The Logic of Quantum Mechanics Derived from Classical General Relativity

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Abstract

For the first time it is shown that the logic of quantum mechanics can be derived from Classical Physics. An orthomodular lattice of propositions, characteristic of quantum logic, is constructed for manifolds in Einstein's theory of general relativity. A particle is modelled by a topologically non-trivial 4-manifold with closed timelike curves - a 4-geon, rather than as an evolving 3-manifold. It is then possible for both the state preparation *and* measurement apparatus to constrain the results of experiments. It is shown that propositions about the results of measurements can satisfy a non-distributive logic rather than the Boolean logic of classical systems. Reasonable assumptions about the role of the measurement apparatus leads to an orthomodular lattice of propositions characteristic of quantum logic.

1 Comment

This paper has been published in Foundations of Physics Letters [1]. The work forms the basis of my doctoral thesis [2]. A short, less formal talk about my work is archived in quant-ph/9609021 [3].

2 INTRODUCTION

Quantum logic is characterised by the propositions of an orthomodular lattice, the distinguishing feature of which is the failure of the distributive law which is replaced with the weaker orthomodularity condition[4]. From this orthomodular lattice it is then possible to generate the Hilbert space structure of quantum mechanics[5]. That quantum systems satisfy a non-Boolean logic is an experimental fact that has never been satisfactorily explained.

This paper suggests an origin for quantum logic and formally constructs an orthomodular lattice of propositions about manifolds in classical general relativity.

Since the formulation of general relativity, attempts[6, 7] have been made to model elementary particles as topologically non-trivial structures in spacetime

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called geons. These models exhibited interesting particle-like properties, such as mass and charge without apparent sources, but could not reproduce the features of quantum mechanics which the particles must obey. These models were, however, based on *three*-manifolds evolving in time, in the sense that there was a global hypersurface permitting the definition of a global time coordinate. Once the manifold was defined at a given time its evolution was deterministic - independent of subsequent measurements that may, or may not, be made.

There has recently been much speculation about the existence of closed timelike curves (CTCs), their stability and the possibility of time travel with its associated inconsistencies [8]. Although CTCs appear unphysical, there is nothing in the theory of general relativity to exclude them[9]. General relativity treats spacetime as a manifold, which is locally diffeomorphic to $\Re^4$, but does not prescribe its topology[7]. Interacting classical objects (or fields) in a spacetime with CTCs require additional boundary conditions to uniquely determine their evolution [8]. Data that would normally be sufficient to define a unique trajectory may, in the presence of CTCs, permit more than one possibility.

This work suggests a model for a particle as a non-trivial topological structure in four dimensions (space and time) not just space. While it is an obvious extension of the old ideas on geons, it makes fuller use of the richness of general relativity than did the earlier work. Thus the extension from topologically non-trivial 3-spaces (3-geons) to spacetime (4-geons) gives rise to the possibility of CTCs and the associated *impossibility* of defining a global time coordinate.

It is well known that if states are modelled as projections of a complex Hilbert space then the symmetries of the spacetime (Galilean or Poincaré) together with the appropriate internal symmetries of the object leads inevitably to the familiar equations of non-relativistic and relativistic quantum mechanics respectively, together with commutation relations and a universal constant with the dimensions of Planck's constant[10].

The conjectured 4-geon description of particles is speculative but this single assumption is able to unify the particle and field descriptions of nature, explain quantum logic and in doing so reconciles general relativity and quantum mechanics.

3 4-GEON

The present analysis is based upon a model of an elementary particle as a distortion of spacetime, (a four dimensional semi-Riemannian manifold with non-trivial topology). The manifold includes both the particle and the background metric, and being four dimensional without a *global* time coordinate, the particle and its evolution are inseparable - they are both described by the four-manifold. We now express the properties, which we require of a particle, in the language of manifolds.

Axiom 1 (Asymptotic flatness) *Far away from the particle spacetime is topologically trivial and asymptotically flat with an approximately Lorentzian metric.*

In mathematical terms - spacetime is a 4-manifold, $\mathcal{M}$ and there exists a 4-manifold K, such that $\mathcal{M}/K$ is diffeomorphic to $\Re^4/(\mathbf{B}^3 \times \Re)$ and the metric on $\mathcal{M}/K$ is asymptotically Lorentzian.[‡] K or $(\mathbf{B}^3 \times \Re)$ can be regarded as the world-tube within which the 'particle' is considered to exist.

This axiom formally states the fact that we experience an approximately Lorentzian spacetime, and that if space and time are strongly distorted and convoluted to form a particle then that region can be localised. (It may be noted that asymptotic flatness is not a reasonable property to require for a quark because it cannot be isolated [there is no evidence of an isolated quark embedded in a flat spacetime] therefore the present work cannot be applied automatically to an isolated quark.)

The position of a distortion of spacetime is not a trivial concept - it implies a mapping from the 4-manifold, which is both the particle and the background spacetime, onto the flat spacetime used within the laboratory. There is in general no such map that can be defined globally, yet a local map obviously cannot relate the relative positions of distant objects. This axiom gives a practical definition of the position of a particle - it is the region where the non-trivial topology resides. Any experimental arrangement which confines (with barriers of some sort) the $\mathbf{B}^3$ region of non-trivial topology, defines the position of the particle. From this axiom, the region outside the barrier is topologically trivial and therefore *does* admit global coordinates.

Using the asymptotic flatness axiom it is now possible to define what is meant by a particle-like solution:

Axiom 2 (Particle-like) *In any volume of 3-space an experiment to determine the presence of the particle will yield a true or a false value only.*

This is consistent with the non-relativistic indivisibility of the particle. By contrast, a gravitational wave may be a diffuse object with a density in different regions of space which can take on a continuous range of values. An object which did not satisfy this axiom (at least in the non-relativistic approximation) would not be considered to be a particle. The axiom is clearly satisfied by classical particles and, because it refers only to the result of a position measurement, it conforms also with a quantum mechanical description of a particle.

The particle-like axiom requires the property of asymptotic flatness, defined above, to give meaning to a 3-space. The three space is defined in the global asymptotically flat, topologically trivial region, $\mathcal{M}/K$, which is diffeomorphic to $\Re^4/(\mathbf{B}^3 \times \Re)$ as defined above.

We are now able to state the required properties of a 4-geon.

Conjecture 1 (4-Geon) *A particle is a semi-Riemannian spacetime manifold, $\mathcal{M}$, which is a solution of Einstein's equations of general relativity. The manifold is topologically non-trivial, with a non-trivial causal structure, and is asymptotically flat and particle-like (Axiom 2).*

It would be very appealing if $\mathcal{M}$ was a solution of the vacuum equations[6], but for the arguments that follow this is not essential; unspecified non-gravitational sources could be part of the structure. The assumed existence of CTCs as an integral part of the structure (rather than as a passive feature of the background topology) is an essential feature of the manifold; when they exist additional boundary conditions may be required to define the manifold[8]. This aspect of the structure provides a causal link between measurement apparatus and state

[‡] $\mathbf{B}^3$ is a solid sphere

preparation, permitting *both* to form part of the boundary conditions which constrain the field equations.

The axioms formally state conditions that any description of a particle must reasonably be expected to satisfy. In contrast, the 4-geon (Conjecture 1 above) is novel and speculative since it is not known whether such solutions exist - either to the vacuum or the full field equations of general relativity; however, *there no reason to suppose that they cannot exist.* It will be shown that this single speculative element not only yields quantum logic, but is sufficient to derive the equations of quantum mechanics and in doing so reconciles general relativity with quantum mechanics. Although this work proposes novel and unproven structures in general relativity it requires neither a modification, nor any addition to Einstein's equations; the number of spacetime dimensions remains $3 + 1$. The work does not require extraneous particle fields (as used in conventional quantum field theory), nor does it impose a quantum field of unknown origin (as does Bohm's theory). In short, this single conjecture is sufficient to unify particle and gravitational field descriptions of Nature, quantum and classical logic and quantum mechanics with general relativity.

4 STATE-PREPARATION AND MEASUREMENT

The role of both the measurement and state preparation in defining the 4-geon is crucial. It is self-evident that state preparation sets boundary conditions. Whether we regard a particle as a classical billiard ball, a quantum of a quantum field, or a classical field, the state preparation limits the possibilities; it restricts the possible solutions to those consistent with the apparatus. Systems with slits, collimators and shutters provide obvious boundary conditions which any solution must satisfy. For a geon, or a 4-geon, a barrier is a region which the topologically non-trivial region cannot traverse. Such barriers can be used to form slits and collimators *etc.*. and they obviously restrict the space of possible solutions. We state this formally as an axiom:

Axiom 3 (State preparation) *The state preparation sets boundary conditions for the solutions to the field equations.*

The exact nature of these boundary conditions, and whether they can always be equated with physical barriers such as collimators, is irrelevant to the analysis that follows.

Consider now an apparatus associated with a measurement, which is in many respects similar to that involved in a state preparation. Arrangements of slits, barriers and collimators are common features of the measurement apparatus. They are constructed from barriers which cannot be traversed by the non-trivial topology, which is the particle.

We take as a paradigm for a position measurement that barriers divide space into regions which are then probed (in any manner) to ascertain the existence, or otherwise, of the particle in a region. The particle-like axiom and the asymptotic flatness axiom assures us that the topologically non-trivial region can be confined but not split.

We take the view of Holland [11] that most measurements can be reduced to position measurements. A sequence of shutters and collimators and filters (*eg.*such as used in a Stern-Gerlach apparatus) determines the state preparation, while a very similar system of shutters *etc.*, resulting in confinement to one of a number of regions and subsequent detection, acts as a measurement.

For a classical object there is no causal connection that could allow the measurement conditions to influence the evolution. If the state preparation was insufficient to uniquely specify the trajectory there would be a statistical distribution of possible initial states, each of which would evolve deterministically. By contrast on a spacetime with CTCs extra conditions are required for a unique deterministic evolution [8]. With a particle modelled as a 4-geon however, there *would* be a causal link allowing the measurement conditions to contribute to the definition of the 4-manifold. A 4-geon is a 4-dimensional spacetime manifold which satisfies the boundary conditions set by *both* the state preparation *and* the measurement. This justifies a further axiom:

Axiom 4 (Measurement process) *The measurement process sets boundary conditions for the 4-geon which are not necessarily redundant, in the sense that they contribute to the definition of the 4-manifold.*

This axiom is inevitable if the particle contains CTCs, because the state preparation and the measurement conditions can no longer be distinguished by causal arguments.

Axiom 5 (Exclusive experiments) *Some pairs of measurements are mutually exclusive in the sense that they cannot be made simultaneously.*

This axiom expresses an established experimental fact - see[12, Chapter 7]. The famous examples of two such complementary variables are x-position and x-momentum. The x and y components of spin form another pair of complementary variables with a very simple logical structure. That measurements cannot be made simultaneously is still consistent with classical physics; objects would have a precise position and momentum, but we could only measure one property or the other. Quantum mechanics goes much further and asserts that a particle cannot *even* posses precise values of both properties simultaneously. The present work is unique in explaining why an inability to make simultaneous measurements should lead to incompatible observables in the quantum mechanical sense.

5 PROPOSITIONS AND 4-MANIFOLDS

We now consider the semi-Riemannian manifolds, $\mathbf{M}$, that could satisfy the different boundary conditions imposed by state preparation and measurement: Let $\mathcal{M} \equiv \{\mathbf{M}\}$ denote the set of 4-manifolds consistent with the state preparation conditions; there is no reason to suppose that a $\mathbf{M}$ is unique. The inability to define $\mathbf{M}$ uniquely will result in a *classical* distribution of measurement results.

The 4-manifold describes both the particle and its evolution; for a 4-geon they are inseparable. Consequently, the terms *initial* and *evolution* need to be used with great care. Although valid in the asymptotically flat region (and hence to any observer), they cannot be extended throughout the manifold. Preparation *followed by* measurement is also a concept valid only in the asymptotic region: *within the particle causal structure breaks down.*

Consider first the case of the classical 3-geon. Each $\mathbf{M}$ corresponds to an evolving 3-manifold $\mathbf{M}^3$. Each $\mathbf{M}^3$ will evolve deterministically in a way determined uniquely by the Einstein field equations and the initial condition $\mathbf{M}^3(t_0)$ (the distribution of $\mathbf{M}^3(t_0)$ determines the distribution of $\mathbf{M}^3(t)$ at any later time $t_1 > t_0$). If the geon is particle-like, then any experiment that depends upon a position measurement will give a result for each $\mathbf{M}^3$ at any time. Consequently, the boundary conditions imposed by measurements are necessarily compatible with any 3-geon that satisfies the particle-like proposition; in other words they are redundant.

By contrast, the 4-geon with CTCs as part of its structure cannot be decomposed into a three manifold and a time variable. It is known that further boundary conditions need not be redundant in a spacetime which admits CTCs[8]. In principle, the measurement apparatus itself can provide additional boundary conditions.

Consider measurements P, Q for which the boundary conditions cannot be simultaneously applied. They could be the x-component of spin and y-component of spin, or x-position and x-momentum; for simplicity we will consider two-valued measurements (*eg.* spin up or down for a spin-half particle or x-position > 0 and x-momentum > 0). We will denote the result that *"the state has a +ve P value"* by P^+, (P^-, Q^+, Q^-, are defined similarly). As propositions, P^+ is clearly the complement of P^-; if P^+ is true then P^- is false and *vice versa*, and similarly for Q^+ and Q^-.

As before, let $\mathcal{M} \equiv \{\mathbf{M}\}$ denote the set of 4-manifolds consistent with the state preparation. The measurements define subsets of $\{\mathbf{M}\}$; we denote by $\mathcal{P}$ those manifolds consistent with the state preparation *and* the boundary conditions imposed by a P-measurement. $\mathcal{P}$ is clearly the disjoint union of $\mathcal{P}^+$ and $\mathcal{P}^-$ - the manifolds corresponding to P^+ and P^-, respectively. Where the boundary conditions imposed by the measurement are not redundant $\{\mathbf{M}\}$, $\mathcal{P}$ and $\mathcal{Q}$ need not be the same (see Figure 1). There is a one-to-one correspondence between the sets of manifolds in the Figure and the four non-trivial propositions, p, q, r, s. However, two statements, or experimental procedures correspond to the same proposition if they cannot be distinguished by any state preparation - in other words if they give exactly the same information about each and every state. Therefore the statement that P has a value is always true by the particle-like Axiom 2, as is the statement that Q has a value; hence the subsets $\mathcal{P}$ and $\mathcal{Q}$ correspond to the *same* proposition I and we have the possibility:

$$\mathcal{P}^+ \neq (\mathcal{P}^+ \cap \mathcal{Q}^+) \cup (\mathcal{P}^+ \cap \mathcal{Q}^-) \tag{1}$$

If the boundary conditions are incompatible then $\mathcal{P}$ and $\mathcal{Q}$ are disjoint and the following holds (see Figure 1):

$$0 = (\mathcal{P}^+ \cap \mathcal{Q}^+) = (\mathcal{P}^+ \cap \mathcal{Q}^-) \neq \mathcal{P}^+ \tag{2}$$

Therefore, propositions about a state do not necessarily satisfy the distributive law of Boolean algebra.

6 GENERAL RELATIVITY AND QUANTUM MECHANICS

The significance of this result (Equation 1) is that the failure of the distributive law is synonymous with the existence of incompatible observables[4, Page 126]; it is a definitive property of non-classical systems of which a system obeying the rules of quantum mechanics is an example. To obtain quantum mechanics (as represented by a projections of a Hilbert space) we need to replace the distributive law with the weaker orthomodular condition:

$$a \leq b \Rightarrow b = a \vee (b \wedge \mathrm{NOT}(a)) \tag{3}$$

where $\leq$ is a partial ordering relation which is transitive, reflexive and anti-symmetric; it corresponds to set theoretic inclusion of the manifolds, $\mathcal{A} \subseteq \mathcal{B}$.

For propositions, a and b, the ordering relation can only be applied if they can be evaluated together [4, Chapter 13]. When $a \leq b$ there must be a measurement apparatus which enables a and b to be measured together. Let $\mathcal{P}$ be the subset of $\mathcal{M}$ defined by this measurement (see Figure 2). Then $\mathcal{A}^+ \subseteq \mathcal{B}^+ \subseteq \mathcal{P}$ and the complements with respect to $\mathcal{P}$ satisfy $\mathcal{B}^- \subseteq \mathcal{A}^- \subseteq \mathcal{P}$. Clearly:

$$\mathcal{A}^+ \subseteq \mathcal{B}^+ \Rightarrow \mathcal{B}^+ = \mathcal{A}^+ \cup (\mathcal{B}^+ \cap \mathcal{A}^-) \tag{4}$$

Hence the weaker orthomodularity condition is satisfied by propositions about the 4-geon manifolds. Quantum mechanics (as represented on a complex Hilbert space) is a realization of non-distributive proposition systems which satisfy Equation 3, and is believed to be unique as a representation on a vector space. For a review and further references on the relation between non-distributive proposition systems, quantum mechanics and complex Hilbert spaces see [4, Chapters 21,22].

7 CONSTRUCTION OF A MODULAR LATTICE

By considering the measurements of the x and y components of spin of a 4-geon with spin-half it is possible to construct a modular lattice of propositions. It has been reported by Friedman and Sorkin[13] that manifolds with the transformation properties of a spinor can be constructed. For the construction which follows, we require the 4-geon to have more than one possible outcome from a Stern-Gerlach apparatus. We will consider two possible outcomes ($> 0, \leq 0$); the exact spectrum, whether it is finite or infinite, continuous or discrete is not important. The choice of x and y-spin and the restriction to two outcomes is made to give a simple model of the spin for a spin-half particle; momentum and position could equally well have been used.

The relationship between orthomodular lattices and complex Hilbert spaces described in References[4, 5], means that once we have constructed an orthomodular lattice of propositions we can apply the internal symmetries and the symmetries of space-time in the usual way[10, Chapter 3] to determine the form of the operators and the eigenvalues for spin, momentum *etc.* The fact that a spin-half particle has two possible values for the x, y or z component of spin need not be assumed.

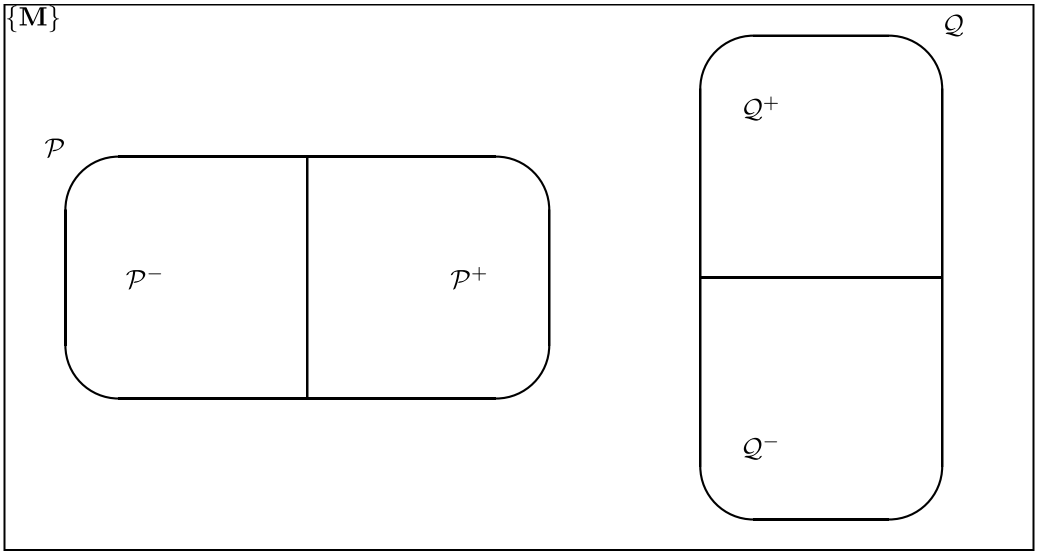

Figure 1: Sets of 4-manifolds consistent with both state preparation and the boundary conditions imposed by different measurement conditions.

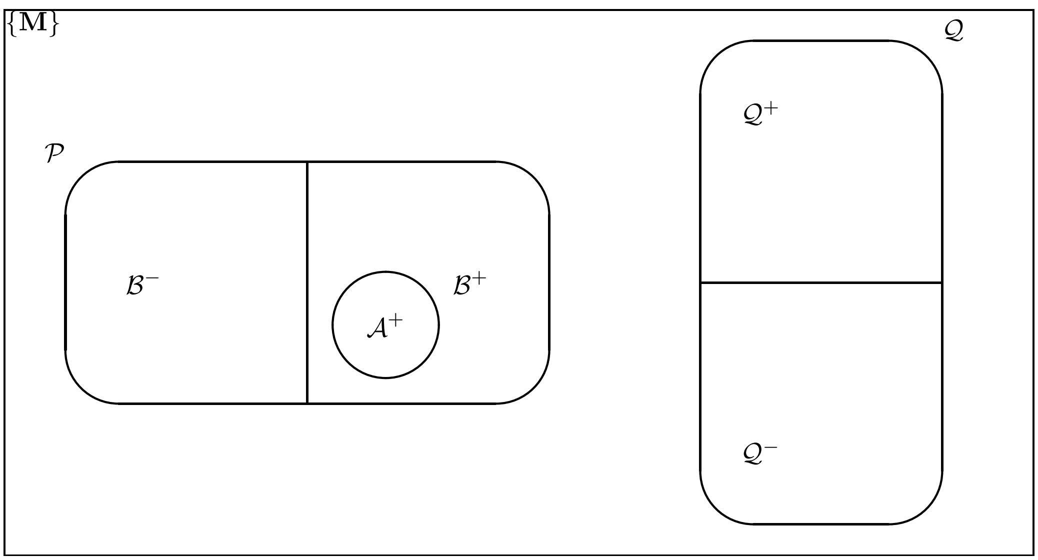

Figure 2: Sets of 4-manifolds illustrating the orthomodular condition for compatible propositions, a and b.

The set of all possible 4-geon manifolds, $\tilde{\mathcal{M}}$, is not very useful, since it includes manifolds compatible and incompatible with every experimental arrangement. Let us constrain the possible manifolds by setting up the state preparation apparatus as depicted in Figure 3. By Axiom 3, the apparatus imposes boundary conditions which limit the set of relevant manifolds to $\mathcal{M} \subset \tilde{\mathcal{M}}$, *i.e.* to those 4-geons compatible with the apparatus of Figure 3.

Next we can set up a Stern-Gerlach apparatus aligned with the x-axis, followed by an x-position detector which here gives a value for the spin (see Figure 4). By Axiom 2, the particle will certainly be detected at one position and only one position. We denote by $\mathcal{X}$ the 4-geon manifolds consistent with the state-preparation and the x-oriented Stern-Gerlach equipment. Clearly $\mathcal{X} \subseteq \mathcal{M}$; because of the 4-geon postulate we can have $\mathcal{X} \neq \mathcal{M}$. Of all the manifolds in $\mathcal{X}$, some will correspond to $x > 0$, and the remainder to $x \leq 0$; these will be denoted $\mathcal{X}^+$ and $\mathcal{X}^-$, respectively. Note that the same measurement apparatus determines $x > 0$ and $x \leq 0$; therefore $\mathcal{X} = \mathcal{X}^+ \cup \mathcal{X}^-$.

A y-axis measurement may be made in a similar way (see Figure 5) which defines subsets $\mathcal{Y}$, $\mathcal{Y}^+$ and $\mathcal{Y}^-$ of $\mathcal{M}$. An x and y-oriented Stern-Gerlach apparatus clearly cannot both be set in the *same* place at the *same* time; they are incompatible, and by Axiom 4 the boundary conditions which they set are incompatible. Hence $\mathcal{Y}$ and $\mathcal{X}$ are disjoint subsets of $\mathcal{M}$.

7.1 The Propositions

The propositions are the equivalence classes of outcomes of yes/no experiments, two experiments being equivalent if there is no state preparation that can distinguish them. Four non-trivial propositions, p,q,r and s, can be stated about the x and y-spin of 4-geon manifolds, $\mathbf{M}$. They listed in Table 1, together with the subsets of manifolds in the equivalence class and the experimental results which they relate to.

In addition, there are the two trivial propositions 0 and I. Axiom 2 ensures that there exists at least one 4-geon manifold consistent with any measurement ($\exists \mathbf{M} \in \mathcal{X}$). Equivalently, given the state-preparation and measurement boundary conditions then $\mathbf{M} \in \mathcal{X}$. The trivial propositions, I (which is always true) and 0 (which is always false), correspond to this Axiom and its converse:

$$
\begin{aligned}
0 \;&\equiv\; \mathbf{M} \in \emptyset & \qquad I \;&\equiv\; \mathbf{M} \in \mathcal{X} \text{ for an } x\text{-spin measurement} \\
&\equiv\; \mathcal{X} = \emptyset & &\equiv\; \mathbf{M} \in \mathcal{Y} \text{ for a } y\text{-spin measurement} \\
&\equiv\; \mathcal{Y} = \emptyset & &
\end{aligned} \tag{5}
$$

The fact that the trivial propositions have more than one interpretation is common to classical mechanics. For example, the propositions that *the momentum is a real number* and that *the position is a real number* are both always true for a classical object. What is non-classical here is that these two physical descriptions correspond to two different (and disjoint) sets of possible results. Classically the measurements are different ways of partitioning the common set defined by the initial conditions alone. Here the measurements define two different sets, but the propositions are identical because the sets give the same information.

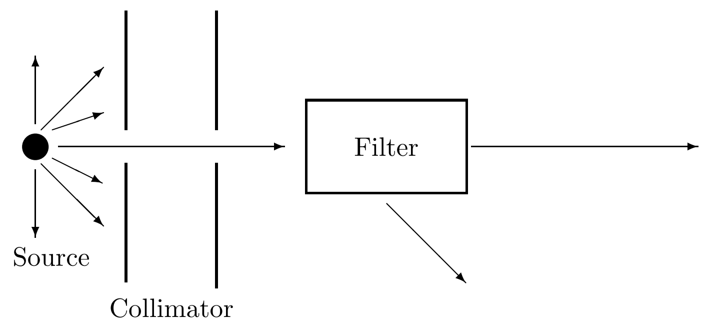

Figure 3: The boundary conditions imposed by state-preparation

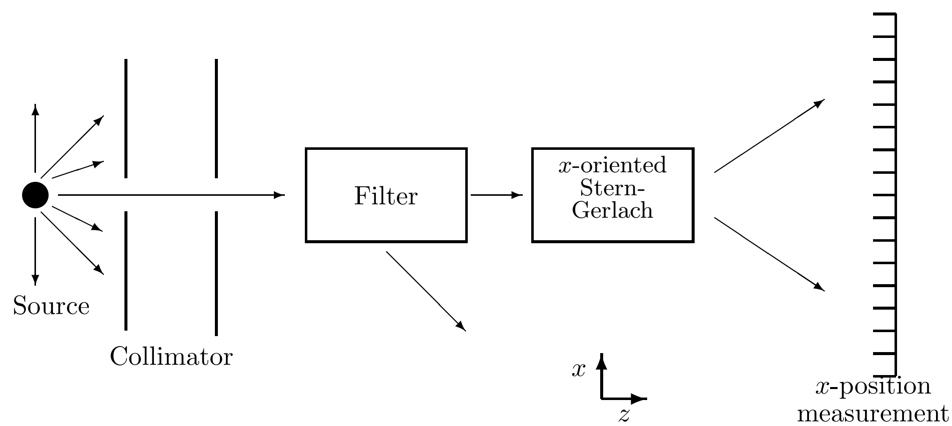

Figure 4: The boundary conditions imposed by state-preparation and an x-spin measurement

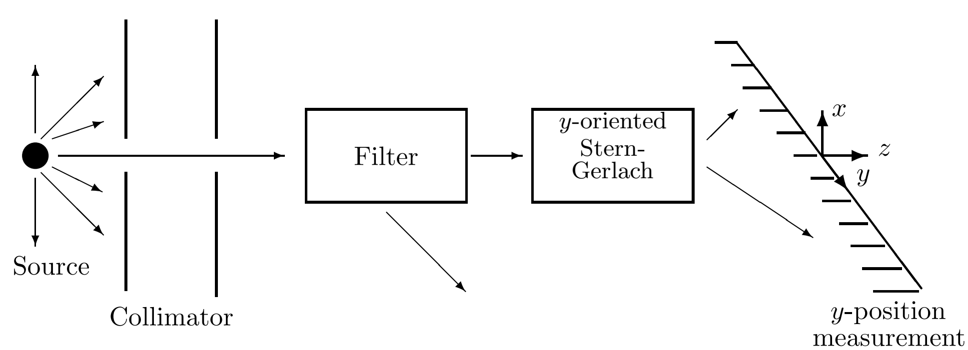

Figure 5: The boundary conditions imposed by state-preparation and a y-spin measurement

7.2 Partial Ordering

The ordering relation for two propositions, a and b, is $a \leq b$ which means that a true implies that b is true. For the spin-half system the partial ordering is almost trivial:

$$0 \leq p \leq I, \qquad 0 \leq q \leq I, \qquad 0 \leq r \leq I, \qquad 0 \leq s \leq I \tag{6}$$

In this case there can be no ordering between p and r *etc.* when they are in different directions, because $\mathcal{X}$ and $\mathcal{Y}$ are disjoint (and can clearly be distinguished by some state preparations) and so a manifold cannot be in both. The propositions of the system therefore form a poset (partially ordered set). Generally, the ordering relation can only be applied to propositions if there exists at least one experimental arrangement which evaluates both of them together.

7.3 Meet and Join

The meet of two propositions, $a \wedge b$, is the largest proposition, the truth of which implies that both a and b are true. For any poset it follows that: $a \wedge a = a$, $a \wedge I = a$ and $a \wedge 0 = 0$. For this system we have in addition:

$$a \wedge b = 0, \qquad \forall a \neq b \tag{7}$$

For a 4-geon manifold, $\mathbf{M}$, to be in the meet of p and r, it would have to be in $\mathcal{X}^+$ and $\mathcal{Y}^+$ which is not possible; the solution set is therefore the empty set which corresponds to 0. Membership of the subsets $\mathcal{X}^+$ and $\mathcal{Y}^+$ corresponds to physically distinguishable statements about the state preparation so the equivalence relation does not affect this conclusion.

The join of two propositions, $a \vee b$, is the smallest proposition which is true whenever either a or b is true. For any poset it follows that: $a \vee a = a$, $a \vee 0 = a$ and $a \vee I = I$. For this system we have in addition:

$$a \vee b = I \qquad \forall a \neq b \tag{8}$$

In this small system there is no other acceptable choice for $p \vee r$ *etc.*

7.4 Orthocomplementation

As in classical mechanics we consider the orthocomplementation $a^\perp$ of a proposition a by taking the set-theoretic complement with respect to all possible outcomes of the same experiment. We define the complements of our system in Table 2

From Table 2 and Table 1, it is clear that the required properties of orthocomplementation are satisfied:

1. $(a^\perp)^\perp = a$
2. $a \vee a^\perp = I$ and $a \wedge a^\perp = 0$
3. $a \leq b \Rightarrow b^\perp \leq a^\perp$

The first two follow directly from set theory, while the third only applies in the cases: $a < I$ or $0 < a$, because of the simple structure of this poset.

The definition given satisfies DeMorgan's laws:

$$(a_1 \wedge a_2)^{\perp} = a_1^{\perp} \vee a_2^{\perp} \tag{9}$$

$$(a_1 \vee a_2)^{\perp} = a_1^{\perp} \wedge a_2^{\perp} \tag{10}$$

$$\tag{11}$$

Thus we have an orthocomplemented poset. DeMorgan's Laws can be used to define the join of two incompatible propositions in terms of the meet and orthocomplementation *eg.*:

$$p \vee r = (q \wedge s)^{\perp} = 0^{\perp} = I \tag{12}$$

7.5 Lattice

A lattice is a poset where the meet and join always exist. The meet and join of any two elements of this system always exist, these being 0 and I respectively, for any two different propositions. Table 3 shows the meet and join for all the propositions.

The poset is thus seen to be an orthocomplemented *Lattice*.

7.6 Orthomodularity

The orthomodularity condition:

$$a \leq b \Rightarrow b = a \vee (b \wedge a^{\perp}) \tag{13}$$

is satisfied by the simple spin-half poset, as can be seen by considering each case, $\forall a \in \{p, q, r, s\}$:

$$0 \leq a \quad \Rightarrow \quad a = 0 \vee (a \wedge I) \tag{14}$$

$$a \leq I \quad \Rightarrow \quad I = a \vee (I \wedge a^{\perp}) \tag{15}$$

$$a \leq a \quad \Rightarrow \quad a = a \vee (a \wedge a^{\perp}) \tag{16}$$

$$0 \leq I \quad \Rightarrow \quad I = 0 \vee (I \wedge I) \tag{17}$$

7.7 Modularity

That this lattice is modular can be seen by examining it case by case. The failure of the modularity law, as required for a strictly orthomodular lattice, will only occur for systems with an infinite spectra [14, Page 220].

7.8 Distributivity

A simple counterexample suffices to show that the distributive rule fails for propositions about different directions:

$$p \wedge (r \vee r^{\perp}) \neq (p \wedge r) \vee (p \wedge r^{\perp}) \tag{18}$$

the LHS is $p \wedge I = p$, while the RHS is $0 \vee 0 = 0$; thus p and r are not compatible. The result can be checked from Table 3 of meets and joins or by noting that the subsets $\mathcal{X}^{+}, \mathcal{Y}^{+}, \mathcal{Y}^{-}$ corresponding to the propositions p,r and s, respectively, are all disjoint and not related by the equivalence relation.

Proposition	Manifolds	Measurement
0	$\emptyset$	Always False
p	$\mathbf{M} \in \mathcal{X}^+$	The x-Spin is measured to be > 0
q	$\mathbf{M} \in \mathcal{X}^-$	The x-Spin is measured to be ≤ 0
r	$\mathbf{M} \in \mathcal{Y}^+$	The y-Spin is measured to be > 0
s	$\mathbf{M} \in \mathcal{Y}^-$	The y-Spin is measured to be ≤ 0
I	$\mathbf{M} \in \mathcal{X}$	The x-Spin is measurable
I	$\mathbf{M} \in \mathcal{Y}$	The y-Spin is measurable

Table 1: The propositions and sets of manifolds of the spin-half system

Complement	Manifolds
$0^\perp = I$	Always True
$p^\perp = q$	$\mathbf{M} \in (\mathcal{X} \setminus \mathcal{X}^+ \equiv \mathcal{X}^-)$
$q^\perp = p$	$\mathbf{M} \in (\mathcal{X} \setminus \mathcal{X}^- \equiv \mathcal{X}^+)$
$r^\perp = s$	$\mathbf{M} \in (\mathcal{Y} \setminus \mathcal{Y}^+ \equiv \mathcal{Y}^-)$
$s^\perp = r$	$\mathbf{M} \in (\mathcal{Y} \setminus \mathcal{Y}^- \equiv \mathcal{Y}^+)$
$I^\perp = 0$	Always False

Table 2: The complements of the propositions of the spin-half system

$\wedge$	0	p	q	r	s	I
0	0	0	0	0	0	0
p	0	p	0	0	0	0
q	0	0	q	0	0	0
r	0	0	0	r	0	0
s	0	0	0	0	s	0
I	0	0	0	0	0	I

$\vee$	0	p	q	r	s	I
0	0	I	I	I	I	I
p	I	p	I	I	I	I
q	I	I	q	I	I	I
r	I	I	I	r	I	I
s	I	I	I	I	s	I
0	I	I	I	I	I	I

Table 3: The meets and joins of the propositions of the spin-half system

7.9 Atomicity

An atom is a proposition, different from 0, which does not have any smaller proposition. The propositions p, q, r, s are clearly the atoms.

7.10 Covering Property

We say that a covers b if $a > b$, and $a \geq c \geq b$ implies either $c = a$ or $c = b$. A lattice has the covering property if the join of any element, a, with an atom not contained in a covers a. Clearly $\forall a, b \in \{p, q, r, s\}$:

$$0 \vee a = a \quad \text{which covers } 0 \tag{19}$$

$$a \vee b = I \quad \text{which covers } a \tag{20}$$

This establishes that the system has the covering property.

Starting with propositions about sets of manifolds in classical general relativity, we have constructed a non-distributive, orthomodular lattice, which is atomic and has the covering property. The significance is not just that such a lattice is a feature of quantum mechanics, but that it is *the* distinguishing feature of quantum mechanics. It has previously been thought that a non-distributive lattice of propositions could never be constructed from a classical theory and hence that no classical explanation of quantum mechanics was possible; this is shown to be false. The present work gives a classical explanation for the origin of quantum mechanics and because it is based on the accepted theory of general relativity, it offers the most economical interpretation.

8 CONCLUSIONS

By modelling particles as 4-geons in general relativity (rather than evolving 3-manifolds), features characteristic of quantum mechanics can be derived. This work therefore offers a novel possibility for a classical basis for quantum mechanics, and in doing so offers a way to reconcile general relativity and quantum mechanics. Some of the implications and unresolved issues are:

1. Time is an asymptotic approximation as expected by workers in quantum gravity[15].

2. The theory does not exclude classical objects. Measurements of a 3-geon, if one existed, could not satisfy the logic of quantum mechanics. Gravitational waves are also described as evolving 3-manifolds and, although there may be some problems defining a global hypersurface [16], they do not have the topological structure to exhibit the measurement-dependent effects characteristic of quantum mechanics.

3. It follows from the previous comment that there is no graviton. Potentially, this is a testable prediction of the theory.

4. Even in the asymptotic region, the metric associated with our 4-geon model of a single particle is not well-defined by state preparation alone, since each manifold consistent with the state preparation ($\mathbf{M} \in \mathcal{M}$) can

have different asymptotic properties. This is an almost inevitable consequence of reconciling quantum mechanics and general relativity[17]. However, the present perspective on the origin of quantum mechanics accounts for the lack of a well-defined metric as being due to incompleteness of boundary conditions imposed by state preparation, rather than as an inherent feature of the gravitational field.

5. Non-local effects - as exemplified by the EPR experiments - can be explained by theories with non-trivial topologies[11, Page 481], since with CTCs there can exist causal routes from one arm of the experiment to the other. Quantum mechanics itself requires only a failure of weak-causality (statistical correlations of a non-local character between spacelike separated events[18]); with CTCs as an essential part of the structure of an elementary particle however, it is not clear why a failure of strong-causality (communication between spacelike separated events) is not apparent.

6. Like all theories of quantum gravity and interpretations of quantum mechanics, this work is speculative. The theory can only be proven if exact solutions to Einstein's equations with the required properties are found. Considering the difficulty of finding exact solutions with non-trivial topology, a more practical way of confirming these ideas is to examine the predictions, the first of which is the absence of a graviton.

It would indeed be ironic if the interpretation quantum theory with which Einstein was so dissatisfied could be seen to be a consequence of his general theory of relativity.

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