ZEROES OF HOLOMORPHIC VECTOR FIELDS AND GROTHENDIECK DUALITY THEORY

(and applications to the holomorphic fixed-point formula of Atiyah and Bott)

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Summary

The holomorphic fixed-point formula of Atiyah and Bott is discussed in terms of Grothendieck's theory of duality for algebraic varieties. The treatment is valid for an endomorphism of a compact complex-analytic manifold with arbitrary isolated fixed points. An expression for the fixed-point indices is then derived for the case where the endomorphism belongs to the additive group generated by a holomorphic vector field with isolated zeroes. An application and some examples are given. Two generalisations of these results are also proved. The first deals with holomorphic vector bundles having sufficient homogeneity properties with respect to the action of the additive group on the base manifold, and the second with additive group actions on algebraic varieties.

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Notational conventions

Number underlined in square brackets refer to the list of references; if $\underline{0}$ is a sheaf of rings, a sheaf of $\underline{0}$-modules will be referred to as an $\underline{0}$-Module (similarly for an $\underline{0}$-Ideal); the complex numbers are denoted by $\mathbb{C}$, the real numbers by $\mathbb{R}$, and if $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, then $|z|$ will denote $\{\sum_{i=1}^{n} |z_i|^2\}^{1/2}$.

0. Introduction

This paper is concerned with certain problems which arise when the holomorphic Lefschetz fixed-point formula of Atiyah-Bott [2] is applied to the one-parameter group of endomorphisms of a compact complex-analytic manifold generated by a holomorphic vector field with isolated zeroes.

When these zeroes are non-degenerate, i.e. the section of the holomorphic tangent bundle defined by the vector field intersects the zero section transversally, then it is easy to calculate the fixed-point indices. In this case the fixed-point formula as it appears in [2] has been used by Kosniowski [18] and Lusztig [20] to prove several interesting results.

However, it was pointed out [2] that the fixed-point theorem can be formulated for arbitrary isolated fixed points, i.e. where the graph does not necessarily intersect the diagonal transversally. The fixed-point contributions will then involve the Grothendieck residue. Proofs of the formula in this case have been given by Toledo [25] and Tong [26, 27]. This then poses the problem of generalising the above results on holomorphic vector fields to this situation and studying the case of a vector field with arbitrary isolated zeroes. However, it is by no means clear what the fixed-point indices are in this case. The main difficulty is that an explicit calculation of the appropriate residue at a fixed point involves the use of the complex-analytic analogue of the Hilbert Nullstellensatz, a highly non-constructive result. It is easy to show that the fixed-point indices are meromorphic functions of the parameter t with a pole at $t = 0$, but to obtain further information is more difficult.

This problem was solved in [22], using an integral formula expression for the residue and a perturbation to reduce to the case of non-degenerate zeroes. The results obtained there also allowed one to prove an analogue of a result of Lusztig given in [20]. In particular a partially affirmative answer was given to a conjecture of Lusztig that if the zero of the vector field is 'totally degenerate', i.e. all the eigenvalues of the Lie bracket action of the vector field on the tangent space at the zero (the characteristic roots) are zero, then the fixed point index is of the form $t^{-n}P(t)$, where n is the dimension of the manifold and $P(t)$ is a polynomial of degree $\leq n$. This is true except that the bound on the degree of the polynomial does not hold in the most general situation; see section 6.

The main results presented here are essentially those of [22] but the method of proof which will be used is superior for several reasons. Firstly, the basic theorem of [22] is that two local cohomology elements (see below) have the same image under the Grothendieck residue. Here it is proved that they are actually the same element -- a considerably stronger result. Secondly, the present method allows generalisations in two directions; to more general coefficients for the cohomology groups occurring in the fixed point formula (5.2), and to algebraic varieties over more general fields (7).

The Grothendieck residue was introduced as part of Grothendieck's theory of duality for algebraic varieties [7], and this seems to be the natural context for some of the results given here. For this reason the first two sections are given over to a discussion of the relevant parts of Grothendieck duality theory and the closely related theory of local cohomology. This is then applied in later sections in the context of the holomorphic fixed-point formula.

The material is organised as follows.

Section 1.1 introduces the local cohomology functors and the related local and global Ext functors. These are defined as the derived functors of certain functors from sheaves to sheaves and abelian groups, and are somewhat analogous to the relative cohomology functors of topology. An interpretation of local cohomology in terms of Čech theory is also discussed.

Section 1.2 goes on to introduce some algebraic machinery, the Koszul complex, which is used to prove one of the basic results of Grothendieck duality theory, the fundamental local isomorphism. This gives a particularly convenient interpretation of the local Ext sheaves in certain special cases.

Section 1.3 applies the above techniques to analytic sheaves on complex manifolds, and also shows how the Koszul complex may be used to relate local Čech cohomology to the local Ext sheaves.

All of the results of section 1 are implicit in the various accounts [1,7,9,10,11,13] where they are discussed mainly in the context of schematic algebraic geometry.

Section 2 uses the above machinery to introduce certain constructions relevant to the holomorphic fixed-point formula. This approach was discovered in the course of several interesting conversations with Professor M. S. Narasimhan, and is rather different from the methods used in other published versions [2,25,26,27]. It appears that these ideas are essentially already known, but the description given here serves to place some of the concepts involved in their natural setting. A similar approach has been used recently by D. Toledo and Y. L. L. Tong in the context of a holomorphic Lefschetz formula for non-isolated fixed points.

As in the topological Lefschetz fixed-point theorem, the idea is to construct a cohomology class in the product of the manifold with itself which is represented by a "delta-function on the diagonal." This is done here by using the fundamental local isomorphism, and the resulting class can be identified with the delta-function via a construction of Harvey [14] involving the Cauchy kernel, the Bochner-Martinelli kernel and Dolbeault's isomorphism. The usefulness of Harvey's construction in this context has already been shown in [25,27].

Section 3 deals with the situation in which the fixed-point formula can be formulated and section 4 introduces the Grothendieck residue and shows how the fixed point indices can be calculated by applying the residue to elements of a certain Ext group which occurs at an isolated fixed point.

The remaining sections consist of new results concerning the nature of the fixed-point indices when the endomorphism of the manifold belongs to the one-parameter group generated by a holomorphic vector field with isolated zeroes. Section 5.1 contains the main theorem which gives an expression for the fixed-point index as a meromorphic function of the parameter, again in terms of the Grothendieck residue. This expression involves the Todd polynomials in "Chern classes" associated to the zero of the vector field and has a much more explicit dependance on the parameter than the "ordinary" formula. The method of proof involves the use, in the context of Grothendieck duality theory, of a standard iterative procedure often employed in proving the existence of integral curves of a vector field.

Section 5.2 generalises these results to calculate the fixed-point indices for the case of coefficients in any bundle which satisfies certain homogeneity conditions with respect to the action of the additive group on the base manifold. The results obtained are similar to those of section 5.1; the formula for the fixed-point index being modified by a factor analogous to the Chern character in "Chern classes" associated to the bundle at a zero of the vector field.

In section 6 the fixed point indices are shown to be rational functions in t and certain exponentials in t, and therefore have analytic continuations as entire meromorphic functions. The form of these meromorphic functions then allows one to prove an analogue of a theorem of Lusztig [20] by comparing the two sides of the fixed-point formula and concluding that both must be constant. This section also contains some examples.

In section 7 it is shown that similar results hold for additive group actions on algebraic varieties defined over an algebraically closed field of characteristic zero. This is done by replacing the analytic criteria for convergence used in the preceding work by convergence arguments in the m-adic topology of the local ring at a fixed point.

1.1 Local cohomology

The purpose of this section is to set out the basic concepts of the theory of local cohomology as developed, for example, in [9,10]. The groups and sheaves of local cohomology are defined as derived functors, and some of the basic properties are given. Full details and proofs can be found in [loc. cit.]. In the next paragraph it will be shown how the theory can be expressed in terms of local Čech cohomology, but only the simple case which will be useful later is treated here.

In practice the local cohomology groups are difficult to work with, and it will be convenient to use related concepts involving the functors 'local Ext' and 'global Ext' introduced in [7] and studied in detail in [9,10]. The final paragraph in this section briefly explains the connection of these functors with local cohomology.

1.1.1. Derived functors

Let X be a topological space, and $\underline{O}_X$ a sheaf of rings on X. Given any left-exact functor T from the category of $\underline{O}_X$-Modules to some other abelian category, its right derived functors R^iT may be defined as in [6;ch.7]. We recall the procedure.

Given an $\underline{O}_X$-Module $\underline{F}$, construct a resolution of $\underline{F}$ by $\underline{O}_X$-Modules $\{\underline{I}^k\}$:

$$0 \longrightarrow \underline{F} \longrightarrow \underline{I}^0 \longrightarrow \underline{I}^1 \longrightarrow \cdots$$

where each $\underline{I}^k$ is injective in the category of $\underline{O}_X$-Modules. We then define $R^iT(\underline{F})$ to be the ith cohomology group of the complex

$$0 \longrightarrow T(\underline{I}^0) \longrightarrow T(\underline{I}^1) \longrightarrow \cdots$$

Since T is left-exact it is immediate that $R^0T = T$.

Consider the following examples of functors from the category of $\underline{O}_X$-Modules to the category of abelian groups.

<u>1.1.1.1.</u> The left-exact functor $\underline{F} \longmapsto \Gamma(X,\underline{F})$ which assigns to a sheaf $\underline{F}$ the group of its global sections. Its derived functors give the <u>cohomology groups</u> of X with coefficients in the sheaf $\underline{F}$, which are denoted as usual by $H^i(X,\underline{F})$.

<u>1.1.1.2.</u> Let Z be a locally closed subspace of X, i.e. Z is closed in some open subspace U of X. Then the functor $\underline{F} \longmapsto \Gamma_Z(X,\underline{F})$ which assigns to $\underline{F}$ the subgroup of $\Gamma(U,\underline{F}|U)$ consisting of those sections which are zero on the complement of Z in U, i.e. sections with support in Z, is left-exact. This definition is easily shown to be independant of the choice of U, and the derived functors are the <u>local cohomology groups</u> of X with coefficients in $\underline{F}$ and support in Z, written as $H^i_Z(X,\underline{F})$.

<u>1.1.1.3.</u> Once can also define a left-exact functor from the category of $\underline{O}_X$-Modules to itself,

$$\underline{F} \mapsto \underline{\Gamma}_Z(\underline{F}) .$$

by taking $\underline{\Gamma}_Z(\underline{F})$ to be the sheaf associated to the presheaf

$$U \mapsto \Gamma_{Z\cap U}(U,\underline{F}|U)$$

for U open in X. The derived functors of $\underline{\Gamma}_Z$ are written $\underline{H}^i_Z(\underline{F})$ and are the local cohomology sheaves with coefficients in $\underline{F}$ and support in Z.

Using the exactness of the functor which assigns to a presheaf the corresponding sheaf, it is easy to show [10; prop. 1.2] that $\underline{H}^i_Z(\underline{F})$ is the sheaf associated to the presheaf

$$U \mapsto H^i_{Z\cap U}(U,\underline{F}|U).$$

1.1.2 Properties of the local cohomology functors

1.1.2.1 Excision [10; prop. 1.3]

If Z is locally closed in X, and V is an open subset of X containing Z, then

$$H^i_Z(X,\underline{F}) = H^i_Z(V,\underline{F}|V)$$

for any $\underline{O}_X$-Module $\underline{F}$.

1.1.2.2 Long exact sequence [10; cor. 1.9]

If Z is closed in X, and $\underline{F}$ is an $\underline{O}_X$-Module, there is an exact sequence:

$$0 \to \Gamma_Z(X,\underline{F}) \to \Gamma(X,\underline{F}) \to \Gamma(X-Z,\underline{F}) \to H^1_Z(X,\underline{F})$$
$$\to H^1(X,\underline{F}) \to H^1(X-Z,\underline{F}) \to H^2_Z(X,\underline{F}) \to \cdots$$

The fundamental concept relating the local cohomology functors is:

1.1.2.3 The local cohomology spectral sequence

Since $\Gamma(X,\underline{\Gamma}_Z(\underline{F})) = \Gamma_Z(X,\underline{F})$, and $\underline{\Gamma}_Z(\underline{F})$ has trivial cohomology when $\underline{F}$ is injective, there is a spectral sequence of composition of functors, as in [8; Th. 2.4.1].

1.1.2.4 $$E_2^{pq} = H^p(X,\underline{H}_Z^q(\underline{F})) \Longrightarrow H_Z^{p+q}(X,\underline{F}).$$

For further details, see [10; prop. 1.4]. We shall be particularly interested in the edge-homomorphisms, for $q \geq 0$

1.1.2.5 $$H_Z^q(X,\underline{F}) \xrightarrow{\text{edge}} H^0(X,\underline{H}_Z^q(\underline{F})).$$

This map can be made explicit by the following argument. If $\underline{I}$ is an injective $\underline{O}_X$-Module and U is open in X, then $\underline{I}|U$ is an injective $\underline{O}_X|U$-Module [8; prop. 3.1.3], and so it is easy to see that if $U \subset V$ are open in X, then the restriction map induces a transformation of spectral sequences:

$$\begin{array}{ccc} H^p(V,\underline{H}_Z^q(\underline{F})) & \Longrightarrow & H_Z^{p+q}(V,\underline{F}) \\ \downarrow & & \downarrow \\ H^p(U,\underline{H}_Z^q(\underline{F})) & \Longrightarrow & H_Z^{p+q}(U,\underline{F}) \end{array}$$

and a corresponding commutative diagram involving the edge-homomorphisms. (For brevity I have written $H_Z^p(U,\underline{F})$ instead of $H_{Z\cap U}^p(U,\underline{F}|U)$ etc.) Furthermore, one can take the direct limit of the spectral sequences over all U containing some $x \in X$. The limit spectral sequence then degenerates and the edge-homomorphism 1.1.2.5 is simply the identity map on $\underline{H}_Z^q(\underline{F})_x$. Thus for each $q \geq 0$ there is a commutative diagram:

$$\begin{array}{ccc} H^q_Z(X,\underline{F}) & \xrightarrow{\text{edge}} & H^0(X,\underline{H}^q_Z(F)) \\ & \searrow \quad \swarrow & \\ & \underline{H}^q_Z(\underline{F})_x & \end{array}$$

in which the lower two maps are the natural ones.

1.1.2.7 Behaviour under morphisms of ringed spaces

Let $f : (Y,\underline{O}_Y) \longrightarrow (X,\underline{O}_X)$ be a morphism of ringed spaces. (For definitions and notation, see [11; ch. 0] or [12; 0.4].) Then if $\underline{F}$ is an $\underline{O}_X$-Module, there we induced homomorphisms of abelian groups, for all $p \geq 0$:

$$H^p(X,\underline{F}) \longrightarrow H^p(Y,f^*\underline{F}).$$

In an analogous way, one obtains homomorphisms

$$H^p_Z(X,\underline{F}) \longrightarrow H^p_{f^{-1}Z}(Y,f^*\underline{F})$$

and for U open in X:

$$H^p_Z(U,\underline{F}) \longrightarrow H^p_{f^{-1}Z}(f^{-1}U,\ f^*\underline{F}).$$

This latter defines a homomorphism of presheaves to which there corresponds a homomorphism of $\underline{O}_X$-Modules:

$$\underline{H}^p_Z(\underline{F}) \longrightarrow f_*\underline{H}^p_{f^{-1}Z}(f^*\underline{F}).$$

Using the property 1.1.2.6. of the edge-homomorphisms it is then easy to check that the following diagram commutes:

1.1.2.8

$$\begin{array}{ccc} H^p_Z(X,\underline{F}) & \xrightarrow{\text{edge}} & H^0(X,\underline{H}^p_Z(\underline{F})) \\ & & \downarrow \\ \Big\downarrow & & H^0(X,f_*\underline{H}^p_{f^{-1}Z}(f^*\underline{F})) \\ & & \| \\ H^p_{f^{-1}Z}(Y,f^*\underline{F}) & \xrightarrow{\text{edge}} & H^0(Y,\underline{H}^p_{f^{-1}Z}(f^*\underline{F})) \end{array}$$

1.1.3 Some local Čech cohomology

Suppose now Z is a closed subspace of X, and let $\mathcal{U} = \{U_i\}_{i=1,\ldots,n}$ be an open cover of $U = X-Z$. Then $\mathcal{U}$ can be extended to an open cover of X by adding the set X itself. Let $\mathcal{U}' = \mathcal{U} \cup \{X\}$. For $\underline{F}$ an $\underline{O}_X$-Module, let $C^\bullet(\mathcal{U},\underline{F}|U)$, $C^\bullet(\mathcal{U}',\underline{F})$ be the corresponding cochain complexes (for precise definitions of these objects see [6; ch. 5]).

Define $C^p_Z(\mathcal{U}',\underline{F}) \subset C^p(\mathcal{U}',\underline{F})$ to be the subgroup consisting of those cochains which are zero on the simplices $U_{i_0} \cap \cdots \cap U_{i_p}$, all of whose vertices lie in $\mathcal{U}$.

There is an obvious split short exact sequence for each $p \geq 0$:

$$0 \longrightarrow C^p_Z(\mathcal{U}',\underline{F}) \longrightarrow C^p(\mathcal{U}',\underline{F}) \longrightarrow C^p(\mathcal{U},\underline{F}|U) \longrightarrow 0 .$$

Furthermore, this sequence is compatible with the boundary operators (the splitting is not compatible) and the resulting short exact sequence of cochain complexes gives a long exact cohomology sequence in the usual way. Note that since the trivial cover $\{X\}$ is a refinement of $\mathcal{U}'$, then $C^\bullet(\mathcal{U}',\underline{F})$ is acyclic except in dimension zero [6; foot of p. 222]. Therefore, if $H^\bullet_Z(\mathcal{U}',\underline{F})$ is the cohomology of $C^\bullet_Z(\mathcal{U}',\underline{F})$, we have the following:

There is an exact sequence

1.1.3.1 $$0 \longrightarrow H_Z^0(\mathcal{U}',\underline{F}) \longrightarrow \Gamma(X,\underline{F}) \longrightarrow \Gamma(U,\underline{F}) \longrightarrow H_Z^1(\mathcal{U}',\underline{F}) \longrightarrow 0$$

and for $p > 0$ there are isomorphisms

1.1.3.2 $$H^p(\mathcal{U},\underline{F}|U) = H_Z^{p+1}(\mathcal{U}',\underline{F}).$$

In particular 1.1.3.1 implies that $H_Z^0(\mathcal{U}',\underline{F}) = \Gamma_Z(X,\underline{F})$.

1.1.3.3 Lemma. If $\underline{F}$ is injective, $H_Z^p(\mathcal{U}',\underline{F}) = 0$ for all $p > 0$.

Proof. Any injective sheaf is flasque [10; lemma 1.5] and the result for $p > 1$ follows from 1.1.3.2 and [6; Th. 5.2.3]. For $p = 1$, the the result follows from 1.1.3.1 since the restriction map $\Gamma(X,\underline{F}) \longrightarrow \Gamma(U,\underline{F})$ is surjective for $\underline{F}$ flasque.

These concepts are easily related to the definitions of section 1.1.1 as follows.

1.1.3.4 Proposition.

There exists a natural homomorphism

$$H_Z^p(\mathcal{U}',\underline{F}) \longrightarrow H_Z^p(X,\underline{F}).$$

Proof. Let $\underline{I}^\bullet$ be an injective resolution of $\underline{F}$ as in 1.1.1. Form the bicomplex

$$C_Z^p(\mathcal{U}',\underline{I}^q)$$

with its Čech boundary operator and the boundary operator of the complex $\underline{I}^\bullet$. Then there are canonical inclusions for $p,q \geq 0$:

1.1.3.5 $$C^p_Z(\mathcal{U}',\underline{F}) \xrightarrow{j_1} C^p_Z(\mathcal{U}',\underline{I}^q) \xleftarrow{j_2} \Gamma_Z(X,\underline{I}^q)$$

which become, on passing to cohomology

$$H^p_Z(\mathcal{U}',\underline{F}) \xrightarrow{j_1^*} H^p(C^\bullet_Z(\mathcal{U}',\underline{I}^\bullet)) \xleftarrow{j_2^*} H^p(\Gamma_Z(X,\underline{I}^\bullet)).$$

But j_2^* is now an isomorphism since by 1.1.3.3. $H^p_Z(\mathcal{U}',\underline{I}^q) = 0$ for all $p > 0$, and all q. The composition $j_2^{*-1} \circ j_1^*$ is the required homomorphism.

1.1.3.6 As before, we can consider the corresponding local situation. Thus, if V is an open set in X, there are induced, by restriction, open covers $\mathcal{U} \cap V$, $\mathcal{U}' \cap V$ of V, and we can form the cochain complex of presheaves:

$$V \longmapsto C^\bullet_{Z\cap V}(\mathcal{U}' \cap V, \underline{F}|V).$$

The corresponding complex of sheaves will be denoted by $\underline{C}^\bullet_Z(\mathcal{U}',\underline{F})$, and passing to cohomology at either the presheaf or sheaf level gives sheaves which we shall write as $\underline{H}^p_Z(\mathcal{U}',\underline{F})$.

Then by analogy with proposition 1.1.3.4 there is a natural homomorphism of $\underline{O}_X$-Modules for $p \geq 0$:

$$\underline{H}^p_Z(\mathcal{U}',\underline{F}) \longrightarrow \underline{H}^p_Z(\underline{F}).$$

1.1.4 Global Ext and local Ext

There is a third approach to local cohomology, involving the functor 'Ext'. The great advantage of this approach is that there are two distinct methods of calculating the Ext groups, i.e. by means of either injective or free resolutions. In certain cases this will allow the introduction of an important computational device, the Koszul complex (see [1]). A more detailed treatment of the material in this section can be found in [1] or [7].

1.1.4.1 Let $\underline{E}$ be a fixed $\underline{O}_X$-Module, and let

$$\underline{F} \longmapsto \mathrm{Hom}(X,\underline{E},\underline{F})$$

be the left-exact functor which takes an $\underline{O}_X$-Module $\underline{F}$ into the group of $\underline{O}_X$-homomorphisms of $\underline{E}$ into $\underline{F}$. Its derived functors are the global Ext groups

$$\underline{F} \longmapsto \mathrm{Ext}^i(X,\underline{E},\underline{F}).$$

1.1.4.2 Similarly, we can consider the functor

$$\underline{F} \longmapsto \underline{\mathrm{Hom}}_{\underline{O}_X}(\underline{E},\underline{F})$$

which associates to $\underline{F}$ the sheaf of germs of $\underline{O}_X$-homomorphisms from $\underline{E}$ to $\underline{F}$, i.e. the sheaf associated to the presheaf

$$U \longmapsto \mathrm{Hom}(U,\underline{E}|U,\underline{F}|U).$$

Its derived functors are the local Ext sheaves

$$\underline{F} \longmapsto \underline{\mathrm{Ext}}^i_{\underline{O}_X}(\underline{E},\underline{F}).$$

Note that $\underline{\mathrm{Ext}}^i_{\underline{O}_X}(\underline{E},\underline{F})$ is also the sheaf associated to the presheaf $U \longmapsto \mathrm{Ext}^i(U,\underline{E}|U,\underline{F}|U)$.

1.1.4.3 Since $\mathrm{Hom}(X,\underline{E},\underline{F}) = \Gamma(X,\underline{\mathrm{Hom}}_{\underline{O}_X}(\underline{E},\underline{F}))$ there is a spectral sequence of composition of functors, as in 1.1.2.3,

$$E_2^{pq} = H^p(X, \underline{Ext}^q_{\underline{O}_X}(\underline{E},\underline{F})) \Longrightarrow Ext^{p+q}(X,\underline{E},\underline{F}).$$

The relationship with the groups and sheaves of local cohomology lies in the following constructions.

<u>1.1.4.4</u> Let $\underline{I}$ be an $\underline{O}_X$-Ideal, and let $\underline{O}_Z$ be the quotient Module $\underline{O}_X/\underline{I}$, with

$$Z = \text{supp } \underline{O}_Z = \{x \in X \mid \underline{I}_x \neq \underline{O}_{X,x}\}.$$

Then, if Z is closed in X, there are natural transformations of functors induced by the quotient map $\underline{O}_X \longrightarrow \underline{O}_Z$ which for any $\underline{O}_X$-Module $\underline{F}$ form a commutative diagram:

$$\begin{array}{ccc} Hom(X,\underline{O}_X,\underline{F}) & \longrightarrow & \Gamma(X,\underline{F}) \\ \uparrow & & \uparrow \\ Hom(X,\underline{O}_Z,\underline{F}) & \longrightarrow & \Gamma_Z(X,\underline{F}) \end{array}$$

and

$$\underline{Hom}_{\underline{O}_X}(\underline{O}_Z,\underline{F}) \longrightarrow \underline{\Gamma}_Z(\underline{F})$$

where the horizontal morphisms are defined by 'evaluation on the unit section.'

There are then induced natural transformations of the derived functors [<u>10</u>; p. 30].

<u>1.1.4.5</u>
$$\begin{array}{ccc} Ext^i(X,\underline{O}_X,\underline{F}) & \longrightarrow & H^i(X,\underline{F}) \\ \uparrow & & \uparrow \\ Ext^i(X,\underline{O}_Z,\underline{F}) & \longrightarrow & H^i_Z(X,\underline{F}) \end{array}$$

1.1.4.6 $$\underline{\mathrm{Ext}}^i_{\underline{O}_X}(\underline{O}_Z,\underline{F}) \longrightarrow \underline{H}^i_Z(\underline{F})$$

and a morphism of spectral sequences.

1.1.4.7
$$\begin{array}{ccc} H^p(X,\underline{\mathrm{Ext}}^q_{\underline{O}_X}(\underline{O}_Z,\underline{F})) & \Longrightarrow & \mathrm{Ext}^{p+q}(X,\underline{O}_Z,\underline{F}) \\ \downarrow & & \downarrow \\ H^p(X,\underline{H}^q_Z(\underline{F})) & \Longrightarrow & H^{p+q}_Z(X,\underline{F}) \end{array}.$$

1.2 The Koszul complex

This section is given over to a description of some algebraic apparatus which will be useful in dealing with the local Ext sheaves.

1.2.1 Definition of the Koszul complex

Let A be a ring (commutative with 1, as always) and let $\underline{x} = [x_1,\ldots,x_m]$ be any m-tuple of elements of A. Let $\Lambda^\bullet(A^m)$ be the exterior algebra on the free A-module $A^m = A \oplus \cdots \oplus A$, so that for $1 \leq k \leq m$, $\Lambda^k(A^m)$ is a free A-module with generators $e_{i_1} \wedge \cdots \wedge e_{i_k}$ for $1 \leq i_1 < \cdots < i_k \leq m$, and $\Lambda^o(A^m) = A$. We can then define A-module homomorphisms

$$i_{\underline{x}} : \Lambda^k(A^m) \longrightarrow \Lambda^{k-1}(A^m)$$

given on generators by

$$i_{\underline{x}}(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_{j=1}^{k} (-1)^j x_j e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_k}.$$

It is easily checked that $i_{\underline{x}}^2 = 0$ and the resulting chain complex is the Koszul complex $K_\bullet(\underline{x})$.

1.2.2 Regular A-sequences

With the same notation, $\underline{x} = [x_1, \ldots, x_m]$ is said to be a regular A-sequence if it has the following properties:

(i) x_1 is not a zero divisor in A

(ii) For $1 < i \leq m$, x_i is not a zero divisor in $A/(x_1, \ldots, x_{i-1})$, where $(x_1, \ldots, x_{i-1})$ is the ideal $x_1A + \cdots + x_{i-1}A$.

The importance of these conditions lies in the following result, the proof of which is given in [11; III.1].

1.2.2.1 Proposition

Let $\underline{x} = [x_1, \ldots, x_m]$ be a regular A-sequence; let $(x_1, \ldots, x_m) = I$. Then the Koszul complex $K_o(\underline{x})$ is a resolution of A/I by free A-modules; i.e. the chain complex

$$0 \longrightarrow \Lambda^m(A^m) \xrightarrow{i_{\underline{x}}} \cdots \xrightarrow{i_{\underline{x}}} \Lambda^1(A^m) \xrightarrow{i_{\underline{x}}} A \longrightarrow A/I \longrightarrow 0$$

is exact.

1.2.2.2 If M is an A-module, define a cochain complex $K^\bullet(\underline{x},M)$ by:

$$K^i(\underline{x},M) = \mathrm{Hom}_A(K_i(\underline{x}),M)$$

with corresponding coboundary operators $i'_{\underline{x}}$. Let $H^\bullet(\underline{x},M)$ be the cohomology of this complex. By the definition of the functors Ext by projective resolutions [6; ch. 5] we have that

$$\mathrm{Ext}^i_A(A/I,M) = H^i(\underline{x},M).$$

1.2.2.3 Proposition

Suppose M is a flat A-module. Then

$$\mathrm{Ext}_A^i(A/I,M) \cong \begin{cases} M/IM & i = m \\ 0 & i \neq m \end{cases} .$$

Proof (see [1; III.3.8]. In fact the Koszul complex shows that $\mathrm{Ext}_A^i(A/I,M) \cong \mathrm{Tor}_{m-i}^A(A/I,M)$ and the result follows from the flatness of M.

Note, however, that the isomorphism is non-canonical and depends on the particular set of generators chosen for I.

For $i = m$, the isomorphism is constructed explicitly by first defining

$$\varphi'_{\underline{x}} : K^m(\underline{x},M) \longrightarrow M$$

$$\varphi'_{\underline{x}}(a) = a(e_1 \wedge \cdots \wedge e_m).$$

Then the induced map

$$\varphi_{\underline{x}} : H^m(\underline{x},M) \longrightarrow M/IM$$

is the required isomorphism.

The following result is central for the applications of the theory given in later sections.

1.2.3 The fundamental local isomorphism

For $x \in I$, let $\bar{x}$ be its class mod I^2. Then under the conditions of proposition 1.2.2.3, we have:

(i) I/I^2 is a free A/I-module with generators $\bar{x}_1,\ldots,\bar{x}_m$. In particular, this implies that $\Lambda^m(I/I^2)$ is a free A/I-module with generator $\bar{x}_1 \wedge \cdots \wedge \bar{x}_m$.

(ii) There is a canonical isomorphism

$$\varphi : \mathrm{Ext}_A^m(A/I,M) \xrightarrow{\sim} \mathrm{Hom}_{A/I}(\Lambda^m(I/I^2),M/IM)$$

which is independant of the choice of generators for I and for $a \in H^m(\underline{x},M) = \mathrm{Ext}_A^m(A/I,M)$ is given by

$$\varphi(a)(\bar{x}_1 \wedge \cdots \wedge \bar{x}_m) = \varphi_{\underline{x}}(a).$$

Proof. For (i) see [1; I.4.5 and III.3.4]. For (ii), let

$$\psi_{\underline{x}} : M/IM \longrightarrow \mathrm{Hom}_{A/I}(\Lambda^m I/I^2, M/IM)$$

be the isomorphism resulting from the (non-canonical) isomorphism $A/I \cong \Lambda^m(I/I^2)$ given by $y \longmapsto y\bar{x}_1 \wedge \cdots \wedge \bar{x}_m$. Then it is easily checked that $\psi_{\underline{x}} \circ \varphi_{\underline{x}}$ is independant of the regular A-sequence $\underline{x}$. See [13; ch. II, §7] or [1; ch. I, §4].

The next section will give concrete applications of the above theory to analytic sheaves on complex manifolds.

1.3 Application to complex manifolds

Let M be a complex analytic manifold of complex dimension n, and let $\underline{O}_M$ be the sheaf of germs of holomorphic functions on M. The results of the previous section are going to be used to calculate the sheaves $\underline{\mathrm{Ext}}^i(\underline{O}_Z,\underline{F})$ for certain analytic subspaces $(Z,\underline{O}_Z)$ of $(M,\underline{O}_M)$ and any locally free $\underline{O}_M$-Module $\underline{F}$.

1.3.1 Local complete intersections

1.3.1.1 Let $(Z,\underline{O}_Z)$ be an analytic subspace of $(M,\underline{O}_M)$. We say that $(Z,\underline{O}_Z)$ is locally a complete intersection of codimension m if

$\underline{O}_Z = \underline{O}_M/\underline{I}$ for some $\underline{O}_M$-Ideal $\underline{I}$, and for each $z \in Z$ there exists a neighbourhood U of z in M, and $f_1,\ldots,f_m \in \Gamma(U,\underline{O}_M)$ such that at each point $x \in U \cap Z$ the germs $f_{1,x},\ldots,f_{m,x}$ form a regular $\underline{O}_{M,x}$-sequence generating the ideal $\underline{I}_x$.

The inclusion $(Z,\underline{O}_Z) \longrightarrow (M,\underline{O}_M)$ is then sometimes called a <u>regular immersion</u>. (C.f. [<u>11</u>; 0.15.2].)

<u>1.3.1.2</u> <u>Examples</u>: (1) Any non-singular complex submanifold Z of M with its usual structure sheaf is locally a complete intersection in M. Simply choose local holomorphic coordinates $f_1,\ldots,f_n$ for M at each point of Z, so that Z is defined by the vanishing of $f_1,\ldots,f_m$.

(2) Note that $(Z,\underline{O}_Z)$ is not required to be reduced; its structure sheaf may contain nilpotent elements. For example, let $f_1,\ldots,f_n$ be holomorphic functions defined on a neighbourhood U of $0 \in \mathbb{C}^n$, such that 0 is the only common zero of the f_i in U. Let $\underline{I}$ be the $\underline{O}_U$-Ideal generated by $f_1,\ldots,f_n$. Then $(\{0\},\underline{O}_{U,0}/\underline{I}_0)$ is a local complete intersection in $(U,\underline{O}_U)$. In fact the f_i form a regular sequence in U. For a proof see [<u>28</u>; p. 194] and the references given there.

<u>1.3.1.3</u> In the situation of 1.3.1.1 we see that in the neighbourhood of each point $z \in Z$ there is a resolution of the sheaf $\underline{O}_Z$ by free $\underline{O}_M$-Modules; namely, the Koszul complex $\underline{K}_\bullet(\underline{f})$, constructed precisely as in 1.2.2.1, with $\underline{K}_\bullet(\underline{f})_z = K_\bullet(\underline{f}_z)$.

Let $i : (Z,\underline{O}_Z) \longrightarrow (M,\underline{O}_M)$ be the inclusion, with $i_*\underline{O}_Z = \underline{O}_M/\underline{I}$, and regard $\underline{I}/\underline{I}^2$ as an $\underline{O}_Z$-Module. Then, by 1.2.2.3 and the fundamental local isomorphism, there is a canonical isomorphism of $\underline{O}_M$-Modules, for any locally free $\underline{O}_M$-Module $\underline{F}$:

$$i_* \underline{Hom}_{\underline{O}_Z}(\Lambda^m \underline{I}/\underline{I}^2, i^* \underline{F}) = \underline{Ext}_{\underline{O}_M}(\underline{O}_Z, \underline{F})$$

and $\underline{Ext}^k_{\underline{O}_M}(\underline{O}_Z, \underline{F}) = 0$ for $k \neq m$.

Note that since the isomorphisms are independant of any generating sequence for $\underline{I}$, this is a global isomorphism.

The above can be applied immediately to the spectral sequence of 1.1.4.3 to deduce from the vanishing of the Ext sheaves that this spectral sequence degenerates to give an isomorphism:

$$\Gamma(M, i_* \underline{Hom}_{\underline{O}_Z}(\Lambda^m \underline{I}/\underline{I}^2, i^* \underline{F})) \cong \mathrm{Ext}^m(M, \underline{O}_Z, \underline{F}).$$

1.3.2 Local Čech cohomology and the Koszul complex

It is very illuminating to see how the results of the preceding section are related to the local Čech cohomology of 1.1.3. By analogy with 1.2.2.2 there is a cochain complex of sheaves in a neighbourhood U of each point $z \in Z$:

$$\underline{K}^\bullet(\underline{f}, \underline{F}) = \underline{Hom}_{\underline{O}_M}(\underline{K}_\bullet(\underline{f}), \underline{F})$$

with $\underline{f} = (f_1, \ldots, f_n) \in \Gamma(U, \underline{F})^n$. Let $\mathcal{U}$ be the cover of $U - Z$ formed by the open sets

$$U_i = \{x \in U \mid f_i(x) \neq 0\}$$

and let $\mathcal{U}'$ be the cover of U obtained by adding the open set $U_0 = U$.

Then there is a canonical morphism of cochain complexes

$$c : \underline{K}^\bullet(\underline{f}, \underline{F}) \longrightarrow \underline{C}^\bullet_Z(\mathcal{U}', \underline{F})$$

defined, for $i_0 < i_1 < \cdots < i_p$ by

$$c(a)\langle i_0 \cdots i_p \rangle = a(e_{i_1} \wedge \cdots \wedge e_{i_p})/f_{i_1} \cdots f_{i_p} \quad \text{if} \quad i_0 = 0$$

$$= 0 \quad \text{otherwise.}$$

It is straightforward to check that this defines a cochain map, and by passing to cohomology we obtain homomorphisms over the set U;

$$\bar{c} : \underline{\mathrm{Ext}}^{\bullet}_{\underline{O}_M}(\underline{O}_Z, \underline{F}) \longrightarrow \underline{H}^{\bullet}_Z(\mathcal{U}', \underline{F}).$$

It is easy to check that this is compatible with the mappings of both sides into the local cohomology sheaves. The proof is given in the following lemma.

1.3.2.2 Lemma.

There is a computative diagram, for all $k \geq 0$

$$\begin{array}{ccc} \underline{\mathrm{Ext}}^{k}_{\underline{O}_M}(\underline{O}_Z, \underline{F}) & \xrightarrow{\bar{c}} & \underline{H}^{k}_Z(\mathcal{U}', \underline{F}) \\ & {\scriptstyle (1.1.4.6)} \searrow \quad \swarrow {\scriptstyle (1.1.3.6)} & \\ & \underline{H}^k_Z(\underline{F}) & \end{array}$$

Proof. Let $\underline{I}^{\bullet}$ be an injective resolution of $\underline{F}$ over the open set U. Then there is a commutative diagram for $p,q \geq 0$;

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_{\underline{O}_M}(\underline{K}_p(\underline{f}), \underline{F}) & \xrightarrow{c} & \underline{C}^p_Z(\mathcal{U}', \underline{F}) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}_{\underline{O}_M}(\underline{K}_p(\underline{f}), \underline{I}^q) & \xrightarrow{c} & \underline{C}^p_Z(\mathcal{U}', \underline{I}^q) \\ \uparrow & & \uparrow \\ \underline{\mathrm{Hom}}_{\underline{O}_M}(\underline{O}_Z, \underline{I}^q) & \longrightarrow & \underline{\Gamma}_Z(\underline{I}^q). \end{array}$$

The right column consists of 1.1.3.5 and the left column is the usual construction used to relate the definitions of the functor Ext by projectives and injectives [6; ch. 5]. The required result follows immediately by passing to cohomology.

2.1 The holomorphic Lefschetz class

If M is a smooth compact oriented manifold of real dimension n, then it is well known [21; p. 124] that there is a canonically defined cohomology class in $H^n(M \times M, M \times M - \Delta, \mathbb{Z})$ whose image in $H^n(M \times M, \mathbb{Z})$ is the Poincaré dual of the homology class defined by the diagonal, and which under the Künneth formula and Poincaré duality gives the "alternating sum of the identities" in the endomorphism groups of the $H^i(M, \mathbb{Z})$. This is the basic fact needed to prove the classical Lefschetz fixed-point formula.

The techniques of local cohomology developed in the preceding sections will now be used to show the existence, for any compact complex-analytic manifold of complex dimension n, of a canonical element in $H^n_\Delta(M \times M, p_1^* \underline{\Omega}^n)$ which will play the same role as the topological class mentioned above. The Künneth formula for sheaf cohomology and Serre duality replace the analogous constructions for integer cohomology. In fact the construction will allow for cohomology with coefficients in any locally free $\underline{O}_M$-Module and it is this general case which is treated below.

2.1.1 Let M be a compact complex-analytic manifold with $\dim_{\mathbb{C}} M = n$; let $\underline{O}_M$ be the sheaf of germs of holomorphic functions and $\underline{\Omega}^p$ the sheaf of germs of holomorphic p-forms on M.

If E is a holomorphic vector bundle on M, let E^* be the holomorphic dual bundle. The corresponding $\underline{O}_M$-Modules will be denoted by $\underline{E}$ and $\underline{E}^*$ respectively.

Also we define $\underline{E}' = \underline{E}^* \otimes_{\underline{O}_M} \underline{\Omega}^n$, and let

$$Tr : \underline{E}' \otimes_{\underline{O}_M} \underline{E} \longrightarrow \underline{\Omega}^n$$

be the natural pairing.

We may also form the product $M \times M$ with structure sheaf $\underline{O}_{M\times M}$, and there are holomorphic maps

$$M \xrightarrow{\Delta} M \times M \xrightarrow{p_i} M \qquad (i = 1,2)$$

with $\Delta(x) = (x,x)$ and $p_i(x_1,x_2) = x_i$.

If E and F are locally free sheaves on M, we use the notation

$$\underline{E} \boxtimes \underline{F} = p_1^*\underline{E} \otimes_{\underline{O}_{M\times M}} p_2^*\underline{F}.$$

It is easily checked that $(\underline{E}')' = \underline{E}$ and $(\underline{E} \boxtimes \underline{E}')' = \underline{E}' \boxtimes \underline{E}$.

2.1.2 Serre Duality

For further details of the following constructions the reader is referred to the original paper [23].

First we recall the following consequence of the Künneth formula for sheaves. With the notation of the preceding paragraph, there is a natural isomorphism of finite-dimensional vector spaces

$$\varkappa : \sum_{k=0}^{n} H^{n-k}(M,\underline{E}') \otimes H^k(M,\underline{E}) \longrightarrow H^n(M \times M, \underline{E}' \boxtimes \underline{E}).$$

(In fact $\Delta^* \circ \varkappa$ gives the cup-product pairing.) The element $\varkappa(\alpha \otimes \beta)$ will also be written as $\alpha \times \beta$.

Then (Serre duality) there are perfect (i.e. non-singular) bilinear pairings

$$H^k(M,\underline{E}) \times H^{n-k}(M,\underline{E}') \xrightarrow{<\,,\,>_M} \mathbb{C}$$

$$H^n(M \times M,\ \underline{E} \boxtimes \underline{E}') \times H^n(M \times M,\ \underline{E}' \boxtimes \underline{E}) \xrightarrow{<\,,\,>_{M\times M}} \mathbb{C}$$

which can be explicitly described (in the first example) by

$$<\,,\,>_M = \int_M \circ \operatorname{Tr} \circ \Delta^* \circ \varkappa$$

where $\int_M : H^n(M,\underline{\Omega}^n) \longrightarrow \mathbb{C}$ is defined by representing the cohomology class by a form (resp. current) of type (n,n) and integrating (resp. evaluating on 1).

2.1.2.2 If $a,d \in H^k(M,\underline{E})$ and $b,c \in H^{n-k}(M,\underline{E}')$ we obtain immediately from the definition of the Serre duality pairing:

$$< a \times b,\ c \times d >_{M\times M} = (-1)^k < a,c >_M < b,d >_M .$$

If we let $\delta \in H^n(M \times M,\ \underline{E}' \boxtimes \underline{E})$ be the element defined uniquely by:

$$< a \times b,\ \delta >_{M\times M} = < a,b >_M$$

then it follows by elementary linear algebra that under the identification

$$\sum_{k=0}^{n} H^{n-k}(M,\underline{E}') \otimes H^k(M,\underline{E}) = \sum_{k=0}^{n} \operatorname{End}_{\mathbb{C}} H^k(M,\underline{E})$$

we have:

2.1.2.3 $$\varkappa^{-1}(\delta) = \sum_{k=0}^{n} (-1)^k \times \text{identity on } H^k(M,\underline{E})$$

so that, for example

$$\int_M \operatorname{Tr} \circ \Delta^*(\delta) = \sum_{k=0}^{n} (-1)^k \dim_{\mathbb{C}} H^k(M,\underline{E}).$$

It is easy to represent the cohomology class δ as a current on $M \times M$. In fact, suppose that u is a C^∞ form on $M \times M$ of type $(0,n)$ with coefficients in $\underline{E} \boxtimes \underline{E}'$. The topological dual of the space of such forms is the space $D^n(M \times M, \underline{E}' \boxtimes \underline{E})$ of currents of type $(0,n)$ with coefficients in $\underline{E}' \boxtimes \underline{E}$, so that the linear form

2.1.2.3 $$i_\Delta : u \longmapsto \int_M \mathrm{Tr} \circ \Delta^*(u)$$

defines such a current, which is evidently $\bar\partial$-closed. Comparison with 2.1.2.1 shows that the cohomology class defined by this current is δ.

2.1.2.4 It can now be shown that δ actually lies in the local cohomology group $H^n_\Delta(M \times M, \underline{E}' \boxtimes \underline{E})$. If we write $\underline{D}^m$ for the sheaf of germs of currents of type $(0,m)$ with coefficients in $\underline{E}' \boxtimes \underline{E}$, there is a fine resolution (see [23]):

$$0 \longrightarrow \underline{E}' \boxtimes \underline{E} \longrightarrow \underline{D}^0 \xrightarrow{\bar\partial} \underline{D}^1 \xrightarrow{\bar\partial} \cdots .$$

If now $\underline{J}^\bullet$ is an injective resolution of $\underline{E}' \boxtimes \underline{E}$, then by a standard property of injective resolutions the identity map on $\underline{E}' \boxtimes \underline{E}$ can be lifted to a cochain map between the two complexes:

2.1.2.5 $$\underline{D}^\bullet \longrightarrow \underline{J}^\bullet .$$

Moreover, if we apply the functor $\Gamma(M \times M, \bullet)$ to the two complexes then it is well-known that this cochain map induces isomorphisms on the cohomology [6; th. 4.7.1]. However, it is more interesting to apply the functor $\Gamma_\Delta(M \times M, \bullet)$. The section of $\underline{D}^n$ defined by the current i_Δ clearly has support in the diagonal Δ and so its image lies in $\Gamma_\Delta(M \times M, \underline{J}^n)$ and therefore represents an element of $H^n_\Delta(M \times M, \underline{E}' \boxtimes \underline{E})$ which we denote by δ_Δ.

In a similar way i_Δ defines an element of $\mathrm{Ext}^n(M \times M, \underline{O}_\Delta, \underline{E}' \boxtimes \underline{E})$, where $\underline{O}_\Delta = \Delta_* \underline{O}_M$. To see this let $\underline{I}$ be the $\underline{O}_{M\times M}$-Ideal defined by the diagonal, so that $\underline{O}_\Delta = \underline{O}_{M\times M}/\underline{I}$. Then $\underline{I}$ is contained in the Annihilator of the section i_Δ and so i_Δ can be regarded as an element of $\mathrm{Hom}(M \times M, \underline{O}_\Delta, \underline{D}^n)$. Now use 2.1.2.5 to obtain the required element of the Ext group.

Alternatively we could carry out all these constructions at the sheaf level and obtain sections of $\underline{H}^n_\Delta(\underline{E}' \boxtimes \underline{E})$ and $\underline{\mathrm{Ext}}^n_{\underline{O}_{M\times M}}(\underline{O}_\Delta, \underline{E}' \boxtimes \underline{E})$. All four of these interpretations are then mapped to each other in the commutative diagram obtained from 1.1.4.5,

$$\begin{array}{ccc} \mathrm{Ext}^n(M \times M, \underline{O}_\Delta, \underline{E}' \boxtimes \underline{E}) & \xrightarrow{\text{edge}} & H^0(M \times M, \underline{\mathrm{Ext}}^n_{\underline{O}_{M\times M}}(\underline{O}_\Delta, \underline{E}' \boxtimes \underline{E})) \\ \downarrow & & \downarrow \\ H^n_\Delta(M \times M, \underline{E}' \boxtimes \underline{E}) & \xrightarrow{\text{edge}} & H^0(M \times M, \underline{H}^n_\Delta(\underline{E}' \boxtimes \underline{E})) \end{array}$$

in which the upper horizontal morphism is an isomorphism since the diagonal is a local complete intersection of codimension n in the product (1.3.1.4). In fact the lower horizontal morphism is also an isomorphism, (although we do not need this). This follows from the fact that the sheaves $\underline{H}^k_\Delta(\underline{E}' \boxtimes \underline{E})$ vanish for $k < n$ [24; th. 3.3].

2.1.3 The class δ_Δ and the fundamental local isomorphism

The principal objection to the methods of representing the cohomology class δ_Δ described above is that it is not clear how the class behaves under mappings. To be more precise, it is necessary in order to prove the fixed point formula for a holomorphic map $f : M \longrightarrow M$ to be able to construct the class $\mathrm{Tr} \circ \Gamma^*(\delta_\Delta) \in H^n_F(M, \underline{\Omega}^n)$, where F is the set of fixed points of f and Γ is the graph morphism $x \mapsto (x, f(x))$. However, there is no natural way to pull back currents, and so we adopt the following alternative description of δ.

2.1.3.1 In the notation of section 2.1 there is a canonical isomorphism as in 1.3.1.3

$$\underline{\mathrm{Ext}}^n_{\underline{O}_{M\times M}}(\underline{O}_\Delta, \underline{E}' \boxtimes \underline{E}) = \Delta_* \underline{\mathrm{Hom}}_{\underline{O}_M}(\Lambda^n \underline{I}/\underline{I}^2, \underline{E}' \otimes \underline{E})$$

where $\underline{I}/\underline{I}^2$ is now regarded as an $\underline{O}_M$-Module.

Now if $\underline{N}$ is the locally free sheaf associated to the (holomorphic) normal bundle of the diagonal in $M \times M$, with dual $\underline{N}^*$, we have:

$$\underline{I}/\underline{I}^2 = \underline{N}^*$$

- the isomorphism being given by mapping the class of a germ f mod $\underline{I}^2$, which we denote by $[f]$, to its differential df. However, we also have:

$$\underline{\Omega}^1 = \underline{N}^*$$

since both appear as the kernel in the short exact sequence

$$0 \longrightarrow \underline{N}^* \longrightarrow \Delta^* \underline{\Omega}^1_{M\times M} \longrightarrow \underline{\Omega}^1 \longrightarrow 0$$

$$(dx, dy) \mapsto dx + dy$$

or

$$0 \longrightarrow \underline{\Omega}^1 \longrightarrow \Delta^* \underline{\Omega}^1_{M\times M} \longrightarrow \underline{\Omega}^1 \longrightarrow 0$$

$$dx \mapsto (dx, -dx).$$

Thus we have the canonical isomorphism

2.1.3.2 $$\underline{\mathrm{Ext}}^n_{\underline{O}_{M\times M}}(\underline{O}_\Delta, \underline{E}' \boxtimes \underline{E}) = \Delta_* \underline{\mathrm{Hom}}_{\underline{O}_M}(\underline{\Omega}^n \otimes \underline{E}, \underline{\Omega}^n \otimes \underline{E}).$$

Claim: The global section of $\underline{\mathrm{Ext}}^n_{\underline{O}_{M\times M}}(\underline{O}_\Delta, \underline{E}' \boxtimes \underline{E})$ obtained by taking the identity section of the right-hand side of 2.1.3.2 is the same as that defined by the current i_Δ (2.1.2.5).

In fact it will be shown that the following four constructions are essentially identical. For the sake of brevity the discussion will be limited to the case where E is the trivial line bundle, and the modifications needed in the general case will be indicated later.

(1). The identity section in $\Delta_* \underline{\mathrm{Hom}}_{\underline{O}_M}(\underline{\Omega}^n, \underline{\Omega}^n)$.

(2). If U is an open subset of M, on which there exist local coordinates $z_1, \ldots, z_n$, let $(z_1, \ldots, z_n, \zeta_1, \ldots, \zeta_n)$ be local coordinates for $U \times U$, where $z_i = p_1^* z_i$ and $\zeta_i = p_2^* z_i$. Let U_i be the open subset of $U \times U$ on which $z_i - \zeta_i$ is non-zero, and let $\mathcal{U}'$ be the open cover given by $U \times U$ itself and the U_i. Then as in 1.1.3 the Cauchy kernel

$$\frac{dz_1 \wedge \cdots \wedge dz_n}{(z_1-\zeta_1)\cdots(z_n-\zeta_n)} \in \Gamma(U \times U \cap U_1 \cap \cdots \cap U_n, p_1^* \underline{\Omega}^n)$$

defines a local Čech cohomology class in $H^n_\Delta(\mathcal{U}', p_1^* \underline{\Omega}^n)$, or alternatively a section of the sheaf $\underline{H}^n_\Delta(\mathcal{U}', p_1^* \underline{\Omega}^n)$.

(3). If U is as in (2) then the Bochner-Martinelli kernel

$$k(z,\zeta) = C_n \sum_{k=1}^{n} (-1)^{k+1} \frac{\bar{z}_k - \bar{\zeta}_k}{|z-\zeta|^{2n}} d(\bar{z}_1 - \bar{\zeta}_1) \cdots \widehat{d(\bar{z}_k - \bar{\zeta}_k)} \cdots d(\bar{z}_n - \bar{\zeta}_n) dz_1 \cdots dz_n$$

$$\left(\text{where } C_n = (-1)^{n(n-1)/2} \times \frac{(n-1)!}{(2\pi i)^n}\right)$$

is a $\bar{\partial}$-closed C^{∞}-form in $U \times U - \Delta$, of type $(0,n-1)$ with coefficients in $p_1^*\underline{\Omega}^n$. Moreover k can be regarded as a <u>current</u> on $U \times U$, in which case $\bar{\partial}k$ is a $\bar{\partial}$-closed current with support on the diagonal. It is a basic property of k that $\bar{\partial}k$ is precisely the restriction of the current i_Δ of 2.1.2.3 to the open set $U \times U$. A corresponding section of the sheaf $H^n(\underline{\Gamma}_\Delta \underline{D}^\bullet(p_1^*\underline{\Omega}^n))$ may be obtained by sheafifying and passing to cohomology.

(4). As we have already seen the "alternating sum of the identities" defines a class in $H^n_\Delta(M \times M, p_1^*\underline{\Omega}^n)$ which by the edge homomorphism 1.1.2.5 of the local cohomology spectral sequence gives a section of the sheaf $\underline{H}^n_\Delta(p_1^*\underline{\Omega}^n)$.

The claim will be put into the form of the following proposition.

<u>2.1.3.2</u> <u>Proposition</u>

There is a (naturally defined) commutative diagram of $\underline{O}_{M\times M}$-Modules:

$$\begin{array}{ccc}
\Delta_*\underline{\mathrm{Hom}}_{\underline{O}_M}(\Omega^n,\Omega^n) & \xrightarrow{\alpha_{12}} & \underline{H}^n_\Delta(\mathcal{U}', p_1^*\underline{\Omega}^n) \\
\downarrow{\scriptstyle \alpha_{14}} & & \downarrow{\scriptstyle \alpha_{23}} \\
\underline{H}^n_\Delta(p_1^*\underline{\Omega}^n) & \xleftarrow[\alpha_{34}]{} & H^n(\underline{\Gamma}_\Delta \underline{D}^\bullet(p_1^*\underline{\Omega}^n))
\end{array}$$

in which the sections (1) to (4) are mapped to each other.

<u>Proof</u>. Let $J^\bullet$ be an injective resolution of $p_1^*\Omega^n$. There is commutative diagram obtained from 1.3.2.2 and 2.1.2.5;

$$\begin{array}{ccccc}
\underline{\mathrm{Hom}}_{\underline{O}_{M\times M}}(\underline{K}_p(\underline{z}-\underline{\zeta}),p_1^*\underline{\Omega}^n) & \longrightarrow & \underline{C}^p_\Delta(\mathcal{U}',p_1^*\underline{\Omega}^n) & = & \underline{C}^p_\Delta(\mathcal{U}',p_1^*\underline{\Omega}^n) \\
\downarrow & & \downarrow & & \downarrow \\
\underline{\mathrm{Hom}}_{\underline{O}_{M\times M}}(\underline{K}_p(\underline{z}-\underline{\zeta}),\underline{J}^q) & \longrightarrow & \underline{C}^p_\Delta(\mathcal{U}',\underline{J}^q) & \longleftarrow & \underline{C}^p_\Delta(\mathcal{U}',\underline{D}^q) \\
\uparrow & & \uparrow & & \uparrow\beta \\
\underline{\mathrm{Hom}}_{\underline{O}_{M\times M}}(\underline{O}_\Delta,J^q) & \longrightarrow & \underline{\Gamma}_\Delta(\underline{J}^q) & \longleftarrow & \underline{\Gamma}_\Delta(\underline{D}^q)
\end{array}$$

Passing to cohomology and using the fundamental local isomorphism 2.1.3.2 we obtain the required commutative diagram. However, in order that the four sections should correspond in this diagram, we make the minor modification of replacing the boundary operator $\bar{\partial}$ of the complex $\underline{D}^\bullet$ by $\hat{\partial} = (2\pi i)^{-1}\bar{\partial}$.

Then α_{12} is the composition of the fundamental local isomorphism and the map $\bar{c}$ of 1.3.2.1 and it is easy to check that (1) and (2) correspond under this map.

Also α_{14} is the composition of the fundamental local isomorphism and 1.1.4.6. The only non-trivial part of the proof is to define the map α_{23} and show that the Cauchy kernel and the Bochner-Martinelli kernel correspond under this map. This is essentially proved by F. R. Harvey in [14] using an analogue of the Dolbeault isomorphism. To make clear how this construction fits into the present situation the method used by Harvey is described below.

First recall that since $\underline{D}^q$ is a fine sheaf, $\underline{H}^p_\Delta(\mathcal{U}',\underline{D}^q) = 0$ for all q and all $p > 0$ [6; th. 5.2.3]. Thus the map β induces an isomorphism on cohomology. The problem is to make explicit the inverse of this isomorphism. This will be done in the usual way by constructing a homotopy operator in the complex $\underline{C}^p_\Delta(\mathcal{U}',\underline{D}^q)$.

The result is local, so we work in $\mathbb{C}^n \times \mathbb{C}^n$ with coordinates $(z_1,\ldots,z_n, \zeta_1,\ldots,\zeta_n)$ and open cover $\mathcal{U}'$ consisting of the sets $U_i = \{(z,\zeta) \in \mathbb{C}^n \times \mathbb{C}^n | z_i \neq \zeta_i\}$ and $U_0 = \mathbb{C}^n \times \mathbb{C}^n$. For a multi-index $I = (i_1,\ldots,i_p)$ we use the following notation

$$(z-\zeta)^I = (z_{i_1}-\zeta_{i_1})\cdots(z_{i_p}-\zeta_{i_p}); \quad dz^I = dz_{i_1}\cdots dz_{i_p};$$

$$U_I = U_{i_1} \cap \cdots \cap U_{i_p}$$

and if $f \in C^p_\Delta(\mathcal{U}',\underline{D}^q)$ then

$$f_I = f < 0,i_1,\ldots,i_p > \in \Gamma(U_I,\underline{D}^q).$$

Define $g_i(z,\zeta) = \bar{z}_i-\bar{\zeta}_i/|z-\zeta|^2$. Then the required homotopy operator is given by

$$e_g : C^p_\Delta(\mathcal{U}',\underline{D}^q) \longrightarrow C^{p-1}_\Delta(\mathcal{U}',\underline{D}^q)$$

where $e_g(f)_I = \sum_{i=1}^n (z_i-\zeta_i)g_i(z,\zeta)f_{iI}$. Since $\sum_{i=1}^n (z_i-\zeta_i)g_i(z,\zeta) = 1$, it is easy to check that

2.1.3.3 $$\delta \circ e_g + e_g \circ \delta = \text{identity}.$$

Now the Cauchy kernel is given by $f^0 \in C^n_\Delta(\mathcal{U}',\underline{D}^0)$ where $f^0_{1\ldots n} = dz_1\cdots dz_n/(z_1-\zeta_1)\cdots(z_n-\zeta_n)$. Thus if we define a sequence of elements $f^k \in C^{n-k}_\Delta(\mathcal{U}',\underline{D}^k)$ by $f^{k+1} = \hat{\partial} \circ e_g(f^k)$, then by property 2.1.3.3 of the map e_g each f^k will be $\delta + \hat{\partial}$ cohomologous to f^0. Therefore the $\hat{\partial}$-cohomology class of f^n will be the image of the Cauchy kernel under the isomorphism

$$\beta^{*-1} : H^n(C^\bullet_\Delta(\mathcal{U}',\underline{D}^\bullet)) \xrightarrow{\sim} H^n(\Gamma_\Delta(\mathbb{C}^n \times \mathbb{C}^n, \underline{D}^\bullet))$$

induced by β.

2.1.3.4 Lemma. Let J be the multi-index $(i_1,\ldots,i_{n-p})$. Then

$$f^p_J = (\hat{\partial} \circ e_g)^p (f^0)_J$$
$$= (-1)^{p(p-1)/2}(2\pi i)^{-p} \sum_{|I|=p} (\bar{\partial}g)^I dz^I dz^J \Big/ (z-\zeta)^J.$$

Proof by induction ($p = 0$ trivial).

$$(\hat{\partial} \bullet e_g)(f^p)_K = (-1)^{p(p-1)/2}(2\pi i)^{-p-1} \sum_{|I|=p} \sum_i \bar{\partial}g_i (\bar{\partial}g)^I dz^I dz_i dz^K \Big/ (z-\zeta)^K$$
$$= (-1)^{p(p+1)/2}(2\pi i)^{-(p+1)} \sum_{|L|=p+1} (\bar{\partial}g)^L dz^L dz^K \Big/ (z-\zeta)^K$$

(by putting $L = Ii$).

2.1.3.5 Corollary

$$f^n = (-1)^{n(n-1)/2} \times \frac{n!}{(2\pi i)^n} \times \bar{\partial}g_1 \cdots \bar{\partial}g_n dz_1 \cdots dz_n$$
$$= \bar{\partial}k(z,\zeta)$$

say where

$$k(z,\zeta) = (-1)^{n(n-1)/2} \times \frac{(n-1)!}{(2\pi i)^n} \times \sum_k (-1)^{k+1} g_k \bar{\partial}g_1 \cdots \widehat{\bar{\partial}g_k} \cdots \bar{\partial}g_n dz_1 \cdots dz_n$$

and k is in fact the Bochner-Martinelli kernel [14; p. 87]. Note that k is $\bar{\partial}$-closed on any point not on the diagonal since there

$$\bar{\partial}g_1 \cdots \bar{\partial}g_n = \bar{\partial}(g_1 + \cdots + g_n)\bar{\partial}g_2 \cdots \bar{\partial}g_n = \bar{\partial}1\bar{\partial}g_2 \cdots \bar{\partial}g_n = 0.$$

2.1.3.6 Proposition. The kernel k has the following properties

(a) k is $\bar{\partial}$-closed on $\mathbb{C}^n \times \mathbb{C}^n - \Delta$

(b) Let $i_\xi : \mathbb{C}^n \longrightarrow \mathbb{C}^n \times \mathbb{C}^n$ be the inclusion

$$i_\xi(z) = (\xi, z + \xi)$$

and for $\varepsilon > 0$ let E_ε be the ellipsoid in $i_\xi \mathbf{C}^n$ given by $\zeta = \xi$ and $|z-\zeta| = \varepsilon$. Then

$$\int_{E_\varepsilon} i_\xi^* k = 1.$$

(c) Let u be a form of type $(0,n)$ with compact support in $\mathbf{C}^n \times \mathbf{C}^n$, and with coefficients in $p_1^*\Omega^n$. Then

$$\int_{\mathbf{C}^n \times \mathbf{C}^n} k \wedge \bar{\partial} u = \int_{\mathbf{C}^n} \Delta^* u.$$

Proof. (a) follows from the above remark. For (b) put $h_i(z) = g_i(\xi, z+\xi)$ and obtain

$$\int_{E_\varepsilon} i_\xi^* k = (-1)^{n(n-1)/2} \times \frac{(n-1)!}{(2\pi i)^n} \times \int_{E_\varepsilon} \sum_k (-1)^{k+1} h_k \bar{\partial} h_1 \cdots \widehat{\bar{\partial} h_k} \cdots \bar{\partial} h_n dz_1 \cdots dz_n.$$

The integrand is $\bar{\partial}$- and hence d-closed on $\mathbf{C}^n - \{0\}$ so the surface of integration can be deformed into the unit $2n - 1$ sphere S about the origin and the integral becomes

$$\text{const.} \times \int_S \sum_k (-1)^{k+1} \bar{z}_k d\bar{z}_1 \cdots \widehat{d\bar{z}_k} \cdots d\bar{z}_n dz_1 \cdots dz_n$$

$$= \text{const.} \times (-1)^{n(n-1)/2} \times (2i)^n \times \frac{1}{2} \times \int_S \underline{N} \lrcorner \, dV$$

where $\underline{N}$ is the unit normal on S and $dV = dx_1 dy_1 \cdots dx_n dy_n$ is the volume form for the canonical orientation on $\mathbf{C}^n$. The symbol $\lrcorner$ denotes interior product. But this is just 1 since the volume of the unit $2n - 1$ sphere is $2\pi^n/(n-1)!$.

For part (c) we have

$$\int_{\mathbf{C}^n \times \mathbf{C}^n} k \wedge \bar{\partial} u = \int_{\mathbf{C}^n \times \mathbf{C}^n} d(k \wedge u) = \lim_{\varepsilon \to 0} \int_{|z-\zeta|=\varepsilon} k \wedge u$$

$$= \int_{\mathbf{C}^n} \Delta^* u.$$

The first equality follows from (a) and considerations of type. The second follows from Stokes' theorem. For the third, integrate first over the fibre ζ = constant and use (b).

This also completes the proof of proposition 2.1.3.2.

2.1.4 Generalisation to arbitrary holomorphic vector bundles

The discussion in the preceding section was of a local nature, so if E is a holomorphic vector bundle over M we may locally choose generating sections $s_1, \dots, s_m$ of $\underline{E}$, and let $s_1^*, \dots, s_m^*$ be the corresponding dual sections in $\underline{E}^*$. We can then tensor everything by the section $\sum_i s_i^*(z) \otimes s_i(\zeta)$ of $\underline{E} \boxtimes \underline{E}^*$ and obtain the required generalisation.

3.1 Holomorphic geometrical endomorphisms

We now discuss the situation in which the holomorphic fixed point theorem of Atiyah-Bott can be formulated. As before M is a compact complex manifold of complex dimension n, and E is a locally free $\underline{O}_M$-Module. Then a holomorphic geometrical endomorphism of E consists of a pair (f, φ) where $f : M \longrightarrow M$ is a holomorphic map and $\varphi : f^*\underline{E} \longrightarrow \underline{E}$ is a homomorphism of $\underline{O}_M$-Modules.

Under these circumstances there are induced homomorphisms of cohomology groups

$$H^k(M, \underline{E}) \xrightarrow{f^*} H^k(M, f^*\underline{E}) \xrightarrow{\varphi} H^k(M, \underline{E})$$

where the first map is the standard pull-back, and the second is induced, by functionality, by φ. The composition gives a $\mathbf{C}$-linear endomorphism of the finite-dimensional complex vector space $H^k(M,\underline{E})$ which we denote by $H^k(f,\varphi)$.

Examples. (1) If $\underline{E} = \underline{O}_M$, then $f^*\underline{E} = \underline{E}$ and so φ can be taken to be the identity.

(2) If $\underline{E}$ is the sheaf of germs of holomorphic p-forms $\underline{\Omega}^p$ then φ can be taken to be the pth exterior power of the natural bundle map

$$df : f^*(T^*M) \longrightarrow T^*M$$

where T^*M is the holomorphic cotangent bundle of M.

3.1.1 The Lefschetz number of a geometrical endomorphism

With the notation of the preceding paragraph, one obtains a geometrical endomorphism $(1 \times f, 1 \boxtimes \varphi)$ of the $\underline{O}_{M\times M}$-Module $\underline{E}' \boxtimes \underline{E}$, and it is a standard property of the Künneth decomposition 2.1.2.1 that the induced endomorphism $H^n(1 \times f, 1 \boxtimes \varphi)$ of $H^n(M \times M, \underline{E}' \boxtimes \underline{E})$ is given by:

$$\kappa^{-1} \circ H^n(1 \times f, 1 \boxtimes \varphi) \circ \kappa = \sum_{k=0}^{n} \{\text{identity on } H^{n-k}(M,\underline{E}')\} \otimes H^k(f,\varphi)$$

and by elementary linear algebra one obtains, by analogy with 2.1.2.3,

$$\int_M \mathrm{Tr} \circ \Delta^* \circ H^n(1 \times f, 1 \boxtimes \varphi)(\delta) = \sum_{k=0}^{n} (-1)^k \mathrm{trace}_{\mathbb{C}}\, H^k(f,\varphi)$$

$$= \chi(f,\varphi) \quad \text{say.}$$

However we may also write:

$$\mathrm{Tr} \circ \Delta^* \circ H^n(1 \times f, 1 \boxtimes \varphi) = \mathrm{Tr} \circ (1 \otimes \varphi) \circ \Delta^* \circ (1 \times f)^*$$

$$= \mathrm{Tr} \circ (1 \otimes \varphi) \circ \Gamma^*$$

where $\Gamma : M \longrightarrow M \times M$ is the <u>graph</u> <u>morphism</u> $\Gamma(x) = (x, f(x))$.

$\chi(f,\varphi)$ will be called the <u>holomorphic</u> <u>Lefschetz</u> <u>number</u> of the geometrical endomorphism (f,φ). In the case where f has isolated fixed points it was proved in [2] that $\chi(f,\varphi)$ is determined completely by the behaviour of f and φ in the neighbourhood of the fixed points. But this now follows immediately from the preceding work, since if F is the set of fixed points of f, then $\Gamma^{-1}\Delta = F$ and so $\Gamma^*\delta_\Delta$ lies in $H^n_F(M, \underline{E}' \otimes f^*\underline{E})$. Applying the map $Tr \circ (1 \times \varphi)$ we obtain an element of $H^n_F(M, \underline{\Omega}^n)$ which by the excision formula 1.1.2.1 is the same as:

$$\bigoplus_{p \in F} H^n_{\{p\}}(V_p, \underline{\Omega}^n)$$

where V_p is a small open set containing p.

To complete the proof of the fixed-point formula as it appears in [25], [26] it is only necessary to determine the element $\Gamma^*\delta_\Delta$ of $H^n_F(M, \underline{\Omega}^n)$ and to make explicit the maps Res_p for which the following diagram commutes:

$$\begin{array}{ccc} H^n_{\{p\}}(V_p, \underline{\Omega}^n) & & \\ \downarrow & \searrow^{Res_p} & \\ & & \mathbb{C} \\ & \nearrow_{\int_M} & \\ H^n(M, \underline{\Omega}^n) & & \end{array} .$$

(The vertical map is the natural one from $H^n_{\{p\}}(M, \underline{\Omega}^n)$ to $H^n(M, \underline{\Omega}^n)$).

In this situation it is known that Res_p can be described purely algebraically and is in fact the <u>Grothendieck</u> <u>residue</u> of [13]. The next section briefly describes the concepts involved.

4.1 Local cohomology and the Grothendieck residue

Let $h_1, \ldots, h_n$ be holomorphic functions defined in a neighbourhood V of $p \in M$, such that p is the only common zero of the h_i in V. If $\underline{J}$ is the $\underline{O}_M$-Ideal generated by the h_i in V, then as in 1.2 any holomorphic n-form w on V defines a section of $\underline{\mathrm{Ext}}^n_{\underline{O}_M}(\underline{O}_M/\underline{J}, \underline{\Omega}^n)$ which under the fundamental local isomorphism is given by the section of $\underline{\mathrm{Hom}}_{\underline{O}_M/\underline{J}}(\Lambda^n \underline{J}/\underline{J}^2, \underline{\Omega}^n/\underline{J}.\underline{\Omega}^n)$ which takes the generator $[h_1] \wedge \cdots \wedge [h_n]$ of $\Lambda^n \underline{J}/\underline{J}^2$ to the class of $w \bmod \underline{J}.\underline{\Omega}^n$. The corresponding section of $\underline{\mathrm{Ext}}^n_{\underline{O}_M}(\underline{O}_M/\underline{J}, \underline{\Omega}^n)$ will be denoted by

$$\left[\frac{w}{h_1, \ldots, h_n}\right] .$$

Note that in terms of local Čech cohomology (1.3.2), if $V_i \subset V$ is the open set on which $h_i \neq 0$, then this cohomology element is also represented by the section $w/h_1 \cdots h_n \in \Gamma(V_1 \cap \cdots \cap V_n, \underline{\Omega}^n)$.

In the above notation, the Cauchy kernel can be expressed locally as the section

$$\left[\frac{dz_1 \cdots dz_n \otimes \sum_{i=1}^{m} s_i^*(z) \otimes s_i(\zeta)}{z_i - \zeta_1, \ldots, z_n - \zeta_n}\right]$$

of $\underline{\mathrm{Ext}}^n_{\underline{O}_{M \times M}}(\underline{O}_\Delta, \underline{E}' \boxtimes \underline{E})$. Then, just as the Cauchy kernel can be represented by a current with support on the diagonal so it is also easy to represent this class by a current of type (n,n) with support at p by means of the method of proposition 2.1.3.2. In fact, let $\alpha : U \longrightarrow \mathbb{C}^n \times \mathbb{C}^n$ be defined by $\alpha(z) = (z, z+h(z))$. Then it is clear from the description of the class in terms of Čech cohomology that it is obtained by pulling back the Cauchy kernel along α. By the functoriality of the Dolbeault construction the $\bar{\partial}$-cohomology class of the required current is determined by the pull-back α^*k of the Bochner-Martinelli kernel. Thus if $w = \varphi\, dz_1 \wedge \cdots \wedge dz_n$ and $f \in \underline{O}_{M,p}$, the map

$$f \longmapsto \int_S f\varphi\, \alpha^*k$$

defines the appropriate class, where S is a small, smooth $2n-1$ sphere around p. Hence we recover the integral formula of [25], [26]:

$$\mathrm{Res}_p\left[\frac{w}{h_1,\ldots,h_n}\right] = \int_S \varphi\, \alpha^*k.$$

4.2 The fixed point contributions in the holomorphic Lefschetz formula

Since F is discrete, the sheaf $\underline{H}^n_F(\underline{\Omega}^n)$ has discrete support and so the edge-homomorphism of the local cohomology spectral sequence, i.e.

$$H^n_F(M,\underline{\Omega}^n) \longrightarrow H^0(M,\underline{H}^n_F(\underline{\Omega}^n))$$

is essentially just the identity map in this case. Now using the behaviour of local cohomology under mappings (1.1.2.7) we can determine the element $\mathrm{Tr} \circ (1 \otimes \varphi) \circ \Gamma^*(\delta_\Delta)$ explicitly as follows.

Let $f_i(z)$ be the local coordinates of $f(z)$ in a neighbourhood V_p of the point $p \in F$, and let $\mathcal{V}$ be the open cover of $V_p - \{p\}$ consisting of the set $V_i = \{z \in V_p \,|\, f_i(z) \neq z_i\}$. Since the pull-back of Čech cohomology is compatible with the pull-back of cohomology defined by injective resolutions, it is possible to conclude that $\mathrm{Tr} \circ 1 \times \varphi \circ \Gamma^*(\delta_\Delta)$ is given in the neighbourhood of p by the section of $\underline{H}^n_{\{p\}}(\mathcal{V}', \underline{\Omega}^n)$ corresponding to

$$\frac{\mathrm{trace}\ \varphi(z) dz_1 \wedge \cdots \wedge dz_n}{z_1 - f_1(z), \ldots, z_n - f_n(z)} \in \Gamma(V_1 \cap \cdots \cap V_n, \underline{\Omega}^n).$$

Here $\mathrm{trace}\ \varphi(z) = \sum_{i=1}^{m} < s_i^*(z), \varphi(s_i(f(z))) >$.

Thus, in terms of the Ext functors, the class $\mathrm{Tr} \bullet 1 \otimes \varphi \bullet \Gamma^*(\delta_\Delta)$ can be expressed a direct sum of elements $\alpha_p \in \mathrm{Ext}^n(M, \underline{O}_{\{p\}}, \underline{\Omega}^n)$ where $\underline{O}_{\{p\}}$ is the quotient of $\underline{O}_M$ by the ideal generated by the germs of the functions $z_i - f_i(z)$, and

$$\alpha_p = \left[\frac{\mathrm{trace}\ \varphi(z) dz_1 \wedge \cdots \wedge dz_n}{z_1 - f_1(z), \ldots, z_n - f_n(z)} \right].$$

Thus we obtain the Atiyah-Bott formula for a holomorphic geometrical endomorphism with arbitrary isolated fixed points:

$$\chi(f, \varphi) = \sum_{p \in F} \mathrm{Res}_p(\alpha_p).$$

Note that the elements α_p are independant of choices of coordinates and generating sections since they are essentially just the restriction to the graph of the canonically defined Cauchy kernel.

4.3.1 An algorithm for calculating $\mathrm{Res}_p(\alpha_p)$

For the sake of completeness the method described in [3] by which the Grothendieck residue can be calculated algebraically is given below. The relation of this algorithm to the integral formula for the residue (4.1.1) is discussed in [27]. The following applies to the situation of paragraph 4.1. We wish to calculate the composition:

$$\mathrm{Ext}^n(M,\underline{O}_M/\underline{J},\underline{\Omega}^n) \longrightarrow H^n(M,\underline{\Omega}^n) \xrightarrow{\int_M} \mathbb{C}$$

where the first map is obtained from the natural transformation of functors $\mathrm{Hom}(M,O_M/J,\bullet) \longrightarrow \Gamma(M,\bullet)$.

By the Nullstellensatz for germs of analytic functions [see R. C. Gunning and H. Rossi, Analytic Functions of Several Complex Variables, Prentice-Hall 1965, p. 97], there exist positive integers m_i for $0 \leq i \leq n$ and analytic functions c_{ij} for $0 \leq i, j \leq n$, defined on some neighbourhood $W \subset V$ of p, such that for $0 \leq i \leq n$,

$$z_i^{m_i} = \sum_{j=1}^{n} c_{ij}(z)h_j(z).$$

Then $\mathrm{Res}_p\left[\dfrac{w(z)dz_1\wedge\cdots\wedge dz_n}{h_1,\ldots,h_n}\right]$ is equal to the coefficient of $z_1^{m_1-1}\cdots z_n^{m_n-1}$ in the power series expansion of $w(z)\det(c_{ij}(z))$.

We note that the analytic Nullstellensatz is a highly non-constructive result, and in general it is impossible to determine the multipliers c_{ij} explicitly. However in many special cases which are of interest the situation is simple enough to allow the c_{ij} to be found by trial and error.

In the case where the h_i form local coordinates in a neighbourhood of p the situation is particularly simple. Then all the m_i can be taken to be 1 and if

$$z_i = \sum_{j=1}^{n} c_{ij}(z)h_j(z)$$

then the residue is simply $w(p)\det(c_{ij}(p)) = w(p)\det(\partial h_i/\partial z_j(p))^{-1}$. This is the situation which arises in the fixed point formula at a <u>transversal</u> fixed point.

Thus the formula can be written

$$\chi(f,\varphi) = \sum_{p\in F} \text{trace } \varphi(p)\big/\det(I-df(p))$$

provided that all the fixed points are transversal. Here $df(p)$ denotes the endomorphism of the holomorphic tangent space at p induced by f.

Finally we remark that in the case $n = 1$ the Grothendieck residue coincides with the classical Cauchy residue. In fact, from the integral formula expression for the residue 4.1.1 it can be seen that

$$\operatorname{Res}_p\left[\frac{w(z)dz}{h}\right] = \frac{1}{2\pi i}\oint \frac{w(z)dz}{h(z)} .$$

5.1 The fixed-point formula and holomorphic vector fields

The techniques developed in the preceding sections will now be applied to the following situation.

As before, let M be a compact, complex-analytic manifold of complex dimension n, and let X be a holomorphic vector field on M; i.e. a holomorphic section of the holomorphic tangent bundle TM of M. Then X generates a one-parameter group of endomorphisms of M,

$$f : M \times \mathbb{C} \longrightarrow M$$

such that for $z \in M$, and $s,t \in \mathbb{C}$,

$$f(f(z,s),t) = f(z,s+t).$$

In fact, if $t = uv$ with u in the unit circle in the complex plane and v real, then $f(z,t) = f_u(z,v)$, where f_u is the one-parameter group generated by the real vector field $uX + \overline{uX}$ on M.

A method often used to demonstrate the existence of such a flow is the iterative procedure found, for example, in [19]. This device will be applied in the context of the fixed-point formula in order to calculate the fixed-point index at an isolated zero of the vector field, as a function on the complex plane.

It is not difficult to show that this index is a meromorphic function of t in the neighbourhood of $t = 0$, with a pole at the origin. This can be done as follows (this proof was shown to me by G. Lusztig):

Let $z_1,\ldots,z_n$ be local holomorphic coordinates centred at the isolated zero p of X. Since $f(z,0) = z$ it makes sense to consider the coordinates $f_i(z,t)$ for small t, and z near p.

We make the abbreviation $f_i(z,t) = z_i^t$ and note that on a small neighbourhood of $(p,0)$ in $M \times \mathbb{C}$ the only common zeroes of the functions $z_i - z_i^t$ are the union of the set $z = 0$, and the set $t = 0$. Then we can apply the analytic Nullstellensatz to obtain analytic functions $c_{ij}(z,t)$ on a neighbourhood of $(p,0)$, and positive integers m_i, such that for $1 \leq i \leq n$

$$t^{m_i} z_i^{m_i} = \sum_{j=1}^{n} c_{ij}(z,t)(z_j - z_j^t).$$

Then by properties of the residue [3], the fixed-point index is

$$\operatorname{Res}_p \left[\frac{dz_1 \wedge \cdots \wedge dz_n}{z_1 - z_1^t, \ldots, z_n - z_n^t} \right] = \frac{1}{t^M} \operatorname{Res}_p \left[\frac{\det c_{ij}(z,t)\, dz_1 \wedge \cdots \wedge dz_n}{z_1^{m_1}, \ldots, z_n^{m_n}} \right]$$

$= \frac{1}{t^M} \times$ coefficient of $z_1^{m_1-1} \cdots z_n^{m_n-1}$ in the expansion of $\det c_{ij}(z,t)$ as a power series in z, with coefficients analytic functions of t (where $M = \sum_i m_i$).

However, since the Nullstellensatz is a nonconstructive result, it is very difficult to obtain further information about the resulting meromorphic function of t by this method and so we adopt instead the following approach.

With the above notation, let $\underline{O}_p$ be the local ring at $p \in M$, and let $\underline{O}'_p$ be the local ring at $(p,0) \in M \times \mathbb{C}$. Then $\underline{O}_p$ may be identified with a subring of $\underline{O}'_p$ via the projection $(z,t) \mapsto z$. Similarly an element of $\underline{O}'_p$ may be regarded as a family of elements of $\underline{O}_p$ parametrized by t, for t sufficiently small.

Then X may be expressed in a neighbourhood of p by:

5.1.1 $$X(z) = \sum_{i=1}^{n} a_i(z)\partial/\partial z_i$$

with each a_i holomorphic. In the following, holomorphic functions will be identified with their germs whenever this is convenient. The following theorem holds for an arbitrary holomorphic vector field (not necessarily with isolated zeroes), and all $p \in M$.

5.1.2 **Theorem.** Let $\underline{I}_p(z-z^t) \subset \underline{O}'_p$ be the ideal generated by the germs of the functions $z_i - z_i^t$ for $1 \leq i \leq n$, and let $\underline{I}_p(a) \subset \underline{O}_p$ be the ideal generated by the germs of the $a_i(z)$. Then $\underline{I}_p(z-z^t) = t.\underline{I}_p(a)$.

Before stating the next theorem we introduce the following notation. Let $A(z)$ be the $n \times n$ matrix over the ring $\underline{O}_p$ given by the partial derivatives of the a_i:

5.1.3 $$A_{ij}(z) = \partial a_i/\partial z_j(z).$$

We note that

5.1.4 $$\tau = \sum_{k=0}^{\infty} t^k A^k/(k+1)!$$

is an invertible element in the ring of $n \times n$ matrices over $\underline{O}'_p$. Formally we may write

$$\tau^{-1} = tA/(I-e^{tA}).$$

Then we can define an element $T(z,t) \in \underline{O}'_p$ by:

5.1.5 $$T(z,t) = \det(tA/(I-e^{tA})).$$

Theorem 5.1.2 is of course trivial at any point where the vector field does not vanish. In order to apply the theorem, we again make the restriction that p is an <u>isolated</u> zero of X. We then obtain the following:

5.1.6 <u>Theorem</u>. Let $\underline{\Omega}^k$ be the sheaf of germs of holomorphic k-forms on M, and let dz be the element $dz_1 \wedge \cdots \wedge dz_n$ of $\underline{\Omega}^n_p$. Then (see 4.1 for notation) for $t \in \mathbb{C} - \{0\}$ sufficiently small, the following equality holds in $\underline{\mathrm{Ext}}^n_{\underline{O}_p}(\underline{I}_p(a), \underline{\Omega}^n_p)$:

$$\left[\frac{dz}{z_1 - z_1^t, \ldots, z_n - z_n^t}\right] = \left[\frac{t^{-n} \times T(z,t)\,dz}{a_1, \ldots, a_n}\right].$$

Note that if we apply the residue to the left side we obtain the fixed-point index at p in the Atiyah-Bott formula for the sheaf $\underline{O}_M$, and the right side gives an alternative expression for this index. Note also that the right side has a much more explicit dependance on t

than does the left side. This fact will be exploited in section 6 in order to investigate the properties of the meromorphic function of t obtained by applying the residue map.

We shall first prove these two theorems, and then show how the results may be generalized to prove a fixed-point formula for cohomology with values in the sheaf of germs of holomorphic sections of any vector bundle which satisfies certain homogeneity properties compatible with the action of X on M.

__Proof of theorems__ 5.1.2 __and__ 5.1.6. Let U be an open neighbourhood of $p \in M$ on which the vector field has the form 5.1.1. Then the resulting flow f is characterized by the conditions, for $1 \leq i \leq n$,

$$\partial f_i/\partial t(z,t) = a_i(f(z,t))$$

and

$$f_i(z,0) = z_i.$$

Let $V \subset U$ be open and let W be a disc centred on the origin in $\mathbb{C}$. Then if V and W are chosen to be sufficiently small [__19__; IV.1] we may inductively define functions $f^{(n)} : V \times W \longrightarrow U$ by:

$$f^{(0)}(z,t) = z$$

and for $m > 0$:

$$f^{(m)}(z,t) = z + \int_0^t a(f^{(m-1)}(z,s))ds$$

where $a : U \longrightarrow \mathbb{C}^n$ is the function $a(z) = (a_1(z),\ldots,a_n(z))$. Note that each $f^{(m)}$ is holomorphic so that the integral may be taken along any smooth path from 0 to t in W.

Again, as in [19; IV.1], if V and W are small enough, the sequence $f^{(m)}$ converges uniformly on $V \times W$ and the limit f is the local flow associated to the vector field. This also shows that f is holomorphic, being the uniform limit of holomorphic mappings [16; 2.2.4].

The elements of $\underline{O}'_p$ can be expressed as convergent power series, and $\underline{O}'_p$ will be given the topology of simple convergence of the coefficients [16; 2.2.4]. Thus $f^{(m)} \longrightarrow f$ if each coefficient in the power series expansion of $f^{(m)}$ converges to the corresponding coefficient in the expansion of f.

We first prove the following lemma.

5.1.7 Lemma. Let A be the matrix of partial derivatives 5.1.3. Then in matrix notation, the following holds in $\underline{O}'^{\,n}_p$ for $m > 0$:

5.1.8
$$f^{(m)}(z,t) - z \equiv \sum_{i=1}^{m} \frac{t^k}{k!} . A^{k-1} . a(z) \quad \mathrm{mod}(t . \underline{I}_p(a))^2 . \underline{O}'^{\,n}_p .$$

Proof. The proof is a simple induction. The result is trivially true for $m = 1$, and if true for $m = N$ then, if $\underline{I}_p(z-f^{(N)})$ is the ideal generated by the germs of the functions $z_i - f_i^{(N)}(z,t)$ for $1 \leq i \leq n$:

$$\begin{aligned} a(f^{(N)}(z,t)) &= a(z + f^{(N)}(z,t) - z) \\ &\equiv a(z) + A(z) \bullet (f^{N}(z,t) - z) \\ &\quad \mathrm{mod}(\underline{I}_p(z-f^{(N)}))^2 . \underline{O}'^{\,n}_p . \end{aligned}$$

The congruence follows from the fact that it is true for every finite partial sum of the Taylor expansion of $a(z + f^{(N)}(z,t) - z)$, and the fact that ideals are closed in $\underline{O}'_p$ for the topology of simple convergence [16; 6.3.5].

The result for $m = N + 1$ then follows from the case $m = N$ and the definition of $f^{(N+1)}$.

We are now in a position to prove the theorem. First note that since the functions $f_i^{(m)}$ converge uniformly in a neighbourhood of $(p,0)$ in $V \times W$, they also converge in $\underline{O}'_p$ by the Cauchy inequalities [16; 2.2.7]. The same applies to the partial sums on the right-hand side of 5.1.8. Then, again using the fact that ideals of $\underline{O}'_p$ are closed, we can let $m \longrightarrow \infty$ in 5.1.8 and conclude, with τ defined by 5.1.4, that

$$z-f(z,t) \equiv \tau(z,t) \bullet ta(z) \quad \mathrm{mod}(t.\underline{I}_p(a))^2\underline{O}'^n_p .$$

Because τ is invertible, this means that

$$t.I_p(a) = I_p(z-z^t) + (t.I_p(a))^2$$

and the theorem follows from Nakayama's lemma [see e.g. M. F. Atiyah and I. G. Macdonald, Introduction to Commutative Algebra, Addison-Wesley 1969, prop. 2.6].

In order to prove theorem 5.1.6, simply observe that

5.1.9 $$z-z^t \equiv ((I-e^{tA})/tA) \bullet ta \quad \mathrm{mod}\ \underline{I}_p^2.\underline{O}'^n_p$$

where $\underline{I}_p = t.\underline{I}_p(a)$. The theorem then follows from the definitions (see 4.1).

5.2 A generalisation

We recall that the fixed-point theorem can be proved for cohomology with coefficients in the sheaf of germs of sections of any holomorphic vector bundle E for which there exists a suitable geometrical endomorphism (see 3.1). In the present situation we require the existence of a family of geometrical endomorphisms of E corresponding to the group of endomorphisms of M induced by the vector field. Sufficient conditions for the existence of such a family are obtained in the situation discussed below, which will be formulated in the context of an arbitrary group G acting (on the left) on the complex-analytic manifold M by holomorphic transformations.

In this case M will be said to be a holomorphic G-space, and for $g \in G$ the corresponding endomorphism of M will also be denoted by g.

First recall the following definition.

5.2.1 Definition. The holomorphic vector bundle E is said to be a holomorphic G-bundle if:

(i) E is a holomorphic G-space.

(ii) The projection $E \longrightarrow M$ commutes with the action of G.

(iii) For $g \in G$ and $x \in M$ the map $E_x \longrightarrow E_{g(x)}$ is complex linear.

For example, any combination of tensor or exterior powers of the holomorphic tangent bundle of M is a holomorphic G-bundle. Other examples occur on the homogeneous spaces of Lie groups [2,5].

5.2.2 Remark. If E is a holomorphic G-bundle then so is the holomorphic dual E^*. To see this, for $x \in M$ let $g_x^{-1} : E_{g(x)} \longrightarrow E_x$ be the map induced by the action of g^{-1} on E. Then the adjoint maps $E_x^* \longrightarrow E_{g(x)}^*$ give an action of g on E^* which makes E^* into a holomorphic G-bundle. Thus tensor and exterior powers of the holomorphic cotangent bundle are holomorphic G-bundles.

The point of this definition is that for G-vector bundles there always exist geometrical endomorphisms compatible with the action of G; by definition 5.2.1 and the universal property of the pull-back there exists a holomorphic bundle map over M for all $g \in G$,

$$E \longrightarrow (g^{-1})^* E.$$

By taking the pull-back relative to the map $g : M \longrightarrow M$ we obtain the required endomorphism, which will be denoted by

$$\varphi_g : g^* E \longrightarrow E.$$

Note that if E is a G-vector bundle this gives a (right) representation of G on the space $C^\infty(M,E)$ of smooth global sections of E, which will be written $s \longmapsto s^g$ for $s \in C^\infty(M,E)$, where $s^g(x) = \varphi_g(s(g(x)))$. It is easy to check that this defines a representation.

Let $x \in M$ be a fixed point for the action of G, i.e. $g(x) = x$ for all $g \in G$. Then if $\underline{E}_x$ is the space of germs of holomorphic sections of E at x, there is a (right) representation of G on this space which will again be denoted by $s \longmapsto s^g$, where as before $s^g = \varphi_g \circ s \circ g$.

In the case where E is the trivial line bundle the resulting representation on the local ring $\underline{O}_x$ is simply $f \longmapsto f^g$ where $f^g(x) = f(g(x))$. This is therefore compatible with the notation of section 5.1.

5.2.3 We now return to the case where $G = \mathbf{C}^+$ and the $\mathbf{C}^+$-action on M is induced by a holomorphic vector field X with isolated zeroes. Let p be a zero of the vector field and let $s_1,\ldots,s_m$ be holomorphic sections generating the $\mathbf{C}^+$-vector bundle E in a neighbourhood of p, with corresponding dual sections $s_1^*,\ldots,s_m^*$ of E^*. Recall that the expression which enters into the fixed-point index at p is the class in $\underline{O}'_p/\underline{I}_p$ of

$$\text{trace } \varphi_t(z) = \sum_{i=1}^{m} < s_i^*, s_i^t > .$$

As before, it is not clear how this expression depends on t. This dependance can be clarified as follows. We identify sections of E with their germs at p whenever this is convenient.

For $1 \leq i \leq m$ we may differentiate s_i^t in $\underline{E}_p$ with respect to t and obtain

$$\partial s_i^t/\partial t \Big|_{t=0} = \sum_{j=1}^{n} L_{ij} s_j$$

with $L_{ij} \in \underline{O}_p$. Since $s \longmapsto s^t$ is a representation of $\mathbf{C}^+$, this implies that

$$\partial s_i^t/\partial t \Big|_{t=u} = \sum_{j=1}^{n} L_{ij}^u s_j^u .$$

This is a system of first-order differential equations for the s_j^t which may be integrated in precisely the same way as the vector field of 5.1. We write $s = (s_1,\dots,s_m)$ and, using matrix notation, define inductively

$$s^t_{(0)} = s$$

and

$$s^t_{(k+1)} = s + \int_0^t L^u \bullet s^u_{(k)}\, du.$$

As before the $s^t_{(k)}$ converge in the topology of simple convergence in $\underline{E}'_p = \underline{E}_p \otimes_{\underline{O}_p} \underline{O}'_p$ to s^t. We first prove:

<u>5.2.4</u> <u>Lemma</u>. The following holds in $\underline{E}'_p$ for $k \geq 0$:

$$s^t_{(k)} \equiv \sum_{i=0}^{k} \frac{1}{i!} (tL)^i \bullet s \quad \text{mod } \underline{I}_p . \underline{E}'_p$$

(as before $\underline{I}_p$ denotes the ideal $t.\underline{I}_p(a)$).

<u>Proof</u>. This is trivially true for $k = 0$, and if true for $k = N$, then as in lemma 5.1.7,

$$\begin{aligned} s^t_{(N+1)} &\equiv s + \sum_{i=0}^{N} \int_0^t \frac{1}{i!} L^u (uL)^i \bullet s\, du \\ &\equiv \sum_{i=0}^{N+1} \frac{1}{i!} (tL)^i \bullet s \quad \text{mod } I_p . \underline{E}'_p \end{aligned}$$

as required.

Thus if we write formally $e^{tL} = \sum_{i=0}^{\infty} (tL)^i/i!$ and define an element $ch(E,z,t) \in \underline{O}'_p$ by

$$ch(E,z,t) = \text{trace } e^{tL(z)}$$

we obtain the following theorem:

5.2.5 Theorem. Let dz be the element $dz_1 \wedge \cdots \wedge dz_n$ of $\underline{\Omega}^n_p$. Then for $t \in \mathbb{C} - \{0\}$ sufficiently small, the following equality holds in $\underline{\mathrm{Ext}}^n_{\underline{O}_p}(\underline{O}_p/\underline{I}_p(a), \underline{\Omega}^n_p)$:

$$\left[\frac{\operatorname{trace} \varphi_t(z)\, dz}{z_1 - z_1^t, \ldots, z_n - z_n^t}\right] = \left[\frac{t^{-n} \times ch(E,z,t) \times T(z,t)\, dz}{a_1, \ldots, a_n}\right].$$

5.2.6 Remark. The expression for the fixed-point index obtained by applying the Grothendieck residue to the right-hand side of the equality given by theorem 5.1.6 can also be derived by purely analytic methods, using the integral formula expression for the residue, and a local perturbation of the vector field to obtain non-degenerate zeroes; see [22]. However the generalisation given by the above theorem appears to be less amenable to this approach, except in the case where E is some tensor or exterior power of the holomorphic cotangent bundle. This is because it is not clear in general how to extend the perturbation on M to a compatible perturbation of the $\mathbb{C}^+$ action on E, although this may be possible in particular cases.

5.2.7 Remark. Note that if $\underline{E} = \underline{\Omega}^1$, the sheaf of germs of sections of holomorphic cotangent bundle of M with the corresponding geometric endomorphism given by the differential of the endomorphism of M, then taking the usual generating sections $dz_1, \ldots, dz_n$,

$$L_{ij} = \partial^2 z_i^t / \partial t \partial z_j = \partial a_i / \partial z_j = A_{ij}.$$

Therefore if we define elements $T^k(z,t) \in \underline{O}'_p$ for $0 \le k \le n$:

$$T^k(z,t) = \operatorname{trace} \Lambda^k(e^{tA}) \times \det(tA/(I - e^{tA}))$$

we can calculate the fixed-point indices for the case $\underline{E} = \underline{\Omega}^k$ by using:

5.2.8 $$\left[\frac{\text{trace } \Lambda^k(\partial z_i^t/\partial z_j)\ dz}{z_1 - z_1^t, \ldots, z_n - z_n^t}\right] = \left[\frac{t^{-n} \chi T^k(z,t)\ dz}{a_1, \ldots, a_n}\right].$$

Of course, this result could also be obtained directly from the work of 5.1. If 5.1.9 is differentiated with respect to z it gives

$$(\partial z_i^t/\partial z_j) = e^{tA} \quad \text{mod } \underline{I}_p$$

which immediately implies the above equality.

5.2.9 Remark. One or two observations are in order concerning the matrix L, which may be regarded as an endomorphism $L : \underline{E}_p \longrightarrow \underline{E}_p$ where, for $s \in \underline{E}_p$,

$$L(s) = \partial s^t/\partial t \Big|_{t=0} .$$

Note that if E is the trivial line bundle, the corresponding endomorphism of $\underline{O}_p$ is simply

$$f \longmapsto X \bullet f$$

where $X \bullet f$ is the derivative of f along the vector field, i.e. $\sum_{i=1}^{n} a_i\ \partial f/\partial z_i$. Thus

$$L(fs) = (X \bullet f)s + fL(s).$$

This implies that L induces a well-defined linear transformation of the fibre of E at the zero p of X, which with respect to the basis $\{s_i(p)\}$ of the fibre is given by the matrix $L_{ij}(p)$. In case

$E = \Omega^1$, then L is essentially the Lie bracket action and the eigenvalues of the matrix $A_{ij}(0)$ are the characteristic roots of the vector field at p.

Compare the situation with that of [4], where closely related results on zeroes of holomorphic vector fields are given.

5.2.10 Remark. The explanation for the notation T and ch is as follows.

Define elements $c_i \in \underline{O}_p$ for $1 \leq i \leq m$ by

$$\det(I + xL(z)) = 1 + \sum_{i=1}^{m} x^i c_i(z)$$

where x is an indeterminate. In fact the c_i depend on the particular coordinates chosen around p, but it is not difficult to check that their classes mod $\underline{I}_p(a)$ are in fact independant of the coordinates. In any case when $L = A$, then $T(z,t)$ is essentially the (dual) Todd class in the "Chern classes" $c_i(z)$ and when L is the matrix associated to the $\mathbb{C}^+$ - vector bundle E, then $ch(E,z,t)$ is the Chern character in the classes $c_i(z)$. See [15].

6. Properties of the fixed point index at a zero of the vector field

We now investigate more closely the form of the fixed-point contributions in the Atiyah-Bott formula for a one parameter group, i.e. the functions

$$\nu_p(E,t) = \frac{1}{t^n} \operatorname{Res}_p \left[\frac{ch(E,z,t)T(z,t)\, dz}{a_1,\ldots,a_n} \right] .$$

In case $E = \Omega^k$ with the usual geometrical endomorphisms induced by the kth exterior power of the derivative we write

$$\nu_p(\Omega^k, t) = \nu_p^k(t).$$

We prove the following results.

<u>6.1</u> <u>Theorem.</u> Let λ_i for $1 \leq i \leq n$ be the characteristic roots of the vector field at p, i.e. the eigenvalues of the matrix $\partial a_i/\partial z_j(p)$, and let $Y_i(t) = (1-e^{\lambda_i t})^{-1}$ if $\lambda_i \neq 0$, and $Y_i(t) = 0$ otherwise. Similarly, for $1 \leq i \leq m$, let μ_i be the eigenvalues of the matrix $L(p)$. Then for t sufficiently small and non-zero:

(i) $$\nu_p(E,t) = t^{-n} \sum_{i=1}^{m} e^{\mu_i t} Q_{p,i}(t, Y_1(t), \ldots, Y_n(t))$$

for certain polynomials $Q_{p,i}$ in $n+1$ variables with coefficients in $\mathbb{C}$.

(ii) $$\nu_p^k(t) = t^{-n} Q_p^k(t, Y_1(t), \ldots, Y_n(t))$$

for certain polynomials Q_p^k in $n+1$ variables with coefficients in $\mathbb{C}$.

<u>Proof</u>. For a multi-index $I = (i_1, \ldots, i_n)$ let d_z^I denote $\partial^I/\partial z^I$. In view of the algorithm 4.3 for calculating the residue it is sufficient to prove that for each I:

(a) $$d_z^I \operatorname{ch}(E,p,t) = \sum_{i=1}^{m} e^{\mu_i t} P_i(t)$$

for certain polynomials P_i in one variable, and for $0 \leq k \leq n$:

(b) $d_z^I T^k(p,t)$ is a polynomial in t and the $Y_i(t)$.

We first prove (a); the proof of (b) will be similar. As in 5.2.10 we write

<u>6.2.1</u> $$\det(I + tL(z)) = 1 + \sum_{i=1}^{m} c_i(z)t^i$$

and then $ch(E,z,t)$ can be expressed in terms of the c_i as

$$ch(E,z,t) = F(c(z),t)$$

where $c = (c_1,\ldots,c_n)$. Then we can write also

<u>6.2.2</u> $$F(c,t) = S(x,t)$$

where $S(x,t) = \sum_{i=1}^{m} e^{x_i t}$ and c_k is regarded as the kth elementary symmetric function in the x_i.

Now differentiation by the chain rule shows that it is sufficient to show that all the $d_c^I F(c(p),t)$ are of the required form. The purpose of the following lemmas is to show that we can go further and consider only $d_x^I S(x,t)\big|_{x=\mu}$.

<u>6.2</u> <u>Lemma</u>. Let $D(x_1,\ldots,x_n) = \det(\partial c_i/\partial x_j)$. Then $D(x_1,\ldots,x_n) = \prod_{i<j}(x_i - x_j)$.

<u>Proof</u>. Let c_j^i be the jth elementary symmetric function in the n-1 variables $x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n$, and let $c_0^i = 1$ for all i. Then $\partial c_i/\partial x_j = c_{i-1}^j$. We first prove that $D(x_1+t,\ldots,x_n+t) = D(x_1,\ldots,x_n)$. To see this note that $c_i^j(x+t) = c_i^j(x) + \sum_{k=1}^{i} B_k c_{i-k}^j(x)t^k$ for certain integers B_k depending only on k, and then use standard properties of determinants.

Thus $D(x_1,\ldots,x_n) = D(x_1-x_n,\ldots,x_{n-1}-x_n,0) = \prod_{k=1}^{n-1}(x_k-x_n)$ $\times \prod_{0\leq i<j<n}(x_i-x_j)$ by induction, assuming the result true for n - 1. This is the required result.

Let us say an analytic function defined on the open neighbourhood $U \times W$ of the origin in $\mathbb{C}^n \times \mathbb{C}$ has property (M,N) if it can be expressed as

$$D(x)^{-M} \times \sum_{|I| \leq N} P_I(x) d_x^I S(x,t)$$

with each $P_I(x)$ a polynomial in $x_1, \ldots, x_n$, on the open dense subset of $U \times W$ where $D(x) \neq 0$. The proof of the following lemma is immediate.

<u>6.3</u> <u>Lemma</u>. If $w(x,t)$ has property (M,N) in $U \times W$, then $\partial w/\partial x_k$ has property $(M+1, N+1)$ in $U \times W$ for $1 \leq k \leq n$.

<u>6.4</u> <u>Lemma</u>. If F and S are related by 6.2.2, then for each I, $d_c^I F(c,t)$ has property $(2|I|, |I|)$.

<u>Proof</u>. The proof is by induction on $|I|$. The case $|I| = 0$ is simply $F(c,t) = S(x,t)$, so suppose the lemma holds for $|I| \leq N$. Then by lemma 6.3, for $0 \leq k \leq n$,

$$\partial/\partial x_k [d_c^I F(c,t)] = \sum_{j=1}^{n} \partial/\partial c_j [d_c^I F(c,t)] \partial c_j/\partial x_k$$

has property $(2N+1, N+1)$ if $|I| = N$. Therefore by solving for $\partial/\partial c_j [d_c^I F(c,t)]$ we see that for each j this function has property $(2N+2, N+1)$. This completes the induction step and the proof of the lemma.

<u>6.5</u> <u>Lemma</u>. Note that $c_k(p)$ is the kth elementary symmetric function of $\mu_1, \ldots, \mu_n$. For all I the function $d_c^I F(c(p),t)$ of t is a certain finite linear combination of the functions $d_x^J S(\mu, t)$.

<u>Proof</u>. If the μ_i are distinct, then $D(\mu) \neq 0$ and the lemma follows simply by setting $x = \mu$ in lemma 6.4. If there exist equalities between the μ_i, say $\mu_1 = \mu_2$, then if

$$d_c^I F(c,t) = D(x)^{-M} \times \sum_{|J| \leq N} P_J^I(x) d_x^J S(x,t)$$

we can apply L'Hôpital's rule and obtain

$$d_c^I F(c(p),t) = \frac{1}{M!} \times \prod_{i<j}{}' (\mu_i - \mu_j)^M \times \frac{\partial^M}{\partial x_1^M} \left(\Sigma\, P_J^I(x) d_x^J S(x,t)\right)\Big|_{x=\mu}$$

where the prime indicates that $(\mu_1 - \mu_2)$ is omitted in the product. This expression is still of the required form and we may proceed similarly if there exist further equalities between the μ_i.

Result (a) will now follow if each $d_x^J S(\mu,t)$ is of the form $\sum_{i=1}^{m} e^{\mu_i t} P_{I,i}(t)$ for certain polynomials $P_{I,i}(t)$, but this is clearly true.

The proof of (b) is the same except that we set

$$F(c(z),t) = T^k(z,t)$$

where this time the c_i are defined by setting $L = A$ in 6.2.1. Then $F(c,t) = S(x,t)$ where, if σ_k is the kth elementary symmetric function,

$$S(x,t) = \sigma_k(e^{x_i t}) \prod_{i=1}^{n} (x_i t/(1-e^{x_i t})).$$

As before we reduce the problem to the direct differentiation of $S(x,t)$ where the result is easily checked.

These results have the following interesting corollary which generalises the results of [18] and [20] to include the case of isolated degenerate fixed points.

If $f : M \longrightarrow M$ is complex analytic we denote the induced endomorphism of $H^j(M,\Omega^k)$ by $H^{j,k}(f)$ and write

$$\chi^k(f) = \sum_{j=0}^{n} (-1)^j \operatorname{trace}_{\mathbb{C}} H^{j,k}(f)$$

for the corresponding Lefschetz number. If f is the one-parameter group generated by the holomorphic vector field X with isolated zeroes, we write the Lefschetz number as $\chi^k(t)$ for $t \in \mathbb{C}$.

6.6 Theorem. For $0 \le k \le n$,

$$\chi^k(t) = \chi^k(0) = \chi^k(M) = \sum_{j=0}^{n} (-1)^j \dim_{\mathbb{C}} H^j(M,\underline{\Omega}^k)$$

for all $t \in \mathbb{C}$.

Proof. This is a simple extension of the proof given in [20] for the non-degenerate case. First note that each fixed point index $\nu_p^k(t)$ has an analytic continuation as a meromorphic function on the whole complex plane, and that $\chi^k(t) = \sum_{i=1}^{M} \pm e^{\alpha_i t}$ for certain $\alpha_i \in \mathbb{C}$. By uniqueness of analytic continuation the fixed-point formula

$$\chi^k(t) = \sum_p \nu_p^k(t)$$

then holds for all t, except where the right-hand side has poles. Now if $t \in \mathbb{R}$ and s lies on the unit circle in the complex plane, one observes that each $\nu_p^k(t)$ has the property that there exists some integer N such that $\nu_p^k(st)t^{-N} \longrightarrow 0$ as $t \longrightarrow \infty$ for all but a finite number of s, i.e. those s such that λs is purely imaginary for some characteristic root λ of the vector field at one of its zeroes. But this behaviour is incompatible with the behaviour of $\chi^k(t)$ unless this function is actually constant, as is shown by the following lemma.

<u>6.7</u> <u>Lemma</u>. In the expression $\chi^k(t) = \sum_{i=1}^{M} \pm e^{\alpha_i t}$ the exponentials either cancel in pairs or, with the above notation, $\chi^k(st)t^{-N} \longrightarrow \infty$ as $t \longrightarrow \infty$ for s running in an <u>open</u> set of the unit circle.

<u>Proof</u>. Suppose there exists an α of largest modulus among the α_i which cannot be cancelled. Choose s such that $s\alpha$ is real and positive. Then as $t \longrightarrow \infty$ through positive real values it is clear that the term $e^{u\alpha t}$ dominates all others for all u in an open neighbourhood of s in the unit circle, and for any N we have that $\chi^k(ut)t^{-N} \longrightarrow \infty$ as required.

<u>6.8.1</u> <u>Remark</u>. The above argument cannot be applied to the situation involving more general bundles due to the occurrence of extra factors in the fixed-point indices. Note however the remark at the end of the next section which shows that if the situation is wholly algebraic, i.e. M is also an algebraic variety over $\mathbf{C}$ and the action of $\mathbf{C}^+$ is rational, then the above argument becomes valid in general since the eigenvalues of the matrix L(p) will be zero at all the fixed points.

<u>6.8.2</u> <u>Remark</u>. If M is <u>Kähler</u>, then theorem 6.6 will follow also from the fact that the cohomology group $H^p(M,\underline{\Omega}^q)$ is embedded in the

cohomology group $H^{p+q}(M,\mathbb{C})$ for all p and q, and these groups are of course acted on trivially by any connected Lie group operating continuously on M.

6.9 Examples

By analogy with [15] we may write, using the notation of 5.2.7,

$$T(y;z,t) = \sum_{k=0}^{n} T^k(z,t)y^k$$

where y is an indeterminate. Write also

$$\alpha_p(y;t) = \begin{bmatrix} \dfrac{t^{-n}T(y;z,t)\ dz}{a_1,\ldots,a_n} \end{bmatrix}$$

so that

$$\alpha_p(y;t) = \sum_{m=0}^{\infty} \alpha_{p,m}(y)t^{m-n}.$$

With this notation, theorem 6.6 may be rewritten as

$$\sum_p \mathrm{Res}_p(\alpha_{p,m}(y)) = \begin{cases} \chi_y(M) & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

where $\chi_y(M)$ is the χ_y-genus of M, as defined in [15].

Example 1. Setting $y = -1$, we see that

$$\alpha_{p,m}(-1) = \begin{bmatrix} \dfrac{da_1\wedge\cdots\wedge da_n}{a_1,\ldots,a_n} \end{bmatrix}$$

and it follows from properties of the residue [13; III.9.R6] that $\mathrm{Res}_p(\alpha_{p,n}(-1)) = \dim_{\mathbb{C}}(\underline{O}_p/\underline{I}_p)$. But this is the multiplicity of the zero of X, and since $\chi_{-1}(M)$ is simply the Euler-Poincaré characteristic, we recover a special case of the classical Hopf theorem.

Example 2. In the case where $\dim_{\mathbf{C}} M = 1$, the vector field may be written locally as $a(z)\partial/\partial z$, and

$$\alpha_p(y;t) = [a'(1+ye^{ta'})(1-e^{ta'})^{-1}dz/a]$$

where $a' = \partial a/\partial z$. In this case the Grothendieck residue coincides with the classical Cauchy residue and if $a'(p) = \lambda \neq 0$ then

$$\mathrm{Res}_p(\alpha_p(y;t)) = (1+ye^{\lambda t})/(1-e^{\lambda t})$$

while if $a(z)$ has a zero of order $M > 1$ at p;

$$\mathrm{Res}_p(\alpha_p(y;t)) = -(1+y)(2\pi i t)^{-1}\oint dz/a + (1-y)M/2.$$

In dimension n we recall the following results from the earlier work of section 6. The first also appears in [20].

a) If the zero of X is non-degenerate, i.e. $A(p)$ is non-singular, then $\mathrm{Res}_p(\alpha_p(y;t))$ is a polynomial in y whose coefficients are bounded as $t \longrightarrow \infty$ radially in all but a finite number of directions.

b) At the other extreme, if the transformation given by $A(p)$ is nilpotent, then the coefficients of the powers of y in $\mathrm{Res}_p(\alpha_p(y;t))$ are of the form

$$t^{-n} \times \text{polynomial in } t.$$

One might then ask if the behaviour of a) occurs in case b). In other words, are the polynomials in t always of degree $\leq n$? The previous example shows that this is certainly the case in dimension 1, but the vector field described below gives a counterexample in dimension 3. However, I know of no counterexample in dimension 2, or in the case of a holomorphic vector field which is defined globally on a compact manifold.

Example 3. Let (x,y,z) be coordinates for $\mathbf{C}^3$, and take X to be the vector field

$$X(x,y,z) = a\partial/\partial x + b\partial/\partial y + c\partial/\partial z$$

where

$$a(x,y,z) = x^2 + ye^x$$
$$b(x,y,z) = y^2 + z$$
$$c(x,y,z) = z^2.$$

This has an isolated zero at the origin, and $A(0)$ is clearly nilpotent. The algorithm described in 4.3 will be used to find the coefficient of t in $\mathrm{Res}_0(\alpha_0(0;t))$. First note that

$$x^8 = (x^6-x^4ye^x+x^2y^2e^{2x}-y^3e^{3x})a + e^{4x}(y^2-z)b + e^{4x}c$$
$$y^4 = (y^2-z)b + c$$
$$z^2 = c.$$

Let $C(x,y,z)$ be the determinant of the matrix of multipliers:

$$C(x,y,z) = (x^6-x^4ye^x+x^2y^2e^{2x}-y^3e^{3x})(y^2-z).$$

Setting $a_x = 2x+ye^x$ it is easily checked that

$$T^0(x,y,z,t) \equiv -(1-a_xt/2+a_x^2t^2/12-a_x^4t^4/720) \times (1-yt+y^2t^2/3) \times (1-zt) \mod(x^8,y^4,z^2,t^5).$$

Then, applying the algorithm, the required number is the coefficient of $x^7y^3zt^4$ in $C(x,y,z) \times T^0(x,y,z,t)$, and this may be checked to be $-1/90$.

7. Additive group actions on algebraic varieties

Analogous results to those of section 5 hold in the algebraic category, provided we stay in zero characteristic. Suppose the additive group A^+ of the algebraically closed field k acts rationally on the smooth variety M, defined over k, by

$$f : M \times A^+ \longrightarrow M.$$

Assume that $p \in M$ is an isolated fixed point for the action of A^+. Note that it is proved in [17] that if M is complete and connected then the fixed-point set is connected, so that p will be the *only* fixed point. Perhaps the simplest example of this situation is the action of A^+ on $P_1(k) = A^+ \cup \{\infty\}$ given by $(z,t) \longmapsto z+t$ with fixed point $\{\infty\}$.

As before let O_p and O'_p be the local rings at $p \in M$ and $(p,0) \in M \times A^+$, with maximal ideals m_p and m'_p respectively. Let $z_1,\ldots,z_n \in O_p$ be regular parameters for M at p and let $f_i = f^* z_i \in O'_p$. The corresponding "germ of vector field" can then be defined by $a_i = \partial f_i/\partial t\big|_{t=0}$. The problem then comes down to constructing a solution for the formal differential equations

$$\partial f_i/\partial t(z,t) = a_i(f(z,t))$$

in the ring O'_p, subject to the initial conditions $f_i(z,0) = z_i$. It is then necessary to prove uniqueness in order to identify the solution with the given A^+ action. As in the analytic case this can be done by defining an endomorphism α of the m_p'-adic completion $(\hat{O}_p')^n$ of $(O_p')^n$ by

$$\alpha(g) = z + \int_0^t a(g(z,u))\, du$$

for $g \in m_p'(\hat{O}_p')^n$. (The formal integration needs the hypothesis that k is of zero characteristic.) It is then trivial to prove that α is a "contraction map" for the m_p'-adic topology, and the existence and uniqueness of the solution in $(\hat{O}_p')^n$ follows immediately. However, it is more convenient to prove the following slightly stronger result.

Let $\tilde{O}_p'$ be the completion of O_p' with respect to the m_p-adic topology (this topology is finer than the m_p'-adic topology). Then α defines an endomorphism of $m_p(\tilde{O}_p')^n$ and in order to show that a solution of the differential equations exists in $\tilde{O}_p'$ it is only necessary to prove the following proposition.

<u>7.1</u> <u>Proposition</u>

The endomorphism α is a contraction map for the m_p-adic topology.

<u>Proof</u>. For $g,h \in m_p(\tilde{O}_p')^n$ with $g-h \in m_p^k(\tilde{O}_p')^n$ for $k \geq 1$ we have

$$\alpha(g)-\alpha(h) \equiv A \bullet \int_0^t (g(z,u)-h(z,u))\, du \qquad \text{mod } m_p^{2k}(\tilde{O}_p')^n$$

where as before $A_{ij}(z) = \partial a_i/\partial z_j$ in the m_p-adic completion $\hat{O}_p$ of O_p. Now A^+ has a rational representation on the n-dimensional vector space m_p/m_p^2, say $t \longmapsto U_t$. Since A^+ is a unipotent group the matrices U_t are unipotent and $\partial U_t/\partial t\big|_{t=0} = A(p)$ is nilpotent. Thus for sufficiently large N the matrix A^N has entries in $m_p\hat{O}_p$ and

$$\alpha^N(g) - \alpha^N(h) \in m_p^{k+1}(\tilde{O}'_p)^n$$

as required.

Again we obtain the congruence

$$z-f(z,t) \equiv \tau(z,t) \bullet ta(z) \qquad \text{mod } I_p^2(\tilde{O}'_p)^n$$

where as before $I_p = t.I_p(a)$. Note that for k large enough, $m_p^k \subset I_p(a)$, since the zero is isolated. Thus no essential information has been lost by taking the completion.

The generalisation of (5.2) can be carried through similarly in the algebraic case. Note that in all cases the residue at the fixed point will be of the form $t^{-n} \times$ polynomial in t.

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