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Unique normal forms near a degenerate elliptic fixed point in two-parametric families of area-preserving maps

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Abstract

We derive simplified normal forms for an area-preserving map in a neighbourhood of a degenerate resonant elliptic fixed point. Such fixed points appear in generic families of area-preserving maps. We also derive a simplified normal form for a generic two-parametric family. The normal forms are used to analyse bifurcations of n -periodic orbits. In particular, for $n \geq 6$ we find regions of parameters where the normal form has ‘meandering’ invariant curves.

Keywords: area-preserving maps, resonant fixed point, normal forms, bifurcation

Mathematics Subject Classification: 37J40, 37G05, 70K45, 70K4

(Some figures may appear in colour only in the online journal)

1. Introduction

In a Hamiltonian system, small oscillations around a periodic orbit are often described using the normal form theory [2, 14]. In the case of two degrees of freedom, the Poincaré section is used to reduce the problem to studying a family of area-preserving maps in a neighbourhood of



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a fixed point. The Poincaré map depends on the energy level and possibly on other parameters involved in the problem. A sequence of coordinate changes is used to transform the map to a normal form. In the absence of resonances, the normal form is a rotation of the plane, and the angle of the rotation depends on the amplitude. In the presence of resonances, the normal form is more complicated.

Let us describe the normal form theory in more detail. Let $F_0 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an area-preserving map. We assume that F_0 also preserves orientation. Let the origin be a fixed point:

$$F_0(0) = 0.$$

Since F_0 is area-preserving, $\det DF_0(0) = 1$. Therefore the two eigenvalues of the Jacobian matrix $DF_0(0)$ are λ_0 and λ_0^{-1} . These eigenvalues are often called the *multipliers* of the fixed point. We will consider an elliptic fixed point when λ_0 is not real. As the map is real-analytic, the multipliers are complex conjugate, i.e. $\lambda_0^{-1} = \lambda_0^*$, and consequently belong to the unit circle, i.e. $|\lambda_0| = 1$. Note that in our case $\lambda_0 \neq \pm 1$, as it is assumed to be non-real.

There is a linear area-preserving change of variables such that the Jacobian of F_0 takes the form of a rotation:

$$DF_0(0) = R_{\alpha_0} \quad \text{where } R_{\alpha_0} = \begin{pmatrix} \cos \alpha_0 & -\sin \alpha_0 \\ \sin \alpha_0 & \cos \alpha_0 \end{pmatrix}, \tag{1}$$

where the rotation angle α_0 is related to the multiplier: $\lambda_0 = e^{i\alpha_0}$.

The classical normal form theory [2] states that there is a formal area-preserving change of coordinates which transforms F_0 into the resonant normal form N_0 such that the formal series N_0 commutes with the rotation: $N_0 R_{\alpha_0} = R_{\alpha_0} N_0$. Following the method suggested in [17] (see e.g. [5, 9]), we consider a formal series H_0 such that

$$N_0 = R_{\alpha_0} \Phi_{H_0}^1, \tag{2}$$

where $\Phi_{H_0}^1$ is a flow generated by the Hamiltonian H_0 . The Hamiltonian is invariant with respect to the rotation $H_0 \circ R_{\alpha_0} = H_0$. It follows that the normal form preserves the Hamiltonian: $H_0 \circ N_0 = H_0$, since the Hamiltonian flow also preserves H_0 . Returning to the original coordinates, we conclude that the original map has a non-trivial formal integral \hat{H}_0 . This integral provides a powerful tool for analysis of the local dynamics including the stability of the fixed point (see e.g. [1]).

It is natural to describe the normal forms using the symplectic polar coordinates (I, φ) defined by the equations

$$x = \sqrt{2I} \cos \varphi, \quad y = \sqrt{2I} \sin \varphi. \tag{3}$$

A fixed point is called *resonant* if there exists $n \in \mathbb{N}$ such that $\lambda_0^n = 1$. The least positive n is called the *order of the resonance*. In the resonant case, the formal Hamiltonian takes the form:

$$H_0(I, \varphi) = I^2 \sum_{k \geq 0} a_k I^k + \sum_{k \geq 1} \sum_{j \geq 0} b_{kj} I^{kn/2+j} \cos(kn\varphi + \beta_{kj}), \tag{4}$$

where a_k, b_{kj} and β_{kn} are real coefficients. The normal form is not unique. In the paper [9], we proved that in the non-degenerate case, namely if $a_0 b_{10} \neq 0$ for $n \geq 4$ or $b_{10} \neq 0$ for $n = 3$, the Hamiltonian can be simplified substantially: there is a canonical formal change of variables that transforms the Hamiltonian to the form

$$H_0(I, \varphi) = I^2 A(I) + I^{n/2} B(I) \cos(n\varphi), \tag{5}$$

where A and B are formal series in powers of I . We refer the reader to paper [9] for detailed discussions of possible simplifications for the formal series A and B that depend on the order

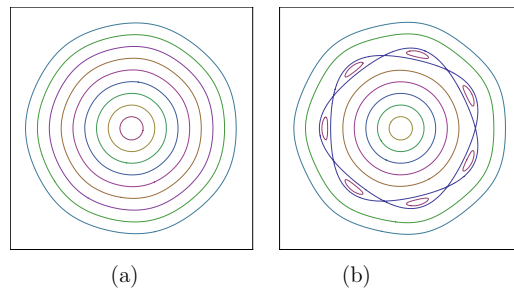


Figure 1. Level lines of the normal form Hamiltonian: typical pictures for $n = 7$.

of the resonance n . In particular, for $n \geq 4$, the series $B(I)$ contains even powers only. We note that (5) involves a formal series in a single variable I only and therefore is substantially simpler than (4), which involves a series in two variables.

In this paper, we consider the case when the fixed point of F_0 is degenerate. Namely, assuming some leading coefficients $a_k = 0$ and $b_{10} \neq 0$, we prove that the normal form Hamiltonian (4) can be transformed to a simplified form that looks similar to (5). The simplified normal form is unique and, therefore, in general there is no room for further simplification. A detailed description of the unique normal form is given in section 2.

A degeneracy of the lowest co-dimension corresponds to the case when in (4)

$$a_0 = 0, \quad a_1 \neq 0, \quad b_{10} \neq 0. \tag{6}$$

In this case, it is natural to consider the map F_0 as a member of a generic two-parameter family. In section 2, we provide unique normal forms for such families.

Of course, in general, the series of the normal form theory are expected to diverge. Nevertheless, they provide rather accurate information about the dynamics of the original map. For example, for any $k \in \mathbb{N}$, the partial sum \hat{H}_0^k (a polynomial that includes all terms of H_0 up to the order k) satisfies

$$\hat{H}_0^k \circ F_0 - \hat{H}_0^k = O_{k+1},$$

where O_{k+1} stands for an analytic function that has a Taylor series without terms of order less than $k + 1$. The implicit function theorem can be used to show that the map F_0 has n -periodic points near critical points of the Hamiltonian \hat{H}_0^k , and KAM theory can be used to establish the existence of invariant curves [1, 2].

In a generic one-parameter family of area-preserving maps, the normal form provides a description for a chain of islands that is born from the origin when the multiplier of the fixed point crosses a resonant value [2, 12, 14, 16]. This analysis distinguishes the cases of weak ($n \geq 5$) and strong ($n \leq 4$) resonances. A typical picture for $n = 7$ is shown in figures 1(a) and (b).

In this paper, we use the normal forms to describe bifurcations of n -periodic orbits that appear in a generic two-parameter family of area-preserving maps near a degenerate resonant elliptic fixed point. In section 3, we analyse bifurcations of n -periodic points using model normal form Hamiltonians for $n \geq 7$, $n = 6$ and $n = 5$, separately. It is interesting to note that bifurcation diagrams contain sectors where the leading part of the normal form does not satisfy the twist condition required by the standard KAM theory, and invariant curves of ‘meandering type’ are observed similar to the ones described in the papers [6, 8, 11, 15].

2. Unique normal forms

Suppose that F_0 is a real-analytic area-preserving map with an elliptic fixed point at the origin. We note the theory repeats almost literally in the C^∞ category. Let λ_0 be the multiplier of the fixed point and suppose that $\lambda_0^n = 1$ for some $n > 2$ and $\lambda_0^k \neq 1$ for all integer values of k , $1 \leq k < n$. Then there is an analytic area-preserving change of variables that eliminates all non-resonant terms from the Taylor series of F_0 up to the order $n + 1$.

It is convenient to rewrite F_0 identifying the plane with coordinates (x, y) and the complex plane of the variable $z = (x + iy)/\sqrt{2}$. In this definition the factor $\sqrt{2}$ is added to simplify the transition to the symplectic polar coordinates as equation (3) implies $z = \sqrt{I}e^{i\varphi}$. In complex notations, the transformed map F_0 takes the form

$$f_0(z, \bar{z}) = a(z\bar{z})z + b(z\bar{z})\bar{z}^{n-1} + O(|z|^{n+1}), \tag{7}$$

where a, b are polynomial in $z\bar{z}$. The coefficients of the polynomials are complex. It is easy to see that $a(0) = \lambda_0 = e^{i\alpha_0}$. If $a'(0) = 0$, the fixed point is degenerate. Let

$$m = 1 + \min\{k : a^{(k)}(0) \neq 0, k \geq 1\}.$$

So $m \geq 2$. We note that if $2m \leq n$, then m does not depend on the coordinate change that transforms F_0 to the form (7).

Theorem 1. *If λ_0 is resonant of order $n \geq 3$ and $b(0) \neq 0$, then there is a formal canonical change of variables that conjugates F_0 with $R_{\alpha_0} \circ \Phi_{H_0}^1$, where the formal Hamiltonian H_0 has the form*

$$H_0(I, \varphi) = A_0(I) + I^{n/2} B_0(I) \cos n\varphi,$$

where $A_0(I) = \sum_{k \geq k_0} a_k I^k$ and $B_0(I) = \sum_{k \geq 0} b_k I^k$ are formal series in I with real coefficients and one of the following two properties holds:

- if $2m \geq n$, then $a_k = b_k = 0$ for $k = -1 \pmod{n}$ and $k_0 \geq \frac{n}{2}$;
- if $2m \leq n$, then $b_k = 0$ for $k = -1 \pmod{m}$ and $k_0 = m$.

The coefficients of the series A_0 and B_0 are defined uniquely by the map F_0 , provided the leading order is normalized to ensure $b_0 > 0$.

The normal form is unique and consequently provides a complete set of formal invariants for the map F_0 . Theorem 1 is a generalization of a result from [9], where we studied the normal forms of a map with $a'(0)b(0) \neq 0$, which corresponds to the case of $m = 2$.

Note that for $n = 2m$, the theorem provides two slightly different alternatives for the normal form. The second one has an advantage, as in that case the transformation to the normal form does not develop a singularity when $a^{(m)}(0) = 0$ vanishes.

Theorem 1 follows from theorem 3 of section 4.1. The proofs are based on a development of the Lie series method [7, 9, 10, 13].

Let F_p be a two-parameter real-analytic family of area-preserving maps that depends on a small parameter $p = (p_1, p_2)$. We assume that

- (A1) F_p has an elliptic fixed point at the origin, i.e., $F_p(0) = 0$ and the multiplier $\lambda_p = e^{i\alpha_p}$ with $\alpha_p \in \mathbb{R}$ for all small p ,
- (A2) $p = 0$ corresponds to a resonance of order n .

Suppose that p is sufficiently small to ensure that $|\alpha_p - \alpha_0| < \frac{\pi}{2n^2}$. Then α_p does not satisfy any resonant condition of order less than $2n$ excepting the original one, which corresponds to $\alpha_p = \alpha_0$. According to the normal form theory, there is an analytic area-preserving change of

variables that eliminates all non-resonant terms from the Taylor series of F_p up to the order $n + 1$. We assume that F_p is already transformed to this form. Then in the complex coordinates, the map F_p has the form

$$f_p = a_p(z\bar{z})z + b_p(z\bar{z})\bar{z}^{n-1} + O(z^{n+1}), \tag{8}$$

where a_p and b_p are polynomial in $z\bar{z}$, which depend analytically on p , and $a_p(0) = \lambda_p = e^{i\alpha_p}$. We introduce new parameters δ and ν by

$$\delta = i \log \frac{\lambda_p}{\lambda_0} = \alpha_0 - \alpha_p \quad \text{and} \quad \nu = \frac{ia'_p(0)}{2\lambda_p}. \tag{9}$$

For $n \geq 4$ the preservation of the area by the map f_p implies that δ and ν are real and both parameters are defined uniquely, i.e., they are independent of the choice of the coordinate change C_p that transforms the original map F_p to the form (8).

If the following property holds, the family F_p is called a generic unfolding of F_0 :

(A3) the Jacobian of the map $(p_1, p_2) \mapsto (\delta, \nu)$ does not vanish at $p = 0$.

Then the inverse function theorem implies that p can be expressed in terms of (δ, ν) . From now on we use $p = (\delta, \nu)$ instead of the original parameters. Note that the assumption (A3) implies that $p = 0$ is not a critical point of the function α_p .

The normal form theory states that there is a formal change of variables that transforms F_p to its normal form $N_p = R_{\alpha_0} \circ \Phi_{H_p}^1$, where the formal sum H_p includes only resonant monomials, i.e., monomials of the form $h_{j,k,l,m} z^j \bar{z}^k \delta^l \nu^m$ with $k = j \pmod{n}$. The transformation to the normal form and the normal form itself are formal power series in z, \bar{z} and δ, ν .

We note that the normal form is not unique and can be simplified using a formal tangent-to-identity change of variables.

Theorem 2. *Let $n \geq 4$. If F_p satisfies assumptions (A1)–(A3), $a'_0(0) = 0$, $b_0(0) \neq 0$ and, only for $n \geq 6$, $a''_0(0) \neq 0$, then there is a formal canonical change of variables that conjugates F_p with $R_{\alpha_0} \circ \Phi_{H_p}^1$, where the formal Hamiltonian H_p has the form*

$$H_{\delta,\nu}(I, \varphi) = \delta I + \nu I^2 + A(I; \delta, \nu) + I^{n/2} B(I; \delta, \nu) \cos n\varphi, \tag{10}$$

where

$$A(I; \delta, \nu) = \sum_{\substack{k \geq 3 \\ m, j \geq 0}} a_{kmj} I^k \delta^m \nu^j, \quad B(I; \delta, \nu) = \sum_{k,m,j \geq 0} b_{kmj} I^k \delta^m \nu^j,$$

and

- if $n \leq 5$, then $a_{kmj} = b_{kmj} = 0$ for $k = -1 \pmod{n}$ and all $m, j \geq 0$;
- if $n \geq 6$, then $b_{kmj} = 0$ for $k = 2 \pmod{3}$ and all $m, j \geq 0$.

Moreover, the coefficients of the series $A(I; \delta, \nu)$ and $B(I; \delta, \nu)$ are defined uniquely.

This theorem follows from the case $m = 3$ of a more general theorem 4 stated and proved in section 4.5.

We note that theorem 4 also covers the case $n = 3$. We exclude this case from the statement of theorem 2 as we lose the uniqueness property: indeed, the definition of the parameter ν relies on a term of degree 4 which, for $n = 3$, depends on the transformation of the original family F_p to the form (8).

3. Bifurcations of n -periodic points

In this section, we use the normal form from theorem 2 to study bifurcations of n -periodic orbits in a two-parameter family of area-preserving maps. Let $H_{\delta,\nu}^n$ be the partial sum of the series (10) that includes all terms up to the order n in $I^{1/2}, \nu, \delta$. Then there is an analytic change of variables such that the n th iterate of the transformed map is close to the time- n Hamiltonian flow: $\tilde{F}_p^n = \Phi_{H_{\delta,\nu}^n}^n + O(I^{n/2}, \delta^n, \nu^n)$. For small I, δ, ν , the error term can be considered as a small perturbation, and a combination of a properly chosen scaling with the implicit function theorem can be used to establish relations between critical points of the Hamiltonian and n -periodic points of the map. Moreover, the KAM theory can be used to make conclusions about the persistence of some invariant curves [1, 6].

Instead of the polynomial $H_{\delta,\nu}^n$, it is convenient to consider a model Hamiltonian which includes only leading terms of the formal series A and B . When possible, the corresponding coefficients are normalized to unity with the help of a scaling applied to the variable I , the parameters δ and ν , and the Hamiltonian function h .

The analysis presented below can be easily extended from the models to the Hamiltonian $H_{\delta,\nu}^n$ and leads to similar-looking bifurcation diagrams and similar conclusions about the types and positions of critical points as well as about the shapes of invariant curves.

3.1. $n \geq 7$

For $n \geq 7$, the model Hamiltonian function takes the form

$$h = \delta I + \nu I^2 + I^3 + I^{n/2} \cos n\varphi, \tag{11}$$

where I, φ are symplectic polar coordinates (3).

If $\delta = \nu = 0$, the Hamiltonian h has a local minimum at the origin. Thus the origin is a stable equilibrium of the normal form. In its neighbourhood, the level lines of h are closed curves around the origin (similar to figure 1(a)).

Since $n \geq 7$, the origin is a stable equilibrium for each δ and ν .

The critical points of h are determined by the equations $\partial_I h = \partial_\varphi h = 0$. After computing the derivatives, we conclude that all equilibria have either $\varphi = 0$ or $\frac{\pi}{n} \pmod{\frac{2\pi}{n}}$, and

$$\delta + 2\nu I + 3I^2 + \sigma_\varphi \frac{n}{2} I^{n/2-1} = 0, \tag{12}$$

where $\sigma_\varphi = \cos n\varphi \in \{+1, -1\}$. The equation (12) can be written in the form

$$\delta = f_\sigma(I, \nu) \tag{13}$$

where

$$f_\sigma = -2\nu I - 3I^2 - \sigma \frac{n}{2} I^{n/2-1}.$$

A typical plot of the functions f_σ is shown in figures 2(a) and (b) for $\nu < 0$ and $\nu > 0$, respectively. Since the function f_σ is independent of δ , the numbers and positions of solutions to (13) can be easily read from the graph. Moreover, as the Hessian of h at a critical point is diagonal, the type of the critical point can be read from the slope of f_σ . A straightforward analysis shows that f_σ with $\nu < 0$ has a single non-degenerate maximum in the neighbourhood of the origin, and it is monotone for $\nu > 0$. So equation (12) has up to two non-negative solutions for $\nu > 0$ and from none to four solutions for $\nu < 0$ (depending on the value of δ).

The results of the above analysis are summarized in the bifurcation diagram for the critical points of h shown in figure 3. We note that qualitatively the bifurcation diagram is the same for all $n \geq 7$. In a neighbourhood of the origin on the (δ, ν) -plane, there are four domains

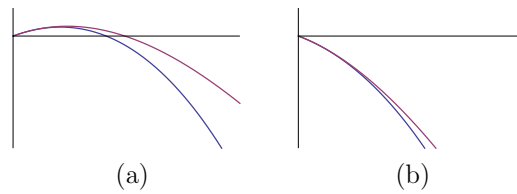


Figure 2. Plots of the functions f_+ and f_- : (a) for $\nu < 0$ and (b) for $\nu > 0$.

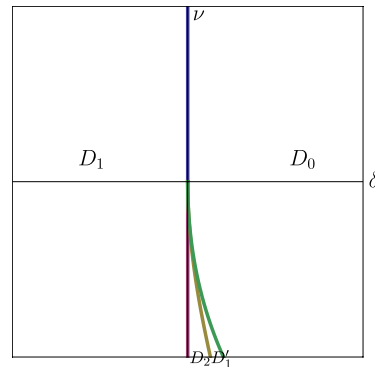


Figure 3. Example of a bifurcation diagram on the plane (δ, ν) ($n \geq 7$).

that correspond to different numbers of saddle critical points of h . If $(\delta, \nu) \in D_k, k = 0, 1, 2$, there are exactly kn saddle critical points of h . The fourth domain is a narrow sector, which separates D_0 and D_2 and which we denote by D'_1 . In this sector, the Hamiltonian has n critical saddle points.

Let us describe the level sets of h . When the parameters are in D_0 , all level sets of h are closed curves that look like the invariant curves shown in figure 1(a). When the parameters are in D_1 , the critical level set of h form a chain of n islands similar to the one shown in figure 1(b). When the parameters cross the positive ν -semiaxes moving from D_0 to D_1 , a chain of islands bifurcates from the origin. When the parameters cross the negative ν -semiaxes moving from D_1 to D_2 , a second chain of n -islands bifurcates from the origin (a typical picture is shown in figure 4(a)). In D_2 , there is a line on which both chains of islands belong to a single level set of h (see figure 4(c)). Near this line, the level sets of h change their topology without any change in the number of critical points (see figure 4(d)). We note that in this region, the Hamiltonian h has ‘meandering’ invariant curves similar to the ones studied in [6, 15]. We will provide a more detailed description of these curves later in this section. Finally, the chains of islands disappear through Hamiltonian saddle-node bifurcations when the parameters cross the boundary of D'_1 : first the outer one and then the other one (see figures 4(e) and (f)).

In order to derive an approximate expression for the curves bounding the domain D'_1 on the bifurcation diagram, we note that the corresponding values of δ coincide with the maximum value of the functions f_σ (compare with equation (13)). The equation $\frac{\partial f_\sigma}{\partial I} = 0$ takes the form

$$2\nu + 6I + \sigma \frac{n}{2} \left(\frac{n}{2} - 1 \right) I^{n/2-2} = 0. \tag{14}$$

Equations (12) and (14) together define a line on the plane of (δ, ν) along which h has a doubled critical point.

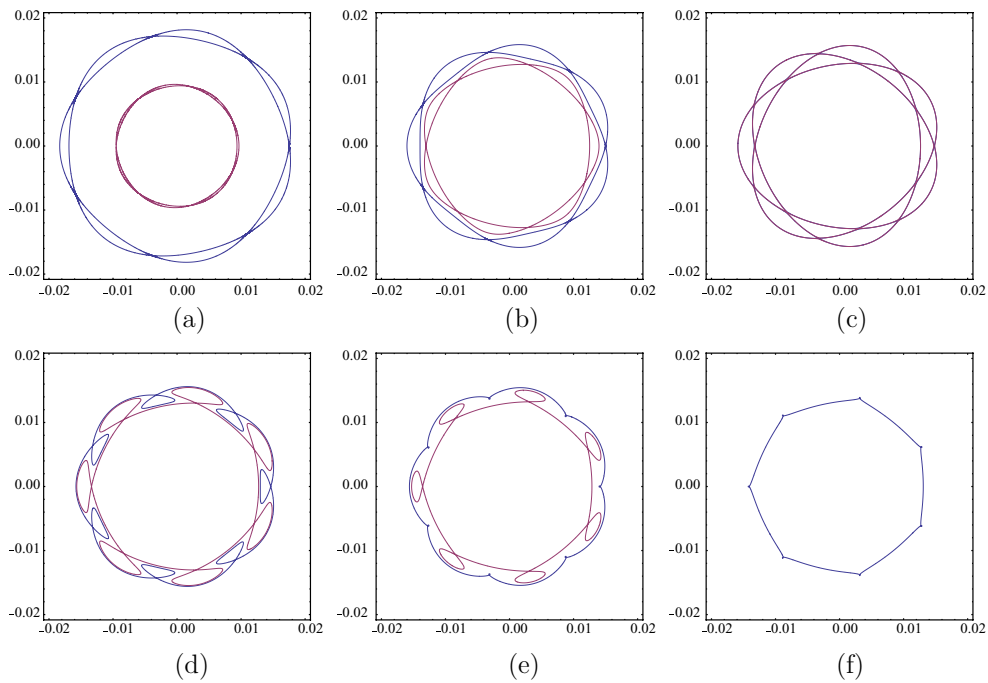


Figure 4. Level lines of the normal form Hamiltonian that correspond to the energy levels of saddle equilibria ($n = 7$). The pictures correspond to the parameters (δ, ν) moving along a circle around the origin in the anti-clockwise direction and crossing the domains D_2 and D'_1 .

In order to solve equation (14), we rewrite it in the form

$$I = -\frac{\nu}{3} - \sigma \frac{n}{12} \left(\frac{n}{2} - 1\right) I^{n/2-2}$$

and apply the method of consecutive approximations starting with the initial approximation $I_0 = -\frac{\nu}{3}$. A standard estimate from the contraction mapping theorem implies

$$I = -\frac{\nu}{3} - \sigma \frac{n}{12} \left(\frac{n}{2} - 1\right) \left(-\frac{\nu}{3}\right)^{n/2-2} + O(\nu^{n-5}).$$

Substituting this approximation into (13), we get

$$\delta = \frac{\nu^2}{3} - \sigma \frac{n}{2} \left(-\frac{\nu}{3}\right)^{n/2-1} + O(\nu^{n-4}).$$

This equation defines two lines on the plane (δ, ν) (one for each value of σ). Both lines enter the zero quadratically and differ by $O(\nu^{n/2-1})$. In this way, we have derived an approximate expression for the lines that bound the domain D'_1 on the bifurcation diagram of figure 3. We note that $\sigma = 1$ leads to a smaller δ for the same ν compared to $\sigma = -1$. Consequently, $\sigma = 1$ corresponds to the boundary between D_2 and D'_1 , and $\sigma = -1$ corresponds to the boundary between D'_1 and D_0 .

Shape of the islands. Let

$$\nu = \mu - 3\epsilon, \quad \delta = 3\epsilon^2 - 2\mu\epsilon, \quad I = \epsilon + \epsilon^{n/4} \mu^{-1/2} J, \quad h = \epsilon^{n/2} \bar{h}. \quad (15)$$

In the new variables, the Hamiltonian (11) takes the form (after skipping a constant term)

$$\bar{h} = J^2 + \epsilon^{n/4} \mu^{-3/2} J^3 + (1 + \epsilon^{n/4-1} \mu^{-1/2} J)^{n/2} \cos n\varphi.$$

Assuming $0 < \epsilon < \mu$, we get

$$\bar{h} = J^2 + \cos n\varphi + O(\epsilon^{(n-6)/4}). \tag{16}$$

Solving equations (15) with respect to ϵ and μ , we find that $\epsilon = \frac{1}{3}(\sqrt{v^2 - 3\delta} - v)$ and $\mu = \sqrt{v^2 - 3\delta}$. These expressions give small values when δ and v are small. In terms of the original parameters, the region $0 < \epsilon < \mu$ has the form $\delta < \frac{v^2}{4}$ for $v \leq 0$ and $\delta < 0$ for $v > 0$. In this region, the chain of islands is approximated by a chain of the pendulum's separatrices. Reversing the scaling (15), we obtain the radius and the width for the chain of islands.

Note that equation (15) gives a second expression for ϵ , namely $\epsilon = \frac{1}{3}(-\sqrt{v^2 - 3\delta} - v)$, which corresponds to the second chain of islands for $v < 0$ and $0 < \delta < \frac{v^2}{3}$.

A model for the changes in critical level sets of h in D₂. In order to study the changes in the level sets of h near the boundary between D_2 and D'_1 , we rescale the variables:

$$v = -3\epsilon, \quad \delta = 3\epsilon^2 + \epsilon^{n/3}a, \quad I = \epsilon + \epsilon^{n/6}J, \quad h = \epsilon^{n/2}\bar{h}, \quad \psi = n\varphi. \tag{17}$$

In the new variables, the Hamiltonian (11) takes the form (after skipping a constant term)

$$\bar{h} = aJ + J^3 + (1 + \epsilon^{n/6-1}J)^{n/2} \cos \psi. \tag{18}$$

Introducing $\epsilon_1 = \frac{n}{2}\epsilon^{n/6-1}$, we can rewrite the Hamiltonian in the form

$$\bar{h} = aJ + J^3 + \cos \psi + \epsilon_1 J \cos \psi + O(\epsilon_1^2).$$

Naturally, we compare the level sets of this Hamiltonian with

$$\bar{h}_0 = aJ + J^3 + \cos \psi + \epsilon_1 J \cos \psi, \tag{19}$$

which does not have any explicit dependence on n . In this equation, ϵ_1 is a small parameter while the parameter a is not necessarily small. Critical level sets of \bar{h}_0 for various values of parameter a are shown in figure 5. For aesthetic reasons, the diagrams show two complete periods of the Hamiltonian \bar{h}_0 . The original Hamiltonian (11) possesses a chain of n islands.

Obviously, the critical points of \bar{h}_0 are located at $\cos \psi = \sigma_\psi = \pm 1$ and

$$J = \pm \sqrt{-(a + \epsilon_1 \sigma_\psi)/3}.$$

This explicit expression for J shows that \bar{h}_0 has exactly two saddle critical points per period for $a < -\epsilon_1$, one critical point for $-\epsilon_1 < a < \epsilon_1$ and no critical points for $a > \epsilon_1$. The critical level sets for $a = \pm\epsilon_1$ are shown in figures 5(e) and (f), respectively, which correspond to a Hamiltonian saddle-node bifurcation.

We note that at $a = -a_c$, $a_c = 3 \cdot 2^{-2/3} + O(\epsilon_1^2)$, the two critical saddle points belong to a single critical level set as shown in figure 5(b).

Summarizing the results, we note that our analysis implies the existence of ‘islands’ as well as ‘meandering’ invariant curves similar to the ones considered in [6, 15]. Indeed, the normal form Hamiltonian (assume the formal series (10) is truncated at an order $N \geq n$) can be written in the form $H_{\delta,v}(I, \varphi) = H_0(I; \delta, v) + H_1(I, \varphi; \delta, v)$. In a small neighbourhood of the origin, $H_{\delta,v}$ can be considered as a small perturbation of H_0 . All invariant curves of H_0 are circles defined by the equation $I = \text{const}$. A combination of scaling and the implicit function theorem can be used to demonstrate that the majority of level lines of $H_{\delta,v}$ are C^1 -close to the level lines of H_0 . Nevertheless, some level lines of $H_{\delta,v}$ are not necessarily C^1 -close to the

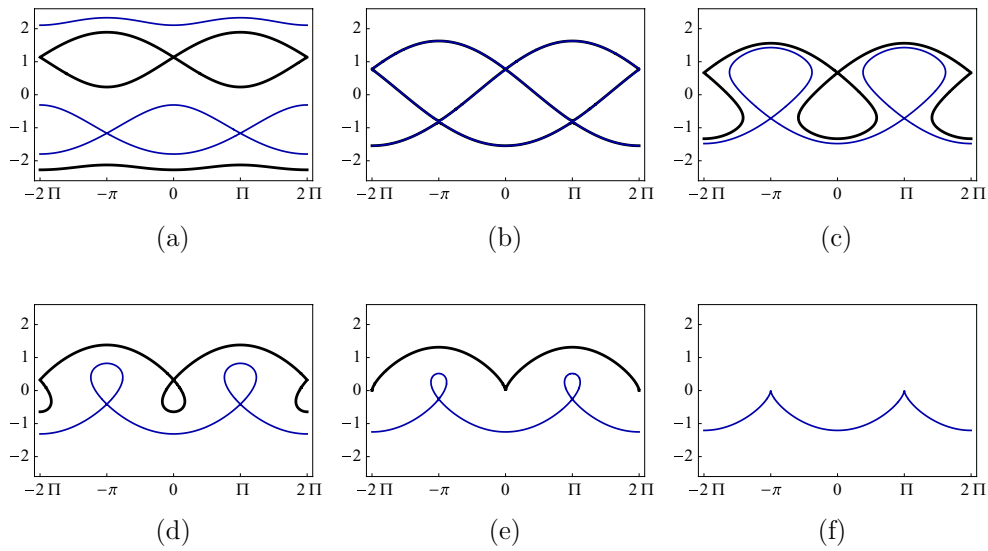


Figure 5. Critical level sets of the Hamiltonian (19). The parameter a increases from left to right and from top to bottom: (a) $a = -5$; (b) $a = a_c$; (e) $a = -\epsilon_1$; (f) $a = \epsilon_1$. For all pictures $\epsilon_1 = \frac{1}{10}$.

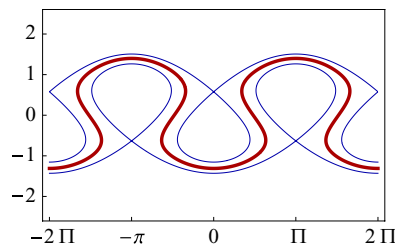


Figure 6. A ‘meandering’ curve (bold line) and two critical level sets for the Hamiltonian (19).

level lines of H_0 , as the implicit function theorem fails near circles defined by the equation $\frac{\partial H_0}{\partial I} = 0$.

The analysis of this section shows that near a simple zero of $\frac{\partial H_0}{\partial I}$, the level lines of $H_{\delta, \nu}(I, \varphi)$ are close (after a properly chosen scaling) to the level lines of the pendulum Hamiltonian (16) and, consequently, there is a ‘chain of islands’ similar to the one shown in figure 1(b).

Our analysis also shows that there is a sector D_2 on the plane of parameters where $\frac{\partial H_0}{\partial I}$ has a double zero. For $(\delta, \nu) \in D_2$, there is an annulus such that level lines of $H_{\delta, \nu}(I, \varphi)$ are close (after a properly chosen scaling) to the level lines of the Hamiltonian \tilde{h}_0 defined by (19). The Hamiltonian \tilde{h}_0 has essential invariant curves that cannot be represented as graphs $I(\varphi)$ in polar coordinates (e.g. level lines located between the two critical level sets shown in figures 4(d) and 5(c)). An example of a ‘meandering’ invariant curve is shown in figure 6.

Such invariant curves are called ‘meandering’ due to their shape. We point out that the KAM theory [6] can be used to demonstrate the presence of similarly shaped invariant curves for the original map.

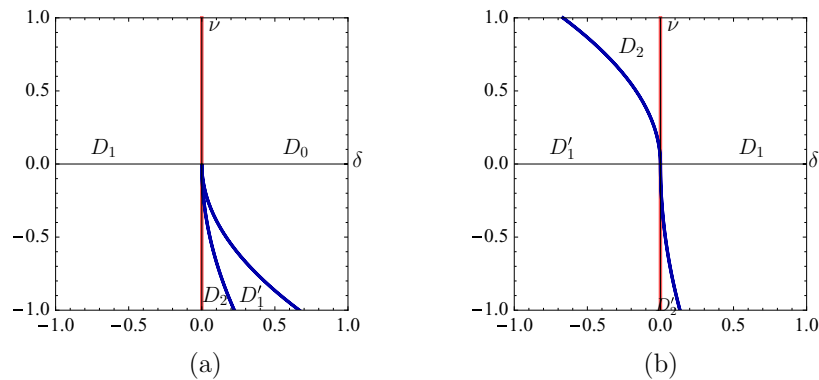


Figure 7. Bifurcation diagrams for $n = 6$: the stable case (a) and the unstable one (b).

3.2. $n = 6$

In the case $n = 6$, the model Hamiltonian takes the form

$$h = \delta I + \nu I^2 + I^3 + b_0 I^3 \cos n\varphi. \tag{20}$$

For $\delta = \nu = 0$, the origin is stable if $|b_0| < 1$ and unstable if $|b_0| > 1$. We assume that $b_0 \notin \{-1, 0, 1\}$, as these three cases correspond to a degeneracy of a higher co-dimension.

The critical points of h have either $\varphi = 0$ or $\frac{\pi}{n} \pmod{\frac{2\pi}{n}}$, and

$$\delta + 2\nu I + 3I^2(1 + \sigma_\varphi b_0) = 0, \tag{21}$$

where $\sigma_\varphi = \cos \varphi = \pm 1$. Equation (21) can easily be solved explicitly, but it is more convenient to write it in the form

$$\delta = f_\sigma(I, \nu), \tag{22}$$

where

$$f_\sigma = -2\nu I - 3(1 + \sigma b_0)I^2.$$

Since the function f_σ is independent of δ , the number and positions of solutions to equation (22) for given δ and ν can easily be read from the graphs of the functions f_+ and f_- . The analysis leads to different conclusions for the stable and unstable cases. Moreover, the behaviour of the graphs essentially depends on the sign of ν . In the rest, this analysis is completely straightforward and its results are summarized in the bifurcation diagrams of figures 7(a) and (b) for the stable and unstable case, respectively.

In the stable case, the bifurcation diagram is similar to the case of a resonance of order $n \geq 7$, although later in this section we will observe some quantitative differences.

In the unstable case, the bifurcation diagram consists of four domains: D_1 , D'_1 , D_2 and D'_2 , which correspond to the existence of n and $2n$ saddle critical points of h , as indicated by the subscript in the domain's name.

In order to derive equations for the boundaries between the domains in the bifurcation diagram, we note that the corresponding value of δ coincides with a critical value of f_σ . For each σ , the equation $\frac{\partial f_\sigma}{\partial I} = 0$ has a unique solution

$$I = -\frac{\nu}{3(1 + \sigma b_0)}.$$

Substituting this value into (22), we get

$$\delta = \frac{\nu^2}{3(1 + \sigma b_0)}.$$

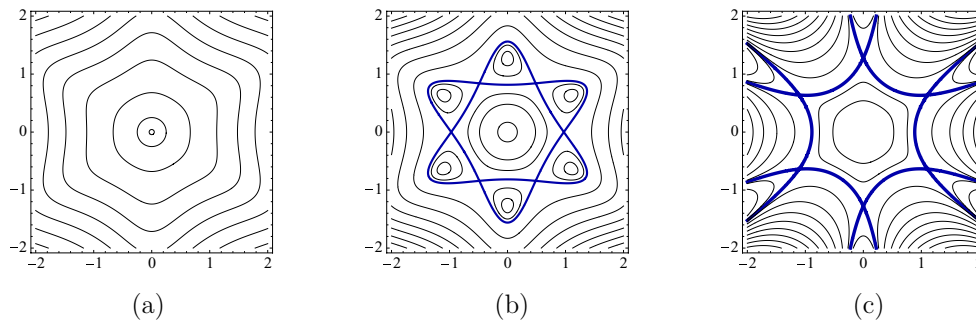


Figure 8. Level sets for the limit Hamiltonian (23): (a) $\text{sign}(\delta) = 1$ and $b_0 = \frac{1}{2}$, (b) $\text{sign}(\delta) = -1$ and $b_0 = \frac{1}{2}$, (c) $\text{sign}(\delta) = 1$ and $b_0 = \frac{3}{2}$.

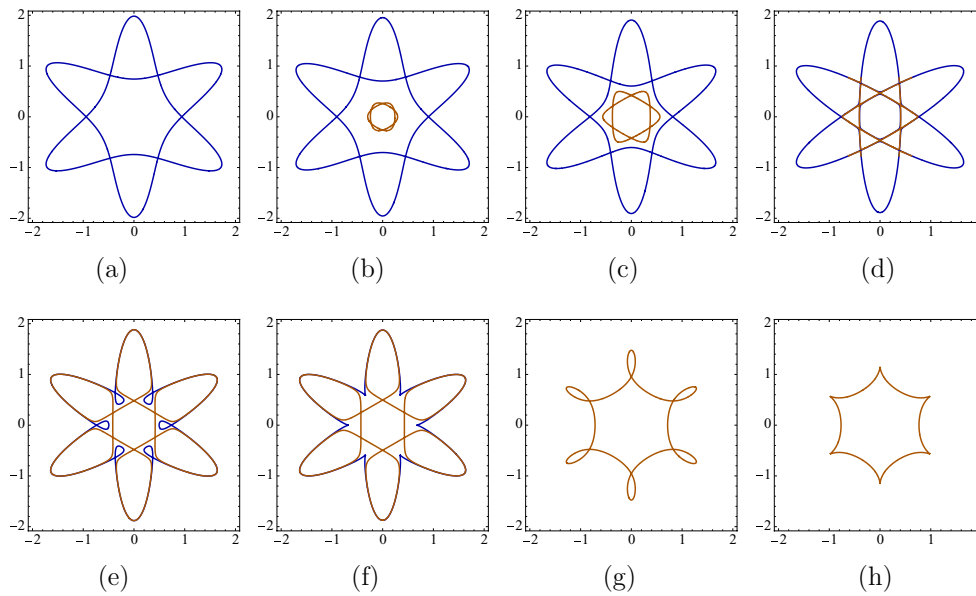


Figure 9. Critical level sets of the Hamiltonian (25) for $b_0 = \frac{1}{2}$ (the stable case).

This equation defines two lines corresponding to double critical points of h (for $\sigma = \pm 1$ respectively). Note that the bifurcation diagrams of figure 7 include only the parts of the parabolas which correspond to positive critical points.

Typical pictures of level sets are shown in figure 8. The bifurcations of islands are illustrated in figures 9 and 10 for the stable and unstable case, respectively.

We note that after the substitution $I = |\delta|^{1/2}J$, $h = |\delta|^{3/2}\bar{h}$, the Hamiltonian (20) takes the form

$$\bar{h} = \text{sign}(\delta) J + |\delta|^{-1/2}\nu J^2 + J^3(1 + b_0 \cos 6\varphi).$$

If $|\delta| \gg \nu^2$, this Hamiltonian is approximated by

$$\bar{h}_0 = \pm J + J^3(1 + b_0 \cos 6\varphi). \tag{23}$$

The Hamiltonian \bar{h}_0 depends on the coefficient b_0 but is independent of both δ and ν . We conclude that outside a narrow sector near the δ -axis, the critical level sets of the model

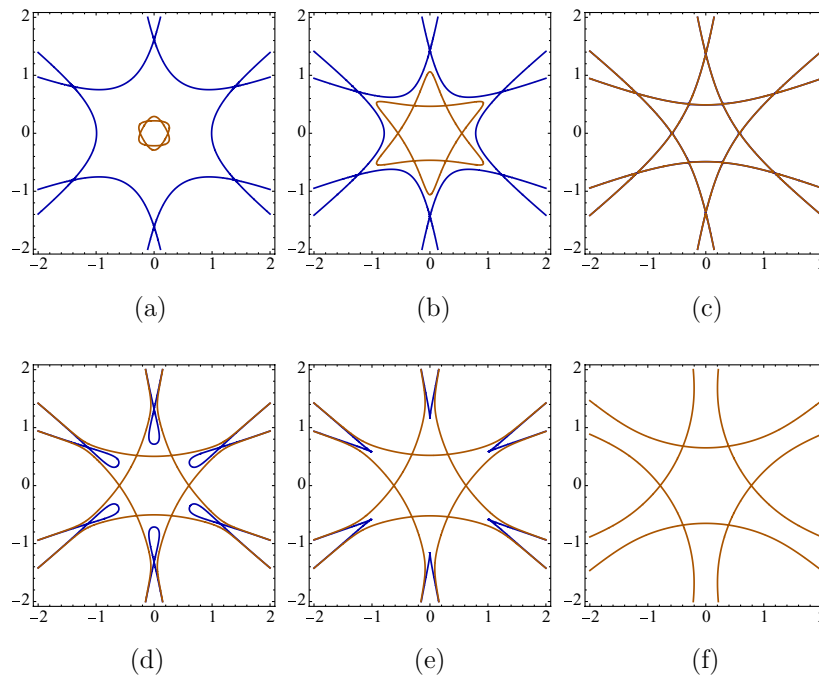


Figure 10. Critical level sets of the Hamiltonian (25) for $b_0 = \frac{3}{2}$ (the unstable case).

Hamiltonian (20) have a shape that is described by the Hamiltonian \bar{h}_0 . The level sets for this Hamiltonian are shown in figure 8.

In order to study bifurcations of the critical level sets of the Hamiltonian (20) for $\nu \neq 0$, we use the scaling

$$\nu = \epsilon, \quad \delta = \epsilon^2 a, \quad I = \epsilon J, \quad h = \epsilon^3 \bar{h}. \tag{24}$$

In the new variables, the Hamiltonian takes the form

$$\bar{h} = aJ + J^2 + J^3(1 + b_0 \cos n\varphi). \tag{25}$$

We note that the transition from (20) to (25) reduces the number of parameters without involving any approximation. The bifurcations of the critical level sets of \bar{h} are illustrated in figures 9 and 10 for the stable and unstable case, respectively. Note that the corresponding analysis takes into account that \bar{h} is to be considered in the domain $\nu J = \epsilon^2 I \geq 0$ only.

3.3. $n = 5$

For $n = 5$, the normal form is modelled by the Hamiltonian

$$h = \delta I + \nu I^2 + I^{5/2} \cos 5\varphi. \tag{26}$$

We note that, in contrast to the cases with $n \geq 6$, in the cases of $n \leq 5$ the cubic term I^3 can be excluded from the model as it is much smaller than the term proportional to $I^{n/2}$ and, consequently, does not affect the qualitative properties of the bifurcation diagram.

The bifurcation diagram for $n = 5$ looks similar to the unstable subcase of $n = 6$. In particular, the equilibrium at the origin is unstable when $\delta = \nu = 0$. The corresponding level set $\{h = 0\}$ is not compact and consists of five straight lines defined by the equation $\cos 5\varphi = 0$.

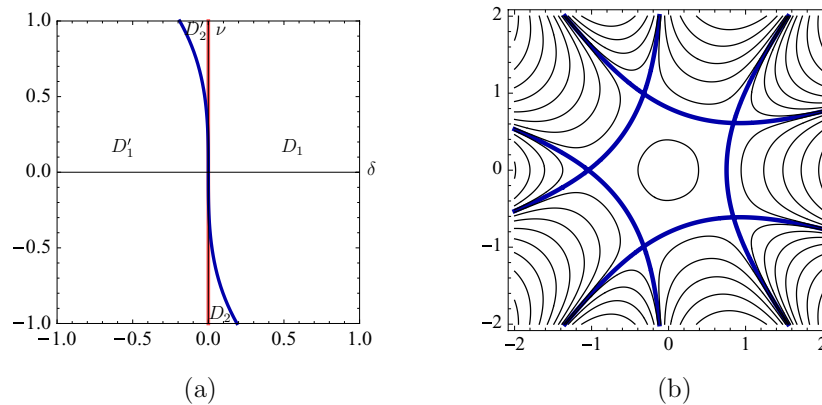


Figure 11. (a) A bifurcation diagram on the plane (δ, ν) for $n = 5$ and (b) the level sets for the limit Hamiltonian (30); the critical level set is shown using the bold lines.

Similar to the previous cases, the critical points of h are determined from the equation $\frac{\partial h}{\partial I} = 0$ and $\varphi = 0$ or $\varphi = \pi/5 \pmod{2\pi}$. Therefore the problem is reduced to finding positive solutions for the equations:

$$\delta + 2\nu I + \sigma \frac{5}{2} I^{3/2} = 0, \quad \sigma = \pm 1. \tag{27}$$

These equations can be rewritten in the form $\delta = f_\sigma(I, \nu)$. For a fixed $\nu \neq 0$, one of the functions f_σ is monotone ($\sigma \nu > 0$) and the other one ($\sigma \nu < 0$) has an extremum at

$$I^{1/2} = -\frac{8}{15} \sigma \nu.$$

Substituting this value into equation (27), we obtain the relationship between the parameters δ and ν , which corresponds to a doubled critical point of the Hamiltonian h :

$$\delta = -\frac{128}{675} \nu^3.$$

This equation defines the boundary between domains D_1 and D_2 , and between D_1' and D_2' on the bifurcation diagram shown in figure 11(a). In D_1 and D_2 , the Hamiltonian h has five critical points (of saddle type) and in D_2 and D_2' it has 15 critical points: five elliptic and ten saddles.

When the parameters (δ, ν) are in D_1 or D_1' , the level sets of h look similar to the ones shown in figure 11(b), where the singular level set is indicated by the bold line.

A chain of n islands is born from the origin and goes through a bifurcation shown in figure 12, as the parameters (δ, ν) move in an anticlockwise direction through D_2 or D_2' on the plane of figure 11(a). We note that the symmetry of the Hamiltonian (26) allows us to consider the case $\delta \geq 0$ only.

In order to study the shape of level sets, we apply the scaling

$$I = \delta^{2/3} J, \quad h = \delta^{5/3} \bar{h}. \tag{28}$$

In the new variables, the Hamiltonian function (26) takes the form

$$\bar{h} = J + \nu \delta^{-1/3} J^2 + J^{5/2} \cos 5\varphi. \tag{29}$$

If $\nu \delta^{-1/3}$ is small, the quadratic term can be ignored and we arrive at the Hamiltonian

$$\bar{h}_0 = J + J^{5/2} \cos 5\varphi. \tag{30}$$

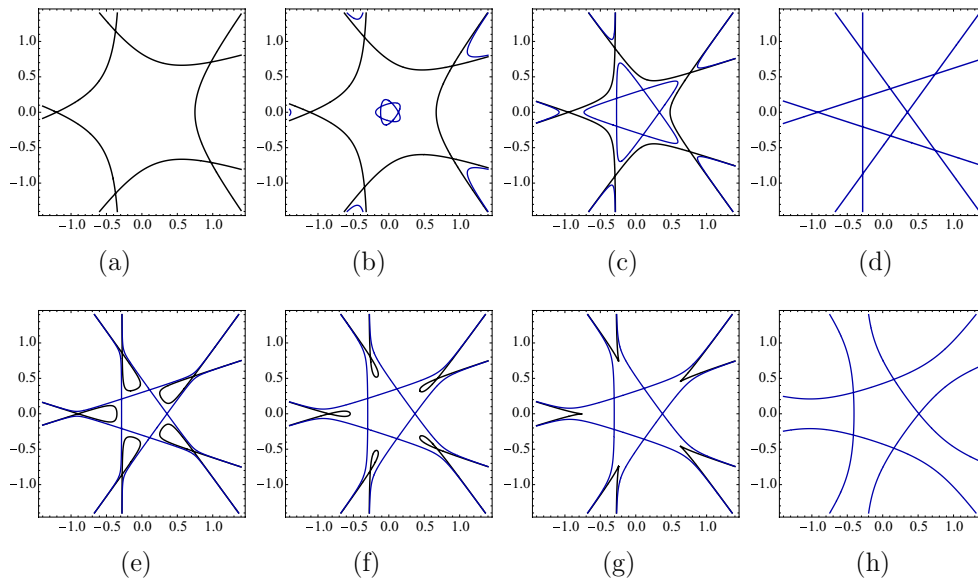


Figure 12. Critical level sets lines for the Hamiltonian (33); the parameter a decreases from left to right and from top to bottom. Special values: (a) $a = 0$, (d) $a = -\frac{4}{25}$ and (g) $a = -\frac{128}{675}$.

Its critical points are located at a point with coordinates $J_{cr} = (\frac{2}{5})^{2/3}$, $\varphi_{cr} = \pi/5 \pmod{2\pi}$, and the corresponding critical value is given by $\bar{h}_0(J_{cr}, \varphi_{cr}) = \frac{3}{5}(\frac{2}{5})^{2/3}$. So the critical set is defined by the equation

$$J + J^{5/2} \cos 5\varphi = \frac{3}{5}(\frac{2}{5})^{2/3}. \tag{31}$$

This critical set and some non-critical level lines of \bar{h}_0 are shown in figure 11(b).

If $|\delta| \gg |v|^3$, the critical points of \bar{h} are close to the critical points of \bar{h}_0 , and the shape of the corresponding critical set is approximately defined by equation (31). Taking into account the scaling, we conclude that the critical counts of the original model (26) are located on the circle $I = J_{cr}\delta^{2/3}(1 + O(v\delta^{-1/3}))$.

Note that on a small disc $\delta^2 + v^2 \leq \epsilon_0^2 \ll 1$, the assumption $|\delta| \gg |v|^3$ is violated only in a narrow zone near the δ -axis. The relative area of this zone converges to zero when ϵ_0 goes to zero. This zone includes the domains D_2 and D'_2 .

In order to study the changes in the critical level sets of h in the region where the above assumption is possibly violated, we introduce another scaling

$$\delta = av^3, \quad I = v^2J, \quad h = v^5\bar{h}. \tag{32}$$

In the new variables, the Hamiltonian function (26) takes the form

$$\bar{h} = aJ + J^2 + J^{5/2} \cos 5\varphi. \tag{33}$$

We note that the Hamiltonian \bar{h} depends on one parameter only. Moreover, the substitution (32) is invertible for $v \neq 0$. On every cubic curve $\delta = av^3$ in the plane (δ, v) , the Hamiltonian (26) is equivalent to the same Hamiltonian (33) up to the scaling of the space, energy and time variables.

The Hamiltonian (33) has n saddle critical points if $a > 0$ or $a < -\frac{128}{625}$. The number of saddle critical points is $2n$ when $0 > a > -\frac{128}{625}$. We note that for $a = -\frac{4}{25}$, both families

of critical points belong to a single critical level set. The changes in the critical level sets are illustrated in figure 12.

4. Simplification of formal Hamiltonians

4.1. Simplification of a degenerate Hamiltonian

It is well known that the complex variables defined by

$$z = \frac{x + iy}{\sqrt{2}} \quad \text{and} \quad \bar{z} = \frac{x - iy}{\sqrt{2}}$$

facilitate manipulation with real formal series in variables x, y . It is important to note that \bar{z} is not necessarily the complex conjugate of z (the latter is denoted by z^* in this paper). Of course, $\bar{z} = z^*$ if both x and y are real.

In these variables, a rotation R_{α_0} takes the form of the multiplication: $z \mapsto \lambda_0 z, \bar{z} \mapsto \lambda_0^* \bar{z}$, where $\lambda_0 = e^{i\alpha_0}$. Let λ_0 be resonant of order n , i.e., $\lambda_0^n = 1$. Any formal power series h invariant with respect to the rotation R_{α_0} has the form

$$h(z, \bar{z}) = \sum_{\substack{k, l \geq 0 \\ k = l \pmod{n}}} h_{kl} z^k \bar{z}^l. \tag{34}$$

We say that the series h consists of resonant terms, as each term of the series is invariant under a rotation by α_0 . It is easy to come back from the variables (z, \bar{z}) to the symplectic polar coordinates by substituting $z = \sqrt{I}e^{i\varphi}$ and $\bar{z} = \sqrt{I}e^{-i\varphi}$.

We note that the formal interpolating Hamiltonian of the resonant normal form theory (described in the introduction) has the form (34) and satisfies two additional properties. First, the series is real-valued, i.e., it has real coefficients when written in terms of x, y variables, which is equivalent to the condition $h_{kl} = h_{lk}^*$ for all $k, l \geq 0$. Second, the series does not contain terms of order two or lower.

We use tangent-to-identity formal canonical transformations to simplify the series h by eliminating as many resonant terms as possible without breaking the symmetry of the series. The simplification procedure depends on the order of the resonance and on some lowest-order terms of the series. The following theorem describes one of the possible simplifications of the series, which still involves infinitely many coefficients. We note that no further simplification is possible as the coefficients of the simplified series are defined uniquely and, consequently, can be considered as moduli of the formal canonical classification.

Theorem 3. *Let h be a formal series of the form*

$$h = \sum_{\substack{k, l \geq 0, k+l \geq 3 \\ k = l \pmod{n}}} h_{kl} z^k \bar{z}^l \tag{35}$$

such that $h_{lk} = h_{kl}^$ and $h_{n0} > 0$. Let $m = \min\{k : h_{kk} \neq 0\}$. There is a formal tangent-to-identity canonical change of variables that transforms h to the form*

$$\tilde{h} = \sum_{k \geq k_0} a_k z^k \bar{z}^k + (z^n + \bar{z}^n) \sum_{k \geq 0} b_k z^k \bar{z}^k, \tag{36}$$

where $b_0 = h_{n0}$ and $k_0 \geq 2$. Moreover, one of the following two properties holds:

1. if $m \leq \frac{n}{2}$, then $k_0 = m, a_m = h_{mm}$ and $b_k = 0$ for all $k = -1 \pmod{m}$;
2. if $m \geq \frac{n}{2}$, then $k_0 \geq \frac{n}{2}$ and $a_k = b_k = 0$ for all $k = -1 \pmod{n}$.

The coefficients of the series \tilde{h} are defined uniquely by these properties.

Note that in the case $m = \frac{n}{2}$, the theorem provides two alternative versions of the unique normal form. The second one has an advantage, as the transformation to the normal form is not singular at $h_{\frac{n}{2}, \frac{n}{2}} = 0$.

Following the traditional approach, we construct the transformation to the normal form as a composition of consecutive canonical coordinate changes. These changes are constructed using the Lie series method: a polynomial (or a formal series) χ generates a canonical change of variables when we consider a time-one map of the Hamiltonian vector-field of χ . This change of variables transforms a series h into

$$\tilde{h} = h \circ \Phi_\chi^1 = h + \sum_{k \geq 1} \frac{1}{k!} L_\chi^k h \tag{37}$$

where

$$L_\chi h = -i\{h, \chi\} = -i \left\{ \frac{\partial h}{\partial z} \frac{\partial \chi}{\partial \bar{z}} - \frac{\partial h}{\partial \bar{z}} \frac{\partial \chi}{\partial z} \right\}.$$

If χ contains neither linear nor quadratic terms, the linear operator L_χ increases the degree of every monomial. Then the transformation is meaningful in the class of a formal series and every term of \tilde{h} can be written in the form of a finite sum.

In order to prove theorem 3, we consider three cases: $m < \frac{n}{2}$, $m = \frac{n}{2}$ and $m > \frac{n}{2}$.

4.2. The case of $m < \frac{n}{2}$

Suppose that there is an integer m , $2 \leq m < \frac{n}{2}$, such that $h_{kk} = 0$ for $k < m$ and $h_{mm} \neq 0$. Then let

$$h_m = a_m z^m \bar{z}^m + b_0(z^n + \bar{z}^n), \tag{38}$$

where $a_m = h_{mm}$ and $b_0 = h_{n0}$. We define a δ -degree of a monomial by

$$\delta(z^k \bar{z}^l) = \delta(z^l \bar{z}^k) = \frac{m}{n}k + \left(1 - \frac{m}{n}\right)l \quad \text{for } k \geq l. \tag{39}$$

This definition is a modification of a grading function introduced earlier by Bider and Sanders [4] for similar purposes. It is convenient to write an equivalent expression for the δ -degree:

$$\delta(z^k \bar{z}^l) = \frac{m}{n}|k - l| + \min\{k, l\}.$$

Since the monomial $z^k \bar{z}^l$ is resonant, $k = l + nj$ for some $j \in \mathbb{Z}$. Consequently, the δ -degree is integer. We note that if $m = \frac{n}{2}$, the δ -degree of a monomial is half the usual degree: $\delta(z^k \bar{z}^l) = \frac{k+l}{2}$.

For $p \geq 2$ and $|j| \leq \frac{p}{m}$, we define

$$Q_{p,j}^\pm = z^k \bar{z}^l \pm z^l \bar{z}^k, \quad \text{where } k = p + (n - m)j \text{ and } l = p - mj. \tag{40}$$

Consider the action of the operator

$$L_{h_m} \chi = -i\{\chi, h_m\} \tag{41}$$

on a monomial:

$$\begin{aligned} L_{h_m}(z^k \bar{z}^l) &= -ia_m\{z^k \bar{z}^l, z^m \bar{z}^m\} - ib_0\{z^k \bar{z}^l, z^n + \bar{z}^n\} \\ &= -ia_m m(k - l)z^{k+m-1} \bar{z}^{l+m-1} - ib_0 n(kz^{k-1} \bar{z}^{l+n-1} - lz^{k+n-1} \bar{z}^{l-1}). \end{aligned}$$

Then for $j \geq 1$, we get $k > l$ and

$$L_{h_m}(Q_{p,j}^\pm) = -ia_m(k - l)mQ_{p+m-1,j}^\mp + ib_0 n l Q_{p+m-1,j+1}^\mp - ib_0 n k Q_{p+n-m-1,j-1}^\mp. \tag{42}$$

Note that the last term has a higher δ -degree than the first two. For $j = 0$, we get $k = l = p$ and

$$L_{h_m}(Q_{p,0}^\pm) = L_{h_m}(z^p \bar{z}^p) \pm L_{h_m}(z^p \bar{z}^p) = \begin{cases} i2b_0npQ_{p+m-1,1}^- \\ 0. \end{cases} \tag{43}$$

In the formal series (35), terms of the lowest δ -degree are given by equation (38), where all monomials are of δ -degree m . The Hamiltonian h_m already has the required form. We continue inductively. Suppose that the Hamiltonian h is already transformed to the desired form at all δ -degrees lower than p for some $p > m$. In order to transform terms of δ -degree p to the normal form, first we eliminate terms proportional to $Q_{p,j}^\pm$ in order of decreasing j for $\frac{p}{m} \geq j \geq 1$ using the changes of variables generated by

$$\chi = \begin{cases} \frac{\Im(h_{kl})Q_{p',j-1}^+ - i\Re(h_{kl})Q_{p',j-1}^-}{b_0n(p - mj + 1)} & \text{for } j \geq 2, \\ \frac{\Im(h_{kl})Q_{p',0}^+}{2b_0n(p - m + 1)} & \text{for } j = 1, \end{cases}$$

where h_{kl} is a coefficient in front of the monomial $z^k \bar{z}^l$ with k, l defined by (40) and $p' = p - m + 1$. For $j \geq 2$, this change of variables eliminates the terms proportional to $Q_{p,j}^-$ and $Q_{p,j}^+$, and changes the coefficient in front of $Q_{p,j-1}^\pm$. Of course, terms of higher δ -degree are also changed, but no other term of δ -degree p or lower is affected. After completing these changes, the terms proportional to $Q_{p,j}^\pm$ with $j \in \{0, 1\}$ are left in the normal form.

If $p \equiv -1 \pmod{m}$, the normal form admits a further simplification. We eliminate the term $Q_{p,1}^+$ by a sequence of coordinate changes that starts with $j = 1$ and runs in the order of increasing j while $1 \leq j \leq \frac{p'}{m}$. Note that in this case, $\frac{p'}{m} = \frac{p+1}{m} - 1$ is an integer. Each of the changes is generated by

$$\chi = \frac{i\Re(h_{kl})Q_{p',j}^-}{a_m m n j}$$

and eliminates the term proportional to $Q_{p,j}^+$. For $j < \frac{p'}{m}$, the change creates a term $Q_{p,j+1}^+$ that is eliminated in the next step in j .

Repeating this procedure inductively in p , we conclude that the Hamiltonian h can be transformed to the normal form (36).

4.3. Case of $m = \frac{n}{2}$

We note that the equality $m = \frac{n}{2}$ is only possible when n is even. In this case, all resonant monomials are of even degree, and the δ -degree is just half the usual degree. The strategy of the proof is similar to the previous section, the lowest-order terms in h have the same form (38) but, in contrast to the previous case, all terms in (42) are of the same degree, so we need a different approach to the analysis of the homological operator L_{h_m} .

Let \mathcal{H}_p be the space of all real-valued resonant homogeneous polynomials of degree $2p$. Equations (42) and (43) imply that $L_{h_m} : \mathcal{H}_p \mapsto \mathcal{H}_{p+m-1}$.

Taking any $\chi \in \mathcal{H}_p$, we transform the Hamiltonian using equation (37). It is easy to check that the terms of degree less than $2(p + m - 2)$ are left unchanged, $[\tilde{h}]_{p'} = [h]_{p'}$ for $p' \leq p + m - 1$, and at the order $2(p + m - 1)$ we get

$$[\tilde{h}]_{p+m-1} = [h]_{p+m-1} - L_{h_m}\chi,$$

where $[\cdot]_{p'}$ is used to denote terms of order $2p'$ in a formal series. Consequently, it is possible to choose χ in such a way that \tilde{h} is in a complement to the image of the operator L_{h_m} .

We can write \mathcal{H}_p in the form of a direct sum $\mathcal{H}_p = \mathcal{H}_p^+ \oplus i\mathcal{H}_p^-$, where \mathcal{H}_p^+ is the real linear span of monomials $Q_{p,j}^+$, with $0 \leq j \leq \frac{p}{m}$, and \mathcal{H}_p^- is the real linear span of monomials $Q_{p,j}^-$, with $1 \leq j \leq \frac{p}{m}$. Since the coefficients in (42) and (43) are purely imaginary, it is convenient to define $\Lambda_p^\pm : \mathcal{H}_p^\pm \rightarrow \mathcal{H}_{p+m-1}^\mp$ by $\Lambda_p^\pm = -i L_{h_m}|_{\mathcal{H}_p^\pm}$. Then equations (42) and (43) imply that the operators Λ_p^\pm are described by matrices with real coefficients.

The complement to the image of Λ_p^+ is empty. Moreover, the kernel of the operator consists of $h_m^{p/m}$ if $p \equiv 0 \pmod{m}$ and is trivial otherwise. Indeed, a straightforward computation shows that

$$\dim \mathcal{H}_{p+m-1}^- = \begin{cases} \dim \mathcal{H}_p^+ & \text{if } p \not\equiv 0 \pmod{m}, \\ \dim \mathcal{H}_p^+ - 1 & \text{if } p \equiv 0 \pmod{m}. \end{cases}$$

Writing down the matrix for the operator Λ_p^+ , we see that since $b_0 \neq 0$, the matrix is of maximal rank and, consequently, the complement to the image is indeed empty.

A similar computation shows that

$$\dim \mathcal{H}_{p+m-1}^+ = \begin{cases} \dim \mathcal{H}_p^- + 2 & \text{if } p \not\equiv 0 \pmod{m}, \\ \dim \mathcal{H}_p^- + 1 & \text{if } p \equiv 0 \pmod{m}, \end{cases}$$

and, since $b_0 \neq 0$, the operator Λ_p^- is of maximal rank and its kernel is trivial.

In order to describe the complement to the image of Λ_p^- , we consider the following two cases separately.

If $p \not\equiv 0 \pmod{m}$, we delete the first two rows of the matrix corresponding to the operator Λ_p^- to get an upper-triangle matrix with non-vanishing terms on the main diagonal (these are proportional to $b_0 \neq 0$). Then the complement to $\Lambda_p^-(\mathcal{H}_p^-)$ coincides with the real linear span of $Q_{p+m-1,j}^+$ with $j = 0, 1$.

If $p \equiv 0 \pmod{m}$, then $p = sm$ for some $s \in \mathbb{N}$. The operator Λ_p^- is described by a matrix with $s + 1$ rows and $s + 2$ columns. We delete the first row and obtain a square tridiagonal matrix.

The non-degeneracy of this tridiagonal matrix is proved using the following well-known rule for the determinant of a tridiagonal matrix. Let $A = (a_{ij})$ be a square $k_0 \times k_0$ matrix with non-zero elements on the three main diagonals only: $a_{jj} = \alpha_j$, $a_{j,j+1} = \beta_j$, $a_{j+1,j} = -\gamma_j$. Then $\det A = K_{k_0}$, where the continuants K_j are defined recursively:

$$K_0 = 1, \quad K_1 = \alpha_1, \quad K_j = \alpha_j K_{j-1} + \beta_{j-1} \gamma_{j-1} K_{j-2} \quad \text{for } j \geq 2.$$

If $\alpha_j, \beta_j, \gamma_j > 0$, the straightforward induction implies $K_{k_0} > 0$.

We conclude that, for $a_m b_0 \neq 0$, the one-dimensional linear subspace generated by $Q_{p+m-1,0}^+$ is a complement to the image of Λ_p^- .

These arguments lead to the normal form described by item 1 in the statement of theorem 3. In order to prove the possibility of the normal form described by item 2, we need to make a minor modification: when s is even, instead of deleting the first row from the matrix of Λ_p^+ , we delete the second row. Then, at the corresponding orders, a complement to the image is generated by $Q_{p+m-1,1}^+$ instead of $Q_{p+m-1,0}^+$.

4.4. The case of $m > \frac{n}{2}$

In this case, the lowest order in the Hamiltonian (35) has the form

$$h_n = b_0(z^n + \bar{z}^n)$$

where $b_0 = h_{n0}$. The operator

$$L_{h_n} \chi = -i\{\chi, h_n\}$$

acts on a monomial in the following way

$$L_{h_n}(z^k \bar{z}^l) = -ib_0\{z^k \bar{z}^l, z^n + \bar{z}^n\} \\ = -ib_0n(kz^{k-1}\bar{z}^{l+n-1} - lz^{k+n-1}\bar{z}^{l-1}).$$

We note that the image is a homogeneous polynomial of degree $k + l + n - 2$. Let

$$j = \frac{k - l}{n} \quad \text{and} \quad p = k + l. \tag{44}$$

Define

$$Q_{p,j}^\pm = z^k \bar{z}^l \pm z^l \bar{z}^k, \quad \text{where } k = \frac{p + nj}{2} \quad \text{and} \quad l = \frac{p - nj}{2} \tag{45}$$

are integers, $p \geq 2$ and $|j| \leq \frac{p}{n}$. Note that if n is even, then $p = 0 \pmod{2}$. If n is odd, then $j = p \pmod{2}$. According to the definition $Q_{p,-j}^\pm = \pm Q_{p,j}^\pm$, so we will use $j \geq 0$ in our arguments.

A straightforward computation shows that

$$L_{h_n}(Q_{p,j}^\pm) = -ib_0n(kz^{k-1}\bar{z}^{l+n-1} - lz^{k+n-1}\bar{z}^{l-1}) \mp ib_0n(lz^{l-1}\bar{z}^{k+n-1} - kz^{l+n-1}\bar{z}^{k-1}) \\ = -ib_0n \left(kQ_{p+n-2,j-1}^\mp - lQ_{p+n-2,j+1}^\mp \right) \quad \text{for } j \geq 1,$$

and, for $p = 0 \pmod{2}$ only,

$$L_{h_n}(Q_{p,0}^\pm) = -ib_0nk \left(Q_{p+n-2,-1}^\mp - Q_{p+n-2,1}^\mp \right) = \begin{cases} ib_0np Q_{p+n-2,1}^- \\ 0. \end{cases}$$

Suppose that the Hamiltonian h is transformed to the desired form at all orders lower than p . First, we eliminate terms in order of decreasing j for $\frac{p}{n} \geq j \geq 1$ using the changes of variables generated by

$$\chi = \begin{cases} \frac{\Im(h_{kl}) Q_{p',j-1}^+ - i\Re(h_{kl}) Q_{p',j-1}^-}{b_0n(p - nj + 2)/2} & \text{for } j \geq 2, \text{ and} \\ \frac{\Im(h_{kl}) Q_{p',0}^+}{b_0n(p - n + 2)} & \text{for } j = 1, \end{cases}$$

where h_{kl} is a coefficient in front of the monomial $z^k \bar{z}^l$ with k, l defined by (45) and $p' = p - n + 2$. For $j \geq 2$, this change of variables eliminates the term proportional to $Q_{p,j}^\pm$ and changes the coefficient in front of $Q_{p,j-2}^\pm$. Of course, higher-order terms are also changed, but no other term of degree p or lower is affected. The terms $Q_{p,j}^+$ with $j \in \{0, 1\}$ are left in the normal form (since $Q_{p,0}^- = 0$ for all p).

If $p = -2 \pmod{n}$ the normal form admits further simplification. Let $j_0 = 0$ if $(p + 2)/n$ is even and $j_0 = 1$ otherwise. We eliminate the term Q_{p,j_0} by a sequence of coordinate changes that starts with $j = j_0$ and is generated by

$$\chi = \frac{i\Re(h_{kl}) Q_{p',j+1}^-}{b_0n(p + nj + 2)/2}.$$

This change of variables eliminates the term proportional to $Q_{p,j}^+$. For $j+1 < \frac{p'}{n} = \frac{p+2}{n} - 1$, the change creates a term $Q_{p,j+2}^+$ that is eliminated in the next step in j .

Repeating this procedure inductively in p , we conclude that the Hamiltonian h can be transformed to the normal form (36).

The uniqueness of the normal form series of theorem 3 is established by arguments similar to the ones used in [3, 10]. The proof is based on the observation that the kernel of the

homological operator L_{h_m} is a real linear span of integer powers of h_m . This property is used to show that any canonical substitution that transforms the normal form Hamiltonian \tilde{h} into a Hamiltonian of the same form is generated by a Hamiltonian that itself is a series in powers of \tilde{h} and, consequently, does not change the coefficients of the series.

4.5. Families with a small twist

We will use the following notations: $\epsilon = (\epsilon_1, \dots, \epsilon_{m-1})$, $\mathbf{j} = (j_1, j_2, \dots, j_{m-1})$, $\epsilon^{\mathbf{j}} = \epsilon_1^{j_1} \epsilon_2^{j_2} \dots \epsilon_{m-1}^{j_{m-1}}$, and $\mathbf{j} \geq 0$ means $j_k \geq 0$ for $k = 1, \dots, m - 1$.

Theorem 4. *Let*

$$h(z, \bar{z}; \epsilon) = \sum_{1 \leq k \leq m-1} \epsilon_k z^k \bar{z}^k + \sum_{k \geq m, j \geq 0} h_{kkj} z^k \bar{z}^k \epsilon^j + \sum_{\substack{k+l \geq n, k \neq l, j \geq 0 \\ k=l \pmod n}} h_{klj} z^k \bar{z}^l \epsilon^j \tag{46}$$

be a formal power series with $2 \leq m \leq n - 1$, $h_{klj} = h_{lkj}^*$ for all $k, l, j \geq 0$, $h_{n00} > 0$ (and additionally $h_{mm0} \neq 0$ for the case of $m \leq \frac{n}{2}$ only). Then there exists a formal canonical change of variables that transforms the Hamiltonian h into

$$\tilde{h} = \sum_{1 \leq k \leq m-1} \epsilon_k z^k \bar{z}^k + \sum_{k \geq m, j \geq 0} a_{kj} z^k \bar{z}^k \epsilon^j + (z^n + \bar{z}^n) \sum_{k \geq 0, j \geq 0} b_{kj} z^k \bar{z}^k \epsilon^j \tag{47}$$

where $b_{00} = h_{n00}$. Moreover, one of the following two properties holds:

1. if $m \leq \frac{n}{2}$, then $a_{m0} = h_{mm0}$ and $b_{kj} = 0$ for all $k = -1 \pmod m$ and all j ;
2. if $m \geq \frac{n}{2}$, then $a_{kj} = b_{kj} = 0$ for all $k = -1 \pmod n$ and all j .

The coefficients of the series \tilde{h} are defined uniquely by these properties.

Proof.

- $m \leq \frac{n}{2}$

Let us define the $\tilde{\delta}$ -degree by

$$\tilde{\delta}(z^k \bar{z}^l \epsilon^j) = \delta(z^k \bar{z}^l) + \|\mathbf{j}\|,$$

where $\delta(z^k \bar{z}^l)$ is defined by (39) and

$$\|\mathbf{j}\| = mj_1 + (m - 1)j_2 + \dots + 2j_{m-1}.$$

Then the lowest $\tilde{\delta}$ -degree in (46) is m . Let

$$h_{(m)}(z, \bar{z}; \epsilon) = h_m(z, \bar{z}) = a_m z^m \bar{z}^m + b_0(z^n + \bar{z}^n),$$

where $a_m = h_{mm0}$ and $b_0 = h_{n00}$.

The Hamiltonian is transformed to the normal form inductively: at each step we normalize terms that have the same $\tilde{\delta}$ -degree. Let $h_{(s)}$ have the desired form up to $\tilde{\delta}$ -degree $p - 1$. The term of $\tilde{\delta}$ -degree p can be written as

$$h_{(p)}(z, \bar{z}; \epsilon) = \sum_{0 \leq \|\mathbf{j}\| \leq p-m} h_{pj}(z, \bar{z}) \epsilon^j,$$

where $h_{pj}(z, \bar{z})$ is the sum of resonant monomials of δ -degree $p - \|\mathbf{j}\|$.

Let $\chi = \chi(z, \bar{z})$ be a sum of resonant monomials of δ -degree $p - \|\mathbf{j}\| - (m - 1)$ for some $\mathbf{j} \geq 0$. After the change of variables $(z, \bar{z}) \mapsto \Phi_\chi^{\epsilon^j}(z, \bar{z})$, the Hamiltonian takes the form

$$\tilde{h} = h + \epsilon^j L_\chi h + \sum_{l \geq 2} \frac{\epsilon^{lj}}{l!} L_\chi^l h. \tag{48}$$

It is not difficult to see that

$$\begin{aligned} \tilde{h}_{(s)}(z, \bar{z}; \epsilon) &= h_{(s)}(z, \bar{z}; \epsilon) \quad \text{for } s \leq p - 1 \\ \tilde{h}_{p\mathbf{j}'}(z, \bar{z}) &= h_{p\mathbf{j}'}(z, \bar{z}) \quad \text{for } \mathbf{j}' \neq \mathbf{j}, \\ \tilde{h}_{p\mathbf{j}} &= h_{p\mathbf{j}} - [L_{h_m} \chi]_{p-\|\mathbf{j}\|}, \end{aligned}$$

where $[\cdot]_p$ denotes terms of δ -order p in a series. It is shown in section 4.2 that there exists such a χ that $\tilde{h}_{p\mathbf{j}}$ has the desired form.

- $\frac{n}{2} < m \leq n - 1$

Now let $\tilde{\delta}$ -degree be

$$\tilde{\delta}(z^k \bar{z}^l \epsilon^j) = k + l + \|\mathbf{j}\|,$$

where

$$\|\mathbf{j}\| = nj_1 + (n - 1)j_2 + \dots + (n - m + 2)j_{m-1}.$$

Then in (46), the term of the lowest $\tilde{\delta}$ -degree has the form

$$h_n(z, \bar{z}) = b_0(z^n + \bar{z}^n).$$

We write the formal series (46) in the form

$$h = \sum_{p \geq n, j \geq 0} h_{p,j}(z, \bar{z}) \epsilon^j,$$

where $h_{p,j}$ is a resonant homogeneous polynomial of order $p - \|\mathbf{j}\|$.

Let $\mathbf{j} \geq 0$ and $\chi(z, \bar{z})$ be a homogeneous resonant polynomial of degree $p' = p - n + 2 - \|\mathbf{j}\|$, $p' \geq 2$. After the change of variables $(z, \bar{z}) \mapsto \Phi_\chi^{\epsilon^j}(z, \bar{z})$, the Hamiltonian takes the form (48). So

$$\tilde{h}_{s\mathbf{j}'} = h_{s\mathbf{j}'} \quad \text{for } \begin{cases} s < p \\ s = p \text{ and } \mathbf{j}' \neq \mathbf{j} \end{cases}$$

and

$$\tilde{h}_{p,\mathbf{j}} = h_{p,\mathbf{j}} - L_{h_n} \chi.$$

It has been shown in section 4.4 that there exists such a χ that $\tilde{h}_{p,\mathbf{j}}$ has the desired form. The proof is completed by induction in p .

The uniqueness can be proved using the same type of arguments as for individual maps. □

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