T}^*, \tag{5.15}$$

$$\widetilde{T}^\infty = \infty \quad \text{a.s.}, \tag{5.16}$$

$$\widetilde{X}^\infty(t) = \widetilde{Y}^\infty(t) \text{ for } t \geq \widetilde{T}^*. \tag{5.17}$$

The bounded vector field $\chi(\widetilde{X}^n(t), \widetilde{Y}^n(t))$ depends continuously on $\widetilde{X}^n(t)$ and $\widetilde{Y}^n(t)$, over $[0, \widetilde{T}^n \wedge \widetilde{T}^{*,n})$, by Proposition 5.1. Hence, by the bounded convergence theorem, we can pass to the limit in (5.11) on the interval $[0, \widetilde{T}^\infty \wedge \widetilde{T}^*)$, with the limit satisfying (5.15) on $[0, \widetilde{T}^\infty \wedge \widetilde{T}^*)$. By Proposition 4.1, for given $X(0)$ and $Y(0)$ with $\mathrm{dist}_{\mathbf{I}}(X(0), Y(0)) < \pi$ and $\{\widetilde{Y}^\infty(t), t \geq 0\}$, (5.15) defines a unique function $\{\widetilde{U}^\infty(t), t \in [0, \widetilde{T}^\infty \wedge \widetilde{T}^*)\} = \{\widetilde{X}^\infty(t), t \in [0, \widetilde{T}^\infty \wedge \widetilde{T}^*)\}$.

Since (5.15) holds for $t \in [0, \widetilde{T}^\infty \wedge \widetilde{T}^*)$, $\widetilde{X}^\infty$ conducts a simple pursuit of $\widetilde{Y}^\infty$ over this time period. As noted above Theorem 4.6, it follows that $t \to \mathrm{dist}_{\mathbf{I}}(\widetilde{X}^\infty(t), \widetilde{Y}^\infty(t))$ is non-increasing on this interval. Consequently, $\sup_{0\leq t\leq \widetilde{T}^\infty\wedge\widetilde{T}^*} \mathrm{dist}_{\mathbf{I}}(\widetilde{X}^\infty(t), \widetilde{Y}^\infty(t)) \leq \mathrm{dist}_{\mathbf{I}}(\widetilde{X}^\infty(0), \widetilde{Y}^\infty(0)) \leq \pi/2$. Since $(\widetilde{X}^n, \widetilde{Y}^n)$ converges a.s. to $(\widetilde{X}^\infty, \widetilde{Y}^\infty)$ uniformly on compact intervals, we conclude that, for large n, $\sup_{0\leq t\leq \widetilde{T}^\infty\wedge\widetilde{T}^*} \mathrm{dist}_{\mathbf{I}}(\widetilde{X}^n(t), \widetilde{Y}^n(t)) \leq 5\pi/8 < 3\pi/4$. This contradicts the definition of $\widetilde{T}^\infty$ unless either $\widetilde{T}^\infty = \infty$ or $\widetilde{T}^* < \widetilde{T}^\infty$.

Suppose that $\widetilde{T}^* < \widetilde{T}^\infty$ and $\widetilde{X}^\infty(t) \neq \widetilde{Y}^\infty(t)$ for some $t \in (\widetilde{T}^*, \widetilde{T}^\infty)$. Since the processes $\widetilde{X}^\infty(t)$ and $\widetilde{Y}^\infty(t)$ are continuous, this implies that there exist $\widetilde{T}^* < t_1 < t_2 < \widetilde{T}^\infty$ such that $0 < \operatorname{dist}_{\mathbf{I}}(\widetilde{X}^\infty(t_1), \widetilde{Y}^\infty(t_1)) < \operatorname{dist}_{\mathbf{I}}(\widetilde{X}^\infty(t_2), \widetilde{Y}^\infty(t_2))$ and $\inf_{t_1 \le t \le t_2} \operatorname{dist}_{\mathbf{I}}(\widetilde{X}^\infty(t), \widetilde{Y}^\infty(t)) > 0$. Therefore, $t_2 < \liminf T^{*,n}$. Arguing in the same way as for (5.15), it follows that $\widetilde{X}^\infty$ conducts a simple pursuit of $\widetilde{Y}^\infty$ over the interval $[t_1, t_2]$, and therefore $t \to \operatorname{dist}_{\mathbf{I}}(\widetilde{X}^\infty(t), \widetilde{Y}^\infty(t))$ is non-increasing on this interval. This is a contradiction, so we conclude that $\widetilde{X}^\infty(t) = \widetilde{Y}^\infty(t)$ for $t \in (\widetilde{T}^*, \widetilde{T}^\infty)$. This completes the proof of (5.15)-(5.17) and shows that $\widetilde{X}^\infty$ conducts a simple pursuit of $\widetilde{Y}^\infty$ over the interval $[0, \widetilde{T}^*)$.

Fix an arbitrarily small $\varepsilon > 0$. Since the CL domain D is CAT(1) and $\operatorname{dist}_{\mathbf{I}}(\widetilde{X}^\infty(0), \widetilde{Y}^\infty(0)) \le \pi/2$, it follows from Theorem 4.6 that it is impossible for $\widetilde{Y}^\infty$ to successfully evade $\widetilde{X}^\infty$ over the time interval $[0, \infty)$. Moreover, since this holds for all such $\widetilde{X}^\infty(0)$ and $\widetilde{Y}^\infty(0)$, application of Proposition 5.2 implies that there exists $t_1 < \infty$, not depending on either $\widetilde{X}^\infty(0), \widetilde{Y}^\infty(0)$, ω or the particular simple pursuit in Λ_s of the pair $(\widetilde{X}^\infty, \widetilde{Y}^\infty)$, such that, for $t \ge t_1$,

$$\operatorname{dist}_{\mathbf{I}}(\widetilde{X}^\infty(t), \widetilde{Y}^\infty(t)) \le \varepsilon/2. \tag{5.18}$$

Because of the uniform convergence of $(\widetilde{X}^n, \widetilde{Y}^n)$ to $(\widetilde{X}^\infty, \widetilde{Y}^\infty)$ over finite intervals, it follows from (5.18) that, for some $n_0 < \infty$ depending on $X(0)$ and $Y(0)$, and all $n \ge n_0$,

$$\mathbb{P}\left[\operatorname{dist}_{\mathbf{I}}(\widetilde{X}^n(t_1), \widetilde{Y}^n(t_1)) \le \varepsilon\right] \quad > \quad 0\,.$$

Changing the clock back to the original pace, we obtain

$$\mathbb{P}\left[\operatorname{dist}_{\mathbf{I}}(X^n(t_1/n), Y^n(t_1/n)) \le \varepsilon\right] \quad > \quad 0\,.$$

By the Cameron-Martin-Girsanov theorem,

$$\mathbb{P}\left[\operatorname{dist}_{\mathbf{I}}(X(t_1/n), Y(t_1/n)) \le \varepsilon\right] \quad > \quad 0\,. \tag{5.19}$$

This implies (5.6) when $\operatorname{dist}_{\mathbf{I}}(X(0), Y(0)) \le \pi/2$.

We now consider (5.6) for $X(0) = x_0$, $Y(0) = y_0$, and arbitrary $x_0, y_0 \in \overline{D}$. The reasoning is similiar to the case where $\operatorname{dist}_{\mathbf{I}}(X(0), Y(0)) \le \pi/2$, after constructing a chain of points, each of which is within distance $\pi/2$ of its immediate neighbors.

Choose a sequence of points $z_1, z_2, \ldots, z_m \in \overline{D}$ such that $z_1 = x_0$, $z_m = y_0$ and $\operatorname{dist}_{\mathbf{I}}(z_k, z_{k+1}) \le \pi/2$ for all $1 \le k \le m-1$. For $n \ge 1$ and $k = 1, \ldots, m$, we define the chain of random processes $Z^{k,n}$ in D, with $Z^{k,n}(0) = z_k$. We set $Z^{1,n} \equiv X^n$ and $Z^{m,n} \equiv Y^n$, but with the drift $n\chi(X^n(s), Y^n(s))$ replaced by $n\chi(X^n(s), Z^{2,n}(s)) = n\chi(Z^{1,n}(s), Z^{2,n}(s))$ for both processes. For $k = 2, \ldots, m-1$, $Z^{k,n}$ denotes the process that conducts a simple pursuit directed toward $Z^{k+1,n}$, but carried out at rate n. The corresponding stochastic system is given by

$$Z^{1,n}(t) = X(0)+B(t)+\int_0^t n\chi(Z^{1,n}(s),Z^{2,n}(s))\,\mathrm{d}\,s-\int_0^t \nu_{Z^{1,n}(s)}\,\mathrm{d}\,L_s^{Z^{1,n}}\,, \tag{5.20}$$

$$\ldots$$

$$Z^{k,n}(t) = Z^{k.n}(0)+\int_0^t n\chi(Z^{k,n}(s),Z^{k+1,n}(s))\,\mathrm{d}\,s\,, \tag{5.21}$$

$$\ldots$$

$$\begin{aligned}Z^{m,n}(t) &= Y(0)+\int_0^t \left(\mathbb{J}_s^\top\,\mathrm{d}\,B(s)+\mathbb{K}_s^\top\,\mathrm{d}\,A(s)\right)\\ &\quad+\int_0^t n\mathbb{J}_s^\top\chi(Z^{1,n}(s),Z^{2,n}(s))\,\mathrm{d}\,s-\int_0^t \nu_{Z^{m,n}(s)}\,\mathrm{d}\,L_s^{Z^{m,n}}\,.\end{aligned} \tag{5.22}$$

Since $Z^{2,n}$, …, $Z^{m-1,n}$ are simple pursuits run at rate n and directed toward adapted processes, they are Lipschitz(n) adapted random processes. Also, for $k=2,\ldots,m-2$, $\mathrm{dist}_{\mathbf{I}}(Z^{k,n}(t),Z^{k+1,n}(t))$ is non-increasing in time. (No reflection term is required in (5.21) since $Z^{k,n}$, for $k=2,\ldots,m-1$, will never attempt to cross the boundary.)

The system (5.20)-(5.22) is run up until the time

$$S^n = \inf\{t\geq 0:\mathrm{dist}_{\mathbf{I}}(\,Z^{1,n}(t),Z^{2,n}(t))\geq 3\pi/4\}\ \wedge\ \inf\{t\geq 0:\mathrm{dist}_{\mathbf{I}}(Z^{m-1,n}(t),Z^{m,n}(t))\geq 3\pi/4\}\,.$$

For $k=2,\ldots,m-2$, we set $Z^{k,n}(t)=Z^{k+1,n}(t)$ when $t\geq\inf\{s:Z^{k,n}(s)=Z^{k+1,n}(s)\}$. We adopt the convention that $\chi(Z^{1,n}(t),Z^{2,n}(t))=0$ when $Z^{1,n}(t)=Z^{2,n}(t)$ and $\chi(Z^{m-1,n}(t),Z^{m,n}(t))=0$ when $Z^{m-1,n}(t)=Z^{m,n}(t)$. (Almost surely, the set of times t at which either of the latter two equalities occurs has measure zero since, in either case, one process is a Brownian motion with drift and the other is a Lipschitz process.)

Rescaling time as in (5.9)-(5.10), we obtain the system

$$\widetilde{Z}^{1,n}(t) = X(0)+\frac{1}{\sqrt{n}}\widetilde{B}^n(t)+\int_0^t \chi(\widetilde{Z}^{1,n}(s),\widetilde{Z}^{2,n}(s))\,\mathrm{d}\,s-\int_0^t \nu_{\widetilde{Z}^{1,n}(s)}\,\mathrm{d}\,L_s^{\widetilde{Z}^{1,n}}\,, \tag{5.23}$$

$$\ldots$$

$$\widetilde{Z}^{k,n}(t) = Z^{k.n}(0)+\int_0^t \chi(\widetilde{Z}^{k,n}(s),\widetilde{Z}^{k+1,n}(s))\,\mathrm{d}\,s\,, \tag{5.24}$$

$$\ldots$$

$$\begin{aligned}\widetilde{Z}^{m,n}(t) &= Y(0)+\frac{1}{\sqrt{n}}\int_0^t \left(\widetilde{\mathbb{J}}_s^\top\,\mathrm{d}\,\widetilde{B}^n(s)+\widetilde{\mathbb{K}}_s^\top\,\mathrm{d}\,\widetilde{A}^n(s)\right)\\ &\quad+\int_0^t \widetilde{\mathbb{J}}_s^\top\chi(\widetilde{Z}^{1,n}(s),\widetilde{Z}^{2,n}(s))\,\mathrm{d}\,s-\int_0^t \nu_{\widetilde{Z}^{m,n}(s)}\,\mathrm{d}\,L_s^{\widetilde{Z}^{m,n}}\,,\end{aligned} \tag{5.25}$$

which is run up until time

$$\widetilde{S}^n = \inf\{t \geq 0 : \mathrm{dist}_{\mathbf{I}}(\widetilde{Z}^{1,n}(t), \widetilde{Z}^{2,n}(t)) \geq 3\pi/4\} \wedge \inf\{t \geq 0 : \mathrm{dist}_{\mathbf{I}}(\widetilde{Z}^{m-1,n}(t), \widetilde{Z}^{m,n}(t)) \geq 3\pi/4\},$$

and which follows the conventions noted earlier when two processes coincide.

We now argue as in the case where $\mathrm{dist}_{\mathbf{I}}(X(0), Y(0)) \leq \pi/2$, letting $n \to \infty$ through a subsequence so that the system of solutions to (5.23)-(5.25) converges weakly to a chain of simple pursuits $\widetilde{Z}^{1,\infty}, \ldots, \widetilde{Z}^{m,\infty}$ commencing at $z_1, \ldots, z_m$. For $k = 1$, with $\widetilde{Z}^{1,n}$ and $\widetilde{Z}^{2,n}$, the reasoning is almost the same as before; although the process $\widetilde{Z}^{2,n}$ is different than $\widetilde{Y}^n$, in both cases their drifts are at most 1, and as $n \to \infty$, both result in a simple pursuit. The steps $k = 2, \ldots, m-1$ are easier to see since, for each n, $\widetilde{Z}^{k,n}$ already conducts a (random) simple pursuit of $\widetilde{Z}^{k+1,n}$. For step $m-1$, $\widetilde{Z}^{m-1,n}$ has drift 1 and the drift of $\widetilde{Z}^{m,n}$ is at most 1 and so, as $n \to \infty$, one again obtains a simple pursuit.

As before, $\widetilde{S}^n \to \infty$. Fixing $\varepsilon > 0$, it follows from Theorem 4.6 and Proposition 5.2, as before, that there exists $t_1 < \infty$, not depending on $X(0), Y(0)$, ω or the particular limiting simple pursuit in Λ_s, such that, for all $t \geq t_1$ and $k = 1, \ldots, m-1$,

$$\mathrm{dist}_{\mathbf{I}}(\widetilde{Z}^{k,\infty}(t), \widetilde{Z}^{k+1,\infty}(t)) \leq \varepsilon/(2(m-1)).$$

It therefore follows that, for some $t_1 > 0$, n_0, and all $n > n_0$,

$$\mathbb{P}\left[\mathrm{dist}_{\mathbf{I}}(Z^{1,n}(t_1/n), Z^{m,n}(t_1/n)) \leq \varepsilon\right] = \mathbb{P}\left[\mathrm{dist}_{\mathbf{I}}(\widetilde{Z}^{1,n}(t_1), \widetilde{Z}^{m,n}(t_1)) \leq \varepsilon\right] \geq$$
$$\mathbb{P}\left[\mathrm{dist}_{\mathbf{I}}(\widetilde{Z}^{k,n}(t_1), \widetilde{Z}^{k+1,n}(t_1)) \leq \varepsilon/(m-1) \text{ for } k = 1, \ldots, m-1\right] > 0.$$

Consequently, by the Cameron-Martin-Girsanov theorem,

$$\mathbb{P}\left[\mathrm{dist}_{\mathbf{I}}(X(t_1/n), Y(t_1/n)) \leq \varepsilon\right] > 0. \tag{5.26}$$

This implies (5.6) for $X(0) = x_0$, $Y(0) = y_0$, and arbitrary $x_0, y_0 \in \overline{D}$. □

Theorem 5.5 states that shyness fails for CL domains. The proof requires establishing a uniform lower bound on the probability that shyness fails, over the different possible starting positions of X and Y. For this, we employ Bramson et al. (2012, Proposition 20), which states the following. Consider a (bounded) ESIC domain. Suppose that $\mathbb{P}\left[\inf_{0 \leq t \leq t_1} |X(t) - Y(t)| \leq \varepsilon\right] > 0$ for some $\varepsilon > 0$ and $t_1 > 0$, for any coupled pair of Brownian motions X and Y with arbitrary starting points $X(0), Y(0) \in D$. Then

$$\mathbb{P}\left[\mathrm{dist}_{\mathbf{I}}(X(t), Y(t)) \leq \varepsilon \text{ for some } t \text{ with } 0 \leq t \leq t_1\right] \geq p_1, \tag{5.27}$$

for some t_1 and $p_1 > 0$ not depending on $X(0)$ and $Y(0)$. The proof is based on an Arzela-Ascoli argument exploiting tightness of the processes and compactness of $\overline{D}$.

Proof of Theorem 5.5. By Proposition 5.3 and Bramson et al. (2012, Proposition 20), (5.27) holds true. The remainder of the argument consists of an elementary iteration argument. Consider processes X and Y starting from any pair of points in $\overline{D}$ and corresponding to any choice of $\mathbb{J}$ and

$\mathbb{K}$. Because of the uniform bound in (5.27), the probability of X and Y not coming within distance ε of each other on the interval $[kt_1,(k+1)t_1]$, conditional on not coming within this distance before kt_1, is bounded above by $1-p_1$ for any k, by the Markov property. Hence, the probability of X and Y not coming within distance ε of each other on the interval $[0,kt_1]$ is bounded above by $(1-p_1)^k$. Letting $k\to\infty$, it follows that X and Y are not ε-shy. Since ε can be taken arbitrarily small, the proof is complete. □

6. Domains with a stable rubber band, but no shy coupling

In this section, we exhibit a family of domains possessing stable rubber bands, but nevertheless supporting no shy couplings. Since these domains are not CL domains, these examples complement Theorem 5.5. The family of domains is constructed by appending to a domain possessing a stable rubber band another (typically much larger) domain, so that the combined domain has the same stable rubber band but supports no shy coupling. The precise result is stated in Theorem 6.1, which is the main result of the section.

For each of our examples, we consider a bounded ESIC domain $D_1\subset\mathbb{R}^d$, $d\geq 2$, that possesses a stable rubber band. The larger domain is produced by appending a long thin cuboid to D_1. Some care needs to be taken to ensure that the resulting domain still satisfies the uniform exterior sphere and uniform interior cone conditions, which requires us to impose some conditions on the boundary of D_1.

Rather than attempting to provide a more general result, for the sake of simplicity, we suppose that there is a point p on the boundary ∂D_1 such that, for some $r>0$, $\partial D_1\cap\mathcal{B}(p,r)$ is the graph of a C^1-function (in an appropriate orthonormal coordinate system), and that D_1 lies totally on one side of the hyperplane that is tangent to ∂D_1 at p; we further suppose that the distance from p to the stable rubber band is at least $2r$. Translating and rotating the domain as necessary, we may suppose that the point p on the boundary is given by $(0,\ldots,0,-a)$ for some $a\in(0,r/\sqrt{d})$, and that the supporting hyperplane is $\{x: x_d=-a\}$, with the open set D_1 lying below this hyperplane. We assume that a is small enough so that

$$\partial D_1\cap\big((-a,a)\times\ldots\times(-a,a)\times(-2a,0)\big)\subset(-a,a)\times\ldots\times(-a,a)\times(-3a/2,0).$$

It is elementary to see that, for arbitrarily large L, there exists a domain D such that

(i) $D\cap(\{x:x_d<-a\}\setminus\mathcal{B}(p,r))=D_1\cap(\{x:x_d<-a\}\setminus\mathcal{B}(p,r))$,
(ii) $\{x\in D: x_d>0\}=(-a,a)\times\ldots\times(-a,a)\times(0,L)$,
(iii) $(-a,a)\times\ldots\times(-a,a)\times(-2a,L)\subset D$,
(iv) D satisfies both the uniform exterior sphere and uniform interior cone conditions.

It follows from the uniform exterior sphere and uniform interior cone conditions that reflected Brownian motion on D is strong Markov, with normalized Lebesgue measure as its equilibrium probability measure (see, e.g., Burdzy and Chen, 1998).

Heuristically speaking, D is created by attaching a long thin cuboid $(-a,a)\times\ldots\times(-a,a)\times(-2a,L)$ to D_1 and smoothing the boundary so that the sharp edges are only pointing outside the domain. Note that L can be increased arbitrarily without altering the construction close to D_1. Re-scaling the domain if necessary, we may suppose that $a = 1/2$, and therefore that the intersection of D with $\{x : x_d > 0\}$ is $D_2 = (-1/2, 1/2) \times \ldots \times (-1/2, 1/2) \times (0, L)$. We will assume that

$$L \quad > \quad 512d^2 \tag{6.1}$$

and that

$$|\{x \in D : x_d < L/16\}| \quad < \quad |D|/8\,. \tag{6.2}$$

We now state the main result of the section.

Theorem 6.1. *Suppose the domain D is defined as above, by enlarging a given ESIC domain D_1 by appending a long cuboid. This new domain D supports no shy co-adapted coupling for reflected Brownian motion.*

Before going into details, we describe the general plan for the proof of Theorem 6.1. Consider a coupling of two reflecting Brownian motions X and Y in D. For sufficently large t, the two processes $X(t)$ and $Y(t)$, when viewed separately, will be approximately in statistical equilibrium, and hence their marginal distributions will each approximate the normalized volume measure. As a consequence of inequality (6.2), it will follow (see Lemma 6.2) that there is a positive probability of both $X(t)$ and $Y(t)$ lying in the part of the cuboid $(-1/2, 1/2) \times \ldots \times (-1/2, 1/2) \times (L/16, L)$.

Next consider the Lion and Man pursuit problem in the long cuboid D_2. For each coordinate $i \leq d-1$, we will produce a pursuit strategy given by a continuous vector field under which the Lion tracks the Man closely in the coordinates $1, \ldots, i-1$, while approaching the Man in coordinate i. This can moreover be done without the Man being able to move very much in the dth coordinate and, in particular, before either the Lion or the Man leaves D_2 (see Lemma 6.3).

A similar strategy (see Lemma 6.4), but with respect to the coordinate $i = d$, results in the Lion approaching the Man in the dth coordinate while tracking the Man closely in the other $d-1$ coordinates, and before either the Lion or the Man leaves D_2.

Employing this pursuit by the Lion of the Man, we will then argue, as in Section 5, that shyness must fail for the Brownian problem.

We now state and prove the three lemmas, in preparation of the proof of Theorem 6.1.

Lemma 6.2. *For large enough u_0, all $(x, y) \in \overline{D}$, and any reflected Brownian motions X and Y on D defined on the same probability space, with $X(0) = x$ and $Y(0) = y$,*

$$\mathbb{P}\left[X_d(u_0) \geq L/16 \text{ and } Y_d(u_0) \geq L/16\right] \geq 1/2\,. \tag{6.3}$$

Proof. As $t \to \infty$, the distributions of $X(t)$ and $Y(t)$ separately converge weakly to the equilibrium measure on $\overline{D}$ of reflecting Brownian motion, which is normalized volume measure. In fact (see Bañuelos and Burdzy, 1999, (2.2)), for given $\varepsilon \in (0, 1]$, there exists u_0 such that, for all $x \in \overline{D}$ and any reflected Brownian motion X on D with $X(0) = x$, the density of the distribution of X

at time u_0 is at most $(1+\varepsilon)/|D| \leq 2/|D|$. The same remark applies to Y and so, in view of (6.2),

$$\mathbb{P}\left[X_d(u_0) < L/16 \text{ or } Y_d(u_0) < L/16\right] \leq 2(|D|/8)(2/|D|) = 1/2.$$

The result follows by taking complements. □

We now describe the pursuit strategies corresponding to each choice of coordinate $i \leq d$ by specifying continuous vector fields $\chi^{(i)}(x,y)$ for the velocity of the Lion, where x and y are the locations of the Lion and of the Man. We allow the Man to choose any evasion strategy as long as his speed satisfies $|y'(t)| \leq 1$ for all t.

We fix $\delta \in (0,1)$, on which $\chi^{(i)}(x,y)$ will depend implicitly; in the proof of Theorem 6.1, we will let $\delta \searrow 0$. For $i = 1,2,\ldots,d$, let Π_i be the orthogonal projection of $\mathbb{R}^d$ onto the hyperplane defined by $x_{i+1} = x_{i+2} = \ldots = x_d = 0$. ($\Pi_0$ is the trivial projection onto $\{0\}$ and Π_d is the identity map.)

We will define $\chi^{(i)}(x,y)$ in three steps: first we will specify $\Pi_{i-1}\chi^{(i)}(x,y)$ (equation (6.4)), then $(1-\Pi_i)\chi^{(i)}(x,y)$ (equation (6.6)) and finally $(\Pi_i - \Pi_{i-1})\chi^{(i)}(x,y)$ (equation (6.7)).

Under the strategy given by $\chi^{(i)}(x,y)$, we wish $\Pi_{i-1}x$ to pursue $\Pi_{i-1}y$ based on simple pursuit, but requiring $\Pi_{i-1}x$ to move at speed at most $\sqrt{1-\delta^2}$, and at a slower speed if x is close to y under the projection Π_{i-1}. Specifically, we set

$$\Pi_{i-1}\chi^{(i)}(x,y) \quad = \quad \min\left\{1, \frac{|\Pi_{i-1}(y-x)|}{\delta}\right\} \times \sqrt{1-\delta^2} \times \frac{\Pi_{i-1}(y-x)}{|\Pi_{i-1}(y-x)|}\,. \tag{6.4}$$

Note that, as $\Pi_{i-1}(x-y) \to 0$, then $\Pi_{i-1}\chi^{(i)}(x,y) \to 0$. Differentiating $|\Pi_{i-1}(y-x)|$ with respect to t, it follows from (6.4) and the constraint $|y'(t)| \leq 1$ that, when $|\Pi_{i-1}(x-y)| \geq \delta$,

$$\left\langle \Pi_{i-1}(y' - \chi^{(i)}(x,y)), \frac{\Pi_{i-1}(y-x)}{|\Pi_{i-1}(y-x)|} \right\rangle \quad \leq \quad 1 - \sqrt{1-\delta^2} \quad \leq \quad \delta^2\,, \tag{6.5}$$

and therefore the distance between $\Pi_{i-1}x$ and $\Pi_{i-1}y$ is either smaller than δ or increases only at rate at most δ^2. (For $i = 1$, we set $\Pi_0 = 0$, in which case (6.5) is vacuous. Note that the bounds in (6.5) do not depend on $(1-\Pi_i)\chi^{(i)}(x,y)$ and $(\Pi_i - \Pi_{i-1})\chi^{(i)}(x,y)$, which have not been defined yet.)

We set

$$(1-\Pi_i)\chi^{(i)}(x,y) \quad = \quad 0\,, \tag{6.6}$$

that is, the only nonzero components of $\chi^{(i)}$ are among its first i coordinates.

We still need to specify the i-th coordinate of $\chi^{(i)}$, i.e., $\chi^{(i)}_i(x,y) = (\Pi_i - \Pi_{i-1})\chi^{(i)}(x,y)$. We define it so that it has the same sign as $y_i - x_i = (\Pi_i - \Pi_{i-1})(y-x)$ and so that $\chi^{(i)}(x,y)$ is a unit vector except when $|y_i - x_i|$ is small. Specifically,

$$(\Pi_i - \Pi_{i-1})\chi^{(i)}(x,y) \;=\; \min\left\{1, \frac{|y_i - x_i|}{\delta}\right\} \times \left(\sqrt{1 - |\Pi_{i-1}\chi^{(i)}(x,y)|^2}\right) \times \operatorname{sgn}(y_i - x_i)\,. \tag{6.7}$$

Because of (6.4), this implies that

$$|(\Pi_i - \Pi_{i-1})\chi^{(i)}(x,y)| \geq \delta \qquad \text{if } |y_i - x_i| \geq \delta. \tag{6.8}$$

Note that, as $|x_i - y_i| \to 0$, then $(\Pi_i - \Pi_{i-1})\chi^{(i)}(x,y) \to 0$. On account of this and the observation after (6.4), it is not difficult to check that $\chi^{(i)}(x,y)$ is continuous in x and y.

A crucial point in the strategy associated with $\chi^{(i)}(x,y)$, for given $i < d$, is that it will force $|x_i - y_i|$ to become small before y_d has the chance to decrease by more than a fixed amount $4\gamma_i$ that is independent of the Man's strategy, where

$$\gamma_i \quad = \quad 1 \vee \frac{1}{\delta}|\Pi_{i-1}(x(0) - y(0))|. \tag{6.9}$$

We will apply Lemma 6.3 in the probabilistic part of the argument, but the following explanation may help elucidate our inductive strategy. Heuristically speaking, at the i-th step, the lemma will be applied with the starting points $x(0)$ and $y(0)$ replaced by $x(u_{i-1})$ and $y(u_{i-1})$, and with the function $y(u_{i-1} + \cdot\,)$ in place of the function $y(\,\cdot\,)$.

Lemma 6.3. *Choose $i \in \{2, \ldots, d-1\}$, and assume $x(0)$ and $y(0)$ lie in the long cuboid $D_2 = (-1/2, 1/2) \times \ldots \times (-1/2, 1/2) \times (0, L)$, with $x(0)$ and $y(0)$ satisfying $x_d(0) > 0$, $y_d(0) > 4\gamma_i$. Assume that x and y move at unit speed or less, with the motion of x being given by $x' = \chi^{(i)}(x,y)$. There exists $t < \infty$ at which $|x_i(t) - y_i(t)| \leq \delta$; denote by u_i the first time t at which this condition is satisfied. Whatever the motion of y, one has $u_i \leq 1/\delta$. Moreover,*

$$|(\Pi_i - \Pi_{i-1})(x(u_i) - y(u_i))| \quad = \quad |x_i(u_i) - y_i(u_i)| \quad \leq \quad \delta\,, \tag{6.10}$$

$$(1 - \Pi_i)x(t) \quad = \quad (1 - \Pi_i)x(0) \quad \textit{for } t \leq u_i\,, \tag{6.11}$$

$$|\Pi_{i-1}(x(u_i) - y(u_i))| \quad \leq \quad |\Pi_{i-1}(x(0) - y(0))| + 2\delta\,, \tag{6.12}$$

$$y_d(t) \quad \geq \quad y_d(0) - 4\gamma_i \quad \textit{for } t \leq u_i\,. \tag{6.13}$$

Note that, in the case where $i = 1$, (6.12) is vacuous and the other formulas hold trivially with $u_1 \leq 1$, since the width of the first component of the cuboid is 1 and $|\chi_1^{(1)}(x(t), y(t))| = 1$ for $t \leq u_1$.

Proof of Lemma 6.3. The formulas (6.10)–(6.13) hold trivially, with $u_i = 0$, when $|y_i(0) - x_i(0)| \leq \delta$. So, we will assume that $|y_i(0) - x_i(0)| > \delta$, with u_i being the time t at which $|y_i(t) - x_i(t)| = \delta$ first occurs.

Assume for the moment that (6.13) holds, but with the weaker $t \leq u_i^*$ in place of $t \leq u_i$, where $u_i^* = u_i \wedge (1/\delta)$. Then, x and y both remain in the long cuboid until time u_i^*.

By (6.8), the speed of the component x_i is at least δ, up until time u_i^*. Since the width of the ith component of the cuboid is 1, it follows that $u_i = u_i^* \leq 1/\delta$. Also, inequality (6.10) follows immediately from the definition of u_i.

Equation (6.11) follows from $(1 - \Pi_i)\chi^{(i)}(x,y) = 0$.

Let t_* be the supremum of $t \leq u_i$ such that $|\Pi_{i-1}(x(t) - y(t))| \leq \delta$; we let $t_* = 0$ if there is no such t. Inequality (6.12) follows from the upper bound δ^2 in (6.5) on the directional derivative of $|\Pi_{i-1}(x - y)|$ on the interval $[t_*, u_i]$, and from the bound $u_i \leq 1/\delta$.

In order to complete the proof, it remains to demonstrate (6.13), with $t \leq u_i^*$ in place of $t \leq u_i$. The argument strongly uses the definition of $\chi^{(i)}(x, y)$, which will ensure that the pursuit by the Lion of the Man is "efficient" with respect to the allowed change of the dth coordinate of the Man. The argument requires some estimation since $\chi^{(i)}(x, y)$ is constructed in terms of the Euclidean metric, whereas we will need bounds with respect to the L^1 metric in order to obtain (6.13).

We choose $0 = a_0 < a_1 < \ldots < a_J = 1$ and let $A_j = (a_{j-1}, a_j)$ such that, for any points x, $\tilde{x}$, y, $\tilde{y}$ with $|\Pi_{i-1}(x - y)|, |\Pi_{i-1}(\tilde{x} - \tilde{y})| \in A_j$, for given j,

$$|\chi_i^{(i)}(x, y)| - |\chi_i^{(i)}(\tilde{x}, \tilde{y})| \leq \delta/4. \tag{6.14}$$

Note that, on $t \leq u_i^*$, $\chi_i^{(i)}(x(t), y(t))$ depends only on $|\Pi_{i-1}(x(t) - y(t))| \wedge \delta$. Let $B_j \subseteq [0, u_i^*]$ denote the time set on which $|\Pi_{i-1}(x(t) - y(t))| \in A_j$. One can choose a_j so that the set where $|\Pi_{i-1}(x(t) - y(t))| = a_j$ has measure 0 and so that $a_{J-1} \in [\delta\gamma_i, 2\delta\gamma_i]$. ((6.14) is satisfied on $[a_{J-1}, a_J]$ since $a_{J-1} \geq \delta$, and so $\chi_i^{(i)}(x, y)$ is constant there.) We claim that

$$\begin{aligned} \int_{B_j} \left(\sqrt{\sum_{k=1}^{i-1} x_k'(t)^2} - \sqrt{\sum_{k=1}^{i-1} y_k'(t)^2} \right) dt &\leq a_j - a_{j-1} \quad \text{for } j \leq J - 1, \\ &\leq 0 \quad \text{for } j = J, \end{aligned} \tag{6.15}$$

which we demonstrate at the end of the proof.

Employing the definition of $\chi_i^{(i)}$ and $|y'(t)| \leq 1$, we have

$$x_i'(t)^2 = 1 - \sum_{k=1}^{i-1} x_k'(t)^2 \quad \text{and} \quad y_d'(t)^2 \leq 1 - \sum_{k=1}^{i-1} y_k'(t)^2$$

for $t \leq u_i^*$. On account of $1 - v \leq \sqrt{1 - v} \leq 1 - v/2$ for $v \in [0, 1]$, it follows from this and (6.15) that

$$\begin{aligned} \int_{B_j} \left(\frac{1}{2} y_d'(t)^2 - x_i'(t)^2 \right) dt &\leq a_j - a_{j-1} \quad \text{for } j \leq J - 1, \\ &\leq 0 \quad \text{for } j = J. \end{aligned} \tag{6.16}$$

Because of (6.8) and (6.14), for $t_1, t_2 \in B_j$,

$$\frac{1}{2} \quad \leq \quad \left(\frac{|x_i'(t_1)|}{|x_i'(t_2)|} \right)^2 \quad \leq \quad 2\,. \tag{6.17}$$

Isolating the term $y_d'(t)^2$ in (6.16), applying the Cauchy-Schwarz inequality to its integral, applying (6.17) and the inequality $\sqrt{v + w} \leq \sqrt{v} + \sqrt{w}$ to the other side, and summing over $j = 1, \ldots, J$

yields

$$\int_0^{u_i^*} |y_d'(t)|\,dt \le 2\sum_{j=1}^{J} |B_j| \min_{t\in B_j} |x_i'(t)| + \sum_{j=1}^{J-1} \sqrt{2(a_j - a_{j-1})|B_j|}\,. \tag{6.18}$$

Since $x_i'(t)$ retains the same sign on $t \le u_i^*$, the first sum on the right side of (6.18) is at most $2\int_0^{u_i^*} |x_i'(t)|\,dt \le 2$. Because $u_i^* \le 1/\delta$ and $a_{J-1} \le 2\delta\gamma_i$, it follows from the Cauchy-Schwarz inequality that the second term on the right is at most

$$\sqrt{2u_i^* \sum_{j=1}^{J-1}(a_j - a_{j-1})} \le 2\sqrt{\gamma_i} \le 2\gamma_i.$$

Hence, $y_d(t) - y_d(0) \ge -2 - 2\gamma_i \ge -4\gamma_i$ for $t \le u_2^*$, as desired.

We still need to demonstrate (6.15). First, note that since each A_j is open, so is each B_j. Let B_j^η denote the subset of B_j consisting of the union of all open intervals in B_j with length at least η, with $\eta \in (0, (a_j - a_{j-1})/2]$. In order to show (6.15), it is sufficient to show its analog

$$\int_{B_j^\eta} \left(\sqrt{\sum_{k=1}^{i-1} x_k'(t)^2} - \sqrt{\sum_{k=1}^{i-1} y_k'(t)^2} \right) dt \le a_j - a_{j-1} \quad \text{for } j \le J-1, \tag{6.19}$$

$$\le 0 \quad \text{for } j = J,$$

for each such η, because the integrands are bounded.

We can assume that $B_j^\eta \ne \emptyset$ in (6.19). We decompose B_j^η into disjoint intervals (b_ℓ, c_ℓ), $\ell = 1, \dots, L$, with b_ℓ and c_ℓ increasing in ℓ. It follows from the definition of $\Pi_{i-1}\chi^{(i)}(x,y)$ and differentiation of $|\Pi_{i-1}(y(t) - x(t)|$ that, for any $\ell \le L$,

$$\int_{b_\ell}^{c_\ell} \left(\sqrt{\sum_{k=1}^{i-1} x_k'(t)^2} - \sqrt{\sum_{k=1}^{i-1} y_k'(t)^2} \right) dt \le |\Pi_{i-1}(x(b_\ell) - y(b_\ell))| - |\Pi_{i-1}(x(c_\ell) - y(c_\ell))|. \tag{6.20}$$

We claim that

$$|\Pi_{i-1}(x(b_{\ell+1}) - y(b_{\ell+1}))| = |\Pi_{i-1}(x(c_\ell) - y(c_\ell))|,$$

with both equalling either a_{j-1} or a_j: these are endpoints of A_j, and the length of any time interval during which the distance between $\Pi_{i-1}x(t)$ and $\Pi_{i-1}y(t)$ crosses A_j must be at least $|A_j|/2 = (a_j - a_{j-1})/2$. Hence, such an interval is included in B_j^η, because $\eta \le (a_j - a_{j-1})/2$. This would contradict the definitions of $b_{\ell+1}$ and c_ℓ if the projected distances between x and y were different at these two times.

Summing ℓ over $1, \dots, L$ in (6.20), the terms from the right side therefore telescope, and so the left side of (6.19) is at most

$$|\Pi_{i-1}(x(b_1) - y(b_1))| - |\Pi_{i-1}(x(c_L) - y(c_L))|. \tag{6.21}$$

Since the difference in (6.21) is dominated by $a_j - a_{j-1}$, the first line on the right side of (6.19) follows immediately. The second line of (6.19) follows by noting that $|\Pi_{i-1}(x(b_1) - y(b_1))| = a_{J-1}$, for $j = J$, since $|\Pi_{i-1}(x(0) - y(0))| \le a_{J-1}$ (by the definition of a_{J-1}), and therefore the second term in (6.21) is at least as large as the first. This completes the proof of the lemma. □

We note that the times u_i, $i = 1, \ldots, d-1$, in Lemma 6.3, depend on the trajectory y taken by the Man. Since u_i can be up to order $1/\delta$, u_i might be larger than the length L of the cuboid when δ is chosen close to 0. Although this could conceivably allow the Man to escape from the cuboid before being approached by the Lion, the bound on $y_d(t) - y_d(0)$ in (6.13) will allow us to show this will not occur.

We also obtain bounds for the case $i = d$; these bounds are much easier to derive than the corresponding bounds in Lemma 6.3.

Lemma 6.4. *Assume $x(0)$ and $y(0)$ lie in the long cuboid $D_2 = (-1/2, 1/2) \times \ldots \times (-1/2, 1/2) \times (0, L)$, with $x(0)$ and $y(0)$ satisfying $0 < x_d(0) \le y_d(0)$. Assume that x and y move at unit speed or less, with the motion of x being given by $x' = \chi^{(d)}(x, y)$. There exists $t < \infty$ at which $|x_d(t) - y_d(t)| \le \delta$; denote by u_d the first time t at which this condition is satisfied. Then, whatever the motion of y, one has $u_d \le L/\delta$. Moreover,*

$$|\Pi_{d-1}(x(u_d) - y(u_d))| \quad \le \quad |\Pi_{d-1}(x(0) - y(0))| + (L+1)\delta \tag{6.22}$$

and

$$x_d(0) \le x_d(t) \le y_d(t) \quad \text{for } t \le u_d. \tag{6.23}$$

Proof. The inequality (6.23) follows immediately from $x_d(0) \le y_d(0)$ and the definition of u_d. The inequality (6.22) holds trivially when $x_d(0) \ge y_d(0) - \delta$, so we will assume that $x_d(0) < y_d(0) - \delta$, with u_d being the time at which $x_d(t) = y_d(t) - \delta$ first occurs.

Since $\operatorname{sgn}(x'_d) > 0$ over the time interval $[0, u_d)$, one has $0 < x_d < y_d - \delta$ there, and x_d can travel no further than $L - \delta < L$ up until time u_d. Also, by (6.8), the speed of the component x_d is at least δ up until time u_d. Consequently, $u_d \le L/\delta$, as required.

Let t_* be the supremum of $t \le u_d$ such that $|\Pi_{d-1}(x(t) - y(t))| \le \delta$; we let $t_* = 0$ if there is no such t. Inequality (6.22) follows from the upper bound δ^2 in (6.5) on the directional derivative of $|\Pi_{d-1}(x - y)|$ on the interval $[t_*, u_d]$, and on the bound $u_i \le L/\delta$.

□

As before, u_d depends on the trajectory y taken by the Man.

We now outline the proof of Theorem 6.1. The reasoning is similar to that employed in the proofs of Proposition 5.3 and Theorem 5.5 in the previous section, where we employed the Lion and the Man problem to demonstrate the absence of shy couplings for Brownian motion; here, we will employ Lemmas 6.2, 6.3 and 6.4 instead of Theorem 4.6. In the present setting, after employing Lemma 6.2, we must piece together analogous results over d time intervals, and the roles of the Lion and the Man for the two Brownian motions may need to be interchanged at the beginning of the last interval.

Proof of Theorem 6.1. Consider a pair of co-adapted reflecting Brownian motions on $\overline{D}$. By Lemma 6.2, there is a nonrandom time u_0 such that, for any pair of initial states $X(0)$ and $Y(0)$,

$$\mathbb{P}\left[X_d(u_0) \geq L/16 \text{ and } Y_d(u_0) \geq L/16\right] \geq 1/2. \tag{6.24}$$

Restarting the process at time u_0, we will apply (6.24), and Lemmas 6.3 and 6.4 to deduce that, for any given $\varepsilon \in (0,1)$,

$$\mathbb{P}\left[\inf_{0\leq s\leq t} \operatorname{dist}_{\mathbf{I}}(X(s), Y(s)) \leq \varepsilon\right] \quad > \quad 0 \tag{6.25}$$

for some t not depending on $X(0)$ and $Y(0)$, where $\operatorname{dist}_{\mathbf{I}}$ is the intrinsic distance metric on $\overline{D}$. This is the analog of (5.6). It is not hard to modify the argument in the proof of Bramson et al. (2012, Proposition 20) to show that (6.25) implies the uniform bound

$$\mathbb{P}\left[\inf_{0\leq s\leq t_1} \operatorname{dist}_{\mathbf{I}}(X(s), Y(s)) \leq \varepsilon\right] \quad \geq \quad p_1\,, \tag{6.26}$$

for some t_1 and $p_1 > 0$ not depending on $X(0)$ and $Y(0)$. The uniform bound in (6.26) permits us to iterate the inequality (6.26) repeatedly, from which it follows that the coupling cannot be shy.

We now provide details for the derivation of (6.25). Consider an arbitrarily small $\delta \in (0,1)$. Assume that $X^n(0) = x(0)$, $Y^n(0) = y(0)$ and $x_d(0), y_d(0) \geq L/16$. For specific stopping times U^i, $i = 0, \ldots, d-1$, to be defined below, with $U^0 = 0$ and $U^i - U^{i-1} \in [0, 1/\delta]$, we let

$$A_i = \left\{ |\Pi_i(X(U^i) - Y(U^i))| \leq 4\delta i, \quad \inf_{U^{i-1}\leq t\leq U^i} X_d(t) \geq L/16 - i, \quad \inf_{U^{i-1}\leq t\leq U^i} Y_d(t) \geq L/16 - 16i^2 \right\}.$$

Note that it immediately follows from the first and third inequalities, and (6.1) that

$$Y_d(U^i) > 4\gamma'_{i+1}, \tag{6.27}$$

where $\gamma'_i = 1 \vee \frac{1}{\delta}|\Pi_{i-1}(X(U^{i-1}) - Y(U^{i-1}))|$. We will show by induction that

$$\mathbb{P}\left[A_1\right] \quad > \quad 0\,, \tag{6.28}$$

$$\mathbb{P}\left[A_i \;\Big|\; \bigcap_{k=1}^{i-1} A_k\right] \quad > \quad 0\,, \qquad i = 2, \ldots, d-1\,. \tag{6.29}$$

We start with the case $i = 1$, and define $(X^n(t), Y^n(t))$ and $(\tilde{X}^n(t), \tilde{Y}^n(t))$ as in (5.7)-(5.8) and (5.9)-(5.10), with $X^n(0) = x(0)$ and $Y^n(0) = y(0)$, and with χ replaced by $\chi^{(1)}$ as defined before Lemma 6.3. The same reasoning as in the proof of Proposition 5.3, but using Lemma 6.3 instead of Theorem 4.6, can be applied to analyze the limiting behavior of $(\tilde{X}^n(t), \tilde{Y}^n(t))$ as $n \to \infty$. The stopping time T^n defined below (5.8) is replaced by the time at which either X^n or Y^n leaves $\overline{D}_2$. (We note that this means we can work throughout this proof with Euclidean distance rather than intrinsic distance $\operatorname{dist}_{\mathbf{I}}$, since the two agree for pairs of points chosen within the convex set $\overline{D}_2$.) As in the proof of Proposition 5.3, there exists a stopping time $\widetilde{T}^* \le 1/\delta$ and processes $\{\widetilde{X}^\infty(t), t \in [0,1/\delta]\}$ and $\{\widetilde{Y}^\infty(t), t \in [0,1/\delta]\}$, with $\widetilde{X}^\infty(0) = x(0)$, $\widetilde{Y}^\infty(0) = y(0)$, $\widetilde{X}^\infty(t) = \widetilde{Y}^\infty(t)$ for $t \in [\widetilde{T}^*, 1/\delta]$, $|\frac{\partial}{\partial t}\widetilde{Y}^\infty(t)| \le 1$ for $0 \le t \le 1/\delta$, and

$$\widetilde{X}^\infty(t) = \int_0^t \chi^{(1)}(\widetilde{X}^\infty(s), \widetilde{Y}^\infty(s))\,\mathrm{d}\,s \qquad \text{for } t < \widetilde{T}^*,$$

such that $(\widetilde{X}^n, \widetilde{Y}^n)$ converges a.s. to $(\widetilde{X}^\infty, \widetilde{Y}^\infty)$ uniformly on $[0,1/\delta]$.

Note that $\gamma_1' = 1$; together with $y_d(0) \ge L/16$ and (6.1), this implies $y_d(0) > 4\gamma_1'$, and so all of the conditions of Lemma 6.3 are satisfied. Applying the lemma, with $\widetilde{X}^\infty$ and $\widetilde{Y}^\infty$ in place of x and y, and denoting by $\widetilde{U}^1$ the first time s at which $|\widetilde{X}_1^\infty(s) - \widetilde{Y}_1^\infty(s)| \le \delta$, it follows that $\widetilde{U}^1 \le 1/\delta$. Moreover, $|\Pi_1(\widetilde{X}^\infty(\widetilde{U}^1) - \widetilde{Y}^\infty(\widetilde{U}^1))| \le \delta$ by (6.10). Since $\widetilde{X}^\infty(0) = x(0)$ and $x_d(0) \ge L/16$, it follows from (6.11) that $\inf_{0\le t\le \widetilde{U}^1} \widetilde{X}_d^\infty(t) \ge L/16$; it also follows from (6.13) that $\inf_{0\le t\le \widetilde{U}^1} \widetilde{Y}_d^\infty(t) \ge L/16 - 4$. These observations and the fact that $(\widetilde{X}^n, \widetilde{Y}^n)$ converges a.s. to $(\widetilde{X}^\infty, \widetilde{Y}^\infty)$ uniformly on $[0,1/\delta]$ imply that, for large enough n,

$$\mathbb{P}\left[|\Pi_1(\widetilde{X}^n(\widetilde{U}^1) - \widetilde{Y}^n(\widetilde{U}^1))| \le 4\delta, \quad \inf_{0\le t\le \widetilde{U}^1} \widetilde{X}_d^n(t) \ge L/16 - 1, \quad \inf_{0\le t\le \widetilde{U}^1} \widetilde{Y}_d^n(t) \ge L/16 - 16\right] > 0.$$

By the same argument as in (5.19), it follows that, for some stopping time $U^1 \le \widetilde{U}^1 \le 1/\delta$,

$$\mathbb{P}\left[|\Pi_1(X(U^1) - Y(U^1))| \le 4\delta, \quad \inf_{0\le t\le U^1} X_d(t) \ge L/16 - 1, \quad \inf_{0\le t\le U^1} Y_d(t) \ge L/16 - 16\right] > 0.$$

This completes the proof of (6.28).

We will next present the induction step. Suppose that (6.28) and (6.29) hold for 1, 2, ..., $i-1$. We define $(X^n(t), Y^n(t))$ and $(\tilde{X}^n(t), \tilde{Y}^n(t))$ as in (5.7)-(5.8) and (5.9)-(5.10), relative to the processes $\{X(U^{i-1}+\cdot)\}$ and $\{Y(U^{i-1}+\cdot)\}$ in place of $\{X(\cdot)\}$ and $\{Y(\cdot)\}$ (using $\chi^{(i)}$ instead of χ). To simplify our presentation, we do not indicate in our notation that $X^n(t)$ and $Y^n(t)$ depend on i; the same remark applies to other processes and random variables used in the induction step. Note that $X^n(0) = X(U^{i-1})$ and $Y^n(0) = Y(U^{i-1})$. Just as in the first step, we can find a stopping time $\widetilde{T}^* \le 1/\delta$ and processes $\{\widetilde{X}^\infty(t), t \in [0,1/\delta]\}$ and $\{\widetilde{Y}^\infty(t), t \in [0,1/\delta]\}$, with $\widetilde{X}^\infty(0) = X(U^{i-1})$, $\widetilde{Y}^\infty(0) = Y(U^{i-1})$, $\widetilde{X}^\infty(t) = \widetilde{Y}^\infty(t)$ for $t \in [\widetilde{T}^*, 1/\delta]$, $|\frac{\partial}{\partial t}\widetilde{Y}^\infty(t)| \le 1$ for $0 \le t \le 1/\delta$, and

$$\widetilde{X}^\infty(t) = \int_0^t \chi^{(1)}(\widetilde{X}^\infty(s), \widetilde{Y}^\infty(s))\,\mathrm{d}\,s \qquad \text{for } t < \widetilde{T}^*,$$

such that $(\widetilde{X}^n, \widetilde{Y}^n)$ converges a.s. to $(\widetilde{X}^\infty, \widetilde{Y}^\infty)$ uniformly on $[0,1/\delta]$.

Assume that $\bigcap_{k=1}^{i-1} A_k$ holds. Then, by (6.27), $Y_d(U^{i-1}) > 4\gamma_i'$. We can therefore apply Lemma 6.3 to $\widetilde{X}^\infty$ and $\widetilde{Y}^\infty$ in place of x and y. Let $\widetilde{U}^i$ be the first time s such that $|\widetilde{X}_i^\infty(s) - \widetilde{Y}_i^\infty(s)| \le \delta$ and note that $\widetilde{U}^i \le 1/\delta$ by Lemma 6.3.

We are assuming that the conditioning event $\bigcap_{k=1}^{i-1} A_k$ in (6.29) holds, so $|\Pi_{i-1}(X(U^{i-1}) - Y(U^{i-1}))| \le 4\delta(i-1)$. This and (6.12) imply that $|\Pi_{i-1}(\widetilde{X}^\infty(\widetilde{U}^i) - \widetilde{Y}^\infty(\widetilde{U}^i))| \le 4\delta(i-1) + 2\delta$. It follows from (6.10) that $|(\Pi_i - \Pi_{i-1})(\widetilde{X}^\infty(\widetilde{U}^i) - \widetilde{Y}^\infty(\widetilde{U}^i))| \le \delta$ so, combining this with the previous estimate, we obtain

$$|\Pi_i(\widetilde{X}^\infty(\widetilde{U}^i) - \widetilde{Y}^\infty(\widetilde{U}^i))| \le 4\delta(i-1) + 3\delta = 4\delta i - \delta. \tag{6.30}$$

From (6.11) and the induction hypothesis, we obtain $\inf_{0 \le t \le \widetilde{U}^i} \widetilde{X}_d^\infty(t) = X_d(U^{i-1}) \ge L/16 - i + 1$; also, in view of (6.13),

$$\begin{aligned}\inf_{0 \le t \le \widetilde{U}^i} \widetilde{Y}_d^\infty(t) &\ge Y_d(U^{i-1}) - 4\gamma_i' \ge L/16 - 16(i-1)^2 - 4\gamma_i' \\ &\ge L/16 - 16(i-1)^2 - 4\left(1 \vee 4(i-1)\right) \ge L/16 - 16i^2 + 1\,.\end{aligned}$$

These observations and the fact that $(\widetilde{X}^n, \widetilde{Y}^n)$ converges a.s. to $(\widetilde{X}^\infty, \widetilde{Y}^\infty)$ uniformly on $[0, 1/\delta]$ imply that, for large enough n,

$$\mathbb{P}\left[|\Pi_i(\widetilde{X}^n(\widetilde{U}^i) - \widetilde{Y}^n(\widetilde{U}^i))| \le 4\delta i, \quad \inf_{0 \le t \le \widetilde{U}^i} \widetilde{X}_d^n(t) \ge L/16 - i, \quad \inf_{0 \le t \le \widetilde{U}^i} \widetilde{Y}_d^n(t) \ge L/16 - 16i^2\right] > 0\,.$$

By the same argument as in (5.19), it follows that, for some $U^i \in [U^{i-1}, U^{i-1} + \widetilde{U}^i]$,

$$\mathbb{P}\left[|\Pi_i(X(U^i) - Y(U^i))| \le 4\delta i, \quad \inf_{U^{i-1} \le t \le U^i} X_d(t) \ge L/16 - i, \quad \inf_{U^{i-1} \le t \le U^i} Y_d(t) \ge L/16 - 16i^2\right] > 0\,.$$

This completes the proof of (6.29).

Application of (6.28) and (6.29) with $i = 2, \ldots, d-1$ yields, for $X(0) = x(0)$, $Y(0) = y(0)$ with $x_d(0), y_d(0) \ge L/16$, that

$$\begin{aligned}\mathbb{P}\Big(&|\Pi_{d-1}(X(U^{d-1}) - Y(U^{d-1}))| \le 4\delta(d-1), \\ &\inf_{0 \le t \le U^{d-1}} X_d(t) \ge L/16 - d + 1, \quad \inf_{0 \le t \le U^{d-1}} Y_d(t) \ge L/16 - 16(d-1)^2\Big) > 0.\end{aligned} \tag{6.31}$$

Our final step is very similar to the inductive step presented above but requires some minor modifications, where we apply Lemma 6.4, in place of Lemma 6.3, to processes $\widetilde{X}^\infty$ and $\widetilde{Y}^\infty$ constructed from the processes $\{X(U^{d-1} + \cdot)\}$ and $\{Y(U^{d-1} + \cdot)\}$. One of the assumptions of Lemma 6.4 is $0 < x_d(0) \le y_d(0)$ whereas, in our setting, $X_d(U^{d-1}) \le Y_d(U^{d-1})$ need not hold. To deal with the situation where $X_d(U^{d-1}) > Y_d(U^{d-1})$, we relabel the Lion and the Man in the Lion and Man problem, exchanging the roles of X and Y in this step if necessary, so that $L/16 - 16(d-1)^2 \le X_d(U^{d-1}) \le Y_d(U^{d-1})$ holds.

Similar reasoning to the inductive step presented above, together with Lemma 6.4 in place of Lemma 6.3, shows that there is a stopping time U^d, with $U^d - U^{d-1} \le L/\delta$, such that

$$\mathbb{P}\Big[|X(U^d) - Y(U^d)| \le 4\delta(d-1) + \delta(L+3)\,,\\ L/16 - 16(d-1)^2 - 1 \le X_d(t) \le Y_d(t) + 1,\ \text{for } U^{d-1} \le t \le U^d \mid \bigcap_{k=1}^{d-1} A_k\Big] \quad > \quad 0.$$

Combining this with (6.31) implies that

$$\mathbb{P}\left[|X(U^d) - Y(U^d)| < \delta(4d+L), \quad \inf_{0\le t\le U^d} X_d(t) > 0, \quad \inf_{0\le t\le U^d} Y_d(t) > 0\right] \quad > \quad 0\,. \tag{6.32}$$

Note that $U^d \le (L+d)/\delta$.

We now complete the proof of the theorem. Combining (6.24) with (6.32), the lower bound in (6.25) follows upon setting $t = u_0 + (L+d)/\delta$ and $\delta = \varepsilon/(4d+L)$. □

7. Various examples

In this section, we present a number of examples involving CL domains and domains with rubber bands. Since we will be interested only in domains that satisfy the uniform exterior sphere and uniform interior cone conditions in this section, we will implicitly assume that all domains discussed here satisfy these boundary regularity conditions.

Example 7.1. *An example of a simply-connected domain that is not a CL domain and yet for which all loops are contractible.* Let $D \subset \mathbb{R}^3$ be the interior of the intersection of the upper half-space $z \ge 0$ with the spherical shell $\mathcal{B}(0,2) \setminus \mathcal{B}(0,1)$. Loops in $\overline{D}$ that do not lie wholly on $\partial\mathcal{B}(0,1)$ can be contracted in $\overline{D}$ along rays emanating from $(0,0,0)$ to smaller loops that lie wholly on $\partial\mathcal{B}(0,1)$. Loops in $\overline{D}$ that lie on $\partial\mathcal{B}(0,1)$ can be contracted in $\overline{D}$ to the point $(0,0,1)$ along great circles passing through $(0,0,1)$ and perpendicular to the boundary of the upper half-space. So, all loops in $\overline{D}$ are contractible. For an example of a rubber band in $\overline{D}$ that is not well-contractible, consider the intersection K of $\partial\mathcal{B}(0,1)$ with the boundary of the upper half-space. Suppose that $K' \in \mathcal{K}$ and $d_H(K,K') \le \varepsilon$. Let K'' be the radial projection of K' onto $\partial\mathcal{B}(0,1)$. It is easy to check that $d_H(K,K'') \le \varepsilon$, and therefore $\ell_{K'} \ge \ell_{K''} \ge \ell_K - c\varepsilon^2$. Hence, no contraction of K satisfies Definition 2.6 (b).

Example 7.2. *Star-shaped domains are CL domains.* Suppose that D is star-shaped, that is, for some $z_0 \in D$ and all $z \in D$, the line segment between z_0 and z is contained in D. Consider any $K \in \mathcal{K}$ and let $T_a(z) = z_0 + a(z - z_0)$. Then, for $t, \gamma \in [0,1)$, $H(e^{2\pi i t}, \gamma) = T_{1-\gamma}(K(t\ell_K))$ defines a contraction $\{T_{1-\gamma}K\}_{\gamma\in[0,1)}$ of K. Elementary calculations based on scaling show that this contraction satisfies Definition 2.6 (b). So D is a CL domain.

Example 7.3. *CAT*(0) *domains are CL domains.* To see this, first note that a given loop can be approximated as closely as desired by a polygon. Choose any fixed point $x_0 \in D$, which will serve as our reference point. Employing x_0 and the endpoints of any of the line segments defining the polygon, since D is assumed to be CAT(0), there is a unique pair of geodesics from x_0 to these endpoints, $\gamma_1 = \{\gamma_1(t), 0 \le t \le t_1\}$ and $\gamma_2 = \{\gamma_2(t), 0 \le t \le t_2\}$. Moreover, there exist geodesics in $\mathbb{R}^2$ (line segments), $\widetilde{\gamma}_1 = \{\widetilde{\gamma}_1(t), 0 \le t \le t_1\}$ and $\widetilde{\gamma}_2 = \{\widetilde{\gamma}_2(t), 0 \le t \le t_2\}$, with $\operatorname{dist}(\widetilde{\gamma}_1(t_1), \widetilde{\gamma}_2(t_2)) = \operatorname{dist}_{\mathbf{I}}(\gamma_1(t_1), \gamma_2(t_2))$, and such that $\operatorname{dist}_{\mathbf{I}}(\gamma_1(at_1), \gamma_2(at_2)) \le \operatorname{dist}(\widetilde{\gamma}_1(at_1), \widetilde{\gamma}_2(at_2))$ for $a \in [0,1]$. These geodesics induce a well-contractible homotopy to the point x_0, with a contractibility constant $c > 0$ that is at least as large as that corresponding to a planar convex domain with the same diameter. By selecting a sequence of polygons that converges uniformly to the given loop and taking limits, one obtains a well-contractible homotopy, with the same contractibility constant c, for the original loop. This implies D is a CL domain. Note that star-shaped domains are not necessarily CAT(0) and CAT(0) domains are not necessarily star-shaped.

Example 7.4. *Construction of CL domains by modification of CAT*(0) *domains, and a further generalization.* For a given CAT(0) domain D, choose a point $x_0 \in D$ as its reference point. For each $x \in D$, there exists a unique geodesic Γ_x from x_0 to x; denote by t_x the value of the parameter at which $\Gamma_x(t_x) = x$. There are many ways in which one can truncate these geodesics at $s_x \le t_x$ so that the remaining region consisting of the portions of the geodesics $\Gamma_x(t)$, with $t \in [0, s_x)$, is open and connected, and hence a domain. The restricted domain D_1 thus defined will still be CL domain, but need not be CAT(0).

An illustration is given in Fig. 4, where the non-CAT(0) domain on the left is obtained from the CAT(0) domain on the right by making a shallow "dent" on the end of one of its spheres. To see that the above recipe works in this case, choose the reference point x_0 to be the point inside the domain but on the boundary of the sphere, at the spot antipodal to the center of the dent. If the dent is shallow, then all geodesics from x_0 to all points of the sphere, with the dent removed, are line segments. In other words, the sphere with the dent removed is star-shaped relative to x_0. Note that new domain is not CAT(0), since it has a point on the surface where both principal curvatures are negative.

A more general family of examples that is related to the previous one can be obtained by considering the CAT(0) domain given in Bramson et al. (2012) and reproduced here in Fig. 5. Making one or more shallow dents in the spheres at the ends of the domain produces a new domain D_1 that is a CL domain, but not CAT(0), for the same reasons as before.

Any star-shaped domain D_1 can be realized by applying the construction at the beginning of the example, and choosing the domain D to be any convex domain containing D_1. The domain D is CAT(0) and its geodesics are the line segments connecting pairs of points. So, Example 7.2 is included in Example 7.4.

We note that, for this construction, the CAT(0) property for the domain D was only employed to ensure that pairs of geodesics emanating from the given reference point x_0 satisfy the CAT(0) property; the behavior of other geodesics was not employed. The reasoning in the first paragraph thus extends to domains D in which the reference point is connected by a single geodesic in $\overline{D}$ to any point in $\overline{D}$, and for which pairs of geodesics starting at the reference point satisfy the

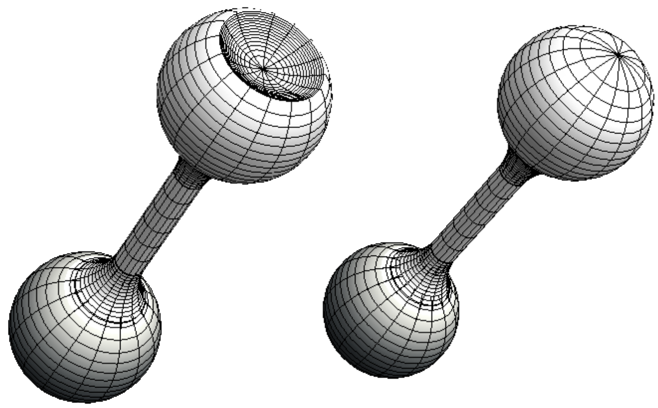

FIGURE 4. The domain on the right that has dumbbells shape is CAT(0). The domain on the left is obtained from the domain on the right by making a shallow spherical dent. The domain on the left is not CAT(0) but it is a CL domain.

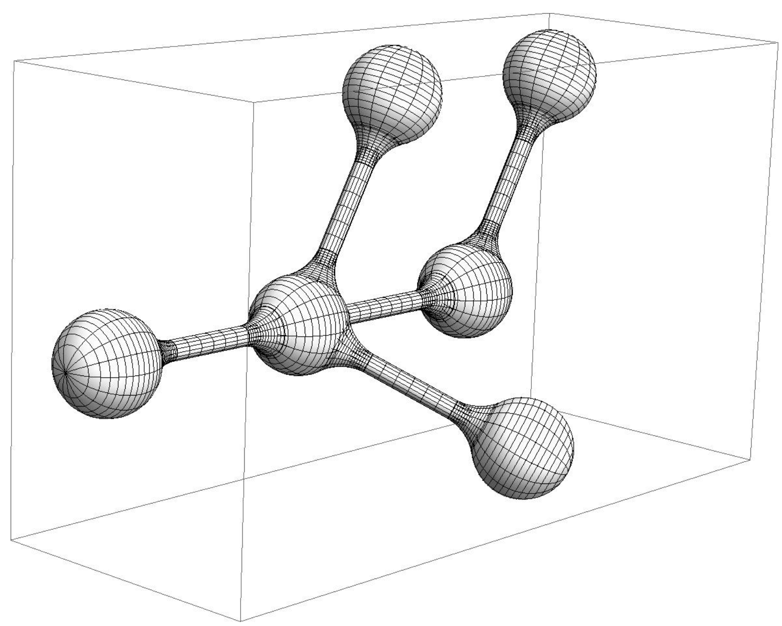

FIGURE 5. This example of a CAT(0) domain appeared in Bramson et al. (2012) where shy couplings and pursuit problems were analyzed in CAT(0) domains.

CAT(0) property. The construction given in Example 7.4 can therefore also be viewed as a natural generalization of that given in Example 7.2.

Example 7.5. *There are domains with semi-stable rubber bands that are not stable.* Consider a

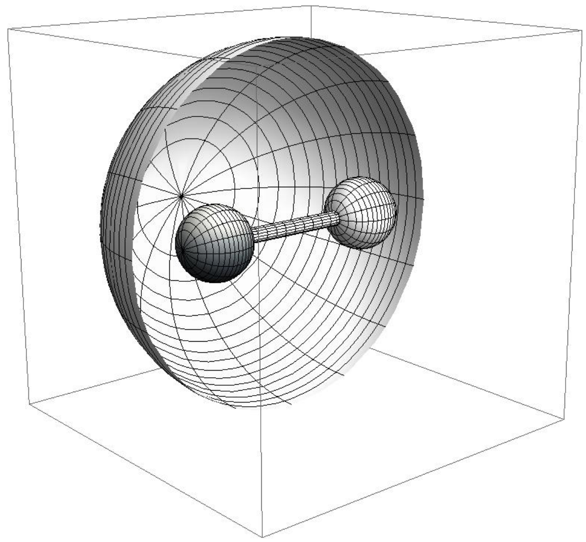

FIGURE 6. "Dumbbells domain" with a semi-stable rubber band and no stable rubber band. One half of the outer boundary is cut away to show the interior part of the boundary. Drawing is not to scale.

domain D that is a ball from which a dumbbell has been removed (see Figure 6):

$$D = \mathcal{B}(0,100) \setminus \left(\mathcal{B}((-10,0,0),2) \cup \mathcal{B}((10,0,0),2) \cup \bigcup_{-10 \le a \le 10} \mathcal{B}((a,0,0),1) \right) .$$

The loop $\{(0, \cos t, \sin t), 0 \le t < 2\pi\}$ is semi-stable. It is not a stable rubber band.

Example 7.6. *If a bounded domain is not simply connected, then it must possess at least one semi-stable rubber band.* Homotopies preserve the homotopy class of a loop. If a domain fails to be simply connected, then there exists a non-trivial homotopy class of loops and, within this class, there will be at least one loop minimizing the length function. This loop will be semi-stable, though not necessarily stable.

Example 7.7. *An explicit construction of a domain with a stable rubber band.* The essence of this construction is given in Figure 7: a doorknob is attached to the interior of a box, and the stable

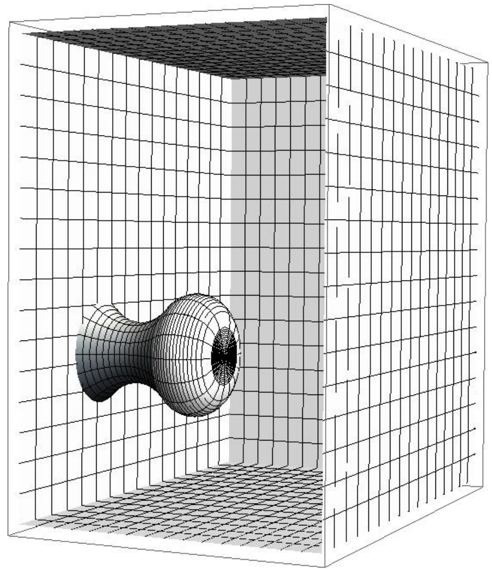

FIGURE 7. "Doorknob domain" with a stable rubber band. Part of the outer boundary is cut to show the interior part of the boundary. Drawing is not to scale.

rubber band fits around the neck of the doorknob. Let

$$\begin{aligned} D = &((0,10) \times (-10,20)^2) \\ &\setminus \Big(\{(x,y,z) \in \mathbb{R}^3 : y^2 + z^2 \leq 1 + (x-1)^2,\ 0 \leq x \leq 2\} \\ &\qquad \cup \{(x,y,z) \in \mathbb{R}^3 : y^2 + z^2 \leq -4(x-3)(x-3/2),\ 2 \leq x \leq 3\} \Big). \end{aligned}$$

The boundary of D is not smooth, although edges can be smoothed without affecting the stable rubber band defined below. We will argue that the following loop is a stable rubber band,

$$K(t) = (1, \cos(t), \sin(t)), \qquad 0 \leq t < 2\pi.$$

Let $\varepsilon = 1/10$ in Definition 2.8 and fix some $0 < \eta \leq \varepsilon$. Suppose that $K_1 \in \mathcal{K}$ is such that $d_H(K, K_1) \geq \eta$ and, for some $n \geq 1$, there exists a continuous mapping $H : \mathcal{S} \times [0,1] \to \mathcal{B}(K, \varepsilon)$ satisfying the conditions in Definition 2.8. It is evident that, for some $\delta > 0$ depending only on D and η, the length of the projection $\widehat{K}_1$ of K_1 on the plane $A = \{(x,y,z) : x = 0\}$ has length greater than $2\pi n + \delta$. Thus K is a stable rubber band in D.

Physical intuition suggests that a "typical domain" either contains a semi-stable rubber band or the domain is a CL domain, although it is easy to construct domains that satisfy neither condition. An intuitive justification is based on part (iv) of Definition 2.6, since a "typical function" on a compact interval is either non-monotone or its slope has one sign and is bounded away from 0. We close this section with Conjecture 7.9, which makes this claim precise. The conjecture employs the following definition.

Definition 7.8. *Consider the family* $\mathbf{D}$ *of all open bounded non-empty sets* $D \subset \mathbb{R}^d$ *with smooth boundary and* $\kappa(D) < \infty$*, where* $\kappa(D)$ *is the supremum over all* $z \in \partial D$ *of the absolute values of the principal curvatures at* z*. The Gromov-Hausdorff distance induces a topology on this family. Let* $\mathcal{D}$ *be the topological space of all pairs* $(D, \kappa(D))$*,* $D \in \mathbf{D}, \kappa(D) > 0$*, equipped with the product topology. Let* $\mathcal{D}_s$ *be the set of* $(D, \kappa(D))$ *such that* D *contains a semi-stable rubber band and let* $\mathcal{D}_c$ *be the set of* $(D, \kappa(D))$ *such that* D *is a CL domain.*

Conjecture 7.9 states that the set of domains with semi-stable rubber bands and the set of CL domains are each open in the above topology, and that the set of domains remaining, after removing these two sets, is nowhere dense.

Conjecture 7.9. *The sets* $\mathcal{D}_s$ *and* $\mathcal{D}_c$ *are open in* $\mathcal{D}$*. The set* $\mathcal{D} \setminus (\mathcal{D}_s \cup \mathcal{D}_c)$ *is nowhere dense.*

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Appendix A. Uniform exterior sphere and uniform interior cone conditions imply CAT(κ)

In this appendix, we will employ a theorem from Lytchak (2004), to show that the uniform exterior sphere and uniform interior cone conditions imply the CAT(κ) property, that is, ESIC domains are CAT(κ), for some $\kappa > 0$. In order to establish this, it is useful to employ the following definition.

Definition A.1 (Lipschitz domain)**.** *Recall that a function $f : \mathbb{R}^{d-1} \to \mathbb{R}$ is* Lipschitz, with constant $\lambda < \infty$, *if $|f(v) - f(z)| \leq \lambda |v - z|$ for all $v, z \in \mathbb{R}^{d-1}$. A domain $D \in \mathbb{R}^d$ is* Lipschitz, with constant λ, *if there exists $\delta > 0$ such that, for every $v \in \partial D$, there exists an orthonormal basis $e_1, e_2, \ldots, e_d$ and a Lipschitz function $f : \mathbb{R}^{d-1} \to \mathbb{R}$, with constant λ, such that*

$$\{w \in \mathcal{B}(v, \delta) \cap D\} \quad = \quad \{w \in \mathcal{B}(v, \delta) : f(w_1, \ldots, w_{d-1}) < w_d\} \,,$$

where we write $w_1 = \langle w, e_1 \rangle$, $\ldots$, $w_d = \langle w, e_d \rangle$.

As noted in Bramson et al. (2012, Section 2), Definition A.1 is equivalent to the uniform interior cone condition 2.2 , with $\lambda = \cot \alpha$. Moreover, if either holds for a given $\delta > 0$, then both hold for that δ and all smaller δ.

Verification of the CAT(κ) property for ESIC domains is most easily done by first establishing that the uniform exterior sphere condition implies a property known as *positive reach*. We begin by stating its definition (adapted from Lytchak, 2004), and then sketch the CAT(κ) implication in Lemma A.3 and Corollary A.5.

Definition A.2. *A set $A \subset \mathbb{R}^d$ has* positive reach greater than or equal to r *if, for all $z \in \mathbb{R}^d$ with $\operatorname{dist}(z, A) \leq r$, there is a unique point $v \in A$ with $\operatorname{dist}(z, v) = \operatorname{dist}(z, A)$.*

Lemma A.3. *Suppose that D is a bounded domain that satisfies a uniform exterior sphere condition and a uniform interior cone condition. Then $\overline{D}$ has positive reach.*

Proof. Suppose r is the uniform exterior ball radius and $\alpha \in (0, \pi/2]$ is the uniform interior cone angle. We shall show that the reach is at least as large as the minimum of $r \sin\alpha$ and a positive constant δ relating to the uniform interior cone condition. Here, $\delta > 0$ is chosen small enough so that, in any ball of radius δ, we may implement the uniform interior cone condition with a fixed axis e_d; moreover, we shall choose δ small enough so that the cones extend sufficiently far so, within the ball, the domain D may be described as the super-level set of a Lipschitz function, with Lipschitz constant $\cot\alpha$, as given in Definition A.1.

If the reach s' is smaller than δ, then there must exist $s \geq s'$ and arbitrarily close to s' such that there exists $z \in \mathbb{R}^d$, with $\operatorname{dist}(z, \overline{D}) = s$, so that, for distinct points $v_1, v_2 \in \partial D$, $\operatorname{dist}(z, v_1) = \operatorname{dist}(z, v_2) = \operatorname{dist}(z, \overline{D}) = s$. We shall show that this will imply $s \geq r \sin\alpha$. From this, it will follow that D has positive reach at least as great as $\min\{\delta, r \sin\alpha\}$.

Suppose that e_d is the d^{th} vector in the orthonormal basis corresponding to points in $\mathcal{B}(z, \delta)$ as in Definition A.1, noting that we have chosen s small enough so that there is a single Lipschitz function representation within $\mathcal{B}(z, \delta)$ based on e_d. Let M be a 2-plane intersecting D and containing v_1, v_2, and $v_1 + e_d$. We note in passing that M will also contain $v_2 + e_d$. As a consequence of Bramson et al. (2012, Lemma 11), any point in $M \cap \partial D \cap \mathcal{B}(z, \delta)$ must be supported by an open disk in M of radius $r \sin\alpha$, such that the disk and $M \cap D$ are disjoint.

The ball $\mathcal{B}(z, s)$ intersects M in an open disk C_1 of radius at most s; moreover, it follows from their definition that v_1 and v_2 must lie on the boundary of C_1. Furthermore, we may use the uniform interior cone condition (based locally on e_d) to argue that the line ℓ through v_1 and v_2 must separate (in M) the center of C_1 from the points $v_1 + e_d$, $v_2 + e_d$.

Let $v_3 \in M \cap \partial D \cap \mathcal{B}(z, \delta)$ be the point with the same first $d - 1$ coordinates as $(v_1 + v_2)/2$. (By the choice of δ, there will be exactly one such point.) As we have noted above, v_3 lies on the boundary of an open disk C_2 with radius $r \sin\alpha$, which is disjoint from $\overline{D}$. Moreover, v_3 must be separated in M from the center of C_1 by ℓ, since otherwise C_1 will intersect with D.

We next argue using plane geometry as follows. Consider the case in which the disk C_2 lies on one side of ℓ. Then the Lipschitz representation of $D \cap \mathcal{B}(z, \delta)$ implies that it is constrained to lie between the lines $v_1 + \mathbb{R}e_d$ and $v_2 + \mathbb{R}e_d$, and hence has diameter no greater than $\operatorname{dist}(v_1, v_2) \leq 2s$. So, in this case, we can deduce that $s \geq r \sin\alpha$.

Consider now the case in which the disk C_2 intersects ℓ at two points. Again using the Lipschitz representation, the intersections must lie on the segment $v_1 v_2$ (since both v_1 and v_2 are in $\partial D \cap M$).

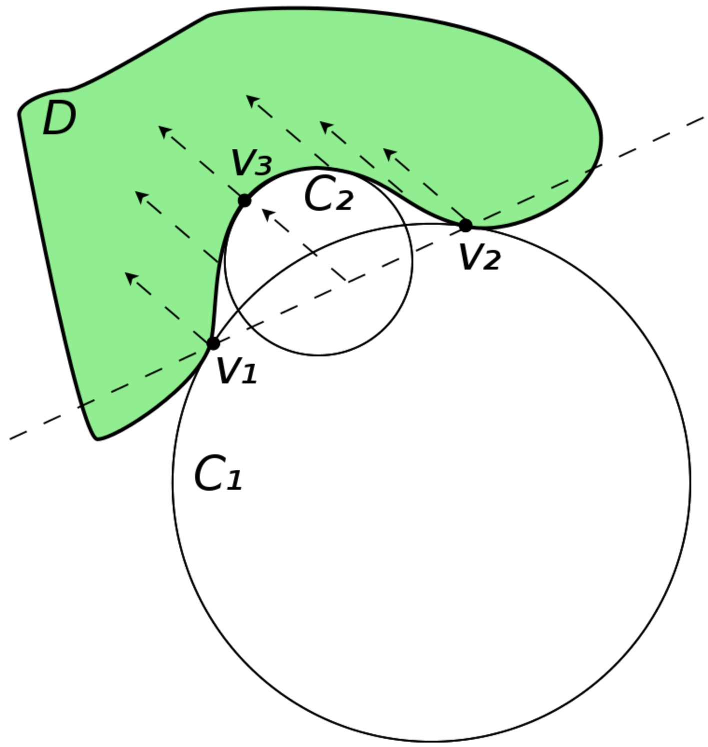

FIGURE 8. Illustration of the argument showing that the uniform exterior sphere and interior cone conditions imply positive reach.

The configuration of C_1, C_2, ... is illustrated in Figure 8. The planar geometry of circles allows us to deduce that the radius $r \sin \alpha$ of C_2 must be smaller than the radius of C_1, which itself is no larger than s. Hence, we deduce that $s \geq r \sin \alpha$ in this case as well. □

We quote the following theorem verbatim from Lytchak (2004, Theorem 1.1).

Theorem A.4. *Let M be a smooth Riemannian manifold, Z a compact subset of M that has positive reach. Then Z has an upper curvature bound with respect to the inner metric.*

Lemma A.3 and Theorem A.4 imply immediately

Corollary A.5. *Suppose that D is a bounded domain which satisfies the uniform exterior sphere and uniform interior cone conditions. Then $\overline{D}$ is a CAT(κ) space for some $\kappa \geq 0$.*

Remark A.6. *(Alexander et al., 2010, Theorem 12) give a short proof of this result in the case where D has a smooth boundary.*

Example A.7. We give two examples illustrating concepts related to Lemma A.3 and Corollary A.5. The easy proofs are left to the reader.

(i) Let $r_n = \frac{1}{3}(\frac{1}{n} - \frac{1}{n+1})$ and let C_n be the circle $\{(z_1, z_2, z_3) \in \mathbb{R}^3 : (z_1 - \frac{1}{n})^2 + z_2^2 = r_n^2, z_3 = 0\}$. Let $\mathcal{B}_n^+$ and $\mathcal{B}_n^-$ be two distinct balls with radii 1 such that the intersection of their boundaries with the plane $\{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_3 = 0\}$ is C_n. Let $D = \mathcal{B}(0, 10) \setminus \bigcup_{n\geq 1}(\overline{\mathcal{B}}_n^+ \cup \overline{\mathcal{B}}_n^-)$. The domain D satisfies a uniform exterior sphere condition, but it does not have a positive reach.

(ii) Let $z_1 = (0, 0, 1)$, $z_2 = (0, 0, -1)$ and $D = \mathcal{B}(0, 10) \setminus (\overline{\mathcal{B}(z_1, 1)} \cup \overline{\mathcal{B}(z_2, 1)})$. The domain D satisfies a uniform exterior sphere condition and has a positive reach, but it is not a Lipschitz domain.

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