Some topics in K-theory

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Abstract .

In Part A (II) the uniqueness theorem for equivariant cohomology theories , (proved in Part A (I)) , is used to calculate the operation rings , $Op(K_G, K_G^{\wedge})$ and $Op(K_G^{\wedge}, K_G^{\wedge})$, ($\wedge$ is $I(G)$-adic completion ; G is a finite group) . Semi-groups , $\tilde{S}_{G(\alpha)} \subset K_G$, are introduced and $Op(\tilde{S}_{G(\alpha)}, K_G)$ is calculated in order to investigate $Op(K_G)$, the ring of self-operations of K_G . Finally $Op(K_G)$ is related to the other two rings of operations and any self-operation of K_G is proved to be continuous with respect to the $I(G)$-adic topology .

In Part B some higher order operations in K-theory , called Massey products , are defined and proved to be the differentials in the Equivariant Kunneth Formula spectral sequence in K-theory .

In Part C the Rothenberg-Steenrod spectral sequences are used (i) to calculate the K-theory of conjugate bundles of Lie groups , (ii) to prove a small theorem on the K-theory of homogeneous spaces of Lie groups , and (iii) to calculate the homological dimension of R(H) as an R(G)-module , for an inclusion of Lie groups , $H \subset G$. As an example of (ii) the algebra $K^*(SO(m))$ and the operation ring , $\varprojlim K^*(SO(m))$, are computed .

CONTENTS

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ON EQUIVARIANT COHOMOLOGY THEORIES and OPERATIONS IN EQUIVARIANT K-THEORY .

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Introduction:

Let G be a group and $\mathcal{E}$ be a category of pairs of G-spaces and equivariant homotopy classes of G-maps , then a cohomology theory defined on $\mathcal{E}$ is called a G-equivariant cohomology.

For G a compact Lie group,equivariant K-theory (K_G) provides a well known example of an equivariant cohomology,the properties of which are developed in the notes of Atiyah and Segal.There are several other examples of equivariant cohomologies and a collection of results on the general theory are to be found in the Borel Seminar on Transformation Groups (Ann.Maths.Studies No46 Also G.E.Bredon has developed the general theory of equivariant cohomologies defined on C.W.complexes,for G a discrete group .

This thesis is in two parts,Part I consisting of a paper submitted to the University of Warwick for a Master's degree.

The purpose of Part I is to prove a'uniqueness' theorem for equivariant cohomologies,satisfying extra axioms of continuity and additivity,defined on the category of locally compact,second countable G-spaces.

§1 contains the basic definitions connected with G-spaces and the axioms for an equivariant cohomology;it will be noticed that the excision axiom is stronger than the usual one for classical cohomologies and,in fact,either will suffice for the purposes of the proofs given here.

§2 consists of several results on G-presheaves and cohomology spectral sequences which are used in obtaining the spectral sequence of §3.This spectral sequence is then employed to prove the 'uniqueness theorem' under either of two additivity axioms (which are treated separately in PartI §§3·1 and 3·2). §4 contains an example of an equivariant cohomology particularly suited to the category of GANR's and deals with a few of its properties related to the results of §3 .

In Part II we examine several operations rings of various functors associated with K_G.

In §1 we define the functor K_G and construct several self-operations of it.

In §2 a representing space ,BU(G),is constructed for K_G for the case of G finite .

In §§3 and 4 we consider the category of finite C.W.complexes acted upon cellularly by a finite group and introduce the functors $\{ K_G^{H\wedge} \}$,obtained from K_G^* by ring completion.Using the 'uniqueness theorem' it is then shown $K_G^{*\wedge}$ is a representable cohomology theory on this category .

The operation rings ,$Op(k_G,\hat{k}_G)$ and $Op(\hat{k}_G,\hat{k}_G)$,for the representable functors (denoted by small type) are calculated in §5 and the relating homomorphism induced by $\lambda : k_G \to \hat{k}_G$ is shown to be an isomorphism .

In §6 we introduce semi-groups $\tilde{S}_{G(\alpha)} \subset K_G$ and determine the operation rings $Op(\tilde{S}_{G(\alpha)},K_G)$ explicitly in terms of the operations constructed in §1;this is done in order to compare the rings $Op(K_G)$ for different finite groups .Finally we calculate $Op(K_G)$ and show it is isomorphic to $Op(k_G)$,which enables us to relate $Op(K_G)$ with the operation rings of §5.

For convenience the necessary calculation of the K_G rings of products of projective spaces has been relegated to a short appendix to Part II .

Throughout the text acknowledgements are made in full to other authors from whom some techniques have essentially been derived, and all proofs of previously published results, used in this thesis, are omitted .

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PART I

§1. Preliminaries.

Throughout this paper G will denote a compact Lie group.

D: A G-space is a pair (X,φ) where

1) X is a topological space.

2) $\varphi: G\times X \to X$ is a continuous map

$(g,x) \to \varphi(g,x) = g.x$

such that $h.(g.x)=(hg.x)$ $h,g\in G$, $x\in X$.

and a G-subspace $A\subset X$ is a subspace of X such that

$\varphi(g,-):A \to A$, $g\in G$.

D: If X,Y are G-spaces then a map $f: X \to Y$ is a G-map (equivariant map) if

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \varphi(g,-)\downarrow & & \downarrow\varphi(g,-) \\ X & \xrightarrow{f} & Y \end{array}$$

is commutative for all $g\in G$.

D: For $x \in X$ the orbit containing x is the G-subspace $\{y\in X; y=g.x \text{ for some } g\in G\}$.

D: The orbit space of (X,φ) is the identification space

$X/\sim = X/G$ where $g.x \sim x$, $g\in G, x\in X$.

We denote by $\Pi: X\to X/G$ the projection.

If I is the unit interval and X a G-space then $X\times I$ will be the product space with action $g.(x,t)=(g.x,t)$; $g\in G, x\in X, t\in I$. Hence we can define a G-homotopy on X and Y as a G-map

$H: X\times I \to Y$

and obtain a category, $\mathcal{C}$, of G-spaces and G-homotopy classes of maps. The corresponding category of pairs

(G-space, closed G-subspace)

will be denoted by $\mathfrak{Z}$.

D: If η is a subcategory of $\mathfrak{Z}$ such that if $(X,A)\in\eta$ then $(X,A)\times I\in\eta$ then a G-equivariant cohomology theory on η is a set of contravariant functors from η to the category of abelian groups -

$$\{h^n_G\} \qquad n\in Z, \text{together with a set of natural}$$

transformations

$$\{\delta^n(X,A): h^n_G(A)=h^n_G(A,\varphi) \to h^{n+1}_G(X,A)\} \qquad (X,A)\in\eta.$$

such that:-

i) If $A,B,A\cup B\in\eta$ and both A and B are closed G-subspaces of $A\cup B$ with $j:(B,B\cap A)\subset(A\cup B,A)$ then $h^n_G(j)$ is an isomorphism for all $n\in Z$.

ii) If $A,X,(X,A)\in\eta$ and $j:(X,\varphi)\subset(X,A)$; $k:(A,\varphi)\subset(X,\varphi)$ then

$$\cdots\to h^n_G(X,A) \xrightarrow[j^*]{} h^n_G(X) \xrightarrow[k^*]{} h^n_G(A) \xrightarrow[\delta^n]{} h^{n+1}_G(X,A) \to \cdots$$

[with $j^*=h^n_G(j), k^*=h^n_G(k)$] is exact.

Remark: Thus an equivariant cohomology theory is an extension of the familiar generalised cohomology theories corresponding to the case $G=1$. Since the orbit is the 'building block' for a G-space and is homeomorphic to a coset space G/H for some closed subgroup $H\subset G$ then there should be a class of G-cohomologies for which there is an analogue of the classical dimension axiom involving $\{h^n_G(G/H)\}$ for $H\subset G$. This is made more precise in §3 where under suitable restrictions the role of $h^*_G(G/H)$ is shown to be that of a coefficient system.

§2. In this section we define the tools to be used in §3, namely G-presheaves, G-sheaves, Cech cohomology with coefficients in a G-presheaf and spectral sequences.

§2.1 The work ofthis section is parallel to the treatment of sheaves to be found in [5] and omitted proofs and further examples are to be found there.

L2.1.0 For any G-space, X, the map $\Pi: X \to X/G$ is both open and closed and if $A \subset X/G$ is compact then so is $\Pi^{-1}(A)$.

Proof: (see [4]).

D: A G-presheaf of abelian groups on a G-space X is a contravariant functor, Γ_G, from the category of open G-subsets of X and inclusion maps to the category of abelian groups; such that $\Gamma_G(\varphi)=0$.

Hence every inclusion of G-subsets, $U \subset V$ open in X, induces a homomorphism $\rho_{UV}: \Gamma_G(V) \to \Gamma_G(U)$

called the restriction map, such that

$$\rho_{UU}=1_{\Gamma_G(U)} \quad \text{and for } U\subset V\subset W \quad \rho_{UV}\cdot\rho_{VW}=\rho_{UW}\ .$$

Given $\gamma \in \Gamma_G(V)$ we write $\gamma|V$ for $\rho_{UV}(\gamma)$.

Examples:

a) For any abelian group K the constant G-presheaf defined by

$K(U)=K$, $U\neq\varphi$ and $K(\varphi)=0$.

$\rho_{UV}=1_K$ for $\varphi\neq U\subset V$.

b) For any G-cohomology definedon X

i) $\Gamma^q_G(U)=h^q{}_G(U)$ $q\in Z$. OR

ii) $\Gamma^q_G(U)=h^q_G(\bar{U})$.

Remark: Any G-presheaf on X, Γ_G, gives a 1-presheaf (simply, a presheaf) on X/G, Γ, defined by

$$\Gamma(\bar{U})=\Gamma_G(\Pi^{-1}(\bar{U})) \qquad \bar{U}\subset X/G,$$

and since if $U\subset X$ is an open G-subset $\Pi(U)\subset X/G$ is open the above correspondence is a bijection.

D: A natural transformation $\beta:\Gamma_G \to \Gamma_G^1$ between two G-presheaves will be called a homomorphism.

Now let Γ_G be a G-presheaf on X and $\Phi=\{U\}$ be a collection of open G-subsets of X, then a compatible Φ-family of Γ_G is an indexed family $\{\gamma_U\in\Gamma_G(U)\}$ $U\in\Phi$, such that

$$\gamma_U|U\cap U^1=\gamma_{U^1}|U\cap U^1 \qquad \text{for all } U,U^1\in\Phi.$$

D: Γ_G on X is called a G-sheaf if

a) Given a collection Φ of open G-subsets of X, with $V=\bigcup_{U\in\Phi}U$; and given $\gamma\in\Gamma_G(V)$ such that $\gamma|U=0$ for all $U\in\Phi$ then $\gamma=0$.

b) Given Φ, V as in (a) and a compatible Φ-family $\{\gamma_U\}$ then there is an element $\gamma_V\in\Gamma_G(V)$ such that

$$\gamma_V|U=\gamma_U \text{ for all } U\in\Phi.$$

(By (a) γ_V is unique.)

To each Γ_G on X we now associate another G-presheaf, $\hat{\Gamma}_G$, which is in fact a sheaf, called the completion of Γ_G.

Given a collection φ of open G-subsets of X let $\Gamma_G(\varphi)$ be the abelian group of all compatible φ-families. If ψ is another collection of open G-subsets refining φ then the compatibility of φ ensures that we have a well-defined map

$\{\gamma_U\}_{U\in\varphi} \to \{\gamma_V^1\}_{V\in\psi}$; where $\gamma_V^1=\gamma_U|V$ for some $U\supset V$. Since Γ_G is a functor this map is in fact a homomorphism $\Gamma_G(\varphi) \to \Gamma_G(\psi)$. For any open G-subset $W\subset X$ as φ varies over open G-coverings of W the $\{\Gamma_G(\varphi)\}$ form a direct system and we define

$$\hat{\Gamma}_G(W)=\varinjlim\{\Gamma_G(\varphi)\}.$$

If $V\subset W$ is an open G-subset of X then a G-cover, φ, of W defines

$\varphi_V=\{V\cap U|U\in\varphi\}$ a G-cover of V which refines φ and hence there is a homomorphism $\Gamma_G(\varphi) \to \Gamma_G(\varphi_V)$ and the direct limit of these homomorphisms is a homomorphism $\hat{\Gamma}_G(W) \to \hat{\Gamma}_G(V)$.

Hence $\hat{\Gamma}_G$ is a G-presheaf and we note that under the bijection G-presheaves on X to presheaves on X/G if $\Gamma_G \to \Gamma$ then $\hat{\Gamma}_G \to \hat{\Gamma}$.

Now the map which assigns $\gamma\in\Gamma_G(V)$ the compatible $\{V\}$-family $\{\gamma\}$, for any open G-subset $V\subset X$, defines a homomorphism

$\alpha:\Gamma_G \to \hat{\Gamma}_G$ and we have

<u>L2.1.1</u>: Γ_G on X is a G-sheaf iff α is an isomorphism.

<u>Proof</u>: (a) is satisfied iff α is mono and (b) iff α is epi.

We now define Cech cohomology with coefficients in Γ_G, written $\hat{H}(-,\Gamma_G)$, and prove some basic properties in the case when X is a locally compact, second countable G-space.

For Γ_G on X, φ an open G-covering of X and $q\geq 0$ define $C^q(\varphi,\Gamma_G)$ to be the free abelian group of functions χ assigning to ordered $(q+1)$-tuples $(U_0,U_1,\ldots,U_q)\in\varphi$ an element

$$\chi(U_0,\ldots,U_q)\in\Gamma_G(U_0\cap U_1\cap\ldots\cap U_q).$$

Then we have a coboundary operation

$$\delta^q:C^q(\varphi,\Gamma_G) \to C^{q+1}(\varphi,\Gamma_G)$$ defined by

$$(\delta^q\chi)(U_0,\ldots,U_{q+1}) = \sum_{i=0}^{q+1} (-1)^i \chi(U_0,\ldots,\hat{U}_i,\ldots,U_{q+1})|U_0\cap\ldots\cap U_{q+1}$$

thus $\delta^{q+1}\circ\delta^q=0$ and $\{C^q(\varphi,\Gamma_G);\delta^q\}$ is a cochain complex with cohomology $H^*(\varphi,\Gamma_G)$.

Now let ψ be a refinement of φ by open G-subsets of X and $\lambda:\psi\to\varphi$ be a function such that $V\subset\lambda(V)$ for all $V\in\psi$ then there is a cochain map $\lambda^*:C^*(\varphi,\Gamma_G)\to C^*(\psi,\Gamma_G)$ defined by

$(\lambda^*\chi)(V_0,\ldots,V_q)=\chi(\lambda\{V_0\},\ldots,\lambda\{V_q\})|V_0\cap\ldots\cap V_q$ and if

$\mu:\psi\to\varphi$ is another such function then

$\{D^q:C^q(\varphi,\Gamma_G)\to C^{q-1}(\psi,\Gamma_G)\}$ defined by

$$(D^q\chi)(V_0,\ldots,V_{q-1})=\sum_{0}^{q-1}(-1)^i\chi(\lambda\{V_0\},\ldots,\lambda\{V_i\},\mu\{V_i\},\ldots,\mu\{V_{q-1}\})|V_0\cap\ldots\cap V_{q-1}$$

is a cochain homotopy between λ^* and μ^*.

Hence we have a well-defined homomorphism

$\lambda^*:H^*(\varphi,\Gamma_G)\to H^*(\psi,\Gamma_G)$

and as φ varies over open G-coverings of X $\{H^*(\varphi,\Gamma_G)\}$ forms a direct system of abelian groups.

We set $\hat{H}(X,\Gamma_G)=\varinjlim\{H^*(\varphi,\Gamma_G)\}$.

Remark: If Γ_G on X corresponds to Γ on X/G then we have a canonical identification $\hat{H}(X,\Gamma_G)\to\hat{H}(X/G,\Gamma)$ which is natural in X and Γ_G.

L2.1.2: There is a covariant functor from the category of short exact sequences of G-presheaves on X to the category of exact sequences of abelian groups which assigns to

$0\to\Gamma_G^1\to\Gamma_G\to\Gamma_G^2\to 0$

the long exact sequence

$$\cdots \to \hat{H}^q(X,\Gamma_G^1) \to \hat{H}^q(X,\Gamma_G) \to \hat{H}^q(X,\Gamma_G^2) \to \hat{H}^{q+1}(X,\Gamma_G^1) \to \cdots$$

Proof: For any open G-cover, φ, we get a short exact sequence

$$0 \to C^*(\varphi,\Gamma_G^1) \to C^*(\varphi,\Gamma_G) \to C^*(\varphi,\Gamma_G^2) \to 0$$

and hence a long exact sequence in $H^*(\varphi,-)$ and the result follows since $\varinjlim$ is an exact functor.

Now suppose (X,A) is a G-pair and Γ_G is a G-presheaf on X, define

$$\Gamma_{G(A)}(U) = \begin{cases} \Gamma_G(U), & U\cap A\neq\varphi \\ 0 & U\cap A=\varphi \end{cases}$$

and

$$\Gamma_G^{(A)}(U) = \begin{cases} \Gamma_G(U) & U\cap A=\varphi \\ 0 & U\cap A\neq\varphi \end{cases}$$

hence $0 \to \Gamma_G^{(X-A)} \to \Gamma_G \to \Gamma_{G(A)} \to 0$ is exact and we have

Cor.2.1.3: For any G-subspace $A\subset X$ and G-presheaf, Γ_G, on X there is a functorial long exact sequence

$$\cdots \to \hat{H}^q(X,A;\Gamma_G) \to \hat{H}^q(X,\Gamma_G) \to \hat{H}^q(A,\Gamma_G) \to \hat{H}^{q+1}(X,A;\Gamma_G) \to \cdots$$

where $\hat{H}(A,\Gamma_G)=\hat{H}(X,\Gamma_{G(A)})$

and $\hat{H}(X,A;\Gamma_G)=\hat{H}(X,\Gamma_G^{(X-A)})$

D: A G-presheaf, Γ_G, will be called locally zero if $\hat{\Gamma}_G$ is the zero G-presheaf; that is iff given any open G-subset $V\subset X$ and $\gamma\in\Gamma_G(V)$ then there is an open G-cover, φ, of V such that $\gamma|U=0$ for all $U\in\varphi$.

Alternatively this is equivalent to the condition that if $A\subset X$ is an orbit then $\varinjlim\{\Gamma_G(U)\}=0$ as U varies over all the open G-neighbourhoods of A.

Now if $\beta:\Gamma_G \to \Gamma_G^1$ is a homomorphism of G-presheaves on X then $\ker\beta:\mathrm{Ker}\beta \to \Gamma_G$ and $\mathrm{coker}\beta:\Gamma_G^1 \to \mathrm{Coker}\beta$ are also homomorphisms of G-presheaves on X.

<u>D:</u> $\beta:\Gamma_G \to \Gamma_G^1$ is called a <u>local isomorphism</u> of G-presheaves if Kerβ and Cokerβ are locally zero. Hence by the previous remark this is equivalent to the condition that for $A\subset X$ an orbit and for U varying over open G-neighbourhoods of A

$$\varinjlim\{\beta(U)\}:\varinjlim\{\Gamma_G(U)\} \to \varinjlim\{\Gamma_G^1(U)\}$$ is an isomorphism.

<u>L2·1·4:</u> $\alpha:\Gamma_G \to \hat{\Gamma}_G$ is a local isomorphism.

<u>Proof:</u> Let $\gamma\in$Ker$(\alpha)(V)$ then by definition $\gamma\in\Gamma_G(V)$ and there is φ, an open G-cover of V such that $\gamma|U=0$ for all $U\in\varphi$.

Now if $\gamma_1\in$Coker$(\alpha)(V)$ then there is an open G-cover, φ, of V and a compatible φ-family $\{\gamma_U\}$ which represents γ_1; for each $U^1\in\varphi$ $\gamma_1|U^1$ is represented by $\{\gamma_U|U^1\}$ and hence by compatibility $\gamma_1|U^1\in$Imα and Cokerα is locally zero.

<u>L2·1·5:</u> If Γ_G is a locally zero G-presheaf on a paracompact, hausdorff space X (e.g. if X is locally compact and second countable) then $\hat{H}(X,\Gamma_G)=0$.

<u>Proof:</u> By virtue of the bijection Γ_G on $X \to \Gamma$ on X/G and L2·1·0 this follows from the corresponding result for presheaves given in [5].

<u>L2·1·6:</u> If $\beta:\Gamma_G \to \Gamma_G^1$ is a local isomorphism of G-presheaves on a paracompact, hausdorff space then $\beta^*:\hat{H}(X,\Gamma_G) \xrightarrow{\approx} \hat{H}(X,\Gamma_G^1)$.

<u>Proof:</u> We have short exact sequences of G-presheaves

$$0 \to \text{Ker}\beta \to \Gamma_G \to \text{Im}\beta \to 0 \qquad \text{and}$$

$$0 \to \text{Im}\beta \to \Gamma_G^1 \to \text{Coker}\beta \to 0$$

thus by L2·1·2 and L2·1·5 β^* is the composition of two isomorphisms, $\hat{H}(X,\Gamma_G) \approx \hat{H}(X,\text{Im}\beta) \approx \hat{H}(X,\Gamma_G^1)$

Cor.2·1·7: For X a paracompact G-space and $\alpha : \Gamma_G \to \hat{\Gamma}_G$

$$\alpha^* : \hat{H}(X,\Gamma_G) \cong \hat{H}(X,\hat{\Gamma}_G).$$

§2·2: In this section we define a cohomology spectral sequence, for brevity several proofs have been omitted and these are to be found in [2],[3] and [5].

D: A bigraded differential abelian group, (E,δ) of bidegree $(r,1-r)$ is a family of abelian groups and homomorphisms $\{E^{p,q} ; \delta^{p,q} : E^{p,q} \to E^{p+r,q+1-r}\}$ $(p,q \in Z)$ such that $\delta^{p,q} \circ \delta^{p+r,q+1-r} = 0$

The cohomology of (E,δ) is the bigraded abelian group $H^*(E)$ in which $H^{p,q}(E) = \dfrac{\mathrm{Ker}(\delta^{p,q})}{\mathrm{Im}(\delta^{p-r,q+r-1})}$.

D: For $k \in Z$ an E_k cohomology spectral sequence is a sequence $\{(E_r,\delta_r)\}$ for $r \geq k$ such that

a) (E_r,δ_r) is a bigraded differential abelian group of bidegree $(r,1-r)$ and

b) For $r \geq k$ there is an isomorphism $H^*(E_r) \approx E_{r+1}$ of bigraded abelian groups.

An E_k spectral sequence homomorphism $\chi : E \to F$ is a collection of homomorphisms $\{\chi_r^{p,q} : E_r^{p,q} \to F_r^{p,q}\}$ which commute with the differentials and such that $\chi_r^* : H^*(E_r) \to H^*(F_r)$ corresponds under the connecting homomorphisms in (b) to χ_{r+1}.

Let $Z_k^{p,q} = \mathrm{Ker}\, \delta_k^{p,q}$ and $B_k^{p,q} = \mathrm{Im}\delta_k^{p-k,q+k-1} \subset Z_k^{p,q}$ then we get bigraded groups Z_k, B_k and identifying $H^*(E_k)$ with E_{k+1}

$$E_{k+1} = \frac{Z_k}{B_k}$$

Now let $Z(E_{k+1})^{p,q} = \text{Ker } \delta_{k+1}^{p,q}$ and $B(E_{k+1})^{p,q} = \text{Im } \delta_{k+1}^{p-k-1,q+k}$ thus by the Noether isomorphism theorem there exist bigraded subgroups Z_{k+1}, B_{k+1} such that $B_k \subset B_{k+1} \subset Z_{k+1} \subset Z_k$ and $\frac{Z_{k+1}}{B_{k+1}} \approx E_{k+2}$ and by induction we get a chain of subgroups of E_k : $B_k \subset B_{k+1} \subset .. \subset B_r \subset .. \subset Z_r \subset .. \subset Z_{k+1} \subset Z_k$ such that $\frac{Z_r}{B_r} \approx E_{r+1}$ as bigraded groups.

If $Z_\infty = \bigcap_r Z_r$ and $B_\infty = \bigcup_r B_r$ then $E_\infty = \frac{Z_\infty}{B_\infty}$ is a bigraded group called the <u>limit</u> of the spectral sequence.

<u>D</u>: An E_k spectral sequence is said to <u>converge</u> if for each (p,q) there exists $r(p,q) \geq k$ such that for $r \geq r(p,q)$ $\delta_r^{p,q}$ is trivial. (Then $E_{r+1}^{p,q}$ is isomorphic to a quotient of $E_r^{p,q}$ and $E_\infty^{p,q}$ is the direct limit of $E_{r(p,q)}^{p,q} \to E_{r(p,q)+1}^{p,q} \to \dots$)

By a <u>strongly convergent</u> spectral sequence we shall mean a pair $(\{E_r\}, M)$ where M is a filtered group with filtration $\{M_p\}$ and $\{E_r\}$ is a spectral sequence such that $E_\infty^p \approx \frac{M_p}{M_{p+1}}$ as graded groups.

For a strongly convergent E_k spectral sequence we write $E_k \Rightarrow M$ and a homomorphism of such spectral sequences is a pair consisting of a spectral sequence homomorphism and a homomorphism of filtered groups which induces on the bigraded groups $\{ \frac{M_p}{M_{p+1}} \}$ a homomorphism corresponding to that between the limits.

<u>L2.2.1</u>: If $\chi : E \to F$ is a homomorphism of E_k spectral sequences which is an isomorphism for some $r > k$ then χ is an isomorphism for all $s > r$. If E and F converge then χ induces an isomorphism of the limits. Further if $\chi : (E,M) \to (F,N)$ is a homomorphism of strongly convergent spectral sequences then in addition χ induces an isomorphism of the filtered groups M and N.

<u>Proof</u>: (see [5]).

Following [2] we now outline in more detail the particular spectral sequence used in §3.

Suppose W is a G-space and that we have a finite, increasing chain of closed G-subsets of W

$$\varphi = W^{-1} \subset W^0 \subset \dots \subset W^{n-1} \subset W^n = W.$$

If h_G^* is an equivariant cohomology defined on all pairs (W^q, W^p) define $\bar{h}_G = \underset{n}{\oplus} h_G^n(W)$ and for $p \leq n$

$$\bar{h}_G(p) = \underset{n}{\oplus} h_G^n(W, W^p) .$$

The inclusions $i_p : (W, \varphi) \to (W, W^p)$ induce $\bar{h}_G(p) \to \bar{h}_G$ thus if $F^p\bar{h}_G = \operatorname{Im} \{ \bar{h}_G(p) \to \bar{h}_G \}$

we have a decreasing filtration of the graded group $\bar{h}_G$.

$$\bar{h}_G = F^{-1}\bar{h}_G \supset \dots \supset F^{p-1}\bar{h}_G \supset F^p\bar{h}_G \supset \dots \supset F^n\bar{h}_G = 0 .$$

Now let $\bar{h}_G(p,q) = \underset{n}{\oplus}\ h^n_G(W^q, W^p)$ for $q > p$ then for $r \geq 1$

$$Z^p_r = \mathrm{Im}\{\ \oplus i^* : \bar{h}_G(p,p+r) \to \bar{h}_G(p,p+1)\ \}$$

$$B^p_r = \mathrm{Im}\{\ \oplus\delta : \bar{h}_G(p-r+1,p) \to \bar{h}_G(p,p+1)\ \}$$

(where $\oplus i^*$ is induced by inclusion and $\oplus\delta$ from the coboundary of the triple $(W^{p+1}, W^p, W^{p-r+1})$) are graded abelian groups such that $B^p_r \subset Z^p_r$.

Defining $E^p_r = \dfrac{Z^p_r}{B^p_r}$ $(1 \leq r \leq \infty)$, graded abelian groups with bigraded differentials and bigrading induced from the cohomology long exact sequences, we obtain a strongly convergent with $E^{p,q}_1 = h^{p+q}_G(W^{p+1}, W^p) \Rightarrow \bar{h}_G = \{ F^p\bar{h}_G \}$.

Consider now the case when E and E^1 are two strongly convergent spectral sequences, with associated filtered groups M and N, obtained from two cohomologies, h_G and h^1_G, defined on $\{W^i\}$. If $\beta : h_G \to h^1_G$ is a natural transformation then β induces a homomorphism of the strongly convergent spectral sequences which is β on the filtered groups. By L2.2.1, if β induces an isomorphism $E_r \approx E^1_r$ for some $r \geq 1$ then it induces an isomorphism of the limits and also $\beta(W) : \bar{h}_G(W) \approx \bar{h}^1_G(W)$ as filtered groups.

§3.

In this section we derive an h_G^* spectral sequence similar to that used in [1] and apply it in conjunction with a further cohomology axiom to obtain a 'uniqueness theorem'.

D: $f: X \to Y$ a G-map is proper if for any compact G-subset $A \subset Y$ $f^{-1}(A) \subset X$ is compact.

§3.1: For convenience throughout the rest of this paper we shall deal only with (a) the full category of locally compact, second countable G-spaces or (b) the subcategory of such spaces and proper maps.

D: Recall that such spaces are paracompact and a space X is said to have finite covering dimension n if each open G-cover of X has a locally finite refinement in which no point lies in more than n sets.

Let $\psi = \{ U_\alpha \}_{\alpha \in S}$ be a countable, locally finite, open G-cover of X whose closure is also locally finite.

For $\sigma \subset S$ let $\bar{U}_\sigma = \bigcap_{\alpha \in \sigma} \bar{U}_\alpha$; a closed G-subset of X.

Then $\{ \sigma \subset S : \bar{U}_\sigma \neq \varphi \}$ defines a simplicial complex N_ψ.

Now let X have finite covering dimension, then N_ψ may be assumed, by refining ψ if necessary, to be finite dimensional.

Thus $|N_\psi|$, the geometrical realisation of N_ψ is locally compact and second countable.

Further we have functions :

$$\zeta : X \to N_\psi \qquad \zeta(x) = \{ \alpha : x \in U_\alpha \}$$

and $\nu : |N_\psi| \to N_\psi$ $\quad \nu(n) = \{ \text{unique open simplex containing } n \}$

Define $W_\psi = \{ (n,x) \in |N_\psi| \times X : \nu(n) \text{ is a face of } \zeta(x) \}$ with G acting trivially on $|N_\psi|$. W_ψ is a closed G-subset of $|N_\psi| \times X$.

We have the G-map projections

$p_1 : W_\psi \to |N_\psi|$ and $p_2 : W_\psi \to X$; in particular p_2 is proper.

This construction is functorial in (X,ψ) in the sense that if $\lambda = \{ V_\beta \}_{\beta \in T}$ is a similar G-cover of a G-space Y ; then for $f : X \to Y$ and $\theta : S \to T$ such that $f(U_\alpha) \subset V_{\theta(\alpha)}$ there is a G-map $|\theta| \times f : W_\psi \to W_\lambda$.

<u>L3.1.1</u>: If (f,θ_0) and $(f,\theta_1) : (X,\psi) \to (Y,\lambda)$ are as above then they induce G-homotopic maps $W_\psi \to W_\lambda$ and if $f : X \to Y$ is proper then so is the homotopy.

<u>Proof</u>: By the standard lemma in Cech theory different choices of a refinement map are contiguous and hence there

is a proper homotopy $H : |N_\psi| \times I \to |N_\lambda|$ between the induced simplicial maps corresponding to θ_0 and θ_1. Then

$H \times f \mid W_\psi \times I$ is the required G-homotopy.

If Y is a G-space and $\{Y_\alpha\}_{\alpha\in A}$ a closed G-cover of Y let $Y_{\alpha\beta} = Y_\alpha \cap Y_\beta$, $Y^1_\alpha = \bigcup_{\alpha\neq\beta} Y_{\alpha\beta}$ and $Y^1 = \bigcup_\alpha Y^1_\alpha$ then clearly we have:

L3.1.2: $\coprod_\alpha (Y_\alpha, Y^1_\alpha) \to (Y, Y^1)$ is a relative homeomorphism.

Now let $|N| = |N_\psi|$ and $|N^p|$ = p-skeleton of $|N|$ then we have $|N| \supset \dots \supset |N^p| \supset \dots \supset |N^1| \supset |N^0|$. Further if $X_r = \{ x \in X : \dim \zeta(x) \geq r \}$ then

$X = X_0 \supset X_1 \supset \dots\dots \supset X_r \supset \dots\dots$

is a finite filtration of X by closed G-subsets.

Put $W^p = p_1^{-1}(|N^p|)$ and $W_r = p_2^{-1}(X_r)$.

D: An equivariant cohomology h^*_G is called additive if for all collections $\{X_\alpha\}_{\alpha\in A}$ of G-spaces on which it is defined $\Pi i^*_\alpha : h^*_G(\coprod X_\alpha) \cong \prod_\alpha h^*_G(X_\alpha)$ where

$i_\beta : X_\beta \subset \coprod_\alpha X_\alpha$.

L3.1.3: If h^*_G is an additive equivariant cohomology theory defined in cases (a) or (b) then the proper map $p_2 : W_\psi \to X$ induces an isomorphism in cohomology.

<u>Proof:</u> $p_2 : (W, W_1) \to (X, X_1)$ is a relative homeomorphism (with inverse $x \in X - X_1 \to (\zeta(x), x)$).

Also $h_G^*(W_r, W_{r+1})$

$$= h_G^*\left(\bigcup_{\dim \sigma \geq r} |\bar\sigma| \times U_\sigma, \bigcup_{\dim \sigma \geq r+1} |\bar\sigma| \times U_\sigma\right)$$

(by definition of W_r)

$$\approx h_G^*\left(\bigcup_{\dim \sigma = r} |\bar\sigma| \times U_\sigma, \bigcup_{\dim \sigma = r} |\bar\sigma| \times U_\sigma^1\right)$$

(by excision and definition of U_σ^1 [see L3·1·2])

$$\approx h_G^*\left(\coprod_{\dim \sigma = r} (|\bar\sigma| \times U_\sigma, |\bar\sigma| \times U_\sigma^1)\right)$$

(by L/ 3·1·2)

$$\approx \prod_{\dim \sigma = r} h_G^*(|\bar\sigma| \times U_\sigma, |\bar\sigma| \times U_\sigma^1) \quad \text{(by additivity)}$$

$$\approx \prod_{\dim \sigma = r} h_G^*(U_\sigma, U_\sigma^1) \quad (\text{by } \Pi p_2^*)$$

$$\approx h_G^*\left(\coprod_{\dim \sigma = r} (U_\sigma, U_\sigma^1)\right) \quad \text{(by additivity)}$$

$$\approx h_G^*(X_r, X_{r+1}) \quad \text{(by L 3·1·2)}$$

Hence by induction on sequences of the triples (X_{r+2}, X_{r+1}, X_r) we have $p_2^* : h_G^*(X, X_r) \xrightarrow{\approx} h_G^*(W, W_r)$ and for large r $X_r = \varphi = W_r$.

Now applying the method of §2 to the finite filtration

$$W_\psi = W \supset \dots \supset W^p \supset \dots \supset W^0 \supset W^{-1} = \varphi$$

we obtain a strongly convergent E_1 spectral sequence whose limit is the bigraded group associated with a finite filtration on $\bigoplus_n h_G^*(W) \approx \bigoplus_n h_G^*(X)$, for which $E_2^{p,q}$ is the p-th cohomology of the complex

$$\text{(A)}\quad h_G^q(W^0) \to h_G^{q+1}(W^1, W^0) \to \dots \to h_G^{q+p}(W^p, W^{p-1}) \to \dots.$$

But

$$h_G^{q+p}(W^p, W^{p-1})$$

$$= h_G^{q+p}\Big(\bigcup_{\dim\sigma \leq p} |\bar\sigma| \times U_\sigma, \bigcup_{\dim\sigma \leq p-1} |\bar\sigma| \times U_\sigma\Big)$$

(by definition of W^p)

$$\approx h_G^{q+p}\Big(\coprod_{\dim\sigma = p} (|\bar\sigma|, |\dot\sigma|) \times U_\sigma\Big)$$

(by excision)

$$\text{(B)}\quad \approx \prod_{\dim\sigma = p} h_G^{q+p}\big((|\bar\sigma|, |\dot\sigma|) \times U_\sigma\big)$$

(by additivity)

Now take an ordering of ψ and thus an orientation on N_ψ ; and choose a sequence of faces of σ, of the dimensions shown, and such that the orientation of σ induced is coherent with that on N_ψ

$$\text{(C)}\quad |\bar\sigma_0| \subset |\bar\sigma_1| \subset \dots \subset |\bar\sigma_p| = |\bar\sigma| .$$

Then, since there is a proper deformation retraction of

$|\bar{\sigma}_p|$ onto $|\dot{\sigma}_p| - |\dot{\sigma}_{p-1}|$ the sequence of the triple $(|\bar{\sigma}_p|, |\dot{\sigma}_p|, |\dot{\sigma}_p| - |\dot{\sigma}_{p-1}|)\times U_\sigma$ gives

$$h_G^{q+p-1}(\ (|\bar{\sigma}_{p-1}|,|\dot{\sigma}_{p-1}|)\times U_\sigma\ )$$
$$\approx\ h_G^{q+p-1}((|\dot{\sigma}_p|,|\dot{\sigma}_p| - |\dot{\sigma}_{p-1}|)\times U_\sigma\ ) \quad \text{(by excision)}$$
$$\approx\ h_G^{q+p}(\ (|\bar{\sigma}_p|,\ |\dot{\sigma}_p|)\times U_\sigma\ )$$

and hence by induction we have an identification

$$\text{(D)} \qquad h_G^{q+p}(\ (|\bar{\sigma}|,|\dot{\sigma}|)\times U_\sigma\ ) \approx h_G^q(\ U_\sigma\ ) \quad \text{and}$$
$$h_G^{q+p}(\ W^p,W^{p-1}\ ) \approx \prod_{\dim\sigma = p} h_G^q(\ U_\sigma\ )\ .$$

Now let $h_G^q(\overline{\cdot})$ be the G-presheaf on X defined by

$$h_G^q(\overline{\cdot})\ (U) = h_G^q(\ \bar{U}\ )$$

then we have $h_G^{q+p}(W^p,W^{p-1}) \approx C^p\ (\psi,\ h_G^q(\overline{\cdot})\ )$ the <u>oriented</u> Cech cochains.

Now the identification (D) depends only on the orientation of σ defined by the sequence (C), thus for $|\bar{\sigma}_p| = \sigma$ a face of $|\bar{\tau}_{p+1}| = \tau$ we have the following diagram :-

$$
\begin{array}{ccccc}
h_G^{q+p}(W^p,W^{p-1}) & \rightleftarrows & h_G^{q+p}((|\bar\sigma|,|\dot\sigma|)\times U_\sigma) & \xleftarrow{\approx} & h_G^q(U_\sigma) \\
 & \searrow & \downarrow i^* & & \downarrow i^* \\
\delta \downarrow & & h_G^{q+p}((|\bar\sigma|,|\dot\sigma|)\times U_\tau) & \xleftarrow{\approx} & h_G^q(U_\tau) \\
 & & \downarrow \approx & & \\
 & & h_G^{q+p}((|\dot\tau|,|\dot\tau|-|\dot\sigma|)\times U_\tau) & & \| \\
h_G^{q+p+1}(W^{p+1},W^p) & & \downarrow \delta & & \\
 & \rightleftarrows & h_G^{q+p+1}((|\bar\tau|,|\dot\tau|)\times U_\tau) & \xleftarrow{\approx} & h_G^q(U_\tau)
\end{array}
$$

which commutes in all but the bottom right rectangle, which commutes or anti-commutes as σ is a coherently or non-coherently oriented face of τ. Hence (A) is the oriented Cech cochain complex $C^*(\psi, h_G^q(\bar{\cdot}))$ by this and L3.1.1 we have proved :

<u>T 3.1.4</u>: There is a strongly convergent spectral sequence

$$\{E_2^{p,q}\} = \{\hat{H}^p(\psi, h_G^q(\bar{\cdot}))\} \Rightarrow h_G^*(X)$$

which is functorial in (X,ψ) .

D: If $(I,<)$ is a directed set then $J \subset I$ is <u>cofinal</u> in I if given $\alpha \in I$ there is $\beta \in J$ such that $\alpha < \beta$.

Hence $\varinjlim I = \varinjlim J$.

Now let h_G^* be an additive equivariant cohomology defined on categories (a) or (b) and let X be a G-space of finite covering dimension n . Let Cov (X) be the collection of open G-coverings of X and let Cov[X] be the subcollection of coverings Ψ such that $N_\Psi = N_{\bar\Psi}$ and $\dim |N_\Psi| = n$. Hence Cov[X] is cofinal in Cov(X) and, since $\varinjlim$ is an exact functor, taking direct limits we obtain :

<u>T 3·1·5</u>: There exists a strongly convergent E_2 spectral sequence, functorial in X,

$$\{ E_2^{p,q} \} = \{ \check{H}(X , h_G^q(\cdot) \} \rightarrow h_G^*(X) .$$

<u>D</u>: If h_G^* is an equivariant cohomology theory defined on categories (a) or (b) , $A \subset X$ an orbit and $U \subset X$ an open G-neighbourhood of A then h_G^* is called <u>continuous</u> if for all A $\varinjlim i^* : \varinjlim_U h_G^*(U) \approx h_G^*(A)$.

If we are in the full category (a) since X is regular and open or closed G-neighbourhoods are cofinal in the G-neighbourhoods of A then this condition is equivalent to $\varinjlim_U h_G^*(U) \approx h_G^*(A)$.

We conclude this section with a 'uniqueness theorem' for continuous, additive equivariant cohomologies.

<u>T 3·1·6</u>: If for i=1,2 $h_{G(i)}^*$ is an additive, continuous equivariant cohomology theory defined on <u>either</u> the full category of locally compact, second countable G-spaces

<u>or</u> the subcategory with only proper maps

and if $\alpha^*: h^*_{G(1)} \to h^*_{G(2)}$ is a natural transformation of cohomology theories such that

(1) For all $H \subset G$, closed subgroup, $\alpha^*(G/H)$ is an isomorphism

then $\alpha^*(X)$ is an isomorphism for all spaces X in the category of finite covering dimension.

<u>Proof:</u> By (1) we have for any orbit $A \subset X$ a commutative diagram of isomorphisms

$$\begin{array}{ccc} \varinjlim \{ h^*_{G(1)}(U) \} & \xrightarrow{\varinjlim \{\alpha(U)\}} & \varinjlim \{ h^*_{G(2)}(U) \} \\ \downarrow{\scriptstyle \varinjlim i^*} & & \downarrow{\scriptstyle \varinjlim i^*} \\ h^*_{G(1)}(A) & \xrightarrow{\alpha^*(A)} & h^*_{G(2)}(A) \end{array}$$

and hence for all q α induces a local isomorphism between the G-presheaves $h^q_{G(i)}(\cdot)$ $(i = 1,2)$.

Thus by L 2·1·(the spectral sequence homomorphism induced by α^* is an isomorphism on the E_2 terms, of the strongly convergent spectral sequences of T 3·1·6 and hence by L2·2·1 $\alpha^*(X)$ is an isomorphism ..

§3·2 : In §3·1 we gave simultaneous proofs for categories (a) and (b) for continuous equivariant cohomologies under the additivity axiom

$$(A1) \qquad h_G^*(\coprod_\alpha X_\alpha) \approx \prod_\alpha h_G^*(X_\alpha) .$$

However, by considering Cech cochains on locally finite coverings that are 'almost everywhere zero ', the methods of §3·1 yield the same results under the additivity axiom

$$(A2) \qquad h_G^*(\coprod_\alpha X_\alpha) \approx \bigoplus_\alpha h_G^*(X_\alpha) .$$

We also note that for compact spaces we require no additivity axiom in §3·1 .

Let (c) denote the full subcategory of compact G-spaces and $\bar{h}_G^*$ be a continuous equivariant cohomology defined there. Then the one-point-compactification , '+' , is a covariant functor from (b) to (c). For $(X,A) \in$ (b)

$$h_G^*(X^+,A^+) = \bar{h}_G^*(X,A)$$

defines a continuous equivariant cohomology on (b) which is, by T3·1·6 , the 'unique' extension of h_G^* on spaces of finite covering dimension.

But if U is an open G-neighbourhood of '*' , the point at infinity , then

$$\bar{h}_G^*(X) = h_G^*(X^+,*) \approx \varinjlim_U h_G^*(X^+,U) .$$

Hence if $X = \coprod_i X_i$ in (b) then a cofinal system of neighbourhoods of * is $\{ X^+ - \coprod X_i \}$. Thus

$$
\begin{aligned}
\tilde{h}_G^*(X) &= h_G^*(X^+, *) \\
&\approx \lim_{\overleftarrow{n}} h_G^*(X^+, X^+ - \beta X_i^+) \\
&\approx \lim_{\overleftarrow{n}} h_G^*(\beta X_i^+) \\
&= \varphi\, \tilde{h}_G^*(X_i^+)
\end{aligned}
$$

Conversely a continuous equivariant cohomology on (b) defines one on (c); thus we have that only the cohomologies defined on (b) by extension from (c) in this way satisfy (A2). An example of such an extended equivariant cohomology is K^*_{Gcp} obtained from K^*_G - (see [1]) .

§4. In this section we give one further example of an equivariant cohomology theory and some of its properties.

D: If $H \subset G$ is a closed subgroup and X a G-space. Let $(H) = \{ gHg^{-1} \mid g \in G \}$ - this will be called an orbit type. Then if $x \in X$ and $G_x = \{ g \in G \mid g.x = x \}$ we say that $x \in X$ is of orbit type (G_x).

Define : $X_{(H)} = \{ x \in X \mid x \text{ is of orbit type } (H) \}$ and $\bar{X}_{(H)} = \Pi_X(X_{(H)})$.

D: A G-map $f : X \to Y$ is isovariant if for all closed subgroups $H \subset G$ $f(X_{(H)}) \subset Y_{(H)}$.

D: A G-pair (X,A) is a G-cofibration if given any G-space Z, and G-maps, f and F, such that

$$\begin{array}{ccccc} A\times 0 & & \subset & & A\times I \\ & & & \swarrow F & \\ \cap & & Z & & \cap \\ & f \nearrow & & & \\ X\times 0 & & \subset & & X\times I \end{array}$$

commutes, then there exists a G-map $H : X\times I \to Z$ which completes the commutative diagram.

4.1.1 : (see [4]) If G is a compact Lie group and $X \underset{\Pi_X}{\to} X/G$ a G-space then

(i) If $O \subset X$ is open (closed or compact) then
$GO = \{ g.x \mid g \in G , x \in O \}$ is also.

(ii) The topology of X/G is uniquely determined by the fact that Π_X is open and continuous

<u>T4.1.2</u>: (see [4])

If X and Y are G-spaces in (a) ;
Given $f_0 : X \to Y$ an isovariant map and $\tilde{F} : X/G \times I \to Y/G$ such that (i) For all closed subgroups $H \subset G$

$$\tilde{F}(\tilde{X}_{(H)} \times I) \subset \tilde{Y}_{(H)}$$

(ii) $$\Pi_Y \cdot f_0 = \tilde{f}_0 \cdot \Pi_X$$

then there exists an isovariant homotopy $f : X \times I \to Y$ of f_0 with induced map $\tilde{F}$.

<u>L4.1.3</u>: If (X, A) is a G-pair in (a) with $\tilde{X} = \Pi_X(X)$, $\tilde{A} = \Pi_X(A)$ and U an, open G-neighbourhood of A , then if there exists $\tilde{\varphi} : \Pi_X(U) \times I = \tilde{U} \times I \to \tilde{U}$ a strong deformation retraction of $\tilde{U}$ onto $\tilde{A}$ such that for all closed subgroups $H \subset G$

$$\tilde{\varphi}(\tilde{U}_{(H)} \times I) \subset \tilde{U}_{(H)}$$

then there exists $\tilde{H} : \tilde{X} \times I \to \tilde{A} \times I \cup \tilde{X} \times 0$, a retraction with $\tilde{H}(\tilde{X}_{(H)} \times I) \subset (\tilde{A} \times I \cup \tilde{X} \times 0)_{(H)}$.

<u>Proof</u>: X, and hence by L4.1.1 $\tilde{X}$, is normal ; thus there exists a function $\chi : \tilde{X} \to I$ which is zero on $\tilde{X} - \tilde{U}$ and one on $\tilde{A}$. $\tilde{H}$ is defined as follows :-

<u>$\chi(x) = 0$</u> $(x, t) \to (x, 0)$

$\underline{0 \leq \chi(x) \leq \frac{1}{2}}$ $(x,t) \to (\tilde{\varphi}(x, 2\chi(x).t), 0)$

$\underline{\frac{1}{2} \leq \chi(x) \leq 1}$ $(x,t) \to (\tilde{\varphi}(x, t/[1-\chi(x)].2), 0)$

if $0 \leq t \leq 2.(1-\chi(x))$

$(x,t) \to (\tilde{\varphi}(x,1), t - 2.[1-\chi(x)])$

if $2.(1-\chi(x)) \leq t \leq 1$

and $\underline{\chi(x) = 1}$ $(x,t) \to (\tilde{\varphi}(x,1), t)$.

$\tilde{H}$ fulfills the condition on orbit type since $\tilde{\varphi}$ does .

<u>L4.1.4</u>: (X,A) as in L4.1.3 is a G-cofibration pair .

<u>Proof</u>: We construct an isovariant retraction $H: X\times I \to A\times I \cup X\times 0$
By T4.1.2 $\tilde{H}$ lifts to H with $H_0 = 1_X$. If $Z = A\times I \cup X\times 0$ then $h = H|Z$ covers the identity on Z and for $z \in Z$ there is a compact G-neighbourhood (by L4.1.1), which is also Hausdorff, and is mapped bijectively onto itself by h ; hence h is a homeomorphism and $h^{-1}H$ is the required retraction .

<u>Remark</u>:

If U is a G-neighbourhood of A such that $U\times 0 \cup A\times I$ is a GANR then there exists a neighbourhood V and an equivariant retraction $\varphi: V\times I \to V\times 0 \cup A\times I$ and thus the construction of L4.1.3 implies that (X,A) is a G-cofibration.

Now if (X,A) is a G-cofibration pair in (a) ; (e.g. an orbit in a G-manifold or a pair satisfying the conditions of L4.1.3) and if '+' is the covariant " disjoint, G-invariant base point adding" functor then we have a coexact sequence of based G-maps

$$(A^+,*) \to (X^+,*) \to (X^+/A^+,*) \to (\Sigma A^+,*) \to (\Sigma X^+,*) \to \ldots$$

Hence for any G-space , L , in (a) we have a Puppe sequence of G-homotopy classes of maps – (as in [5] ,with suspended G-action on ΣA^+ etc.)

$$\ldots \to [\ \Sigma^n(X^+,*);L^+,*]_G \to [\ \Sigma^n(A^+,*);L^+,*]_G \to [\Sigma^{n-1}(X^+/A^+,*);L^+,*]_G$$

<u>Define</u>: $h_G^q(X,A) = \lim\limits_{\to n} [\ \Sigma^n(X^+/A^+,*);\Sigma^{n+q}(L^+,*)\ ]_G$

the direct limit of the suspension system; then the exactness of the Puppe sequence implies :-

<u>L.1.6</u>: $\{\ h_G^q(X,A)\ \}$ is an equivariant cohomology theory on the subcategory of G-cofibration pairs in (a) .

<u>Remarks</u>: (i) h_G^* clearly satisfies (A1) .

(ii) If we define a G-action on the loop space, ΩX^+ , by $(g.\omega)(t) = g.\omega(t)$ then there is a natural bijection between $[\ \Sigma X,\Sigma Y\ ]_G$ and $[\ X,\Omega\Sigma Y\ ]_G$. hence h_G^q is representable by the direct limit of $\Omega^n\Sigma^{n+q}(L^+)$

Now clearly $h_G^q(\ X\ ) \approx h_G^{q+m}(\ \Sigma^m X\ ) \quad (\ m \geq 0)$.

Thus consider $[\ \Sigma^n(X^+,*);\Sigma^{n+p}(L^+,*)]_G \otimes h_G^q(L)$ is isomorphic to $[\ \Sigma^n(X^+,*);\Sigma^{n+p}(L^+,*)]_G \otimes h_G^{n+p+q}(\ \Sigma^{n+p}L^+)$.

For $[f] \in [\ \Sigma^n X^+,\Sigma^{n+p}L^+]_G$ we get

$$f^* : h_G^{n+p+q}(\ \Sigma^{n+p}L\ ) \to h_G^{n+p+q}(\Sigma^n X) \approx h_G^{p+q}(X)$$

and hence $T_n : [\ \Sigma^n X^+,\Sigma^{n+p}L^+] \otimes h_G^q(L) \to h_G^{p+q}(X)$ which gives a map of direct systems,making $h_G^*(X)$ into an $h_G^*(L)$ module.

We now consider the continuity of the functor $[-,Y]_G$ – for convenience we omit the suffix –

Let (X,A) be a G-pair, $\mathfrak{B} = \{U\}$ be a family of open G-neighbhds of A and $B_U = [U,Y]$, then for $U \subset V$ we have $\varphi_{UV} : B_V \to B_U$ defined by $[f] \to [f|U]$. By definition the direct limit of this system is a collection $\{\psi_U : B_U \to B\}$ of morphisms in the category such that

(i)
$$\begin{array}{ccc} B_U & \xrightarrow{\psi_U} & \\ \downarrow \varphi_{VU} & & B \\ B_V & \xrightarrow{\psi_V} & \end{array}$$
is commutative.

(ii) Given a collection of morphisms $\{\chi_U : B_U \to \tilde{X}\}$ such that
$$\begin{array}{ccc} B_U & \xrightarrow{\chi_U} & \\ \downarrow \varphi_{VU} & & \tilde{X} \\ B_V & \xrightarrow{\chi_V} & \end{array}$$
is commutative then there exists a unique $\chi : B \to \tilde{X}$ such that $\chi \circ \psi_U = \chi_U$ for all $U \in \mathfrak{B}$.

Thus $\chi_U : B_U \to [A,Y]$; $[f] \to [f|A]$ defines $\chi : B \to [A,Y]$ which is

(1) an epimorphism if given $f : A \to Y$ there is $U \in \mathfrak{B}$ and $f_1 : U \to Y$ such that $[f_1|A] = [f]$.

(2) a monomorphism if given $f_1, f_2 : U \to Y$ such that $[f_1|A] = [f_2|A]$ then there exists $V \in \mathfrak{B}$ with $[f_1|V] = [f_2|V]$.

Now (1) will be satisfied for all Y, f if there is a G-retract $\lambda : U \times I \to U \times 0 \cup A \times I$ for some open G-neighbourhood U of A. But if A is compact (e.g. an orbit) then there exists a G-neighbourhood $V \subset U$ and a G-map constructed from λ – $V \times I \to A \times I \cup U \times \dot{I}$, which is the identity on $A\times I \cup V\times \dot{I}$ and hence (2) will hold also. Since these conditions are preserved by suspension we get continuity for $\varinjlim [\Sigma^n -, \Sigma^{n+q}L^+]_G$.

Alternatively we can impose conditions on L to ensure continuity:

D: A G-space, Y, is a G-absolute neighbourhood retract (GANR) if for any normal G-space, X, and closed G-subset, A, and $f: A \to Y$ there exists an extension of f to some G-map in a neighbourhood of A. (For details of properties and existence of GANR's see [4])

Suppose L to be a GANR, then (1) and (2) are immediate and since the conditions are preserved by suspension we have:-

$$
\begin{aligned}
&\varinjlim_{U\supset A} \{ \varinjlim_{n} [\Sigma^n U^+, \Sigma^{n+p} L^+]_G \} \\
\cong\ &\varinjlim_{n} \{ \varinjlim_{U} [\Sigma^n U^+, \Sigma^{n+p} L^+]_G \} \\
\cong\ &\varinjlim_{n} [\Sigma^n A^+, \Sigma^{n+p} L^+]_G \\
=\ &h^p_G(A) \quad \text{for an orbit } A \subset X .
\end{aligned}
$$

Finally we have (see [5] p290 proof)

L: If (X,A) is a pair of GANR's (e.g. an orbit in a GANR) then any equivariant cohomology theory is continuous about A.

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§1. Preliminaries: The functor , K_G .

Let G be a compact Lie group ; X , Y etc. compact Hausdorff G-spaces.

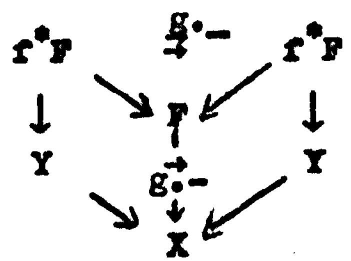
D:1.1 A G-vector-bundle (G.v.b.) over the G-space X consists of a complex vector bundle $p: E \to X$ together with a G-space structure on E such that

i) p is a G-map .

ii) If $g \in G$ $g: p^{-1}(x) \to p^{-1}(g.x)$ is a linear map .

D1.2: For two G.v.b's , E and F , over X a G-homomorphism $E \to F$ is a vector bundle homomorphism which is also a G-map.

Given a G.v.b. , $F \to X$, and a G-map $f: Y \to X$ the induced bundle, $f^*F \to Y$, can be given a unique G.v.b. structure such that the diagram



commutes for all $g \in G$.

Hence, in the usual way, the direct sum , $E \oplus F$, of two G.v.b's over X can be defined as a G.v.b. Also $E \otimes F$ can be given an obvious G.v.b structure , thus forming the Grothendieck ring of ismorphism classes of G.v.b.'s over X yields a contravariant functor , $K_G(X)$.

Alternatively we can use finite complexes of G.v.b's over X

$$0 \to E_0 \to E_1 \to \dots \to E_n \to 0$$

to define elements of $K_G(X)$ by assigning to such a complex the element $\Sigma\ (-1)^i E_i = ({}_i\Sigma_{even}\ E_i) - ({}_i\Sigma_{odd}\ E_i)$;

and under this realisation $K_G(X)$ is the ring of such complexes under the equivalence relation of G-isomorphism and addition of 'elementary' complexes of the form

$$0 \to P \xrightarrow{id} P \to 0 \quad .$$

K_G can be used to define a continuous equivariant cohomology theory by setting

$$K_G^{-n}(X,A) = K_G(\ X\times D^n, X\times S^{n-1} \cup A\times D^n\ )\ ,$$

where $K_G(Y,B)$ is the ring formed from complexes of G.v.b's over Y which are acyclic over B .

[For details of the properties of K_G see [2]] .

<u>D1.3</u>: If A and B are two contravariant functors defined on a category of G-spaces such that B takes its values in the category of rings then a natural transformation $f: A \to B$ will be called an <u>operation</u> .

Op(A,B) will denote the ring of operations from A to B and Op(B) the ring of self-operations of B .

If E,F are G.v.b's over X let Hom(E,F) denote the vector bundle of homomorphisms between E and F .Then Hom(E,F) becomes a G.v.b. by the action

$$(g.\varphi_x)(e) = g.\varphi_x(g^{-1}.e)$$

where $e \in E_{g.x}$; $x \in X$; $g \in G$ and $\varphi_x \in Hom\ (E_x, F_x)$.

Under this action the fixed points of Hom(E,F) are the G-homomorphisms $E \to F$.

Let H be another compact Lie group, X a trivial H-space and $E \to X$ a G×Hv.b. Let $\{ V_\pi \}$ be a complete set of irreducible complex representations of H and denote by $\underline{V}_\pi$ the G×Hv.b. $V_\pi \times X \to X$, with trivial G action on the fibres.

Then we have a G×H.v.b. homomorphism

$$\sum_\pi \underline{V}_\pi \otimes \mathrm{Hom}_H(\underline{V}_\pi, E) \to E$$

which, by Schur's lemma, is an isomorphism on the fibres. Hence this is an isomorphism of G×H.v.b's.

The G×H-isomorphism factors to produce the following:

L1.4: If X is a G×H space with trivial H action then there is a natural isomorphism

$$K_G(X) \otimes R(H) \to K_{G\times H}(X)$$

where $R(H) = K_H(pt)$ is the Grothendieck ring of complex representations of H.

Now take for H the permutation group S_k and let E be a G.v.b. over X, then $\otimes^k E = E \otimes E \otimes \ldots \otimes E$ (k times) has an obvious structure as a $G\times S_k$v.b. over X. Thus $E \to \otimes^k E$ gives rise to an operation $K_G(X) \to K_{G\times S_k}(X)$.

Furthermore, given any $\varphi \in \mathrm{Hom}(R(S_k), Z)$ we can define an element $T(\varphi) \in Op(K_G)$ by the composition

$$K_G(X) \to K_{G\times S_k}(X) \cong R(S_k) \otimes K_G(X) \underset{\varphi\otimes 1}{\to} K_G(X) .$$

Examples:

If $\varphi(V_\pi) = 1$ for the sign representation of S_k
$= 0$ otherwise

then $T(\varphi)(x) = \lambda^k(x)$, $x \in K_G(X)$.

If $\varphi(V_\pi) = 1$ for V_π the trivial one-dimensional representation

$= 0$ otherwise ,

then $T(\varphi)(x) = \sigma^k(x)$. If $G = 1$ λ^k and σ^k are the usual k-th exterior power and symmetric power operations. Similarly [as in [4]] we can define endomorphisms of K_G which correspond to the Adams operations , ψ^k .

As in K-theory the $\{ \lambda^k \}$ make $K_G(X)$ into a λ-ring (see [7]) and we can define operations

$$\gamma^k(X): K_G(X) \to K_G(X)$$

as the coefficient of t^k in the formal power series

$$\gamma_t = \lambda_{t/1-t} \quad , \text{ where } \lambda_s = \sum_0^\infty \lambda^k . s^k \ .$$

Hence, for example, if L is a G-line bundle and 1 is the trivial G-line bundle

$$\gamma_t([L] - 1) = \gamma_t([L]). \{ \gamma_t(1) \}^{-1}$$
$$= 1 + ([L] - 1)t \ .$$

§2: In this section we consider the representability of K_G in the case where G is a finite group.

For any compact Lie group and a G.v.b. $p: E \to X$ it is proved in [2] that there exists a finite dimensional representation, M, of G such that E over X is induced by a G-map of X into a Grassmannian of subspaces of M – $G(M)$ – by pulling back the canonical G.v.b. over $G(M)$.

Hence if G is finite and M_G is the standard $|G|$-dimensional representation any G.v.b. of dimension n can be induced by a G-map into a $G_n(M.M_G)$ – the Grassmannian of n-planes in ∞M_G.

D2·1: We denote by $\mathrm{rank}: K_G \to K_G$ the homomorphism obtained from the semi-rings homomorphism which assigns to a G.v.b $E \to X$ $[E'] \in K_G(X)$, where E' is the trivial G.v.b. over X such that $\dim E'_x = \dim E_x$ for all $x \in X$.

D2·2: Hence $\mathrm{rank}(X)$ is an idempotent ring endomorphism of $K_G(X)$ and we define $\tilde{K}_G(X) = \ker(\mathrm{rank}(X))$.

Therefore if X/G is connected $K_G(X) \cong \tilde{K}_G(X) \oplus Z$; and in general $K_G(X) \cong \tilde{K}_G(X) \oplus \mathrm{Map}(X/G, Z)$. Hence in order to represent the functor K_G it suffices to represent $\tilde{K}_G$ for G-spaces X for which X/G is connected.

Let X be such a G-space, then any element of $\tilde{K}_G(X)$ can be represented in the form $[\xi^{ng}] - nM_G$, where ξ^{ng} is an ng-dimensional G.v.b. over X and $g = |G|$.

Also the canonical inclusions of sums of the standard representation and their subspaces yields a G-homotopy-commutative diagram of Grassmannians and G-maps

$$\begin{array}{ccc} G_{ng}(mM_G) & \to & G_{ng}((m+1)M_G) \\ & \searrow & \downarrow \\ & & G_{(n+1)g}((m+1)M_G) \end{array}$$

if $n < m$.

Now let BU(G) be the direct limit of the 'cofinal' system of Grassmannians

$$\cdots \to G_{ng}(2nM_G) \to G_{(n+1)g}(2(n+1)M_G) \to \cdots\cdot$$

Then if $f: X \to G_{ng}(mM_G) \subset BU(G)$ is a G-map such that $f^*(\gamma^{ng}) \cong \xi^{ng}$ as G.v.b.'s ,where γ^{ng} is the canonical G.v.b. over the Grassmannian, we assign to the G-homotopy class of f the element $([\xi^{ng}] - nM_G) \in \tilde{K}_{G}(X)$.
As in [7] this sets up a bijection between

$$\tilde{K}_G(X) \quad \text{and} \quad [\ X, BU(G)\ ]_G \ .$$

Remark: In fact the G-maps

$$mM_G \times mM_G \to 2mM_G$$

$$[(y_1, y_2, \ldots, y_m):(z_1, z_2, \ldots, z_m)] \to (y_1, z_1, y_2, z_2, \ldots, z_m)$$

induce a G-map of direct systems which gives an addition in $[-,BU(G)]_G$,with an 'inverse' given by the map which takes an n-plane to its orthogonal complement with respect to a standard G-invariant norm (see [7]) .Hence we have,

L2.3: For a G-space X such that X/G is connected there is an isomorphism of abelian groups $\tilde{K}_G(X) \cong [\ X, BU(G)\ ]_G$ and for general X , $K_G(X) \cong [\ X, Z\times BU(G)\ ]_G$.

§3: In this and the following section G will be a finite group acting cellularly on finite CW complex.

D3·1: If X_p is the p-th skeleton of X we have a filtration on $K_G^n(X)$ given by $K^*_{G,p}(X) = \ker\{ K_G^*(X) \to K_G^*(X_{p-1}) \}$.

Remark: By the cellular approximation theorem this filtration is natural when G=1 and then K(X) is a filtered ring.

L3·2: If X is a finite CW complex on which G acts cellularly then

a) $K^0_{G,2k}(X) = K^0_{G,2k-1}(X)$,

b) $K^1_{G,2k}(X) = K^1_{G,2k+1}(X)$ and

c) $K^0_G(X) \oplus K^1_G(X)$ is a finitely generated R(G)-module.

Proof: Setting $H(p,q) = \sum_n K_G^n(X_{q-1}, X_{p-1})$ $(p<q)$ we obtain a strongly convergent spectral sequence

$$E_1^{p,q} = K_G^{p+q}(X_p, X_{p-1}) \to K_G^*(X).$$

Since the action is cellular

$$E_1^{p,q} = K_G^{p+q}\left(\coprod G/H \times (\sigma^p, \dot\sigma^p) \right)$$

where the disjoint union is taken over the orbit types in dimension p and where the action on $G/H \times (\sigma^p, \dot\sigma^p)$ is trivial on the second factor. Thus $E_1^{p,q}$ is a finite sum of terms of the form

$$\begin{aligned} & K_G^{p+q}(G/H \times (\sigma^p, \dot\sigma^p)) \\ \cong\ & K_H^{p+q}(\sigma^p, \dot\sigma^p) \\ \cong\ & R(H) \otimes \tilde K^{p+q}(S^p) \\ =\ & \begin{cases} 0 & \text{if } q \text{ is odd.} \\ \text{A finitely generated } R(G)\text{-module} & \text{if } q \text{ is even.} \end{cases} \end{aligned}$$

Hence (c) follows by standard Mod C theory [see [6]] and (a) and (b) from the fact that $E_r^{p,q} = 0$ if q is odd.

<u>Completions</u> : Let $M = M_0 \supset M_1 \supset M_2 \supset \ldots$ be a filtered abelian group, this filtration on M defines a topological group structure on M using the subgroups M_n as basic neighbourhoods of $0 \in M$.

<u>D3·3</u>: We denote by $M^\wedge$ the completion of M with respect to this topology ; i.e. $M^\wedge = \varprojlim_n M/M_n$.

If M is an R(G)-module let I(G) be the prime ideal defined by

$$I(G) = \ker\{ \operatorname{rank}(pt) : R(G) \to R(\mathbb{Z}) \}$$ then

$$M_n = M.I(G)^n$$

defines a filtration of the module M and $M^\wedge$ is called the completion of M with respect to the I(G)-adic topology .

Thus $M^\wedge$ is an $R(G)^\wedge$- module , and in particular this applies to $M = K_G(X)$.

<u>L3·4</u>: $\{ K_G^n(-)^\wedge \}$ is an equivariant cohomology theory on the category of finite CW complexes acted upon cellularly by G.

<u>Proof</u>: We have only to verify the existence of a long exact sequence for a G-pair (X,A) .This follows from L3·2 since I(G)-adic completion is an exact functor on finitely generated R(G)-modules (see [1])

§4: In this section we define various K-functors which allow us to deal with infinite CW complexes acted upon by a finite group.

Df 4.1: Define
$$k_G^{-n}(X,Y) = [\ \Sigma^n(X/Y), Z\times BU(G)\ ]_G$$
$$k^{-n}(X,Y) = [\ \Sigma^n(X/Y), Z\times BU\ ] \quad \text{and}$$
$$(\varprojlim K)^{-n}(X) = \varprojlim K^{-n}(X_p) \quad .$$

Let $k^* = k^0 \oplus k^1$ and similarly k_G^*, K_G^* and $(\varprojlim K)^*$.

Remark: (a) k_G^* (k^*) define cohomologies which extend K_G^* (K^*) .

(b) It is proved in [5] that if X is a CW complex with $H^*(X)$ finitely generated (f.g.) as a ring the homomorphism $k^*(X) \to (\varprojlim K)^*(X)$ is an isomorphism, and also that there exist two maps $(Z\times BU)\times(Z\times BU) \to Z\times BU$ which give the representable functors $[\Sigma^n -, Z\times BU]$ a ring structure corresponding to that in K^*.

Let $E_G \to B_G$ be the universal G-principal bundle, then E_G and B_G can be represented as CW complexes with finite skeletons, where G acts freely on E_G and $B_G = E_G/G$ (see [9]). Also $H^*(B_G)$ is f.g. Now if X is a finite CW complex with a cellular action and $X_G = (X\times E_G)/G = X\times_G E_G$ then X_G is a CW complex with finite skeletons and, from the Serre spectral sequence of the fibration $X \to X_G \to B_G$, $H^*(X_G)$ is f.g. Given a G.v.b. , V , over X we can form $V\times E_G$ over $X\times E_G$ and then $V\times_G E_G$ is a vector bundle over X_G .

This gives rise to a homomorphism

$$\alpha : K_G(X) \to (\varprojlim K)(X_G) \quad .$$

Furthermore we have the filtration on $(\varprojlim K)(X_G)$ given by

$(\varprojlim K)_n(X_G) = \ker\{ (\varprojlim K)(X_G) \to K(X_{G,n-1}) \}$ making $\varprojlim K$ into a filtered ring, which is complete with respect to this topology (see [1]).

But if $y \in K_G(X).I(G)$ then $\alpha(y) \in (\varprojlim K)_1(X_G) = (\varprojlim K)_2(X_G)$, by L3.2, and thus $\alpha(K_G(X).I(G)^n) \subset (\varprojlim K)_{2n}(X_G)$.

Hence α is continuous with respect to the $I(G)$-adic topology on K_G and the filtration topology on $\varprojlim K$ and we obtain a natural homomorphism

$$\alpha\hat{}(X): K_G(X)\hat{} \to (\varprojlim K)\hat{}(X_G) \cong (\varprojlim K)(X_G) \cong k(X_G) .$$

We will need the result that for G acting cellularly on a finite CW complex $\alpha\hat{}(X)$ is an isomorphism; although this has been proved more generally by Segal in [2] we give a short proof sufficient for our purposes as a corollary of the result for X = point.

<u>T4.2</u>: For X a finite CW complex with cellular action $\hat{\alpha}_G(X)$ is a topological isomorphism.

<u>Proof</u>: By Atiyah's theorem $\hat{\alpha}_G(pt): R(G)\hat{} \to k(B_G)$ is an isomorphism for all finite groups G.

For $H \subset G$ we have a commutative diagram

$$\begin{array}{ccc} K_G(G/H) & \xrightarrow{\hat{\alpha}_G(G/H)} & k(G/H \times_G E_G) \\ \downarrow & & \downarrow \\ R(H) & \xrightarrow{\hat{\alpha}_H(pt)} & k(B_H) \end{array}$$

where the vertical maps are isomorphisms, the right one being induced by a homotopy equivalence (for the left one see [2]).

Furthermore the $I(H)$-adic topology on $R(H)$ coincides with that induced from the $I(G)$-adic topology on $K_G(G/H)$. Thus all the homomorphisms are continuous and on completion we see that $\alpha_G^{\wedge}(G/H)$ is an isomorphism.

Hence we have a natural transformation between the cohomology theories $K_G^{*\wedge}$ and $k^*(-\times_G E_G)$ which is an isomorphism on all G/H's and in the proof of PartI T3·1·6 taking G-coverings of X which are finite CW complexes (cofinal in Cov[X]) the result follows.

Cor4·3: Under the conditions of the theorem the topology induced by α from the filtration topology coincides with the I(G)-adic topology on K_G.

Remark: As $\alpha_G(X)$ commutes with the rank endomorphisms on K_G and k we have $\alpha_G^{\wedge}(X): \tilde{K}_G^{\wedge}(X) \xrightarrow{\cong} \tilde{k}(X_G)$

Representability of $K_G^{\wedge}$

L4·4: (see [10] p.6)Let X,Y,Z be topological spaces such that X is Hausdorff and locally compact and Z is Hausdorff. Then ψ:Map(Z,Map(X,Y)) $\to$ Map(Z×X,Y)

$\psi(f) = e\circ(f\times 1)$,where e is the evaluation map,

is a homeomorphism.

Now let $Z = BU^{EG} = \mathrm{Map}(E_G,BU)$ with the inverse limit topology defined by $\{ BU^{EG} \to BU^{EG_n} \}$, Y=BU and $X = E_{G,n}$.

Then ψ sends G-maps to G-maps and we have

$$\psi: \mathrm{Map}_G(BU^{EG},\mathrm{Map}(E_{G,n},BU)) \to \mathrm{Map}_G(BU^{EG}\times E_{G,n},BU)$$

$$\downarrow \cong$$

$$\mathrm{Map}(BU^{EG}\times_G E_{G,n},BU)$$

Since Ψ respects homotopy classes we obtain a bijection between $[\ BU^{EG}, BU^{EGn}]_G$ and $[\ BU^{EG}\times_G E_{G,n}, BU\ ]$.

D4.5: Define $\hat{K}_G(X) = [\ X, BU^{EG}\ ]_G$ and

$$\hat{k}_G(X) = [\ X, Z\times BU^{EG}]_G\ .$$

Since if X is a finite CW complex we have

$$[\ X\times_G E_G, BU] \cong \varprojlim [\ X\times_G E_{G,n}, BU] \quad \text{(see [5])}$$
$$\cong [\ X, BU^{EG}]_G$$

T4.2 implies that these functors are representable extensions of $\hat{K}_G$ and $\hat{K}_G$.

Remark: Since E_G is contractible to a point, $e_0 \in E_G$, with the inverse limit topology on BU^{EG} the inclusion $BU = BU^{e_0} \subset BU^{EG}$ is an ordinary homotopy equivalence.

§5: Operations:

Since $k_G^\wedge$ is representable as a ring the Yoneda theorem (see [7]) implies that there are ring isomorphisms

$$\text{(A)} \quad \begin{aligned} Op(\tilde{K}_G^\wedge, k_G^\wedge) &\cong k_G^\wedge(BU^{EG}) \quad \text{and} \\ Op(\tilde{K}_G, k_G^\wedge) &\cong k_G^\wedge(BU(G)) . \end{aligned}$$

Furthermore to determine $Op(k_G^\wedge)$ { respectively $Op(k_G, k_G^\wedge)$ } it suffices to know (A) in view of the splitting

$$Op(k_G^\wedge) \cong Map(Z,Z) \oplus Op(\tilde{K}_G^\wedge, k_G^\wedge) \quad \text{and}$$

$$Op(k_G, k_G^\wedge) \cong Map(Z,Z) \oplus Op(\tilde{K}_G, k_G^\wedge) .$$

Similarly $Op(K_G)$ splits in this way ; we now calculate the first two of the operations rings.

T5·1: If X is a finite dimensional CW complex and $k^*(Y)$ is torsion free then

$$\begin{aligned} \mu : k^*(X) \otimes k^*(Y) &\to k^*(X \times Y) \\ a \otimes b &\to \pi_1^*(a).\pi_2^*(b) \end{aligned} \quad , \pi_i \text{ projections ,}$$

is an isomorphism .

Proof: Since $k^*(Y)$ is torsion free

$$\{ k^*(-) \otimes k^*(Y) \}^i = \bigoplus_{j+k \equiv i (\text{mod } 2)} k^j(-) \otimes k^k(Y)$$

is a continuous cohomology theory ,as is $k^*(- \times Y)$ .Hence the result follows from PartI T3·1·6,as $\mu(pt)$ is an isomorphism .

Now if X is a G-space and $\pi : E_G \to B_G$ the universal G-principal bundle then the G-principal bundle $X \times E_G \to X \times_G E_G$ is classified by the commutative diagram of fibre bundles

$$\begin{array}{ccc} X \times E_G & \xrightarrow{p_2} & E_G \\ \downarrow & & \downarrow \pi \\ X \times_G E_G & \to & B_G \end{array}$$

and since E_G is contractible we have a fibre bundle

$$X \to X_G \to B_G \quad .$$

For the fibre bundle $BU^{EG} \to BU^{EG}\times_G E_G \to B_G$ we define a ring homomorphism $\theta : k(BU) \to k(BU^{EG}\times_G E_G)$ as follows ;

A map $f : BU \to Z\times BU$ determines $\tilde{f} : BU^{EG} \to Z\times BU^{EG}$ and hence

$$\bar{f} : BU^{EG}\times_G E_G \to Z\times BU \quad ; \quad \bar{f}[h,e] = \tilde{f}.h(e) = f(h(e))$$

$$(h\in BU^{EG} , e\in E_G) .$$

Furthermore if $\alpha : (Z\times BU)\times(Z\times BU) \to Z\times BU$ is a map

$$\overline{\alpha(f\times g)}\Delta[h,e]$$

$$= \alpha(f\times g)(h(e),h(e)) \qquad (\Delta \text{ diagonal map })$$

$$= \alpha(f(h(e)),g(h(e)))$$

$$= \alpha(\bar{f}\times\bar{g})\Delta([h,e])$$

and since $k(-)$ is representable as a ring we have that θ is a ring homomorphism .

Remark: θ also defines a homomorphism for the restricted bundle over $B_{G,n}$ such that the composition

$$k^*(BU) \to k^*(BU^{EG}\times_G E_{G,n}) \to k^*(\pi_n^{-1}(b)) \text{ is an isomorphism}$$

for all $b\in B_{G,n}$, since $k^1(BU) = 0$, where $\pi_n : BU^{EG}\times_G E_{G,n} \to B_{G,n}$ is the projection map of the bundle .

T5.2: For the bundle $BU \to BU^{EG}\times_G E_{G,n} \to B_{G,n}$ there is an isomorphism $k^*(B_{G,n})\otimes k^*(BU) \to k^*(BU^{EG}\times_G E_{G,n})$ given by

$$a \otimes b \quad \to \quad \pi_n^*(a).\theta(b)$$

Proof: $k^*(BU)$ is free and hence this is an isomorphism over any open set $U \subset B_{G,n}$ over which the bundle is trivial. The result now follows by a Mayer-Victoris argument since $B_{G,n}$ is compact .

<u>L5·3</u>: If $\{\psi_n : A_n \to A_{n-1}\}$ is an inverse system of abelian groups and B is free abelian then

i) $\chi : (\varprojlim A_n) \otimes B \to \varprojlim(A_n \otimes B)$ is a monomorphism and

ii) Given $x \in \varprojlim(A_n \otimes B)$ and an integer q there exists $y \in (\varprojlim A_n) \otimes B$ such that $\chi(y)$ and x are equal in $A_p \otimes B$ for all $p \leq q$.

<u>Proof</u>: i) We have a commutative diagram

$$\begin{array}{ccccc} (\varprojlim A_n)\otimes B & & \to & & \varprojlim(A_n\otimes B) \\ & \searrow & & \swarrow & \\ & & A_n\otimes B & & \\ & \searrow & \downarrow \psi_n\otimes 1 & \swarrow & \\ & & A_{n-1}\otimes B & & \end{array}$$

Let $\{h_\alpha\}$ be a free basis for B, $\alpha \in \underline{A}$. Then if $\chi(\sum_{\alpha\in S} a_\alpha \otimes h_\alpha) = 0$, for some finite subset $S \subset \underline{A}$ and $a_\alpha \in \varprojlim A_n$, $\sum_{\alpha\in S} a_\alpha \otimes h_\alpha = 0$ in all $A_n \otimes B$.

Now for $\alpha_1 \in \underline{A}$ we define the augmentation $\varepsilon_{\alpha_1} : B \to Z$ by

$$\varepsilon_{\alpha_1}(h_{\alpha_1}) = 1 \; ; \; \varepsilon_{\alpha_1}(h_\alpha) = 0 \text{ otherwise} .$$

This yields a commutative diagram

$$\begin{array}{ccccc} (\varprojlim A_n)\otimes B & & \overset{\rightarrow}{\chi} & & \varprojlim(A_n\otimes B) \\ 1\otimes\varepsilon_{\alpha_1} & \searrow & & \swarrow & \varprojlim(1\otimes\varepsilon_{\alpha_1}) \\ & & \varprojlim A_n & & \end{array}$$

Hence if $\chi(\sum_{\alpha\in S} a_\alpha \otimes h_\alpha) = 0$ then $a_\alpha = 0$ for all $\alpha \in S$.

ii) Let $x = \sum_{\alpha\in S} x_\alpha \otimes h_\alpha$ in $A_q \otimes B$ then define

$$y = \sum_{\alpha\in S} (\varprojlim 1\otimes\varepsilon_\alpha)(x)\otimes h_\alpha .$$

<u>D5·4</u>: An inverse system of abelian groups $\cdots \to A_n \to A_{n-1} \to \cdots$ is said to satisfy the Mittag-Leffler condition (M.L.) if for any integer n there exists m(n) such that $\operatorname{im}\{A_m \to A_n\} = \operatorname{im}\{A_{m(n)} \to A_n\}$ for all $m \geq m(n)$.

It is shown in [1] that $\{k^*(B_{G,n})\} = \{K^*(B_{G,n})\} = A_n$ satisfies M.L. and that $\varprojlim$ is an exact functor on inverse systems having M.L. Since $k^*(BU)$ is free the inverse systems $\{k^*(BU)\otimes k^*(B_{G,n})\}$ and $\{k^*(BU^{EG}\times_G E_{G,n})\}$ satisfy M.L. and the first term in the short exact sequence of [8] ,

$$0 \to R^1\varprojlim k^*(BU^{EG}\times_G E_{G,n}) \to k^*(BU^{EG}\times_G E_G) \to \varprojlim k^*(BU^{EG}\times_G E_{G,n}) \to 0$$

the derived functor, is zero .

Now let $A = \varprojlim k^*(B_{G,n}) \cong k^*(B_G) \cong R(G)^\wedge$ have the topology defined by the filtration ${}_nA = \ker\{\, k^*(B_G) \to k^*(B_{G,n-1})\,\}$ and $B = k^*(BU)$ have the ~~indiscrete~~ discrete topology .

<u>D5·5</u>: Define $A\hat{\otimes}B = \varprojlim A\otimes B/{}_nA\otimes B$, the completed tensor product of A and B with respect to these topologies .

Let $A_n = k^*(B_{G,n})$ and $\varprojlim(A_n\otimes B)$ have the obvious topology, with respect to which it is complete , by L5·3 we have a continuous monomorphism $\chi : A\otimes B \to \varprojlim(A_n\otimes B)$.

Clearly $\{A\otimes B/{}_nA\otimes B\}$ satisfies M.L. , as does $\{A_n\otimes B\}$, therefore we have a continuous monomorphism

$$\chi^\wedge : A\hat{\otimes}B \to \varprojlim(A_n\otimes B) \; .$$

<u>T5·6</u>: $\chi^\wedge$ is a topological isomorphism .

<u>Proof</u>: Define $\tilde{A} = \tilde{K}^*(B_G) \cong I(G)^\wedge$ and $\tilde{B} = \tilde{K}^*(BU)$ then from the splitting $A = Z\oplus\tilde{A}$, $B = Z\oplus\tilde{B}$ it suffices to prove that

$$\tilde{A}\hat{\otimes}\tilde{B} \to \varprojlim(\tilde{A}_n\otimes\tilde{B})$$

is a topological isomorphism . From [1] we know that $\tilde{A}$ is a compact Hausdorff group and since $\tilde{B}$ has the indiscrete topology so is $\tilde{A}\hat{\otimes}\tilde{B}$. Also $\varprojlim(\tilde{A}_n\otimes\tilde{B})$ is compact, Hausdorff thus we have a topological monomorphism of one compact Hausdorff group into another whose image is dense, hence this must be a homeomorphism.

Cor5·7: $Op(\tilde{K}_G^{\wedge}, k_G^{\wedge}) \cong R(G)^{\wedge} \otimes Op(\tilde{K}, k)$.

The homomorphism $\lambda : \tilde{K}_G \to \tilde{K}_G^{\wedge}$:

For any finite Grassmannian $G_{ng}(mM_G) \subset BU(G)$ the canonical G.v.b. corresponds to the homotopy class of the inclusion, thus $\gamma^{ng} \times E_G$ corresponds to the map

$$G_{ng}(mM_G) \times E_G \xrightarrow{ip_1} BU(G) \quad .$$

Now λ sends γ^{ng} to $\gamma^{ng} \times_G E_G$ which corresponds to an ordinary map $\bar{\lambda} : G_{ng}(mM_G) \times_G E_G \to BU$, but as ordinary vector bundles $\pi^*(\gamma^{ng} \times_G E_G) \cong \gamma^{ng} \times E_G$, where π is the map $G_{ng}(mM_G) \times E_G \to G_{ng}(mM_G) \times_G E_G$. Thus we have a homotopy commutative diagram of ordinary maps

$$\begin{array}{ccc} BU(G) \times E_G & \xrightarrow{p_1} & BU(G) \\ \pi \downarrow & & \| \\ BU(G) \times_G E_G & \xrightarrow{\bar{\lambda}} & BU \end{array}$$

where $\bar{\lambda}$ is the adjoint of the G-map $BU(G) \to BU^{EG}$ which induces the homomorphism of representable functors $\tilde{K}_G \to \tilde{K}_G^{\wedge}$. Hence we have a diagram of a fibre homotopy equivalence between bundles:

$$\begin{array}{ccc} BU(G) \times E_G & \xrightarrow{1} & BU \times E_G \\ \pi \downarrow & & \downarrow \\ BU(G) \times_G E_G & \xrightarrow{\bar{\lambda} \times \pi} & BU \times B_G \\ & \searrow \quad B_G \quad \swarrow & \end{array}$$

and as in T5·6 and Cor5·7 we have

<u>T5·8</u>: $Op(\tilde{K}_G, k_G^{\wedge}) \cong R(G)^{\wedge} \otimes Op(\tilde{K}, k)$ and there is a commutative diagram of isomorphisms

$$
\begin{array}{ccc}
 & & Op(\tilde{K}_G, k_G^{\wedge}) \\
 & \nearrow & \\
R(G)^{\wedge} \otimes Op(\tilde{K}, k) & & \downarrow (-.\lambda) \\
 & \searrow & \\
 & & Op(\tilde{K}_{\hat{G}}, k_{\hat{G}}^{\wedge})
\end{array}
$$

---

§6: In this section G will be a finite group acting on compact Hausdorff spaces. If $Vect_G(X)$ denotes the additive semi-group of G.v.b.'s over X we denote by $\tilde{S}_G(X)$ the image in $\tilde{K}_G(X)$ of the semi-group homomorphism $(1 - \text{rank}): Vect_G(X) \to \tilde{K}_G(X)$. This is an 'intermediate' functor between $\tilde{K}$ and $\tilde{K}_G$ in the sense that we 'lose' operations in passing from $Op(\tilde{K}, K)$ to $Op(\tilde{S}_G, K_G)$ and 'lose' still more in $Op(\tilde{K}_G, K_G)$.

If $E \to X$ is an n-dimensional G.v.b. we define the <u>flag bundle</u> of E, $F(E)$, as follows:- let P be the principal bundle with fibre $GL(n;C)$ associated with E, then $F(E)$ is obtained from P by dividing out by the action of the triangular matrices of $GL(n;C)$. The action of G on E gives a G-space structure to $F(E)$ such that the projection $\pi: F(E) \to X$ is a G-map. Furthermore $\pi^*(E)$ over $F(E)$ is a sum of G-line-bundles and in [2] it is shown that

<u>L6·1</u>: There exists a natural homomorphism $\pi_*: K_G(F(E)) \to K_G(X)$ such that $\pi_*\pi^* = 1$. Hence π^* is a monomorphism.

Thus if $f, g \in Op(K_G)$ coincide on all K_G classes of the form ([sum of G-line-bundles] − [representation]) then $f=g$; also from §2 we know that a sum of G-line-bundles can be induced, as a pullback of a sum of canonical line bundles, by some G-map into a product of projective spaces, $CP(mM_G)$, constructed from sums of the standard representation.

<u>D6·2</u>: For a representation, M, of G define

$$R(M)(t) = \sum_{0}^{\dim M} (-1)^i \lambda^i(M)(t+1)^{\dim M - i} \in R(G)[t] .$$

Notice that if M, N are representations of G $R(M \oplus N)(t) = R(M)(t)R(N)(t)$

Let $R(t) = R(M_G)(t)$, then $R(t)$ has no 'constant' term for the trivial representation 1 in M_G gives a differential which makes the complex

$$1\wedge : 0 \to \lambda^0(M_G) \to \lambda^1(M_G) \to \cdots \to \lambda^g(M_G) \to 0 \qquad (g=|G|)$$

acyclic.

Thus if σ^i denotes the i-th elementary symmetric function $\sigma^i(R(t_1),\ldots,R(t_k))$ ($i<k$) is a symmetric polynomial in $R(G)[t_1,\ldots,t_k]$ and we can write

$$\sigma^i(R(t_1),\ldots,R(t_k)) = p_k^i(\sigma^1(t_1\ldots t_k),\ldots,\sigma^k(t_1,\ldots,t_k))$$

where $p_k^i \in R(G)[s_1,\ldots,s_k]$ and $p_k^i|s_k=0 \equiv p_{k-1}^i$.

D6·3: We shall also denote by p_k^i the element of $Op(\tilde{S}_G,K_G)$ defined by $p_k^i(\gamma^1,\ldots,\gamma^k)$ (for γ^i see §1) ; and the weight of a monomial $p_k^I = (p_k^{i_1})^{a_1}\ldots(p_k^{i_l})^{a_l}$ is defined as

$$w(I) = \Sigma\ i_j a_j .$$

T6·4: $Op(\tilde{S}_G,K_G)$ is isomorphic to the subring of the ring of formal polynomials in the $\{\gamma^i\}$ which is $\varprojlim_n R(G)[\gamma^1,\ldots,\gamma^n]^\wedge_{p_n}$, where the completions $[\ ]^\wedge_{p_n}$ corresponds to the filtration on $R(G)[\gamma^1\ldots\gamma^n]$ given by the ideals I_l generated by $\{ p_n^I | w(I) > l \}$.

We interpret and prove T6·4 in a series of lemmas.

L6·5: If R is a commutative ring, with 1, and $z_1,z_2,\ldots,z_k;t_1,\ldots,t_k$ are indeterminates then any polynomial $f(\underline{z},\underline{t}) \in R[\underline{z},\underline{t}]$ of degree m which is invariant under the diagonal action of S_k can be expressed as a polynomial in the $\sigma^i(\underline{z})$ and $\sigma^j(\underline{t})$ ($i,j<k$).

Proof: Assume the result for S_p with $p<k$ and for S_k with $\deg(f)<m$. Then $f(\underline{z},\underline{t})|_{(z_k=0=t_k)} = g(\tilde{\sigma}^1(\underline{z}),\ldots,\tilde{\sigma}^{k-1}(\underline{z});\ldots\tilde{\sigma}^{k-1}(\underline{t}))$ where $\tilde{\sigma}^i(\underline{z}) = \sigma^i(\underline{z})|_{(z_k=0)}$ and $\tilde{\sigma}^j(\underline{t}) = \sigma^j(\underline{t})|_{(t_k=0)}$.

Hence every term in $f(\underline{z},\underline{t}) - g(\sigma^1(\underline{z}),\ldots,\sigma^{k-1}(\underline{z});\ldots,\sigma^{k-1}(\underline{t}))$

is divisible by $\sigma^k(\underline{z})$ or $\sigma^k(\underline{t})$ or both and the result follows by induction on $\deg(f)$.

Cor6.6: If $R=R(G)$ any invariant sum, in L6-5, of degree n in $\underline{t}$ can be expressed in a sum of monomials of weight n in the $\sigma^i(\underline{t})$.

D6.7: Let
$$C_n^m = CP(mM_G)\times\ldots\times CP(mM_G) \quad (\text{ n times }) \text{ and}$$
$$CP_n^m = CP^{m-1}\times\ldots\times CP^{m-1}.$$

Then we have $CP_n^m \subset C_n^m$ and using a G-invariant base point corresponding to the trivial one dimensional representation we have a system of G-spaces and maps for $\{C_n^m\}$ and $\{CP_n^m\}$ which give inverse systems of K_G rings.

Now $K_G(C_n^m) \cong (R(G)[z_1,\ldots,z_n])/I$ (see appendix) where I is the ideal generated by $\{R(z_i)^m\}$ ($i=1,\ldots,n$) and $z_i \in K_G(C_1^m)$ is the class of ([canonical line] - 1).

Similarly $K_G(CP_n^m) \cong (R(G)[z_1,\ldots,z_n])/I$ where I in this case is generated by $\{z_i^m\}$ ($i=1,\ldots,n$).

Furthermore the homomorphisms of the inverse system are

$$K_G(C_n^m) \to K_G(C_{n-1}^m) \;:\; z_i \to z_i \; (i\neq n), \quad z_n \to 0.$$
$$K_G(C_n^m) \to K_G(C_n^{m-1}) \;:\; z_i \to z_i .$$

Similarly for CP_n^m.

L6.8: Given $x\in \tilde{S}_G(X)$ there exists an integer N such that if $w(I)>N$ $p_k^I(x) = 0\in K_G(X)$.

Proof: There exists a 'splitting map' $f:F \to X$ such that $f^*:\tilde{S}_G(X) \to \tilde{S}_G(F)$ is a monomorphism and $f^*(x) = g^*(z_1+\ldots+z_n)$ for some $g:F \to C_n^m$. Since $\gamma_t(z_1+\ldots+z_n) = \Pi(1+z_i t)$ we have $\gamma^i(z_1+\ldots+z_n) = \sigma^i(z_1,\ldots,z_n)$ and by definition of p^i if $w(I)>nm$ the $p^I(z_1+\ldots+z_n) = 0$ and $p^I f^*(x) = g^*(p^I(z_1+\ldots+z_n)) = 0$.

This gives a meaning as operations to the elements of the ring in the statement of T6·4 .

Proof of T6·4: Consider the homomorphism

$$\varphi ; Op(\bar{S}_G, K_G) \rightarrow \varprojlim K_G(C_n^m)$$

$$\varphi(f) = \varprojlim f(z_1 + \dots + z_n) .$$

By remark 6·1 this hommomorphism is injective .Hence to complete the proof it suffices to identify the image of φ as that of φ restricted to the operations given in the statement .

Now there exist G-maps $C_n^m \rightarrow C_n^m$ permuting the z_i, hence $\varphi(f)$ is symmetric in the z_i and the inclusions $CP_n^m \subset C_n^m$ induce a commutative diagram

$$\begin{array}{ccc} Op(\bar{S}_G, K_G) & \xrightarrow{\varphi} & \varprojlim_n \{ \varprojlim_m R(G)[z_1,\dots,z_n]/I_m \}^{S_n} = A \\ & \tilde{\varphi} \searrow & \downarrow \\ & & \varprojlim_n \{ \varprojlim_m R(G)[z_1,\dots,z_n]/\{z_i^m\} \}^{S_n} = B \end{array}$$

where $\{ \ \}^{S_n}$ means invariant under the action of S_n .As in [3] we know that $B \cong R(G)[[\sigma]]$, the formal power series ring in the elementary symmetric function, hence A is the subring of formal power series in B which are 'continuous' with respect to the filtration on A .

Now in $R(G)[z_1,\dots,z_n]$ $\varphi(f_n^m) - \varphi(f_n^{m-1})$ is a symmetric polynomial in $z_1,\dots,z_n; R(z_1),\dots,R(z_n)$ in which every term is of degree m in at least one $R(z_i)$; putting $R(z_i)$ as t_i in L6·5 we see that

$\varphi(f_n^m) - \varphi(f_n^{m-1}) = g(\sigma^1(\underline{z}),\dots,\sigma^n(\underline{R(z)}))$ where every term in g has weight between m and mn in the $\sigma^i(\underline{R(z)})$ .Hence the result follows from $\gamma^i(z_1+\dots+z_n) = \sigma^i(z_1,\dots,z_n)$.

Hence if $G=1$ $Op(\bar{K},K) = Op(\bar{k},K) \cong Z[[\gamma]]$, the ring of formal power series in them γ^i's and by 'losing' operations as G gets bigger we mean that $Op(\bar{S}_G,K_G)$ is smaller than $R(G)[[\gamma]]$, for the following results show that there exist power series which do not respect the filtration of A if $G \neq 1$.

L6·9: If for all representations of G $\Lambda_{-1}(E) = \Sigma(-1)^i \lambda^i(E) = 0$ in $R(G)$ then $G = 1$.

Proof: Take E such that $E \oplus 1 = M_G$ then the 'character' of $\Lambda_{-1}(E)$ is $g \to \det(g_E - 1)$ $(g \in G)$ where g_E is the endomorphism of E given by g. Since $(g_E - 1)$ is non-singular $\Lambda_{-1}(E) \neq 0$.

Cor6·10: For this representation of G $(\Lambda_{-1}(E))^n = \Lambda_{-1}(nE) \neq 0$.

L6·11: For $G \neq 1$

i) $z^n \neq 0$ (mod $R(\emptyset z)$).

ii) For $z_1 + z_2 + \dots + z_n \in \bar{K}_G(C^1_n)$ $(\gamma^{i_1})^{\alpha_1} \dots (\gamma^{i_k})^{\alpha_k}(z_1 + \dots) \neq 0$

if $i_j < n$.

Proof: i) By Cor6·10 the relation in $K_G(C^1_1)$ never reduces to z^n.

ii) If $(\sigma^{i_1}(\underline{z}))^{\alpha_1} \dots (\sigma^{i_k}(\underline{z}))^{\alpha_k} = \emptyset$ (mod $R(z_1)$) then putting $z_i = z$ for all i implies that $pz^{w(I)} = 0$ (mod $R(\emptyset z)$) where p is a non-zero integer. Since $R(G)$ is torsion free this contradicts (i).

Hence if $G \neq 1$, $\sum_n (\gamma^1)^n \in B - A$.

Remark: Notice that if $\alpha \in R(G)$ is a one-dimensional representation $f_t = \lambda_{t/(1-\alpha t)}$ defines operations $\{f^i\}$ with the property that if $\{y_j\}$ are G-line-bundles

$$f^i\left(\sum_1^k (y_j - \alpha)\right) = \sigma^i(y_1 - \alpha, \dots, y_k - \alpha).$$

Hence for the semigroup $S_{G(\alpha)}$

which is the image of $1 - \alpha$.rank $: Vect_G \to \tilde{K}_G$ we have, as in T6.4,

$$Op(S_{G(\alpha)}, K_G) \cong \varprojlim_n R(G)[f^1,\dots,f^n]^{\wedge}_{p_n(\alpha)} \subset R(G)[[\gamma]];$$

where the $p^1_{k(\alpha)}(f^1,\dots,f^k)$ is defined by expressing R(t) in terms of $s = t + 1 - \alpha$ and taking elementary symmetric functions of this polynomial .

We observe that for $\alpha \neq 1$ $\Sigma(f^1_{\alpha})^n \in Op(\tilde{S}_{G(\alpha)}, K_G)$
$\notin Op(S_G, K_G)$.

If G is an abelian group with irreducible representations $\alpha_1,\dots,\alpha_g$, we know the operations rings associated with the $S_{G(\alpha_i)}$ and one might expect $Op(\tilde{K}_G, K_G)$ to be the intersection of these rings; however this intersection is not in general explicitly identifiable and another method for examining $Op(K_G)$ is required. The difficulty in calculating $Op(\tilde{K}_G, K_G)$ explicitly by considering its effect on canonical elements over products of projective spaces arises from the existence of non-nilpotent elements in $\tilde{K}_G$ which cannot lie in the image of a γ-power series if the filtration on $R(G)[\gamma]$ is strictly by weights .

Now $K_G(G_{ng}(2nM_G))$ is a quotient ring of
$R(G)[\gamma^1(F-nM_G),\dots,\gamma^{ng}(F-nM_G)]$ where F is the canonical bundle . Also, by §2, we know that

$$\begin{array}{ccc} Op(\tilde{K}_G, K_G) & \to & \varprojlim K_G(G_{ng}(2nM_G)) \\ f & \to & \varprojlim f(F-nM_G) \end{array}$$

is a monomorphism . Hence if we define a filtration on $R(G)[\gamma]$

$$A_n = \ker \{ p \to p(F-nM_G) \}$$

we see that

$$Op(\bar{K}_G, K_G) \cong R(G)[\gamma]_{\{\hat{A}_n\}} \cong \varprojlim K_G(G_{ng}(2nM_G)) .$$

However, this filtration on $R(G)[\gamma]$ is not explicit enough to enable us to relate $Op(K_G)$ and $Op(k_G^{\wedge})$, to do this we examine the homomorphism

$$\mu : Op(\bar{k}_G ; k_G) \cong k_G(BU(G)) \to \varprojlim K_G(G_{ng}(2nM_G)) \cong Op(\bar{K}_G, K_G).$$

By a method similar to that in [8], we show that μ is an isomorphism .

Consider the space $X = \coprod_n G_{ng}(2nM_G) \times [n-1, n] \subset BU(G) \times [0, \infty)$, which is G-homotopy equivalent to $BU(G)$. We also write X for X with an additional disjoint, G-invariant base point which is considered to be the base point for $X_n = G_{ng}(2nM_G) \coprod (pt)$; and $X_n \times [n-1, n]$ has base point $(pt) \times [n-1, n]$.

Now let

$$Y_1 = \coprod_{i \text{ odd}} X_i \times [i-1, i]$$

$$Y_2 = \coprod_{i \text{ even}} X_i \times [i-1, i] ,$$

$$Y_1 \cap Y_2 = \coprod X_i \times \{i\} \quad \text{and} \quad Y_1 \cup Y_2 = X .$$

Define $Z = Y_1 \times \{0\} \cup \dfrac{(Y_1 \cap Y_2) \times I}{(pt) \times I} \cup Y_2 \times \{1\}$ then

$Y_1 \vee Y_2 \subset Z$ is a G-cofibration ; hence we have a coexact sequence of based G-maps

$$Y_1 \vee Y_2 \subset Z \to Z/Y_1 \vee Y_2 \underset{G}{\approx} \Sigma(Y_1 \cap Y_2) \to (\Sigma Y_1) \vee (\Sigma Y_2) \to \cdots$$

Furthermore we have a G-homeomorphism

$q : X \to Z$ defined by

$(2i-2 < s < 2i-3/2)$

$$q(x_{2i-1},s) = (x_{2i-1}, 2s-2i+2, 0)$$

$(2i-3/2 < s < 2i-1)$

$$q(x_{2i-1},s) = (x_{2i-1}, 2i-1, 2(s-2i+3/2))$$

$(2i-1 < s < 2i-\frac{1}{2})$

$$q(x_{2i},s) = (x_{2i}, 2(s-2i+1), 1)$$

$(2i-\frac{1}{2} < s < 2i)$

$$q(x_{2i},s) = (x_{2i}, 2i, 1-2(s-2i+\tfrac{1}{2})) .$$

Hence we obtain an exact sequence (for $n > 0$)

$$\cdots \to k_G^{-n}(X) \xrightarrow[\lambda_{-n}]{} \prod_m k_G^{-n}(X_m) \xrightarrow[\psi_{-n}]{} \prod_m k_G^{-n}(X_m) \xrightarrow[\chi_{-n}]{} k_G^{-n+1}(X) \xrightarrow[\lambda_{-n+1}]{} \cdots\cdots$$

where $\ker(\psi_{-n}) = \varprojlim K_G^{-n}(G_{kg}(2kM_G))$ and

$\operatorname{coker}(\psi_{-n}) = R^1\varprojlim K_G^{-n}(G_{kg}(2kM_G))$, the derived functor.

Thus we have the following exact sequences

$$0 \to R^1\varprojlim K_G^{-n-1}(G_{kg}) \to k_G^{-n}(BU(G)) \xrightarrow[\mu_{-n}]{} \varprojlim K_G^{-n}(G_{kg}) \to 0$$

$$(n > 0)$$

$$0 \to R^1\varprojlim K_G^{-1}(G_{kg}) \to k_G(BU(G)) \xrightarrow[\mu]{} \varprojlim K_G(G_{kg}) .$$

For n even we have $R^1\varprojlim K_G^{-n-1}(G_{kg}) = 0$, also we have a commutative diagram

$$\begin{array}{ccc} k_G^{-2}(BUG) & \xrightarrow{\mu} & \varprojlim K_G^{-2}(G_{kg}) \\ \downarrow & & \downarrow\cong \\ k_G(BU(G)) & \xrightarrow{\mu} & \varprojlim K_G(G_{kg}) \end{array}$$

where the right vertical homomorphism is induced by the periodicity isomorphism, hence μ is an isomorphism.

Now by considering the G-map $\lambda: BU(G) \to BU^{EG}$ of §5 we have the following corollary :-

<u>Cor 6·12</u>: If G is a finite group.

(i) If $f \in Op(K_G)$ then f is continuous with respect to the $I(G)$-adic topology; i.e. there exists $f^\wedge \in Op(k_G^\wedge)$ such that

$$\lambda \circ f = f^\wedge \circ \lambda .$$

(ii) There is a commutative diagram of ring homomorphisms

$$\begin{array}{ccc}
Op(\bar{K}_G ; K_G) & \cong & R(G)[\gamma]_{(\hat{A}_n)} \\
(\lambda \circ -) \downarrow & & \downarrow \\
Op(\bar{K}_G , k_G^\wedge) & \cong & R(G)^\wedge \hat{\otimes} Z[[\gamma]] \\
(- \circ \lambda) \nwarrow \cong & & \nearrow \cong \\
& Op(\bar{K}_G^\wedge , k_G^\wedge) &
\end{array}$$

<u>Appendix</u> : In this section we calculate the ring

$K_G(CP(E)\times\ldots\times CP(E))$ for E a finite dimensional complex representation of a compact Lie group G .

Firstly we recall from [2] that if $H \subset G$ is a closed subgroup and Y is a compact, Hausdorff G-space the H-map

$$Y \cong H\times_H Y \subset G\times_H Y$$

induces an isomorphism $K_G(G\times_H Y) \xrightarrow{\cong} K_H(Y)$.

Now we have a G-map $\mu : G\times_H Y \to G/H\times Y$, $\mu([g,y]) = (gH,gy)$ (this is well defined since $[g,y] = [gh,h^{-1}y]$) which is a G-homeomorphism .Also if $\nu : G\times_H Y \to Y$ is $\nu([g,y]) = g.y$ then

$$\begin{array}{ccc} G\times_H Y & \underset{\cong}{\overset{\mu}{\longrightarrow}} & G/H\times Y \\ & {\scriptstyle\nu}\searrow \quad \swarrow{\scriptstyle P_2} & \\ & Y & \end{array}$$

is commutative .

<u>TA.1</u>: If Y is a compact Hausdorff G-space and G a compact Lie group such that

i) $K_G^*(Y)$ is a torsion free $R(G)$-module .

ii) For all $H \subset G$ $\theta : R(H)\otimes_{R(G)}K_G^*(Y) \to K_H^*(Y)$ is an isomorphism of $R(H)$-modules (where, if ψ is the forgetful homomorphism, θ is the composition

$$R(H)\otimes_{R(G)}K_G^*(Y) \xrightarrow{1\otimes\psi} R(H)\otimes_{R(G)}K_H^*(Y) \to K_H^*(Y) \quad)$$

then for all G-spaces X having finite covering dimension

$$K_G^*(X)\otimes_{R(G)}K_G^*(Y) \to K_G^*(X\times Y) \quad \text{is a ring isomorphism .}$$

<u>Proof</u>: Condition (ii) states that there is a commutative diagram of isomorphisms

$$\begin{array}{ccc} K^*_G(G/H)\otimes_{R(G)}K^*_G(Y) & \to & K^*_G(G\times_H Y) \\ \downarrow & & \downarrow \\ R(H)\otimes_{R(G)}K^*_G(Y) & \to & K^*_H(Y) \end{array}$$

and hence the result follows , applying PartII3·1·6 to the equivariant cohomologies $K^*_G(-\times Y)$ and $K^*_G(-)\otimes_{R(G)}K^*_G(Y)$.

Now for any compact Lie group G we know that

$$K^0_G(\ CP(E)\ ) \cong R(G)[z]/\{\ R(E)(z)\ \} \quad \text{and}$$

$$K^1_G(\ CP(E)\ ) = 0$$

hence CP(E) satisfies the conditions of TA·1 .

<u>CorA·2:</u> $K_G(\ CP(E)\times\ldots\times CP(E)\ ) \cong R(G)[z_1,\ldots,z_n]/\{R(E)(z_i)\}$, where z_i+1 is the class of the canonical G-line bundle .

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MASSEY PRODUCTS IN K-THEORY

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§0: Introduction.

The aim of this paper is to reformulate the work of Massey, (6), and Spanier, (9,10), on higher order cohomology operations, in terms of vector bundles in such a way as to produce geometrically some higher order operations in K-theory, which we will call Massey products.

The motivation for this is the construction by Luke Hodgkin, (4), of a spectral sequence

$$E_2^{*,*} = \mathrm{Tor}^{*,*}_{R(G)}(K_G^*(X) , Z) \Rightarrow K^*(X) \qquad (A)$$

with whose differentials we require better acquaintance.

In (9,10) these differentials are described functionally and it seems of possible use to be able to describe them geometrically in order to return to the level of representations and vector bundles.

The K-theory Massey products are described in terms of complexes of vector bundles. In the important case when $X = G/H$, a homogeneous space, $K_G^*(G/H) = R(H)$ and the differentials are described entirely in terms of representations and isomorphisms between them. In this case a geometric description seems a definite advantage, as the isomorphisms between two representations of a group are well known. However, we do not attempt any more than a description of the operations here.

§1 contains basic preliminary facts about vector bundles. In §2 we describe the join-construction on complexes of vector bundles, which produces a canonical complex of vector bundles used in the formation of the Massey products .

§3 contains the definition of the Massey products operations, and the derivation of a few of their propertiss .

In §4 we digress to describe the Triple Product, which is not specifically needed in subsequent sections, and to list certain familiar properties which is shares with the Toda bracket operation of (9) .

In §5 we show how Massey products of matrices operate in $Tor^{*,*}_{R(G)}(K^*_G(X) , Z)$ and in §6 we apply the results of §5 and (9,10) to the problem of describing the differentials of (A) . §7 contains an example of a non-trivial triple product .

I would like to express my gratitude to Dr.L.Hodgkin for many helpful and encouraging discussions. Also I am much indebted to the reviewer of the original form of this paper, for helpful suggestions, particularly that of using a definition ,(§3), of the type used in (8) .

§1: Preliminaries:

Throughout this paper a vector bundle will mean a complex vector bundle, and all topological spaces will be supposed compact .

Let G be a compact Lie group, X a G-space and Y a closed G-subspace of X .

Definition 1.1: A complex of G-vector bundles, (E, d_E) , over the pair (X,Y) is a finite family of G-vector bundles

$$E^n, E^{n-1}, \ldots\ldots, E^0$$

and homomorphisms of G-vector bundles , $d_i : E^i \to E^{i-1}$ ($i = 1,\ldots,n$) , satisfying the following properties .

(i) $d_i . d_{i+1} = 0$.

(ii) $0 \to E^n/Y \xrightarrow[d_n]{} E^{n-1}/Y \to \ldots\ldots \xrightarrow[d_1]{} E^0/Y \to 0$

is exact .

Definition 1.2: An elementary complex over (X,Y) is one of the form $0 \to P \xrightarrow[Id.]{} P \to 0$ for some G-vector bundle, P , over X .

Definition 1.3: Let E, F be complexes of G-vector bundles over (X,Y) . A morphism $f: E \to F$ is a family of G-vector bundle homomorphism , $\{ f_i : E^i \to F^i \}$, such that $d_F . f = f . d_E$.

Denote by $L_G(X,Y)$ the set of isomorphiam classes of complexes of G-vector bundles over (X,Y) .

Definition 1.4: Let E_0 , E_1 be two complexes of G-vector bundles over (X,Y) . E_0 and E_1 are called homotopic

, $E_0 \simeq E_1$, if there exists an object, F , of $L_G((X,Y)\times I)$, (I is the closed unit interval) , such that $E_i = F/(X,Y)\times\{i\}$, ($i = 0,1$) .

We define an equivalence relation on $L_G(X,Y)$ by

$$E_0 \sim E_1 \Leftrightarrow E_0 \oplus F_0 \simeq E_1 \oplus F_1 ,$$

for some F_0, F_1 which are sums of elementary complexes .

The set of equivalence classes of $L_G(X,Y)$ then becomes a ring , $K_G(X,Y)$, with the multiplication and addition induced from the operations of tensor product and direct sum of (graded) complexes (see (1) Ch.2 ; (3) p.139) .

$K_G^{-n}(X)$ is defined to be $K_G(X\times D^n, X\times S^{n-1})$.

Lemma 1.5: Let $E^n, \ldots, E^0$ be a family of G-vector bundles over X . Let $s_i, t_i: E^i/Y \to E^{i-1}/Y$ be G-vector bundle homomorphisms such that the sequences

$$0 \to E^n/Y \underset{(s_n)}{\overset{t_n}{\longrightarrow}} E^{n-1}/Y \longrightarrow \cdots\cdots \underset{(s_1)}{\overset{t_1}{\longrightarrow}} E^0/Y \to 0$$

are exact. Then the following properties are true .

(i) (E^i/Y, s_i) can be extended to a complex of G-vector bundles over (X,Y) and any two extensions are homotopic rel.(Y) .

(ii) If the complexes (E^i/Y, s_i) and (E^i/Y, t_i) are homotopic, via a homotopy through exact complexes , then the (s_i) extends to an exact differential if and only if (t_i) does .

Proof: (i) is 2.6.13 of (1) and (ii) follows by the methods used in proving 2.6.3 , 2.6.12 of (1) .

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§2: Joins.

If P and Q are topological spaces the join of P and Q is defined as the quotient of

$\{(p,t_1,q,t_2) \; : \; t_1 + t_2 = 1\} \subset P \times I \times Q \times I$.

under the equivalence relation ,

$(p,0,q,1) \sim (p',0,q,1) \quad , \quad (p,1,q,0) \sim (p,1,q',0)$

where $\quad (p,p' \in P \; ; \; q,q' \in Q)$.

The join of P and Q is denoted by $P * Q$.

Let Δ^r be the oriented r-simplex .

Suppose we have complexes of G-vector bundles, (A,d_A) and (B,d_B), over (X,Y) .

Let H_t , (G_t) , ($t \in I$) , be a homotopy of differentials on the family of G-vector bundles $A \times \Delta^r$, ($B \times \Delta^s$) , exact over $Y \times \Delta^r$, ($Y \times \Delta^s$) , such that

$$H_0 = d_A \times \Delta^r \quad , \; (G_0 = d_B \times \Delta^s) .$$

Definition 2.1: The join complex of (A, H_t) and (B, G_t) is the complex over $(X,Y) \times (\Delta^r * \Delta^s)$ which is defined in the following manner.

(i) The underlying family of G-vector bundles is that of

$$(A \otimes B) \times (\Delta^r * \Delta^s) ,$$

(here we write $A \otimes B$ for the family of G-vector bundles $d^*(A \otimes B)$ over X , where $d: X \to X \times X$ is the diagonal map) .

(ii) The differential of the complex over the point

$$(x; t_1, z_1, t_2, z_2) \in X \times (\Delta^r * \Delta^s) ,$$

$$(x \in X;\ t_1, t_2 \in I\ ;\ z_1 \in \Delta^r\ ,\ z_2 \in \Delta^s\ ) ,$$

is given by $H(x; t_1, z_1) \otimes G(x, t_2, z_2)$.

This join complex is denoted by $(A, H_t) * (B, G_t)$ or merely by $H * G$ if no confusion results .

Remark: The join complex is well defined, since H_o , (G_o) , is independent of z_1 , (z_2) .

Definition 2.2: Suppose H_t and G_t are the homotopies of differentials of complexes of G-vector bundles in Definition 2.1 . We define the join homotopy of H and G , denoted by $\underline{H}(H * G)$, to be the following complex of G-vector bundles over $(X,Y)\times(\Delta^r * \Delta^s)\times I$.

(i) The underlying family of G-vector bundles is that of

$$(A\otimes B)\times(\Delta^r * \Delta^s)\times I .$$

(ii) The differential over the point

$$(x; t_1, z_1, t_2, z_2; j) \quad , (j \in I) ,$$

is given by $H(x; j.t_1, z_1) \otimes G(x; j.t_2, z_2)$.

This is a homotopy of differentials from

$$(d_A \otimes d_B)\times(\Delta^r * \Delta^s) \quad , (j = 0)$$

to $d(H * G)$, ($j = 1$) , which is exact over

$$Y\times(\Delta^r * \Delta^s)\times I .$$

Proposition 2.3:

Suppose (A_i, d_i) , $(i = 1,2,3)$, are complexes of G-vector bundles over (X,Y) and $H_{i,t}$ are homotopies on

$$A_i \times \Delta^{r_i} \quad , (i = 1,2,3) ,$$

as in Definitions 2.1,2.2 , then the following properties are true .

(i) If $H_{i,t}$ is exact for $t \geq 1 - e_i$, $(i = 1,2)$,

$H_1 * H_2$ is exact if $e_1 + e_2 \geq 1$.

Also $\underline{H}_j(H_1 * H_2)$ is exact for $j \geq 2 - (e_1 + e_2)$.

(ii) $\underline{H}(H_1 * H_2) * H_3 = H_1 * \underline{H}(H_2 * H_3) = H_1 * H_2 * H_3$.

(iii) $\underline{H}(\underline{H}(H_1 * H_2) * H_3) = \underline{H}(H_1 * \underline{H}(H_2 * H_3))$.

Proof: (i) $H(x;t_1,z_1) \otimes G(x;t_2,z_2)$ is exact if either factor is . Hence (i) follows since

$t_1 < 1 - e_1$ and $t_2 < 1 - e_2$ implies $e_1 + e_2 < 1$, and

$j.t_1 < 1 - e_1$, $j.t_2 < 1 - e_2$ implies $j < 2 - (e_1 + e_2)$.

(ii) Over the point $(x;t_1,z_1,t_2,z_2,t_3,z_3)$ both differentials are $H_1(x,t_1,z_1) \otimes H_2(x,t_2,z_2) \otimes H_3(x,t_3,z_3)$

(iii) Over the point $(x;t_1,z_1,t_2,z_2,t_3,z_3;j)$ both differentials are

$$H_1(x,j.t_1,z_1) \otimes H_2(x,j.t_2,z_2) \otimes H_3(x,j.t_3,z_3)$$

If $(H_{1,t},\ldots,H_{n,t})$ are a set of homotopies of differentials on complexes of G-vector bundles,

$$A_i \times \Delta^{r_i} \quad , (i = 1,\ldots,n) ,$$

as in Proposition 2.3 , this is called an <u>exact set</u> of homotopies if all the iterated joins

$$H_{i_1} * \cdots\cdots * H_{i_r} \quad , (1 \leq i_1 < i_2 < \ldots < i_r \leq n) ,$$

are exact complexes .

<u>Remark 2.4:</u> A sufficient condition for $(H_{1,t}, \ldots, H_{n,t})$ to be an exact set is that there exist e_i , $(i = 1, \ldots, n)$, for $H_{i,t}$ as in Proposition 2.3 with each $e_i \geq m/(m + 1)$ for some $m > n$.

<u>§3: Massey Products.</u>

We are now in a position to give an inductive definition of some higher order operations in K-theory .

Let $(1, \ldots, s+1) = \Delta^s$ be the standard oriented s-simplex and let

$$F_r^s: \begin{array}{c} (1, \ldots, r-1) * (r+1, \ldots, s+1) \\ \parallel \\ (1, \ldots, r-1, r+1, \ldots, s+1) \end{array} \longrightarrow \Delta^s$$

be the canonical inclusion map .

<u>Definition 3.1:</u>

Let (A_r, d_r) be complexes of G-vector bundles over

$$X \times (D^{m_r}, S^{m_r-1})$$

representing elemenets $a_r \in K_G^{-m_r}(X)$, $(1 \leq r \leq n)$

Put $$(L,M)_{i,j} = X \times \prod_{r=1}^{j} (D^{m_r}, S^{m_r-1}) \quad \text{and}$$

$$(P,Q)_{i,j} = (L,M)_{i,j} \times (\Delta^{k-1} , \dot{\Delta}^{k-1})$$

where $k = j-1$ and $1 \leq i \leq j \leq n$

Now let $1 \leq i < j \leq n$.

Suppose there exist homotopies , $H_{i,j}$, of differentials on the complex whose underlying family of G-vector bundles is that of $A_i \otimes \ldots\ldots \otimes A_j \times \Delta^{j-i-1}$, (over $(L,M)_{i,j} \times \Delta^{j-i-1}$) , and that these homotopies have the following properties .

(i) $H_{i,j} | (P,Q)_{i,j} \times \{0\} = d_i \otimes \ldots\ldots \otimes d_j \times \Delta^{j-i-1}$.

(ii) If $k = j-i < n-1$, $H_{i,j} | (P,Q)_{i,j} \times [k/n+2,1]$ is exact .

(iii) $H_{i,j} | Q_{i,j} \times I = \tilde{H}_{i,j}$, where $\tilde{H}_{i,j}$ is defined for $k = j-i \leq n-1$ by requiring that it satisfy the following properties .

(a) $\tilde{H}_{i,j} | (M_{i,j} \times \Delta^{k-1} \times I) = d_i \otimes \ldots\ldots \otimes d_j \times \Delta^{k-1} \times I$.

(b) If $k > 1$: $\tilde{H}_{i,j} \cdot (1 \times F_1^{k-1} \times 1) = d_i \otimes H_{i+1,j}$,

$\tilde{H}_{i,j} \cdot (1 \times F_k^{k-1} \times 1) = H_{i,j-1} \otimes d_j$.

(c) If $k > 2$ and $i < r < j$

$\tilde{H}_{i,j} \cdot (1 \times F_r^{k-1} \times 1) = \underline{H}(H_{i,r} * H_{r+1,j})$.

Then $\tilde{H}_{1,n} | Q_{1,n} \times \{1\}$ is exact , by (ii) and Proposition 2.3 and so can be extended to give a differential , $d_{1,n}$, and a complex $A_{1,n} = (A_1 \otimes \ldots\ldots \otimes A_n \times \Delta^{n-2} , d_{1,n})$ over $(P,Q)_{1,n}$.

The <u>Massey product</u> , $\langle a_1, \ldots, a_n \rangle$, is defined to be the set of elements of $K_G^{-m_1-m_2-\cdots-m_n-n+2}(X)$ that can be represented by complexes, $A_{1,n}$, in this manner .

We call $\{(A_r, d_r), r = 1, \ldots, n ; H_{i,j}, 1 \leq i < j \leq n\}$ a defining system of complexes and homotopies for $(A_{1,n}, d_{1,n})$. Hence $\{(A_r, d_r); H_{i,j}\}$ is a defining system for any complex over $(P,Q)_{1,n}$ which extends $(A_{1,n}, d_{1,n}) | Q_{1,n}$.

Firstly we must verify the following result.

Proposition 3.2:

The homotopy, $\tilde{H}_{1,n}$, and hence the homotopy, $H_{1,n}$, and the (homotopy class of) the complex $(A_{1,n}, d_{1,n})$ are unambiguously defined.

Proof: By induction on $k = j-i$ we may assume that the homotopies, $\tilde{H}_{i,j}$ and $H_{i,j}$ $(j-i < n-1)$, which satisfy the conditions of Definition 3.1 are well defined.

We have to verify that for $1 \leq r_1 < r_2 \leq n$ the two definitions, given by Definition 3.1 (iii), of

$$\tilde{H}_{1,n} | L_{1,n} x (1, \ldots, r_1-1, r_1+1, \ldots, r_2-1, r_2+1, , n-1) x I \cup M_{1,n} x \Delta^{n-2}$$

agree.

Consider the differentials over

$$L_{1,n} x (1, \ldots, r_1-1) * (r_1+1, \ldots, r_2-1) * (r_2+1, \ldots, n-1) \; x \; I .$$

If $r_1 \neq 1$, $r_1+1 \neq r_2 \neq n-1$, the two homotopies of differentials are given by $\underline{H}(\underline{H}(H_{1,r_1} * H_{r_1+1,r_2}) * H_{r_2+1,n})$ and $\underline{H}(H_{1,r_1} * \underline{H}(H_{r_1+1,r_2} * H_{r_2+1,n}))$ which are equal by Proposition 2.3.

Similarly, if $r_1 \neq 1$, $r_1+1 = r_2 \neq n-1$, the two homotopies are $\underline{H}(H_{1,r_1} * (d_{r_1+1} \otimes H_{r_2+1,n})) = \underline{H}((H_{1,r_1} \otimes d_{r_1+1}) * H_{r_2+1,n})$.

For $r_1 = 1$, $r_2 \neq n-1$ both homotopies are

$$d_1 \otimes \underline{H}(H_{2,r_2} * H_{r_2+1,n}) \quad ,$$

for $r_1 \neq 1$, $r_2 = n-1$,

$$\underline{H}(H_{1,r_1} * H_{r_1+1,n-1}) \otimes d_n \quad ,$$

and for $r_1 = 1$, $r_2 = n-1$

$$d_1 \otimes H_{2,n-1} \otimes d_n$$

All these homotopies restrict to $d_1 \otimes \ldots\ldots \otimes d_n$ over $H_{1,n} \times \Delta^{n-2} \times I$.

Remark 3.3:

(a) For two complexes , (A_1, d_1) and (A_2, d_2) ,

$$\widetilde{H}_{1,2} | Q_{1,2} \times \{1\}$$

is homotopic, through exact complexes , to $d_1 \otimes d_2 \,|\, Q_{1,2}$.
Hence $\langle a_1, a_2 \rangle$ is just the element $a_1 . a_2 \in K_G^{-m_1-m_2}(X)$.

(b) If $\{(A_r, d_r)$, $r = 1, \ldots, n$; $H_{i,j}$, $1 \leq i < j \leq n\}$ is a defining system for a complex , $(A_{1,n}, d_{1,n})$, then
$\{(A_r, d_r)$, $r = m_1, \ldots, m_2$; $H_{i,j}$, $m_1 \leq i < j \leq m_2\}$
is a defining system for a complex, $(A_{m_1,m_2}, d_{m_1,m_2})$.
This complex represents the Massey product, $\langle a_{m_1}, \ldots, \ldots, a_{m_2} \rangle$.
For this m_1 , m_2 , Definition 3.1(ii) implies that the complex

$$(A_{m_1,m_2}, d_{m_1,m_2}) \mid Q_{m_1,m_2}$$

can be extended to an exact complex . Hence for $\langle a_1,\ldots,a_n \rangle$ to be defined it is necessary that

$$\langle a_{m_1},\ldots,a_{m_2} \rangle \quad , \ (m_2 - m_1 < n-1 \ , \ 1 \leq m_1 < m_2 \leq n \ ) \ ,$$

should be defined and contain zero .

However, this condition is not sufficient for the higher Massey product to be defined, unless $n = 3$.

(c) Condition (ii) of Definition 3.1 ensures that the join-complexes specifying $\tilde{H}_{1,n} \mid Q_{1,n}$ are exact . The homotopies, $H_{i,j}$ $(1 \leq i < j \leq n)$, are constructed inductively by iterating the join-homotopy construction and by extending the homotopies thus obtained . Hence it suffices, for the construction of $H_{1,n}$, that

$$\{ H_{i,j} \ , \ j - i < n-1 \}$$

be an exact set of homotopies , (e.g. satisfying the condition of Remark 2.4 for a sufficiently large integer m). Definition 3.1(ii) requires that for a fixed i , j , $H_{i,j}$ should be exact for " longer " when involved in a defining system for $A_{1,n+1}$ than when involved in constructing $A_{1,n}$. Nevertheless, by a suitable change of parameter , we may assume that the $H_{i,j}$ are the same . The change of parameter in the homotopies $H_{i,j}$ can be effected without altering the homotopy classes of the lower order Massey products already constructed .

Suppose $\{(A_r, d_r)\ ,\ r = 1,\ldots,n\ ;\ H_{i,j}\ ,\ 1 \leqslant i < j \leqslant n\}$ is a defining system for the complex $(A_{1,n}, d_{1,n})$.

Let (R, d_R) be an exact complex of G-representations.

Define $(B_1, \underline{d}_1) = (A_1 + R\ ,\ d_1 + d_R)$

and $(B_i, \underline{d}_i) = (A_i, d_i)\ ,\ 2 \leqslant i \leqslant n$..

Also define homotopies, $G_{i,j}$, by

$$G_{1,j} = H_{1,j} + \underline{H}(\ (d_R \times I) * H_{2,j})\ ,\ (\ j \geqslant 3\ )$$

$$G_{1,2} = H_{1,2} + (\ d_R \otimes d_2 \times I\ )$$

and $G_{i,j} = H_{i,j}$ otherwise.

Thus $G_{1,j}$ is a homotopy of differentials on the complex whose underlying family of G-vector bundles is that of

$$(\ A_1 + R\ ) \otimes A_2 \otimes \cdots\cdots \otimes A_j \times \Delta^{j-2}.$$

<u>Proposition 3.4:</u>

(I) $\{(B_r, \underline{d}_r)\ ,\ r = 1,\ldots,n\ ;\ G_{i,j}\ ,\ 1 \leqslant i < j \leqslant n\}$ is a defining system for a complex $(B_{1,n}, \underline{d}_{1,n})$.

(II) $(B_{1,n}, \underline{d}_{1,n})$ can be chosen in the form

$$(\ A_{1,n}, d_{1,n}\ ) + (\ \text{exact complex}\ ).$$

<u>Proof</u>: (I) We must verify the conditions of Definition 3.1.

(i) For $j \geqslant 3$,

$$\begin{aligned} & G_{1,j}\,|(P,Q)_{1,j} \times \{0\} \\ = \ & H_{1,j}\,|(P,Q)_{1,j} \times \{0\} + \underline{H}(\ (d_R \times I) * H_{2,j})\,|\,(P,Q)_{1,j} \times \{0\} \\ = \ & d_1 \otimes \cdots\cdots \otimes d_j + d_R \otimes \cdots\cdots \otimes d_j \\ = \ & (d_1 + d_R) \otimes d_2 \cdots\cdots \otimes d_j \\ = \ & \underline{d}_1 \otimes \cdots\cdots \otimes \underline{d}_j . \end{aligned}$$

Clearly $G_{1,2}|(P,Q)_{1,2} \times \{0\} = (d_1 + d_R) \otimes d_2$.

(ii) Since d_R is an exact differential ,

$$G_{1,j}|(P,Q)_{1,j} \times [j-1/n+2,1]$$

$$= H_{1,j}|(P,Q)_{1,j} \times [j-1/n+2,1] + (\text{ exact complex }) .$$

(iii) Since

$$\underline{H}((d_R \times I) * H_{2,j}) |(H_{1,j} \times \Delta^{j-2} \times I)$$

$$= d_R \otimes d_2 \otimes \ldots\ldots \otimes d_j \times \Delta^{j-2} \times I , .$$

Definition 3.1(iii)(a) is true .

(b) If $j > 2$,

$$\tilde{G}_{1,j} \cdot (1 \times F_1^{j-2} \times 1) = d_1 \otimes H_{2,j} + d_R \otimes H_{2,j}$$

$$= \underline{d}_1 \otimes G_{2,j} .$$

Also

$$\tilde{G}_{1,j} \cdot (1 \times F_{j-1}^{j-2} \times 1)$$

$$= H_{1,j-1} \otimes d_j + \underline{H}((d_R \times I) * H_{2,j}) \cdot (1 \times F_{j-1}^{j-2} \times 1)$$

$$= H_{1,j-1} \otimes d_j + \underline{H}((d_R \times I) * H_{2,j-1}) \otimes d_j$$

(by Proposition 2.3(iii))

$$= G_{1,j-1} \otimes \underline{d}_j .$$

(c) If $j > 3$ and $1 < r < j$,

$$\tilde{G}_{1,j} \cdot (1 \times F_r^{j-2} \times 1)$$

$$= \underline{H}(H_{1,r} * H_{r+1,j}) + \underline{H}(\underline{H}((d_R \times I) * H_{2,r}) * H_{r+1,j})$$

(by Proposition 2.3(iii))

$$= \underline{H}(G_{1,r} * H_{r+1,j}) .$$

(II) Since $G_{1,n} = H_{1,n} + \underline{H}((d_R \times I) * H_{2,n})$ it is clear that $(B_{1,n}, \underline{d}_{1,n})$ can be chosen as the sum of $(A_{1,n}, d_{1,n})$ with the exact complex whose differential is given by the end of the join-homotopy .

<u>Definition 3.</u> The defining system $\{(B_r, d_r); G_{i,j}\}$ of Proposition 3.4 is called the <u>defining system obtained from</u> $\{(A_r, d_r); H_{i,j}\}$ <u>by adding</u> (R, d_R) <u>in the first coordinate</u> .

Similarly we can define addition of exact complexes of G-representations in the i-th coordinate .

Suppose we have a defining system , $\{(A_r, d_r); H_{i,j}\}$, for an exact complex $(A_{1,n}, d_{1,n})$, then we know that $0 \in \langle a_1, \ldots, a_n \rangle$.

We have the following converse .

<u>Lemma 3.6</u>: Suppose $0 \in \langle a_1, \ldots, a_n \rangle \subset K_G^*(X)$.

Then there exist complexes , $(B_r, \underline{d}_r)$, representing

$$a_r \in K_G^{-m_r}(X) \quad , (r = 1, \ldots, n) ,$$

and homotopies of differentials , $H_{i,j}$ $(1 \leq i < j \leq n)$, such that $\{(B_r, \underline{d}_r); H_{i,j}\}$ is a defining system for an ~~exist~~ exact complex , $(B_{1,n}, \underline{d}_{1,n})$.

<u>Proof</u>: Let $\{(A_r, d_r); G_{i,j}\}$ be a defining system for $(A_{1,n}, d_{1,n})$.

If $(A_{1,n}, d_{1,n})$ represents zero in $K_G^*(X)$ then there exists an exact complex , (R, d_R) , of G-representations such that

$$(A_{1,n}, d_{1,n}) + (R, d_R)$$

is homotopic , over $(P,Q)_{1,n}$, to an exact complex .

Let (F, d_F) be the elementary complex

$$0 \rightarrow 1 \xrightarrow[\text{Id.}]{} 1 \rightarrow 0$$, (1 is the one-dimensional

trivial representation of G) .

Let $\{ (B_r, d_r) ; H_{i,j} \}$ be the defining system obtained from $\{ (A_r, d_r) ; G_{i,j} \}$ by successively adding (R, d_R) in the first coordinate and (F, d_F) in the i-th coordinate , (i = 2,...,n) .

As in Proposition 3.4 , $(B_{1,n}, d_{1,n})$ can be taken to be of the form $(A_{1,n}, d_{1,n}) + (R,d_R) \otimes ((F,d_F))^{n-1}$ + (an exact complex) .

Since (R,d_R) is exact there is a homotopy, through exact differentials , from $(R,d_R) \otimes ((F,d_F))^{n-1}$ to $(R,d_R) + (E,d_E)$, where (E,d_E) is exact . Thus there is a homotopy, rel. $Q_{1,n}$, from $(B_{1,n}, d_{1,n})$ to an exact complex and, by Lemma 1.5 , $(B_{1,n}, d_{1,n}) | Q_{1,n}$ can be extended to an exact complex .

Remark 3.7:

Lemma 3.6 serves to illuminate the inductive construction contained in Definition 3.1 . The inductive step is as follows .

Suppose we have elements $a_r \in K_G^{-m_r}(X)$, r = 1,,...,n , and defining systems $\{ (A_r, d_r), r = 1,..,n-1 ; H_{i,j} \}$, $\{ (A_r, d_r), r = 2,...,n ; H'_{i,j} \}$ for complexes $(A_{1,n-1}, d_{1,n-1})$ and $(A_{2,n}, d_{2,n})$. If these complexes represent zero in $K_G^*(X)$ we may assume that they are exact and that $H_{1,n-1}$ and $H'_{2,n}$ satisfy Definition 3.1(ii) . If $H_{i,j}$ can be chosen equal to $H'_{i,j}$

for $k = j-i < n-1$, then Definition 3.1 constructs a homotopy , $H_{1,n}$ and hence the complex $(A_{1,n}, d_{1,n})$.

If $f: Y \to X$ is a G-[illegible] map we have the following naturality property .

Lemma 3.8: Let $a_r \in K_G^{-m_r}(X)$, $r = 1,\ldots,n$.

If $\langle a_1,\ldots,a_n\rangle$ is defined then $\langle f^*(a_1),\ldots,f^*(a_n)\rangle$ is defined and $f^*\langle a_1,\ldots,a_n\rangle \subset \langle f^*(a_1),\ldots,f^*(a_n)\rangle$.

If $f : K_G^*(X) \to K_G^*(Y)$ is an isomorphism the above inclusion of Massey products is an equality .

Proof: The first part is a simple consequence of the fact that f^* , (taking induced bundles) , preserves direct sums, tensor products, exact complexes, joins and join-homotopies . If , in addition , f^* is an isomorphism the indeterminacies of both sides are the same .

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§4: The Triple Product.

This can be constructed in the following way .

Let (A_1,d_1) , (A_2,d_2) and (A_3,d_3) be complexes representing a_1 , a_2 and a_3 , homogeneous elements in $K_G^*(X)$.

If $a_1 . a_2 = 0$ we may suppose , (c.f. Lemma 3.6) that there is a homotopy , $H_{1,2}(t)$, (as in Definition 3.1) , such that

$$H_{1,2}(0) = d_1 \odot d_2$$

and $H_{1,2}(1)$ is exact .

Similarly if $a_2 . a_3 = 0$ there exists $H_{2,3}$.

Then $(A_{1,3}, d_{1,3})$ is a complex over

$$(P,Q)_{1,3} = (L,M)_{1,3} \times (I,\dot{I}) \quad \text{which extends}$$

$$d_1 \otimes d_2 \otimes d_3 \quad \text{on} \quad M_{1,3} \times I \;,$$

$$d_1 \otimes H_{2,3}(1) \quad \text{on} \quad L_{1,3} \times \{0\}$$

and $$H_{1,2}(1) \otimes d_3 \quad \text{on} \quad L_{1,3} \times \{1\} \;.$$

This is a direct, vector bundle analogue of the Toda bracket construction. The triple product enjoys all the familiar properties of Toda brackets, we list a few without proof. The proofs are analogues of those in ((9) §4).

Let $a_i \in K_G^{-m_i}(X)$, $i = 1,2,3,4$.

<u>Proposition 4.1</u>: If $a_1 . a_2 = 0 = a_2 . a_3$ then $\langle a_1, a_2, a_3 \rangle$ is defined and is a well-defined element of

$$K_G^*(X)/(a_1 . K_G^*(X) + K_G^*(X) . a_3) \;.$$

<u>Proposition 4.2</u>: If $a_1 . a_2 = 0 = a_2 . a_3 = a_3 . a_4$ then $a_1 . \langle a_2, a_3, a_4 \rangle + \langle a_1, a_2, a_3 \rangle . a_4 = 0$ in $K_G^*(X)/(a_1 . K_G^*(X) . a_4)$.

<u>Proposition 4.3</u>: If $a_1 . a_3 = 0 = a_2 . a_3 = a_3 . a_4$ then $\langle a_1 + a_2, a_3, a_4 \rangle = \langle a_1, a_3, a_4 \rangle + \langle a_2, a_3, a_4 \rangle$ in $K_G^*(X)/(a_1 . K_G^*(X) + a_2 . K_G^*(X) + K_G^*(X) . a_4)$.

There are similar linearity properties in the other variables.

Proposition 4.4: If $a_2 . a_3 = 0 = a_3 . a_4$ then

$$a_1 . \langle a_2, a_3, a_4 \rangle = \langle a_1 . a_2, a_3, a_4 \rangle$$

$$= (-1)^{m_1 . m_2} \langle a_2, a_1 . a_3, a_4 \rangle$$

$$= (-1)^{m_1 . (m_2 + m_3 + m_4 - 1)} \langle a_2, a_3, a_4 . a_1 \rangle ,$$

modulo suitable indeterminacies .

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§ 5: Massey Products and $Tor^{*,*}_{R(G)}(K^*_G(X), Z)$.

Let $a_r \in K_G^{-m_r}(X)$, $r = 1, \ldots, n$, be elements such that $\langle a_1, \ldots, a_n \rangle$ is defined and contains zero . There exists an acyclic complex , $(A_{1,n}, d_{1,n})$, and a defining system , $\{(A_r, d_r) ; H_{i,j} , 1 \leq i < j \leq n \}$, for the complex $(A_{1,n}, d_{1,n})$.

Let $a_{n+1} \in K_G^{-m_{n+1}}(X)$ be another element .

Definition 5.1:

Suppose $(B_r, \underline{d}_r)$ is a complex representing a_r , $r = 1, \ldots, n+1$, and that $\{(B_r, \underline{d}_r) ; G_{i,j} , 1 \leq i < j \leq n+1 \}$ is a defining system for a complex $(B_{1,n+1}, \underline{d}_{1,n+1})$.

$\{(B_r, \underline{d}_r) ; G_{i,j} , 1 \leq i < j \leq n+1$ is called a <u>defining system restricted by</u> $\{(A_r, d_r) ; H_{i,j} , 1 \leq i < j \leq n\}$ if the following is true .

$\{(B_r, \underline{d}_r) ; G_{i,j} , 1 \leq i < j \leq n \}$ is a defining system obtained from $\{(A_r, d_r) ; H_{i,j} , 1 \leq i < j \leq n \}$ by successively adding exact complexes of G-representations in coordinates $1, 2, \ldots\ldots, n$.

The Massey products obtained from these restricted defining systems will be called <u>restricted Massey products</u> and will be denoted by $\langle a_1, \ldots, a_n, \underline{a}_{n+1} \rangle$.

<u>Example 5.2</u>: The restricted triple product , $\langle a_1, a_2, \underline{a}_3 \rangle$, is defined if $a_1.a_2 = 0 = a_2.a_3$ and is a well-defined element of $K^*_G(X)/(K^*_G(X).a_1)$.

<u>Remark 5.3</u>: Let $a_1, \ldots, a_n$ be <u>matrices</u> of homogeneous elements of $K^*_G(X)$ such that all matrix products ,

$$a_i . \cdots\cdots . a_j \quad , \quad 1 \leq i < j \leq n \quad ,$$

are defined and are again matrices of homogeneous elements . Then Definition 3.1 can be formally applied to define Massey products , $\langle a_1, \ldots, a_n \rangle$. These are subsets of the set of matrices of homogeneous elements in $K^*_G(X)$.

Similarly restricted Massey products of matrices , $\langle a_1, \ldots, a_{n-1}, \underline{a}_n \rangle$ can be defined .

Suppose we have a complex

$$Z \underset{c}{\leftarrow} C_0 \underset{f_0}{\leftarrow} C_{-1} \underset{f_1}{\leftarrow} \cdots\cdots \leftarrow C_{-n+1} \underset{f_{n-1}}{\leftarrow} C_{-n} \qquad (I)$$

in which the $\{C_{-i}\}$ are free $R(G)$-modules , (e.g. a free resolution or part of a free resolution of Z , the integers , as an $R(G)$-module) .

Choosing a basis for each C_{-i} as a free $R(G)$-module we can represent f_i as a mtrix , a_i , of elements of $R(G) = K^0_G(\text{point})$.

Hypothesis 5.4:

Suppose $\langle a_1,\ldots,a_{n-1}\rangle$ is defined and contains the zero matrix . (This condition is independent of the choice of bases for the $\{C_{-i}\}$) .

If Hypothesis 5.4 is true we can define a system of restricted Massey products associated with (I) in the following manner .

Choose a defining system for an exact representative of

$$\langle a_1,\ldots,a_{n-1}\rangle$$

Let X be a G-space , $x_i \in C_{-i} \otimes_{R(G)} K_G^*(X)$ a homogeneous elements . Then x_i is represented as a column vector of homogeneous elements of $K_G^*(X)$, (with respect to the basis defining the $\{a_i\}$) .

We have $\langle a_{i-1},\underline{x}_i\rangle$ defined and representing

$$(f_{i-1}\otimes 1)(x_i) \in C_{-i+1} \otimes_{R(G)} K_G^*(X)$$

as a column vector of homogeneous elements .

If $(f_{i-1} \otimes 1)(x_i) = 0$, we have defined

$$\langle a_{i-2},a_{i-1},\underline{x}_i\rangle$$

interpreted as column vectors or as an element of

$$C_{-i+2} \otimes_{R(G)} K_G^*(X) \,/\, (\operatorname{im}(f_{i-2}\otimes 1)) \quad .$$

Similarly we can give meaning to the restricted Massey products $\langle a_p,\ldots,a_{i-1},\underline{x}_i\rangle$, ($0 \leq p < i \leq n$) .

Theorem 5.5: Let (I) be a complex of free $R(G)$-modules for which Hypothesis 5.4 is true.

Then a system of restricted Massey products associated with (I) gives a system of operators, d_r $(1 \leq r \leq n)$, on $C_* \otimes_{R(G)} K^*_G(X)$ satisfying the following properties.

(i) If $x_i \in C_{-i} \otimes_{R(G)} K^j_G(X)$ then

$$d_1(x_i) = (f_{i-1} \otimes 1)(x_i) .$$

(ii) $d_r(x_i)$ is defined if and only if

$$d_p(x_i) \quad , \quad (p = 1,\ldots,r-1) ,$$

is defined and contains zero, and then

$$d_r(x_i) \subset C_{-i-r} \otimes_{R(G)} K^{j-r+1}_G(X) .$$

(iii) If $x_i = (f_i \otimes 1)(y_i)$ then $d_r(x_i)$ is defined and contains zero.

(iv) $(f_{i-r-1} \otimes 1)(d_r(x_i)) = 0$ (mod. zero).

Proof: We define $d_r(x_i) = \langle a_{i-r},\ldots,a_{i-1},\underline{x}_i \rangle$, which clearly satisfies (i).

For (ii), we have to show that $d_{r+1}(x_i)$ is defined if

$d_1(x_i)$, ... , $d_r(x_i)$ are defined and contain zero.

Suppose $\{ (A_s,\underline{d}_s) , i-r \leq s \leq i ; G_{k,m} , i-r \leq k < m \leq i \}$ is a defining system (of matrices of complexes and homotopies) for $(A_{i-r,i},\underline{d}_{i-r,i})$, an exact matrix over $(P,Q)_{i-r,i}$, a matrix of spaces.

A defining system for an element of $d_{r+1}(x_i)$ is given as follows.

Adjoin to the defining system the complex $(A_{i-r-1},\underline{d}_{i-r-1})$, representing a_{i-r-1}, and the homotopies

$H_{i-r-1,j}$, $(i-r-1 < j < i)$,

from the restricted defining system for an acyclic Massey product , $\langle a_{i-r-1},\ldots,a_{i-1}\rangle$.

Then $H_{i-r-1,i}$ can be chosen to fulfill Definition 3.1(i),XXX & (iii) .

For (iii) , if $x_i = (f_i \otimes 1)(y_i) = a_i . y_i$ we have

$$0 = \langle a_{i-r},\ldots,a_i\rangle . y_i \subset \langle a_{i-r},\ldots,\underline{a_i . y_i}\rangle$$

$$= \langle a_{i-r},\ldots,a_{i-1},\underline{x}_i\rangle .$$

To show (iv) we prove that for any representative of $d_r(x_i)$,

$$0 = a_{i-r-1}.d_r(x_i) + \langle a_{i-r-1},\ldots,a_{i-1}\rangle . x_i \qquad (*) .$$

Suppose $\{(A_s,\underline{d}_s) ; G_{k,m}\}$ is a defining system for $(A_{i-r,i},\underline{d}_{i-r,i})$. Adjoin to this the complex

$(A_{i-r-1},\underline{d}_{i-r-1})$ and the homotopies

$H_{i-r-1,j}$ $(i-r-1 < j < i)$

as in (ii) .

This collection of complexes and homotopies is <u>not</u> a defining system , (it does not satisfy Definition 3.1 (ii) for $G_{i-r,i}$) , but it does define (by the formula of Definition 3.1) a matrix of complexes over

$$L_{i-r-1,i} \times \Delta^{r+1} , \quad (\Delta^{r+1} = (i-r-1,\ldots,i)) .$$

This complex , (E,d_E) , is exact over all the principal faces except $(i-r-2,\ldots,i)$ where the complex is

$a_{i-r-1}.\langle a_{i-r},\ldots,\underline{x}_i\rangle$

and $(i-r-1,\ldots,i-1)$ where it is $\langle a_{i-r-1},\ldots,a_{i-1}\rangle x_i$.

[illegible] complex, (E, d_E) , represents a homotopy from a representative of

$$a_{i-r-1}\cdot\langle a_{i-r},\dots,x_i\rangle + \langle a_{i-r-1},\dots,a_{i-1}\rangle\cdot x_i$$

to an exact complex , which proves $(*)$.

<u>Corollary 5.6</u>: Let (I) be part of a free resolution of Z as an $R(G)$-module for which Hypothesis 5.4 is true . There exist operators, d_r $(r \geq 2)$, such that the following is true .

(i) If $x \in \mathrm{Tor}_{R(G)}^{-p,j}(Z, K_G^*(X))$, then $d_r(x)$ is defined if and only if $d_2(x),\dots,d_{r-1}(x)$ are defined and contain zero .

(ii) In this case

$$d_r(x) \subset \mathrm{Tor}_{R(G)}^{-p+r,\,j-r+1}(Z, K_G^*(X)) .$$

<u>§6: The Kunneth Formula Spectral Sequence</u> :

(A) $E_2^{*,*} = \mathrm{Tor}^{*,*}_{R(G)}(Z, K_G^*(X)) \Rightarrow K^*(X)$, (see (4)) .

We recall how this spectral sequence is constructed . Let G be a compact, connected Lie group whose first homotopy group is torsion free .

<u>Definition 6.1</u>: A <u>basic G-space</u> , X , is one such that

(i) $K_G^*(X)$ is a free $R(G)$-module .

(ii) For all compact G-spaces , Y , the product

$$K_G^*(X) \otimes_{R(G)} K_G^*(Y) \rightarrow K_G^*(X \times Y)$$

is an isomorphism .

Definition 6.2: A geometric resolution of Y is a sequence of based, compact G-spaces and maps

$$\text{(I)} \quad Y^+ = Y_0 \xrightarrow[i_0]{} Z_0 \xrightarrow[j_0]{} Y_1 \xrightarrow[i_1]{} Z_1 \xrightarrow[j_1]{} Y_2 \rightarrow \cdots\cdots$$

(Y^+ is the disjoint union of Y and a point) ,

satisfying the following properties .

(i) Z_i is a basic G-space .

(ii) j_r is the cofibre of i_r , ($r \geq 0$) .

(iii) $\tilde{K}_G^*(Y^+) \leftarrow \tilde{K}_G^*(Z_0) \leftarrow \tilde{K}_G^*(Z_1) \leftarrow \cdots\cdots$ is a free resolution of $\tilde{K}_G^*(Y^+)$ as an $R(G)$-module .

The spectral sequence (A) is produced as follows . Let (I) be a geometric resolution for $Y = G$, then the following $H(p,q)$-system (see (4) §§ 3,4) gives the spectral sequence .

$$H^a(-p,-q) = \tilde{K}_G^{p+a+1}(S^{p-q}(X^+ \wedge Y_{q+1}), X^+ \wedge Y_{p+1})$$
$$(-1 \leq q \leq p \leq \infty) .$$

$$H^a(-p,q) = \tilde{K}_G^{p+a+1}(S^{p+1}(X^+ \wedge Y_0) , X^+ \wedge Y_{p+1})$$
$$(-1 \leq p \leq \infty ; 1 < q \leq \infty) .$$

$$H^a(p,q) = 0 \qquad (1 < p \leq q \leq \infty) .$$

$$H^a(-\infty,q) = \varinjlim_p H^a(-p,q) ,$$

where the restriction and coboundary morphisms are induced from the exact sequences of the maps $X^+ \wedge SY_i \rightarrow X^+ \wedge Y_{i+1}$ which come from the Puppe construction .

Suppose we have a sequence of based G-maps

$$A_0 \xrightarrow[a_1]{} A_1 \xrightarrow[a_2]{} A_2 \to \cdots\cdots \to A_n ,$$

in ((10) §6) a sequence of higher order functional operations are defined , denoted by $\langle a_1,\ldots,a_n\rangle$.

These are defined as the obstruction cocycles to the extension of a nullhomotopy of a map , f , carried by a carrier , (M,m) , on $\Delta^{n-2} = (1,\ldots,n-1)$.

The carrier is defined as follows .

If $s = (i_1, i_1+1, \ldots, j_1, i_2, i_2+1, \ldots, i_q, i_q+1, \ldots, j_q) \subset \Delta^{n-2}$

$$M(s) = \mathrm{Map}_G(A_{i_1-1}, A_{j_1+1}) \times \cdots \times \mathrm{Map}_G(A_{i_q-1}, A_{j_q+1}) ,$$

and if $t < s$, $m(t,s) : M(t) \longrightarrow M(s)$ is given by the obvious compositions and compositions with the $\{a_i\}$.

The map , f , carried by (M,m) is defined by

$$f(s) = (a_{j_1+1}\cdot a_{j_1}\cdots a_{i_1} ; \cdots\cdots ; a_{j_q+1}\cdots\cdots a_{i_q}) .$$

If defined , $\langle a_1,\ldots,a_n\rangle$ is a subset of $\mathrm{Map}_G(S^{n-2}A_0, A_n)$.

Definition 6.3: $A_0 \xrightarrow[a_1]{} A_1 \to \cdots\cdots \to A_n$ is called split if there exists a commutative diagram of maps

$$\begin{array}{ccccccccccc} A_0 & \to & A_1 & \to & A_2 & \to & \cdots\cdots\cdots & \to & A_{n-1} & \to & A_n \\ & & & {\scriptstyle b_2}\searrow & \nearrow{\scriptstyle c_2} & \searrow & \nearrow & \searrow & \nearrow & & \\ & & & & B_2 & & B_3 & & B_{n-1} & & \end{array}$$

such that $b_2 \cdot a_1 ; b_3 \cdot c_2 ; \cdots\cdots\cdots ; a_n \cdot c_{n-1}$ are nullhomotopic .

Now suppose

$$G^{*} = Y_o \to Z_o \to Y_1 \to \ldots\ldots\ldots \to Z_n \qquad \text{(I)}$$

is part of a geometric resolution of G by G-spaces .

For the sequence

$$Z_o \xrightarrow{f_o} Z_1 \xrightarrow{f_1} \ldots\ldots\ldots \to Z_n \qquad \text{(II)}$$

$\langle f_o,\ldots,f_{n-1}\rangle$ is defined and contains zero , since the sequence is split ((10) Theorem 6.3) .

Also we have a complex of free $R(G)$-modules

$$\tilde{K}^{*}_{G}(Z_o) \xleftarrow{a_o} \tilde{K}^{*}_{G}(Z_1) \leftarrow \ldots\ldots\ldots \leftarrow \tilde{K}^{*}_{G}(Z_n) \qquad \text{(III)} .$$

A nullhomotopy of f carried by (M,m) , associated with (II) is a family of nullhomotopies

$$H_{i,j}: \Delta^{j-i-2} \times I \to \mathrm{Map}_G(Z_i,Z_j)$$

$$(\Delta^{j-i-2}\ (i+1,\ldots,j-1) \ ;\ 0 \leq i < j \leq n)$$

satisfying the following properties .

(i) $H_{i,j}(x,0) = f_{j-1}\cdot\ldots\ldots\cdot f_i$, ($x \in \Delta^{j-i-2}$) .

(ii) If $j-i > 1$,

$$H_{i,j}\cdot(F^{j-i-2}_{i+1} \times 1) = H_{i+1,j}\cdot f_i$$

$$H_{i,j}\cdot(F^{j-i-2}_{j-1} \times 1) = f_j\cdot H_{i,j-1} \quad .$$

If $j-i > 2$, $i < r < j$

$$H_{i,j}\cdot(F^{j-i-2}_{r} \times 1) = H_{r,j}\cdot H_{i,r} \quad .$$

We now consider the Massey products associated with (III) .
$H_{i,j}$ induces a homotopy of differentials , also denoted by $H_{i,j}$, on the complex whose underlying family of

vector bundles is $a_i \circ \ldots\ldots \circ a_{j-1} \times \Delta^{j-i-2}$.
$H_{i,j}(0)$ is $d_i \circ \ldots\ldots \circ d_{j-1} \times \Delta^{j-i-2}$ and
$H_{i,j}(1)$ is an exact complex . Furthermore , under this correspondence $H_{r,j} \cdot H_{i,r}$ can be made to induce a join-homotopy .

Hence the nullhomotopy of f , carried by (M,m) , gives rise to a defining system (which can be arranged to satisfy Definition 3.1(ii)) for an exact complex representing $\langle a_o, \ldots\ldots, a_{n-1} \rangle$.

Hence we have :

<u>Theorem 6.4</u>: If $G^+ \to Z_o \to Z_1 \to \ldots\ldots \to Z_n$ is part of a geometric resolution of G then the complex

$$\tilde{K}_G^*(Z_o) \leftarrow \tilde{K}_G^*(Z_1) \leftarrow \ldots\ldots\ldots \leftarrow \tilde{K}_G^*(Z_n)$$

of free $R(G)$-modules satisfies Hypothesis 5.4 .

We turn now to the differentials in the spectral sequence (A) . As an E_1-spectral sequence (A) may be written as

$$E_1^{-n,*} = \tilde{K}_G^*(Z_n \wedge X^+) \Rightarrow K^*(X) .$$

An element $x_i \in E_1^{-i,-q}$ can thus be represented as a based G-map , $x_i : Z_i \wedge X^+ \to BU_q(G)$, (where $BU_q(G)$ is a classifying space for $K_G^{-q}(\_)$, (see (23) ((3)p.134)

Consider the sequence of based G-maps

$$Z_o \wedge X^+ \xrightarrow[f_o \wedge 1]{} Z_1 \wedge X^+ \to \ldots\ldots\ldots\ldots \to Z_i \wedge X^+ \xrightarrow[x_i]{} BU_q(G) .$$

From ((10) § 6) we know that the differentials on x_i

are represented by restricted Spanier-products ,

$$\langle f_j \wedge 1, \ldots\ldots, f_{i-1} \wedge 1, \underline{X}_i \rangle \quad ;$$

(here restricted means that the nullhomotopies on the carrier (M,m) coming from products $\langle f_k \wedge 1, \ldots, f_m \wedge 1 \rangle$ are specified - which corresponds to the notion of restricted Massey products) .

By a similar discussion to that preceding Theorem 6.4 we see that $\langle f_j \wedge 1, \ldots, f_{i-1} \wedge 1, \underline{X}_i \rangle$ is contained in $\langle a_j, \ldots, a_{i-1}, \underline{X}_i \rangle$, interpreted as sets of matrices . However for the double and triple products the indeterminacies of these operations are the same .

The indeterminacy for $\langle a_j, \ldots\ldots, a_{i-1}, \underline{X}_i \rangle$ accumulates in the following two ways :

(a) from the different ways of constructing $\langle a_j, \ldots, a_{i-1}, \underline{X}_i \rangle$ from an exact complex representing $\langle a_{j+1}, \ldots, a_{i-1}, \underline{X}_i \rangle$,

and (b) from the different exact representatives of

$$\langle a_{j+1}, \ldots\ldots, a_{i-1}, \underline{X}_i \rangle \quad .$$

In terms of $\langle f_j \wedge 1, \ldots, f_{i-1} \wedge 1, \underline{X}_i \rangle$, (a) corresponds exactly to varying the ~~[illegible]~~ cocycle by changing nullhomotopies in the top dimension , and (b) to altering the nullhomotopies in lower dimensions .

<u>Remark 6.5</u>: This exact correspondence is due to the fact that $BU_q(G)$ is a classifying space and hence there is

a one-one correspondence

(equivalence classes of homotopies of differentials to exact differentials)	$\longleftrightarrow$	(nullhomotopies of maps inducing these complexes) .

For maps into a general space we have only that a nullhomotopy of maps induces a homotopy of differentials to an exact one .

In conclusion we have the following result :

<u>Theorem 6.6</u>: Let G be a compact, connected Lie group whose first homotopy group is torsion free .

There exists a free resolution of Z by $R(G)$-modules ,

$$Z \leftarrow C_0 \underset{f_0}{\leftarrow} C_{-1} \leftarrow \ldots\ldots \leftarrow C_{-n} \leftarrow \ldots\ldots \qquad (C_G)$$

satisfying the following rpoperties .

(i) (C_G) is a complex satisfying Hypothesis 5.4 .

(ii) The differentials in the Kunneth formula spectral sequence (A) are given by a system of restricted Massey products associated with (C_G) .

(iii) If $G = T_r$, a torus , then (C_{T_r}) may be taken as the Koszul resolution (see (5) p.205) .

<u>Proof</u>: In view of the preceding discussion we have only to show (iii) .

If $G = S^1$, it is well-known that

$$(S^1)^+ \subset (D^2)^+ \rightarrow D^2/S^1 \qquad (*)$$

gives the Koszul resolution on applying $\tilde{K}_{S^1}(\_)$.

For $G = S^1 \times \ldots \times S^1$, (r factors), the general Koszul resolution can be realised geometrically by 'multiplying'

(*) r times by the process given in ((4) § 5) .

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§ 7: An example of a non-trivial matrix Massey product .

To produce a non-trivial Massey product it suffices to find an example of a G-space for which the Kunneth formula spectral sequence does not collapse .

Let $G = T_2$, a torus . We produce a spectral sequence

$$E_2^{*,*} = \mathrm{Tor}^{*,*}_{R(T_2)}(\ Z, K^*_{T_2}(X)\ ) \Rightarrow K^*(X) \qquad (A)$$

in which the only differential , d_2 , is non-trivial .

Now (A) can be produced using a geometric resolution for X , $\{Y_i, Z_i\}$, and applying $\tilde{K}^*_{T_2}(\_)$ to $\{Y_i \wedge T_2^+, Z_i \wedge T_2^+\}$, (see (4) § 5) , or equivalently by applying $\tilde{K}^*(\_)$ to $\{Y_i, Z_i\}$.

Furthermore, if (A) collapses then $(A) \otimes Q$, (Q = rationals) collapses . But $K^*(\_) \otimes Q$ is isomorphic , via Chern character , to $\underline{H}^*(\_;Q)$ – where $\underline{H}^*$ is singular cohomology completed with respect to the grading filtration and graded by Z_2 .

Applying $\underline{H}^*(\_;Q)$ to $\{Y_i, Z_i\}$ produces a spectral sequence

$$\{\ \mathrm{Tor}^{*,*}_{H^*(BT_2;Q)}(\ H^*(X \times_{T_2} ET_2;Q),\ Q\ )\}^{\wedge} \Rightarrow \underline{H}^*(X;Q) \qquad (B)$$

(where ^ is completion with respect to the second grading , a $Z \times Z_2$ bigraded spectral sequence) .

If e is the edge homomorphism in (B) the composite

$$(H^*(X\times_{T_2}ET_2;Q) \otimes_{H^*(BT_2;Q)} Q)^\wedge \xrightarrow{e} E_\infty^{0,*} \to \underline{H}^*(X;Q) \qquad (C)$$

is the completion of the homomorphism

$$H^*(X\times_{T_2}ET_2;Q) \otimes_{H^*(BT_2;Q)} Q \longrightarrow H^*(X;Q)$$

induced by the diagram of principal fibrations

$$\begin{array}{ccc} & T_2 & \\ \swarrow & & \searrow \\ X & \longrightarrow & ET_2 \\ \downarrow & & \downarrow \\ X\times_{T_2}ET_2 & \longrightarrow & BT_2 \end{array}$$

Thus it suffices to find a principal fibration

$$T_2 \longrightarrow X \longrightarrow X_T = X\times_{T_2}ET_2$$

such that $H^*(X_T;Q) \otimes_{H^*(BT_2;Q)} Q \longrightarrow H^*(X;Q)$

is not a monomorphism.

Let $Y = S^2 \vee S^2 \vee S^2$ and $f,g,h \in \pi_2(Y)$ be the homotopy classes of the three canonical inclusions of S^2.

Let $t \in \pi_4(Y)$ represent the Whitehead product, $[f,[g,h]]$.

Put $X_T = Y \cup_t e^5$.

In (7) it is shown that there exist indecomposable elements $x_1,x_2,x_3 \in H^2(X_T;Z)$ and $y \in H^5(X_T;Z)$ such that the singular cohomology Massey product

$$\langle x_1,x_2,x_3\rangle = y \neq 0 \in H^*(X_T;Z)/(x_1\cdot H^*(X_T;Z) + H^*(X_T;Z)\cdot x_3).$$

Let $f_i : X_T \longrightarrow K(Z,2) = BS^1_i$ represent x_i, ($i = 1,3$).

Let

$$\begin{array}{ccc} X & \longrightarrow & BT_2 \\ p\downarrow & & \downarrow \\ X_T & \xrightarrow[f_1 \times f_3]{} & BT_2 \end{array}$$

be the diagram of the induced T_2-principal bundle .

In (7) it is shown that $0 = p^*(y) \in H^*(X;Z)$.

However $H^*(X_T;Q) \otimes_{H^*(BT_2;Q)} Q = \dfrac{H^*(X_T;Q)}{\{x_1 . H^*(X_T;Q) + H^*(X_T;Q) . x_3\}}$

and thus the edge homomorphism cannot be a monomorphism .

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On the K-Theory of homogeneous spaces and Conjugate Bundles of Lie groups.

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§0: Introduction.

Homogeneous spaces of Lie groups form an important class of spaces, which occur frequently in algebraic topology. The singular cohomology of homogeneous spaces of compact Lie groups has been investigated at length by A. Borel, P. Baum and J.P. May, (7,8,18,19). In (7,19) the main tool used is the Eilenberg-Moore spectral sequence :

$$E_2^{*,*} = \mathrm{Tor}^{*,*}_{H^*(BG;k)}(k, H^*(BH;k)) \Rightarrow H^*(G/H;k) . \qquad (I)$$

The K-theory analogue of this is the Kunneth formula spectral sequence of (14) :

$$E_2^{*,*} = \mathrm{Tor}^{*,*}_{R(G)}(Z, R(H)) \Rightarrow K^*(G/H) . \qquad (II)$$

In (22) we gave a description of the differentials of (II) . In (19), (I) is proved to collapse, which in particular implies that (II) collapses modulo torsion . As yet we have no collapsing theorem for (II) in general .

In this paper we use a generalisation of a spectral sequence of M. Rothenberg and N.E. Steenrod, (20), (an R-S spectral sequence) to investigate the K-theory of homogeneous spaces of Lie groups and other associated spaces . This paper arose from the existence of an R-S spectral sequence converging to $K^*(G/H)$, and an attempt (Theorem 2.3) to identify this with an algebraic spectral sequence which converged to $\mathrm{Tor}^*_{R(G)}(Z, R(H))$.

§1 is given over to algebraic preliminaries . In §1.1, §1.2 we recall the derived functors Tor and Cotor and in §1.3

relate these derived functors to the geometrical situations in which we are interested.

In §1.4 we give an algebraic spectral sequence,

$$\{ E_s(\Lambda, r, A) \}$$

associated with a homomorphism, $r: \Lambda \to A$, of filtered algebras and derive sufficient properties to characterise it under certain conditions.

In §2 we describe the R-S spectral sequence for a space, X, with an H-action. When $r: H \to G$ is a homomorphism of compact Lie groups the R-S spectral sequence for G converges to $K_H^*(G)^\wedge$ and we derive a small result, (Theorem2.3 on when this is isomorphic to the spectral sequence,

$\{ E_s(R(G)^\wedge, r, R(H)^\wedge) \}$.

In §3 we investigate the K-theory of conjugate bundles, P_c, associated with Lie groups - these are the basic spaces used in the construction of (II). Using an R-S spectral sequence we give a simple calculation of $K_H^*(P_c)^\wedge$, (the I(H)-adic completion of the main theorem of (12)).

In §4 we apply an R-S spectral sequence to prove that, for an inclusion, r, of compact, connected Lie groups

$$\mathrm{Tor}^{-i}_{R(G)}(Z, R(H)) = 0 \quad \text{for} \quad (\text{rank } G - \text{rank } H < i).$$

In §5 we use Theorem2.3 to compute the algebra, $K^*(SO(n))$. The computation could equally well have been made using (II) and we include this example in this form merely to emphasise the existence of an alternative technique. The advantages of the R-S spectral sequence for these computations are really restricted to the cases when $\mathrm{Tor}^{*}_{R(G)}(Z, R(H))$ is generated in dimensions less than minus two, since in these cases we have as yet no effective way of computing the differentials of (II) .

I would like to express my gratitude to Dr. Luke Hodgkin for many helpful and encouraging discussions .

§1: Tor and Cotor:

Throughout this section a graded algebra, coalgebra etc. will mean graded by Z_2 unless otherwise stated. The ring, k, will be the integers (Z), the rational field (Q) or a finite field (Z_p , p a prime).

§1.1: Tor .

Let Λ be a finitely generated, graded, augmented k-algebra and N_Λ , ($_\Lambda M$) be finitely generated right (left) Λ- modules; then we have $Tor^{*,*}_\Lambda(N,M)$, the derived functor of $(\_\otimes_\Lambda\_)$ (see (7,10,16)) . $Tor^{*,*}_\Lambda(N,M)$ is a ZxZ_2 bigraded k-module, with $Tor^{p,*}_\Lambda(N,K) = 0$ ($p > 0$) . If N_Λ , $_\Lambda M$ are k-algebras then $Tor^{*,*}_\Lambda(N,M)$ inherits the structure of a ZxZ_2 - bigraded k-algebra, (see (7,10,16)) .

If $\Lambda = k[\rho_1,\ldots,\rho_r;\alpha_1,\alpha_1^{-1},\ldots,\alpha_s,\alpha_s^{-1}]$; $e(\rho_i) = 0$, $e(\alpha_j) = 1$

($e:\Lambda \longrightarrow k$, the augmentation)

and $deg(\rho_1,\alpha_j) \equiv 0 \pmod 2$ then we have the canonical <u>Koszul free resolution of k as a Λ-module</u> (7,10 & 16 p205) given by the Z_2-graded complex :

$$(C_*, \underline{d}) \equiv (\Lambda\otimes E_k(v(\rho_1),\ldots,v(\rho_r);v(\alpha_1-1),\ldots,v(\alpha_s-1)) , \underline{d});$$

$E_k(v(\rho_1),\ldots)$ is the exterior algebra over k on generators in C_{-1} of even internal degree and $\underline{d}$ is the derivation given by $\underline{d}(1\otimes v(\rho_i)) = \rho_i$; $\underline{d}(1\otimes v(\alpha_j-1) = (\alpha_j -1)$ and $e: C_0 = \Lambda \longrightarrow k$.

Let $I(\Lambda) = ker(e)$, then we have a filtration on N_Λ given by $N_m = N.(I(\Lambda))^m$ and the <u>completion</u> , $N^\wedge$, with respect to this filtration is defined by (see (1))

$$N^{\frown} = \varprojlim (N/N_m)$$

If we regard k as a left Λ-module there is an isomorphism

$$\mathrm{Tor}^{*,*}_{\Lambda^{\frown}}(N^{\frown},k) \cong \mathrm{Tor}^{*,*}_{\Lambda}(N,k) \qquad (\text{see } (1,10))$$

which is an algebra isomorphism if N is a filtered k-algebra .

§1.2:Cotor. This derived functor is perhaps less familiar (see(11)).

A k-coalgebra, Λ, is a finitely generated, flat, graded k-module with homomorphisms $\nabla : \Lambda \to \Lambda \otimes \Lambda$, $\eta : k \to \Lambda$ and augmentation , $e: \Lambda \to k$ such that

$$(e\otimes 1).\nabla = 1_{\Lambda} = (1\otimes e).\nabla$$

, and ∇ is commutative and associative .

A right Λ-comodule, $M_{(\Lambda)}$, is a k-module with a homomorphism $\nabla: M \to M\otimes\Lambda$ such that $(1\otimes e).\nabla = 1_M$.

Similarly we define left Λ-comodules, $_{(\Lambda)}N$;

The cotensor product of $M_{(\Lambda)}$ with $_{(\Lambda)}N$ is

$$M \,\square_{\Lambda} N = \ker(\nabla_M\otimes 1 - 1\otimes\nabla_N : M\otimes N \to M\otimes\Lambda\otimes N) .$$

A comodule of the form $(C\otimes\Lambda, 1_C\otimes\nabla_{\Lambda})$, (C a free k-module), is called an extended right Λ- comodule and a summand of an extended comodule is called an injective Λ- comodule .

For an extended comodule , $M = C\otimes\Lambda$, we have a homomorphism $j = 1_C\otimes\nabla_N : C\otimes N \to C\otimes\Lambda\otimes N = M\otimes N$ which induces an isomorphism $j: C \otimes N \xrightarrow{\cong} M \,\square_{\Lambda} N$, with inverse induced by $1_C\otimes e \otimes 1_N$ (see (11)) .

$\mathrm{Cotor}^{*,*}_{\Lambda}(M,N)$ is the derived functor of $(\_ \,\square_{\Lambda} \_)$ obtained from injective resolutions of $M_{(\Lambda)}$, $_{(\Lambda)}N$ relative to the proper class of split short exact sequences of k-modules ((11),(16) p.368

and it is a $Z\times Z_2$ bigraded k-module, with $\mathrm{Cotor}_{\Lambda}^{p,*}(M,N) = 0$ ($p \geq 0$).
If M, N are k-algebras $\mathrm{Cotor}_{\Lambda}^{*,*}(M,N)$ inherits the structure of a $Z\times Z_2$ bigraded algebra (see (11)).

If $\Lambda = E_k(v_1,\ldots,v_n)$ ($\deg(v_i) \equiv 1 \pmod 2$)

$$\nabla(v_i) = v_i \otimes 1 + 1 \otimes v_i \quad , \quad e(v_i) = 0 \; , (\; i = 1,\ldots,n\;)$$

is the Hopf algebra (see (16)) given by the exterior k-algebra on primitive generators (v_i : $i = 1,\ldots,n$) then we have the canonical <u>Koszul injective co-resolution of k as a Λ-comodule</u> (see (18)) given by :

$$(\; D_* , d \;) \equiv (\; k[w_1,\ldots,w_n] \otimes \Lambda \; , \; d \;) \; ,$$

$$\mathrm{bideg}(w_i) = (1,1) \quad , \quad \mathrm{bideg}(z) = (0,\deg(z)) \quad (z \in \Lambda);$$

$\eta : k \to \Lambda = D_0$, $d : D_p \to D_{p+1}$ a coderivation and

$$d(1 \otimes v_j) = w_j \otimes 1 \quad , \quad d(w_j \otimes 1) = 0 \quad , (\; j = 1,\ldots,n\;) .$$

Hence if $(\Lambda,\nabla) \equiv (E_k(\underline{v}),\nabla)$ is as above $\mathrm{Cotor}_{\Lambda}^{*,*}(k,k)$ is a polynomial algebra generated by (w_j: $\mathrm{bideg}(w_j) = (1,1)$) .

<u>§1.3</u>: In this section we collect together some examples of graded algebras, coalgebras, comodules etc. which naturally arise in connection with compact Lie groups.

Let G,H be compact Lie groups and X, (Y), be compact right, (left), G-spaces. Let $K_G^*(\_;k)$ be Z_2-graded, equivariant, complex K-theory with coefficients in k, (hence $K_G^*(\_;Q) = K_G^*(\_) \otimes Q$ and $K_G^*(\_;Z_p)$ is defined as in (17)).

Put $R(G;k) = K_G^*(pt;k)$, then $K_G^*(X;k)$ and $K_G^*(Y;k)$ are R(G;k)-modules , (see (4)).

In particular, if H is a subgroup of G we have an isomorphism

$$K^*_G(G/H;k) \cong R(H;k) \qquad \text{(see (4))}.$$

Furthermore, if $I(G;k) = \ker(R(G;k) \to k)$ we have a filtration on $K^*_G(X;k)$ and an isomorphism ,(see (4)),

$$K^*_G(X;k)\hat{} \cong K^*(X\times_G EG;k)$$

and from § 1.1 $\mathrm{Tor}^{*,*}_{R(G;k)}(R(H;k),k) \cong \mathrm{Tor}^{*,*}_{R(G;k)\hat{}}(R(H;k)\hat{},k)$.

Now, if $K^*(G;k)$ is k-flat then the action map, $X\times G \to X$, induces a comultiplication $K^*(X;k) \to K^*(X;k)\otimes K^*(G;k)$, which makes $K^*(X;k)$ into a right $K^*(G;k)$-comodule. In particular, if $r: H \to G$ is a homomorphism of compact Lie groups the action map : $(h,g) \mapsto r(h).g \quad (h\in H,\ g\in G)$ makes $K^*(G;k)$ into a left $K^*(H;k)$-comodule and we obtain a $Z\times Z_2$ bigraded algebra, $\mathrm{Cotor}^{*,*}_{K^*(H;k)}(k,K^*(G;k))$.

We are interested in the case when G is a compact, connected Lie group such that $K^*(G;k)$, as a Hopf algebra , is an exterior algebra, $E_k(v_1,\ldots,v_n)$ on primitive generators; we recall that if $k=Q$ this is always true, and if $k=Z$ it is true if $\pi_1(G)$ is torsion free.

When $K^*(G;k) = E_k(v_1,\ldots,v_n)$ then $R(G;k)\hat{} = k[[\rho_1,\ldots,\rho_n]]$ and the two are related by a homomorphism

$$\beta : R(G;k)\hat{} \longrightarrow K^1(G;k) \quad \text{,(see (13,14))},$$

which factors as a monomorphism

$$\beta : I(G;k)\hat{}/(I(G;k)\hat{})^2 \longrightarrow K^1(G;k)$$

whose image is the set of primitive generators of $K^*(G;k)$.

§1.4: The algebraic spectral sequence.

Let A be a Z_2-graded k-algebra with a decreasing, complete and Hausdorff filtration, (F^nA), $(n \geq 0)$. Let $Gr^{*,*}A$ be the associated, $Z \times Z_2$ bigraded object (see (1)) then

$$Gr^{n,\alpha}A = (F^nA^{\alpha+n}/ F^{n+1}A^{\alpha+n}), \; (n, \text{ a positive integer}, \alpha \in Z_2).$$

If $y \in A$, we say that y has weight n, ($w(y) = n$), if

$y \in F^nA$ and $0 \neq [y] \in F^nA^{\alpha+n}/ F^{n+1}A^{\alpha+n}$, ($\deg(y) \equiv n+\alpha \pmod 2$ and $[\,]$ denotes the equivalence class of y) ; and if

$\underline{v} = v(\rho_1)\ldots v(\rho_r)$ is a monomial in $E_k(v(\rho))$ we say that $\underline{v}$ has weight r, ($w(\underline{v}) = r$).

Now suppose that $r : \Lambda = k[[\rho_1,\ldots,\rho_m]] \to A$, ($\deg(\rho_i) \equiv 0$) is a homomorphism of graded, filtered algebras - with the $I(\Lambda)$-adic filtration on the power series algebra - then the complex

$$(C , \underline{d}) \equiv (A \otimes E_k(v(\rho_1),\ldots,v(\rho_m)) , \underline{d}) ,$$

where $\underline{d}$ is the derivation defined by: $\deg(v(\rho_i)) \equiv 1$

and $\underline{d}(a \otimes 1) = 0$, ($a \in A$) ; $\underline{d}(1 \otimes v(\rho_i)) = r(\rho_i) \otimes 1$,

is a differential, graded algebra filtered by the A-filtration.

Furthermore, the comultiplication $\nabla : E_k(v(\rho)) \to E_k(v(\rho)) \otimes E_k(v(\rho))$ given by $(v(\rho_i)) = v(\rho_i) \otimes 1 + 1 \otimes v(\rho_i)$, ($i = 1,\ldots,m$), induces a homomorphism of differential, filtered, graded algebras,

$$1_A \otimes \nabla : (C, \underline{d}) \to (C \otimes E_k(v(\rho)) , \underline{d} \otimes 1) .$$

Thus we have, (see (10,16)) :

Lemma 4.1: (C , $\underline{d}$) gives rise to a strongly convergent spectral sequence of $Z \times Z_2$ bigraded algebras and right $E_k(v(\rho))$-comodules

$$\{ E_s^{p,\alpha} (\Lambda, r, A) \; ; \; d_s \} \qquad , (s \geq 1) ,$$

such that :

(i) $E_1^{n,\alpha} = H(\ Gr^{n,\alpha}C\ )$ $(\ n \in Z,\ \alpha \in Z_2\ )$.

(ii) $\underline{d}_s : E_s^{n,\alpha} \rightarrow E_s^{n+s,\alpha-s+1}$ is a derivation .

(iii) $\{E_s^{*,*}\} \Rightarrow H(\ C,\underline{d}\ ) = \mathrm{TotTor}_\Lambda^*(\ A,k\ )$, (c.f. §1.1)

(Here we use $\mathrm{TotTor}_\Lambda^*(\ A,k\ )$ to distinguish the Z_2-graded homology of $(\ C,\underline{d}\ )$ from the bigraded algebra, $\mathrm{Tor}_\Lambda^{*,*}(\ A,k\ )$) .

We require a characterisation of the spectral sequence , $\{E_s(\Lambda,r,A)\}$, as an E_2-spectral sequence, for this purpose we recall its construction in terms of the following exact couple, (see (10)) :

$$\begin{array}{ccc} \bigoplus_n H^*(\ F^{n+1}C\ ) & \xrightarrow{i} & \bigoplus_n H^*(\ F^nC\ ) \\ \delta \nwarrow & & \swarrow j \\ & \bigoplus_n H^*(\ F^nC/F^{n+1}C\ ) & \end{array}$$

In terms of E_1-representatives the differential, $\underline{d}_s$, on $[x]$, $(\ x \in E_1\ )$ is given by : $\underline{d}_s(\ [x]\ ) = [j.(i^{s-1})^{-1}.\delta(x)]$,

(where $[\ ]$ denotes the relevant equivalence class) .

If $j: F^nC \rightarrow F^nC/F^{n+1}C \cong H^*(\ F^nC/F^{n+1}C\ ) \cong Gr^{n,*}A \otimes E_k(v(\rho))$)

is the projection and $j(z) = x \in E_1$ then

$$\delta(x) = [\ \underline{d}(y)]\ \in H^*(\ F^{n+1}C\ ) \quad .$$

Hence, if $[\ \sum_{w(\underline{y}) \geq n} \underline{y} \otimes \underline{v}] \in E_s^{n,\alpha}$, $(\ \underline{y} \in A\ ,\ \underline{v} \in E_k(v(\rho))\ )$,

then $\underline{d}(\ \Sigma\, \underline{y} \otimes \underline{v}\ ) \in F^{n+s}C$ and $[\ \underline{d}(\ \Sigma\, \underline{y} \otimes \underline{v}\ )]$ is a representation for $\underline{d}_s([\ \Sigma\, \underline{y} \otimes \underline{v}\ ])$.

Lemma 1.4.2: $E_s(\Lambda, r, A)$, ($s \geq 1$) , satisfies the following properties.

P(i) $E_1^{**} \cong Gr^{**}(A) \otimes E_K(v(\rho_1),\ldots,v(\rho_m))$.

P(ii) It is a $Z \times Z_2$ bigraded spectral sequence of algebras and right $E_K(v(\rho))$- comodules, whose differentials are derivations .

P(iii)(a) If $z \in E_s^{n,*}$ has an E_1-representative, $z = [y \otimes 1]$, ($y \in F^n A$) , then z is a permanent cycle .

(b) If $z \in E_s^{n,*}$ has an E_1-representative, $[\sum_{i=1}^{m} y_i \otimes v(\rho_i)]$, ($y_i \in F^n A$, $i = 1,\ldots,m$) such that

$$\underline{d}(\Sigma y_i \otimes v(\rho_i)) = \Sigma y_i . r(\rho_i) \otimes 1 \quad \varepsilon \quad F^{n+s}A$$

then $\underline{d}_s(z) = [\Sigma y_i . r(\rho_i) \otimes 1] \quad \varepsilon \quad E_s^{n+s,*}$.

P(iv) If $z \in E_s^{n,*}$ has an E_1-representative, $[\sum_{w(\underline{v}) \geq 2} \underline{y} \otimes \underline{v}]$, ($\underline{y} \in F^n A$; $\underline{v} \in E_K(v(\rho))$),

then $\underline{d}_s(z)$ has an E_1-representative, $[\sum_{w(\underline{v}') \geq 1} \underline{y}' \otimes \underline{v}']$.

Furthermore, if the homomorphisms

$$F_s : E_s \,\square_{E_K(v(\rho))}\, k \to H(E_s) \,\square_{E_K(v(\rho))}\, k = E_{s+1} \,\square_{E_K(v(\rho))}\, k$$

are epic , ($s \geq 1$) , then P(i)-(iv) characterise the E_1-spectral sequence completely .

Proof: P(i)-(iii) are obvious from the preceding discussion. P(iv) follows from the fact that $\underline{d}: C \to C$ reduces the weights of monomials in $E_K(v(\rho))$ by one .

Now suppose the homomorphisms, F_s , are epic .

$$E_s' \,\square_{E_K(v(\rho))}\, k = \{ z \in E_s : \nabla(z) = z \otimes 1 \in E_s \otimes E_K(v(\rho)) \}$$

and hence , by induction on s , if $z \in E_s$ and $\nabla(z) = z \otimes 1$

then $z = [y \otimes 1]$, $(y \varepsilon A)$.

Let (E_s , d_s) , $(s \geq 1)$, be a second spectral sequence of bigraded algebras and right $E_K(v(\rho))$-comodules.

Suppose we are given an isomorphism of algebras and comodules,

$$\phi_1 : \underline{E}_1 \to E_1$$

with respect to which (E_s, d_s) satisfies P(i)-(Iv) .

Suppose ϕ_1 induces isomorphism as far as .

$$\phi_s : \underline{E}_s \to E_s \quad ,$$

we can view this as two differentials, $\underline{d}_s$ and d_s , on $\underline{E}_s$ satisfying P(i)-(iv) and we can speak unambiguosly of $\underline{E}_1$-representatives ,etc .

We have to show $\underline{d}_s = d_s$.

The behaviour of $\underline{d}$ with respect to $w(\underline{v})$, $(\underline{v} \varepsilon E_K(v(\rho)))$, implies that $\underline{E}_s^{n,*}$ is additively generated by elements,

$$z_m = \Big[\sum_{w(y) \geq n \, , \, w(\underline{v}) = m} y \otimes \underline{v} \Big]$$

and we proceed by induction on m, for each n .

If $z_m \varepsilon \underline{E}_s^{n,*}$, P(iii) implies $\underline{d}_s(z_m) = d_s(z_m)$ for $(m = 0$ or $1)$.

By P(ii) , we have $(d_s \otimes 1).\nabla(z_m) = \nabla . d_s(z_m)$

and $(\underline{d}_s \otimes 1).\nabla(z_m) = \nabla . \underline{d}_s(z_m)$.

However, $\nabla(z_m) = \sum_{0 \leq q < m} a_q + z_m \otimes 1 \ \varepsilon \ \underline{E}_s^{n,*} \otimes E_K(v(\rho))$,

where $a_q = \sum z_q \otimes \underline{v}_{m-q}$ and $w(\underline{v}_{m-q}) = m-q$.

Hence $(d_s \otimes 1).\nabla(z_m) = \sum (d_s \otimes 1)(a_q) + d_s(z_m) \otimes 1$

$= \sum (\underline{d}_s \otimes 1)(a_q) + d_s(z_m) \otimes 1$,

by induction .

Thus $\nabla.(\underline{d}_s(z_m) - d_s(z_m)) = (\underline{d}_s(z_m) - d_s(z_m)) \otimes 1$,
and if $m \geq 2$, P(iv) implies

$$\underline{d}_s(z_m) = d_s(z_m) \quad .$$

Remark: Lemma 1.4.2 has analogues for E_{s_0}-spectral sequences, ($s_0 > 1$) . If we have a second spectral sequence,

$$(E_s, d_s) \quad , \quad (s \geq s_0)$$

of bigraded algebras and right $E_k(v(\underline{\rho}))$-comodules together with an isomorphism

$$\phi_{s_0}: \underline{E}_{s_0} \to E_{s_0}$$

of algebras and comodules we can use this isomorphism to give a meaning to P(i)-(iv) applied to (E_s, d_s) .

Corollary 1.4.3: Suppose the homomorphisms of Lemma 1.4.2 ,

$$F_s: \underline{E}_s \,\square_{E_k(v(\underline{\rho}))}\, k \longrightarrow \underline{E}_{s+1} \,\square_{E_k(v(\underline{\rho}))}\, k \quad , (s \geq 1) ,$$

are epic .

If $r':\Lambda \to A$ is a second homomorphism of filtered, graded k-algebras such that $r(\rho_i) - r'(\rho_i) \in r(I(\Lambda)).F^1A$, ($i = 1,..,n$) then $\{\underline{E}_s(\Lambda, r, A), \underline{d}_s\} \cong \{\underline{E}_s(\Lambda, r', A), \underline{d}'_s\}$, ($s \geq 1$) .

Proof: Clearly both spectral sequences satisfy P(i),(ii),(iv) and P(iii)(a) . Suppose $r(\rho_i) - r'(\rho_i) = \sum_j y_{ij}.r(\rho_j)$, ($i = 1,..,m$) and ($y_{ij} \in F^1A$) .

Then if $z = [\sum_n y_n \otimes v(\rho_n)] \in \underline{E}_s^{p,*}(\Lambda, r', A)$ we have

$$z = [\sum_n y_n \otimes v(\rho_n) - \sum_{n,j} y_n.y_{nj} \otimes v(\rho_j)]$$

and $\underline{d}'_s(z)$ has a representative $, [\sum_n y_n.r(\rho_n) \otimes 1]$. •

We call $E_k(v(\underline{\rho}))$ the exterior Hopf algebra on primitive generators in $I(\Lambda)/(I(\Lambda))^2$, (c.f. §1.3) .

Suppose $A = k[[y_1,\ldots,y_n]]$, ($\deg(y_i) \equiv 0$, $i = 1,\ldots,n$) . Let $E_k(v(y_1),\ldots,v(y_n)) = E_k(v(\underline{y}))$ be the exterior Hopf algebra on primitive generators in $I(A)/(I(A))^2$.

Suppose we have a homomorphism of graded, filtered algebras, with the $I(A)$-adic filtration on A ,

$$r: \Lambda \rightarrow A .$$

Let $r(\rho_i) \equiv \sum_{j=1}^{n} k_{ij}.y_j$ (modulo $I(A)^2$) ,

($k_{ij} \in k$; $i = 1,\ldots,m$) .

Then we can define a homomorphism of Hopf algebras, (also denoted by r) , $r: E_k(v(\underline{\rho})) \rightarrow E_k(v(\underline{y}))$ by

$$r(v(\rho_i)) = \sum_{j=1}^{n} k_{ij}.v(y_j) \quad , (i = 1,\ldots,m) .$$

This makes $E_k(v(\underline{\rho}))$ into a left $E_k(v(\underline{y}))$- comodule .
By Corollary 1.4.3 , when the homomorphisms , F_s , are epic , we may rename $\{ \underline{E}_s(\Lambda, r, A) , \underline{d}_s \}$ as

$$\{ \underline{E}_s(E_k(v(\underline{\rho})) , r , E_k(v(\underline{y}))) , \underline{d}_s \} , (s \geq 1) ,$$

without ambiguity .

<u>Lemma 1.4.4</u>: There is an isomorphism of $Z\times Z_2$ bigraded algebras and right $E_k(v(\underline{\rho}))$-comodules

$$\underline{E}_2^{*,*}(E_k(v(\underline{\rho})),r,E_k(v(\underline{y})) \cong \mathrm{Cotor}^{*,*}_{E_k(v(\underline{y}))}(k , E_k(v(\underline{\rho}))) .$$

<u>Proof</u>:
From §1.2 we know that $\mathrm{Cotor}^{**}_{E_k(v(\underline{y}))}(k, E_k(v(\underline{\rho})))$ is given by the homology of the bigraded algebra and comodule, $Gr^{*,*}(A)\otimes E_k(v(\underline{\rho})) = \underline{E}_1^{*,*}$, under the differential

$$d' : Gr^{n,*}A \otimes E_k(v(\rho)) \xrightarrow[1\otimes\nabla]{} Gr^{n,*}A \otimes E_k(v(\underline{y})) \otimes E_k(v(\rho))$$

$$\downarrow d$$

$$Gr^{n+1,*}A \otimes E_k(v(\rho)) \xleftarrow[1\otimes e\otimes 1]{} Gr^{n+1,*}A \otimes E_k(v(\underline{y})) \otimes E_k(v(\rho))$$

(where d is described in §1.2) .

Hence $d'(y \otimes 1) = 0$, $(y \in Gr^{n,*}A)$ and if

$r(\rho_i) \equiv \sum_j \lambda_{ij} . y_j \pmod{I(A)^2}$, $(\lambda_{ij} \in k)$

$$d'(v(\rho_i)) = (1 \otimes e \otimes 1).d(\sum_j \lambda_{ij} \otimes v(y_j) \otimes 1 + 1 \otimes 1 \otimes v(\rho_i))$$
$$= \sum_j \lambda_{ij} . y_j \otimes 1 \quad ;$$

thus $d' = \underline{d}_1$.

§ 2: The Rothenberg - Steenrod spectral sequences.

Throughout this section let H be a compact Lie group such that $K^*(H;k)$ is a flat k-module .We briefly recall the construction of the spectral sequence of (20) .

Define EH to be a free,right H-space filtered by closed subspaces: $(pt.) = D_0 \subset E_0 \subset D_1 \subset E_1 \subset \ldots\ldots$,(with the E_i H-invariant subspaces) such that

(i) $EH = \bigcup_n E_n$, with the topology of the union .

(ii) Each n , E_n is contractible in E_{n+1} .

(iii) Each n , the action map $\psi_n : E_n \times H \rightarrow E_n$ restricts to a relative homeomorphism

$$\phi_n : (D_n, E_{n-1}) \times H \rightarrow (E_n, E_{n-1}) \quad ;$$

$(n = 0 , \phi_0$ a homeomorphism $H \rightarrow E_0)$.

(iv) Each n , there exists $u_n : D_n \rightarrow I$ (I , unit interval) and $h_n : I \times D_n \rightarrow D_n$

representing E_{n-1} as a neighbourhood deformation retract in D_n, (i.e. $E_{n-1} \subset D_n$ is a cofibration (see (20,21 II.4.I , such that the unique maps, $\underline{u}_n$ and $\underline{h}_n$, defined by the following commutative diagrams are continuous:

$$\begin{array}{ccc} D_n \times H & \xrightarrow{p_1} & D_n \\ \phi_n \downarrow & & \downarrow u_n \\ E_n & \xrightarrow{\underline{u}_n} & I \end{array} \qquad \begin{array}{ccc} I \times D_n \times H & \xrightarrow{h_n \times 1} & D_n \times H \\ 1 \times \phi_n \downarrow & & \downarrow \phi_n \\ I \times E_n & \xrightarrow{\underline{h}_n} & E_n \end{array}$$

Setting $BH = EH/H$, $B_n = E_n/H$, with projection $p: EH \to BH$,

$p: (D_n, E_{n-1}) \to (B_n, B_{n-1})$ is a relative homeomorphism.

The complex: $0 \to k \xrightarrow{\eta} K^*(E_0;k) \xrightarrow{\delta} K(E_1,E_0;k) \xrightarrow{\delta} \ldots$ (C)

is an exact sequence of right $K^*(H;k)$-comodules, by (ii) .

Furthermore, with the right action

$$(D_n,E_{n-1}) \times H \times H \to (D_n,E_{n-1}) \times H$$

$$(z,h,h_o) \mapsto (z,h.h_o) , (z \in D_n ; h,h_o \in H)$$

the isomorphisms

$$K^*(D_n,E_{n-1};k) \otimes K^*(H;k) \cong K^*((D_n,E_{n-1}) \times H;k) \underset{\phi_n^*}{\overset{\cong}{\leftarrow}} K^*(E_n,E_{n-1};k)$$

represent $K^*(E_n,E_{n-1};k)$ as an extended, right $K^*(H;k)$-comodule .

In fact, for the Milnor realisation of EH , (see (20)) , the complex, (C) , is the cobar resolution of k as a comodule over $K^*(H;k)$.

Now let X be a compact, left H-space; define

$$X_H = EH \times_H X \quad , \quad X_{H,n} = E_n \times_H X \quad ,$$

then the action map gives a relative homeomorphism,

$$\phi_n: (D_n,E_{n-1}) \times H \times X \to (E_n,E_{n-1}) \times X : (z,h,x) \mapsto (\phi_n(z,h),h.x)$$

$$(z \in D_n , h \in H , x \in X) .$$

We call the spectral sequence obtained by applying $K^*(\_;k)$ to the filtered space $(X_{H,n})$, the <u>Rothenberg-Steenrod spectral sequence for X</u>. It is readily verified, (c.f. (20)), that

<u>Lemma 2.1</u>: The Rothenberg-Steenrod spectral sequence for X,

$$\{E_s, d_s\} \quad (s \geq 2),$$

is a strongly convergent, $Z \times Z_2$ bigraded spectral sequence of algebras such that :

(i) $E_2^{p,\alpha} = \mathrm{Cotor}^{p,\alpha}_{K^*(H;k)}(k, K^*(X;k)) \Rightarrow K^*(X_H;k) \cong K_H^*(X;k)\hat{}$

(ii) $d_s : E_s^{p,\alpha} \to E_s^{p+s,\alpha-s+1}$ is a derivation.

(iii) If $r : H \to G$ is a homomorphism of compact Lie groups, such that $K^*(G;k)$ is k-flat, then the multiplication in G induces a <u>right</u> $K^*(G;k)$-comodule structure on $\{E_s, d_s\}$.

Further if the homomorphism, r, is an inclusion we have $K^*(G_H;k) \cong K_H^*(G;k) \cong K^*(G/H;k)$.

<u>Remark</u>: If $X = pt.$, we have the spectral sequence of (20),

$$E_2^{*,*} = \mathrm{Cotor}^{*,*}_{K^*(H;k)}(k,k) \Rightarrow R(H;k)\hat{},$$

and if $K^*(H;k)$ is an exterior algebra $\mathrm{Cotor}^{*,*}_{K^*(H;k)}(k,k)$ is generated in even total degree, hence the spectral sequence collapses. Also if H is a finite cyclic group it is well known that $\mathrm{Cotor}^{*,*}_{K^*(H;k)}(k,k)$ is generated in bidegrees $(2n,0)$, and again the spectral sequence collapses. Hence, in both these cases we have a filtration on the k-algebra, $R(H;k)\hat{}$, such that

$$Gr^{*,*}R(H;k)\hat{} \cong \mathrm{Cotor}^{*,*}_{K^*(H;k)}(k,k).$$

The case when H is a finite abelian group is important because if G is a compact, connected Lie group, ($\pi_1(G)$ not necessarily torsion free) , there is a finite covering homomorphism $\pi: G_0 \to G$, by a simply-connected, compact, connected Lie group such that $\ker(\pi)$ is central, (see (13,14)), representing G as $G_0/(\ker(\pi))$; (c.f. Example §5) .

Let $r: H \to G$ be a homomorphism of compact Lie groups such that $K^*(G;k) = E_k(v(\underline{\rho}))$; $R(G;k)\hat{} = k[[\rho_1;\ldots,\rho_m]]$, (where $v(\rho_i) = \beta(\rho_i)$; $i = 1,\ldots,m$ and $\beta: R(G;k)\hat{} \to K^1(G;k)$ is as in §1.3,(13) & (14)) .

Also suppose either: (i) $K^*(H;k) = E_k(v(y_1),\ldots,v(y_n))$

or (ii) The Rothenberg-Steenrod spectral sequence for a point as an H-space collapses and $K^*(G;k)$ is a trivial left $K^*(H;k)$-comodule.

In these two cases we have , (where r denotes the 'restriction')

$$\underline{E}_2^{*,*}(R(G;k)\hat{}, r, R(H;k)\hat{}) \cong \mathrm{Cotor}^{*,*}_{K^*(H;k)}(k, K^*(G;k)) .$$

Theorem 2.2: The E_2-Rothenberg-Steenrod spectral sequence

$$\underline{E}_2^{*,*}(R(G;k)\hat{}, r, R(H;k)\hat{}) \cong E_2^{*,*} = \mathrm{Cotor}^{*,*}_{K^*(H;k)}(k, K^*(G;k)) \Rightarrow K_H^*(G;k)\hat{}$$

satisfies P(i)-(iii) ; (c.f. §1.4) .

Proof: By L2.1 we have only to show P(iii)(a)&(b) ;however, the Rothenberg-Steenrod spectral sequence for X is a module over that for a point (coinciding in the limit with the module structure from $R(H;k)\hat{} \to K_H(X;k)\hat{}$), hence P(iii)(a) follows from the collapsing of the spectral sequence for a point.

For P(iii)(b) we first consider the differentials on

$1 \otimes v(\rho_i) \in E_1(\, v(\rho)\,) = E_1^{0,*}$, $(\, i = 1,\dots,n\,)$.

We have a commutative diagram:

$$\begin{array}{ccccccc}
 & & K^0(EG,G;k) & \xleftarrow{p_0} & \tilde{K}^0(BG;k) & \cong & I(G;k)^\wedge \\
K^1(G;k) & \overset{\Sigma}{\nearrow} & \downarrow f & & \downarrow r & & \downarrow r \\
 & \overset{\delta}{\searrow} & K^0(G_H,G;k) & \xleftarrow{p_1} & \tilde{K}^0(BH;k) & \cong & I(H;k)^\wedge \\
 & & (i)^s \uparrow & & \uparrow (i)^s & & \\
 & & K^0(G_H,G_{H,s};k) & \xleftarrow{p_2} & K^0(BH,B_s;k) & & \\
 & & j \downarrow & & \downarrow j & & \\
 & & K^0(G_{H,s-1},G_{H,s};k) & \xleftarrow{p_3} & K^0(B_{s-1},B_s;k) & & \\
 & & \downarrow \cong & & & & \\
 & & K^0(\,(B_{s-1},B_s)\times G;k) & & & &
\end{array}$$

where d_{s+1} (when defined) is represented by $j.((i)^s)^{-1}.\delta$.

If $\rho_i \in I(G;k)^\wedge$, $(\, i = 1,\dots,n\,)$, then

$v(\rho_i) = \beta(\rho_i) = (\Sigma)^{-1}\, p_0(\rho_i)$ and hence

$\delta(v(\rho_i)) = f.\Sigma.(\Sigma)^{-1}.p_0(\rho_i) = p_1.r(\rho_i)$, thus if ρ_i has weight $(s+1)$ then $d_{s+1}(1\otimes v(\rho_i))$ is defined and is as described in P(iii)(b) .

Now, if $z \in K^1(G_{H,n},G_{H,n-1};k)$ is such that $[z] \in E_2^{n,*} \cong E_2^{n,*}$ has an E_1-representative , $[\sum_j x_j \otimes v(\rho_j)]$ $(\, x_j \in F^n R(H;k)^\wedge\,)$, an argument similar to that used in deriving the multiplicative properties of the spectral sequence, (c.f. (20)§16) , shows that $\delta(z) \in K^0(G_H,G_{H,n};k)$ is in $\operatorname{im}(\, p: K^0(BH,B_n;k) \to K^0(G_H,G_{H,n};k))$ and that there exists a representative of $d_s(z)$, (s as in P(iii)(b)) , which is the image of $\sum_j y_j \cdot r(\rho_j)$ under the homomorphism

$$j \cdot p_2 : K^0(BH, B_{n+s-1}; k) \longrightarrow K^0(G_{H,n+s}, G_{H,n+s-1}; k) .$$

Theorem 2.3: Suppose that the homomorphisms

$$F_s : \underline{E}_s(R(G;k)^\wedge, r, R(H;k)^\wedge) \square_{E_k(v(\rho))} k$$
$$\downarrow$$
$$\underline{E}_{s+1}(R(G;k)^\wedge, r, R(H;k)^\wedge) \square_{E_k(v(\rho))} k$$

are epic for $(s \geq 1)$.

If the spectral sequence, (E_s, d_s), of Theorem 2.2 satisfies

either (i) P(iv), (c.f. § 1.4).

or (ii) E_s, $(s \geq 2)$, is generated as an algebra by elements with E_1-representatives of the form

$$\Big[\sum_{W(\underline{v}) \leq 2} y \otimes \underline{v}\Big], \quad (y \in F^n R(H;k)^\wedge ; \underline{v} \in E_k(v(\rho))).$$

Then $\{E_s, d_s\} \cong \{\underline{E}_s(R(G;k)^\wedge, r, R(H;k)^\wedge), \underline{d}_s\}$, $(s \geq 2)$.

Proof: We have only to show (ii).

Suppose that the isomorphism, $E_2 \cong \underline{E}_2$, extends as far as $E_s \cong \underline{E}_s$; we consider $\underline{d}_s$ and d_s on $\underline{E}_s^{n,*}$.

By induction on n, Theorem 2.2 and the differential algebra structure of $\underline{E}_s$ we need only consider elements

$$z_2 = \Big[\sum_{W(\underline{v}) = 2} y \otimes \underline{v}\Big] .$$

On such elements we have, (c.f. proof Lemma 1.4.2),

$$\underline{d}_s(z_2) - d_s(z_2) = [y' \otimes 1], \quad (y' \in F^{n+s} R(H;k)^\wedge),$$

but this is impossible because of the behaviour of the differentials with respect to the total grading.

Remark: The hypothesis that the F_s are epic, although sufficient for our applications of Theorem 2.3 , seems far too strong . However, one can not readily find examples of <u>embeddings</u> , $r: H \to G$, which do not satisfy it . This hypothesis implies, (for $k = Z$ or Q) that

$$\underline{E}_s = \underline{B}_s \otimes E_k(v(\rho_1),\ldots,v(\rho_q)) \quad , (\omega(r(\rho_i)) \geq s \ , \ i = 1,\ldots,q)$$

and $\underline{d}_s(\underline{B}_s) = 0$. Hence in this case $\underline{d}_s = d_s$ if and only if there do not exist elements $z \in \underline{B}_s$ such that

$$d_s(z) = [y \otimes 1] \quad , (y \in R(H;k)^{\wedge}) .$$

Given that the F_s are epic , these <u>transgressive</u> <u>elements</u> , (the term derives from (8 §20)) , $z \in \underline{B}_s$ can be located by means of the Adams operations , ψ^p , (see ((2)p.135 ;(15)p159 and (17)) . These operate (unstably) in both algebraic and geometric spectral sequences and for $k = Z$ or Q effectively strengthen the 'weak' $Z\times Z_2$ grading to a $Z\times Z$ grading .

Note that the F_s epic , ($s \geq 1$) , implies that

$$\mathrm{Tor}^{*,*}_{R(G;k)}(R(H;k),k) \,\square_{\mathrm{Tor}^{*,*}_{R(G;k)}(k,k)}\, k = \mathrm{Tor}^{0,0}_{R(G;k)}(R(H;k),k) .$$

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§3: <u>The K-theory of conjugate bundles</u> .

In this section we employ the Rothenberg-Steenrod spectral sequence to calculate the (completed) K-theory of conjugate bundles associated to G-principal bundles. The importance of these spaces is that they provide, together with Grassmannians , the necessary examples of 'basic' G-spaces used in the construction of the Kunneth formula spectral sequence (see (14 §3)) :

The result proved here is the I(H)-adic completion of the main theorem of (12) , (postponed from (14)) . However this result suffices to construct , as in (14) , the Kunneth formula spectral sequence , since

$$\mathrm{Tor}^{*,*}_{R(G)}(K^*_G(X), Z) \approx \mathrm{Tor}^{*,*}_{R(G)^\wedge}(K^*_G(X)^\wedge, Z) ,$$

(see §1 ; (1) §3 and (10)) .

Throughout this section G will be a compact, connected Lie group such that $\pi_1(G)$ is torsion free , (c.f. §1.3) .

The map $\phi : G\mathrm{x}G \to G$

$$\phi(g, g') = g.g'.g^{-1} , (g, g' \in G)$$

defines a left actions of G (the group) on G (the space) - the <u>conjugation action</u> ; we write G_c for G with this action :

Suppose H is a second compact Lie group . Let

$$\pi : P \longrightarrow P/G$$

be a principal G-bundle with a left H-action such that π is an H-map : The associated <u>conjugate bundle</u> to π is

$$G_c \longrightarrow P_c = P\mathrm{x}_G G_c \xrightarrow[\pi_1]{} P/G ,$$

$\pi_1[p, g] = \pi(p)$, ($p \in P$; $g \in G$) . π_1 is an H-map .

<u>Example</u>: Let E be an H-vector bundle over B , with ground field k (= $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$) . Suppose E is provided with an H-invariant metric of the appropriate type , (positive definite k-conjugate linear form) , and so we can reduce the group to G O(n), U(n) or Sp(n) .

Let $P(E)$ be the associated principal G-bundle , thus

$$P(E)_x = I(k^n, E_x) \quad , (x \in B) ,$$

the space of isometric k-linear maps $k^n \to E_x$.

Let $Aut(E)$ be the bundle of isometric automorphisms of E , whose fibre at $x \in B$ is $I(E_x, E_x)$, with H-action

$$(h.f)(\_) = h.f(h^{-1}.\_) .$$

Then $\psi : P(E) x_G G_c \to Aut(E)$

$$\psi(u x_G v) = u.v.u^{-1} , (u \in I(k^n, E_x) ; v \in I(k^n, k^n)),$$

defines an H-isomorphism of bundles , (see (12)) .

For $G = U(n)$ this is the 'basic' H-space required in (14) .

Now let V be a complex vector space with a left G-action , (a representation of G) . We have two G-vector bundles over GxG ;

(i) GxGxV with action $g.(g_1, g_2, v) = (g_1.g^{-1}, g_2.g^{-1}, g.v)$

(ii) GxVxG with action $g.(g_1, v, g_2) = (g_1.g^{-1}, v, g_2)$.

The vector bundle isomorphisms

$$\alpha_i : GxGxV \to GxVxG \quad , (i = 1,2) ,$$

$$\alpha_1(g_1, g_2, v) = (g_1, g_2.v, g_1.g_2^{-1})$$

$$\alpha_2(g_1, g_2, v) = (g_2, g_1.v, g_1.g_2^{-1})$$

commute with the G-action and hence induce vector bundle isomorphisms $\alpha_i : (GxG) x_G V \to G x_G (VxG) \cong VxG$, ($i = 1,2$) , of vector bundles over G .

If $r : H \to G$ is a homomorphism of compact Lie groups we can endow these vector bundles with the following H-actions .

$Hx((GxG) x_G$

$$H\times((G\times G)\times_G V) \to (G\times G)\times_G V$$
$$(h,[g_1,g_2,v]) \mapsto [r(h).g_1, r(h).g_2, v] \quad \text{and}$$
$$H\times V\times G \to V\times G$$
$$(h,v,g) \mapsto (r(h).v,\ r(h).g.r(h^{-1}))\ .$$

Then we have the following commutative diagrams of H-maps

$$\begin{array}{ccc} (G\times G)\times_G V & \xrightarrow{\alpha_i} & V\times G_c \\ & \pi \searrow \quad \swarrow p_2 & \\ & G_c & \end{array}$$

$(i = 1,2)$ and $\pi([g_1,g_2,v]) = g_1.g_2^{-1}$.

Define $D(V) = D(V,\alpha_1,\alpha_2) \in K_H^{-1}(G_c)$ to be the difference element, (see (4) §2.8) ,

$$d(\ (G\times G)\times_G V\ ,\ (G\times G)\times_G V\ ;\ (\alpha_2)^{-1}.(\alpha_1)\ )\ .$$

Since the squares of the universal difference elements in $K^{-1}(U)$, (U is the infinite unitary group) , are zero it follows that the image of $D(V)^2$ in $K_H^*(G_c)^\wedge \cong K^*(EH\times_H G_c)$ is zero .

<u>Lemma 3.1</u>: Let V be a basic irreducible representation of G . If $\psi: K_H^*(G_c) \to K^*(G)$ is the 'forgetful', homomorphism then $0 \neq \psi(\ D(V)\ ) \in K^{-1}(G)$.

<u>Proof</u>: Let $G*G = G\times G\times I/(\sim)$ be the join of G with itself , ($\sim$ is the equivalence relation defined by $(g_1,g_2,1)\sim(g_1,g_2',1)$ and $(g_1,g_2,0)\sim(g_1',g_2,0)$) .

We have a principal G-bundle

$$f: G*G \to S(G)\ ,\ (\text{ unreduced suspension })\ ,$$
$$f(\ [g_1,g_2,t]\ ) = [g_1.g_2^{-1},t]\ ,\ (\ t \in I\ )$$

such that $\{ G * Gx_G V \underset{\Gamma_1}{\to} S(G) - \dim V\} = x \in K^{-1}(G)$ is non-zero, (see (3) Lemma 3 ; $\dim V$ is the trivial bundle of the same dimension as V).

Now $\preceq_1$, ($\preceq_2$) , induce isomorphisms of vector bundles

$\{(G * G)x_G V \mid [g,t] \in S(G)$ such that $t = 1$ ($t = 0$)$\}$ and $\dim V$, such that $\psi(D(V)) = x \in K^{-1}(G)$.

<u>Theorem 3.2</u>: Let $r: H \to G$ be a homomorphism of compact Lie groups and ρ_i , ($i = 1;\dots;n = \text{rank } G$) , be the basic irreducible representations of G . Then the algebra homomorphism $\mu : E_{R(H)^\wedge}(D(\rho_1),\dots,D(\rho_n)) \to K^*_H(G_c)^\wedge$

is an isomorphism .

<u>Proof</u>: We have $K^*_H(G_c)^\wedge \cong K^*(EHx_H G_c)$ and a strongly convergent spectral sequence

$$E_1^{n,*} = K^*(G_{c_{H,n}}, G_{c_{H,n-1}}) \Rightarrow K^*_H(G_c) \quad ;$$

(since $K^*(H)$ may not be torsion free we do not treat this as an E_2-spectral sequence starting at Cotor) .

However, since $K^*(G)$ is torsion free we have

$$K^*(G_{c_{H,n}}, G_{c_{H,n-1}}) \cong K^*(EH_n, EH_{n-1}) \otimes K^*(G)$$

and the composite

$$K^*_H(G_c)^\wedge \to E_\infty^{0,*} \to E_1^{0,*} \cong K^*(G)$$

is the forgetful homomorphism .

By Lemma 3.1 , $E_1^{0,*}$ consists of permanent cycles and if

$F_1^{*,*} \Rightarrow R(H)^\wedge$ is the spectral sequence obtained from filtering BH , μ induces a map of spectral sequences

$$\{ F_B^{*,*} \otimes E_Z(\psi(D(\rho_1)),\ldots,\psi(D(\rho_n))) \} \longrightarrow \{ E_s^{*,*} \} \quad , (s \geq 1) ,$$

which is an E_1-isomorphism .

Now let $\pi: P \to P/G$ be a principal G-bundle with an H-action . For V, a representation of G , we have an element , $D_P(V) \in K_H^{-1}(P_c)$, defined by $P\times_G D(V)$ and a natural homomorphism of algebras .

$$\mu_P : E_{K_H^*(P/G)^\wedge}(D_P(\rho_1),\ldots,D_P(\rho_n)) \longrightarrow K_H^*(P_c)^\wedge \quad .$$

<u>Theorem 3.3</u>: If P/G is locally H-contractible and of finite covering dimension , (see (4)§3 ; (14)§2) , (i.e. P/G is in the category $\mathfrak{A}_H$ of (14)) , then μ_P is an isomorphism .

<u>Proof</u>: The functors $E_{K_H^*(\_)^\wedge}(D(\rho_1),\ldots,D(\rho_n))$ and $K_H^*(\pi^{-1}(\_))^\wedge$ give two presheaves on the H-invariant closed sets of P/G . We have two strongly convergent spectral sequences , (see (4)§3 ; (21) PartI §3) ,

$$E_2^{*,*} = \hat{H}(P/G; E_{K_H^*(\_)^\wedge}(D(\rho))) \Rightarrow E_{K_H^*(P/G)^\wedge}(D(\rho))$$

$$F_2^{*,*} = \hat{H}(P/G;K_H^*(\pi^{-1}(\_))^\wedge) \Rightarrow K_H^*(P_c)^\wedge \quad .$$

μ_P induces an homomorphism of spectral sequences, hence we must show that μ is an isomorphism of the stalks of the presheaves .

For an orbit, H/H' , in P/G the action map

$$H\times G = H\times G\times(\text{point}) \longrightarrow \pi^{-1}(H/H')$$

induces a homeomorphism , $H \times_{H'} G \to \pi^{-1}(H/H')$, for some homomorphism , $r: H' \to G$.

Hence, by the naturality of μ , we have a commutative diagram

$$\begin{array}{ccc} E_{K_H^*(H/H')^\wedge}(D(\rho)) & \xrightarrow{\mu} & K_H^*((\pi^{-1}(H/H'))_c)^\wedge \\ \cong\downarrow & & \downarrow\cong \\ E_{R(H')^\wedge}(D(\rho)) & \longrightarrow & K_H^*((H\times_{H'} G)_c)^\wedge \\ & \searrow^{\mu} & \swarrow\cong \\ & & K_{H'}^*(G_c)^\wedge \end{array}$$

and, by Lemma 3.2 , μ is an isomorphism on the stalks .

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§ 4: In this section we apply the spectral sequence of § 2 to obtain some information about $\mathrm{Tor}^{*,*}_{R(G)}(R(H), Z)$ and about the forgetful homomorphism

$$\alpha : R(H) \cong K^0_G(G/H) \longrightarrow K^0(G/H) \text{ , (see (5) § 5) .}$$

<u>Theorem 4.1</u>: Let $r: H \to G$ be an inclusion of compact, connected Lie groups such that the following properties are true.

(i) rank H = rank G ;

(ii) $K^*(H)$, ($K^*(G)$) , is the exterior Hopf algebra on primitive generators in $I(H)^\wedge/(I(H)^\wedge)^2$, ($I(G)^\wedge/(I(G)^\wedge)^2$) , (c.f. § 1.3) .

Then (a) α is epic ;

(b) $R(H)^\wedge$ is a free $R(G)^\wedge$-module .

Proof: Consider the Rothenberg-Steenrod spectral sequence for the G-space, $X = G/H$,

$$E_2^{*,*} = \mathrm{Cotor}_{K^*(G)}^{*,*}(\ Z, K^*(G/H)\ ) \Rightarrow K^*(EG\times_G G/H) \cong K^*(BH) \cong R(H)^\wedge .$$

From ((5) Theorem 3.6) we know that $K^{-1}(G/H) = 0$ and that $K^0(G/H)$ is a free abelian group.

Since $K^*(G/H)$ is generated in even dimensions and $K^*(G)$ in odd dimensions the comultiplication,

$$\nabla : K^0(G/H) \to K^0(G/H) \otimes K^{-1}(G)$$

is given by $\nabla(x) = x\otimes 1$, ($x \in K^0(G/H)$) ; (either by considering the dual multiplication or by embedding the K-groups in rational, singular cohomology, via Chern character (see (2) p.193 ; (5) §2.5 ; (15) Ch.18), and using homology-cohomology duality).

Hence $E_2^{*,*} = \mathrm{Cotor}_{K^*(G)}^{*,*}(\ Z, Z\ ) \otimes K^0(G/H)$ is generated in even total degree and the spectral sequence collapses.

Thus the composite

$$\alpha^\wedge : R(H)^\wedge \to E_\infty^{0,0} \to E_2^{0,0} \cong K^0(G/H)$$

is epic, which proves (a) since $K^0(G/H)$ is complete.

Let $x_1, \ldots, x_q$ be a base for $K^0(G/H)$ and choose elements in $R(H)^\wedge$, $y_1, \ldots, y_q$, such that

$$\alpha^\wedge(\ y_i\ ) = x_i \qquad (\ i = 1, \ldots, q\ ).$$

Define $\mu : R(G)^\wedge \otimes \{z_1, \ldots, z_q\} \to R(H)^\wedge$

by $\mu(\ \sum_{i=1}^{q} a_i \otimes z_i\ ) = \sum_{i=1}^{q} r(\ a_i\ ).y_i$

($a_i \in R(G)^\wedge$; $\{z_1, \ldots, z_q\}$ is the free ablian group on the z_i).

We have a complete, Hausdorff filtration on $R(H)^\wedge$, given by the collapsed spectral sequence, such that the $R(G)^\wedge$-module homomorphism, μ, is continuous with respect to this and the $I(G)$-adic filtration on $R(G)^\wedge \otimes \{z_1, \ldots, z_q\}$.

Furthermore, $Gr(\mu): Gr(R(G)^\wedge) \otimes \{z_1, \ldots, z_q\} \rightarrow Gr(R(H)^\wedge)$ is an isomorphism, hence so is μ.

<u>Theorem 4.2</u>: Let $r: H \rightarrow G$ be an inclusion of compact, connected Lie groups such that $K^*(G)$ and $K^*(H)$ are exterior Hopf algebras, as in Theorem 4.1.

Then $\mathrm{Tor}^{-i,*}_{R(G)}(R(H), Z) = 0$ if

$$i > (\operatorname{rank} G - \operatorname{rank} H).$$

<u>Proof</u>: Let T, (T_1), be a maximal torus for H, (G), such that T is included in T_1.

Since $R(H)^\wedge \rightarrow R(T)^\wedge$ is a split monomorphism of $R(H)^\wedge$-modules, by Theorem 4.1, it is also split as an $R(G)^\wedge$-module homomorphism.

Thus $\mathrm{Tor}^{-i,*}_{R(G)^\wedge}(R(T)^\wedge, Z) = 0$ implies that

$$\mathrm{Tor}^{-i,*}_{R(G)^\wedge}(R(H)^\wedge, Z) = 0.$$

Since $R(T_1)^\wedge$ is free over $R(G)^\wedge$ an elementary change of rings theorem, (see (7)§2.4 ; (10)p.116 ; (16)), implies

$$\mathrm{Tor}^{-i,*}_{R(G)^\wedge}(R(T)^\wedge, Z) \cong \mathrm{Tor}^{-i,*}_{R(T_1)^\wedge}(R(T)^\wedge, R(T_1)^\wedge \otimes_{R(G)^\wedge} Z).$$

However, the inclusion $T \rightarrow T_1$ splits as a direct productx

and hence $R(T)^\wedge$ as an $R(T_1)^\wedge$-module has homological dimension equal to (rank T_1 - rank T) = (rank G - rank H) .

Finally we have the isomorphism , (see §1) ,

$$\mathrm{Tor}^{*,*}_{R(G)}(R(H) , Z) \cong \mathrm{Tor}^{*,*}_{R(G)^\wedge}(R(H)^\wedge , Z) .$$

<u>Remark:</u> Since there exist no even differentials in the Kunneth formula spectral sequence

$$E_2^{*,*} = \mathrm{Tor}^{*,*}_{R(G)}(R(H) , Z) \Rightarrow K^*(G/H)$$

Theorem 4.2 implies that this spectral sequence collapses if (rank G - rank H) $\leq$ 2 .

In particular, if rank G = rank H , the homomorphism α induces an isomorphism

$$R(H) \otimes_{R(G)} Z \xrightarrow{\cong} K^0(G/H) .$$

Also, if (rank G - rank H) $\leq$ 2 the homomorphism

$$K^*(G/H) \cong K^*_H(G) \longrightarrow K^*_T(G) \cong K^*(G/T) ,$$

(T a maximal torus in H) , is a monomorphism . This is to be expected in view of the general result that

$$K^*_H(X) \longrightarrow K^*_T(X)$$

is a split monomorphism for all compact H-spaces, X ,(see (6)§4).

<u>§5: $K^*(SO(n))$.</u>

In this section we make an elementary application of Theorem 2.3 in order to calculate the Z_2-graded algebra, $K^*(SO(n))$.

Recall that we have an exact sequence

$$1 \rightarrow Z_2 \rightarrow Spin(m) \longrightarrow SO(m) \rightarrow 1 ,$$

(c.f. Remark (Lemma 2.1)) , which represents $SO(m)$ as $Spin(m)/Z_2$.

The representation rings of Z_2 and $Spin(m)$ are as follows, (see (9)p.54 ; (15)p.187) :

$R(Z_2) = Z[x]/(x^2 + 2x = 0)$, (where $x = y-1$, and y is the canonical irreducible representation of Z_2) ;

$R(Spin(2n+1)) = Z[\rho_1,\ldots,\rho_{n-1}, \Delta_{2n+1}]$.

$R(Spin(2n)) = Z[\rho_1,\ldots,\rho_{n-2}, \Delta^+_{2n}, \Delta^-_{2n}]$,

where the generators are in the kernel of the augmentation .

The homomorphism $r: R(Spin(m)) \to R(Z_2)$ is given by

$r(\rho_i) = 0$, (for all i) ,

$r(\Delta_{2n+1}) = 2^n.x$

$r(\Delta^+_{2n}) = r(\Delta^-_{2n}) = 2^{n-1}.x$.

Now $K^*(Spin(m))$ is an exterior Hopf algebra , and since $Spin(m)$ is connected the comultiplication

$\nabla : K^*(Spin(m)) \to K^*(Z_2) \otimes K^*(Spin(m))$ is $\nabla(z) = 1 \otimes z$;

Hence, for the Rothenberg-Steenrod spectral sequence of $Spin(m)$, the isomorphisms exist

$E_2^{*,*}(R(Spin(m))^\wedge, r , R(Z_2)^\wedge) \approx E_2^{*,*} = Gr^{*}(R(Z_2)^\wedge) \otimes E_K(v(\rho_1),\ldots)$.

The algebraic spectral sequence, $\{\underline{E}_s^{*,*}\}$, satisfies the hypotheses of Theorem 2.3, in fact it has only one non-trivial differential and thereafter $\underline{E}_s^{*,*}$ is generated by elements which are permanent cycles, by properties P(iii)(a) and (b) .

Hence the geometric spectral sequence has only one non-trivial differential and in terms of representatives in

$$Gr^{**}(\; R(Z_2)\hat{}\;) \otimes E_Z(v(\rho_1),\ldots.)$$

a system of generators for the algebra , $K^*(SO(m))$, is given by

$$\mathrm{im}(\; R(Z_2) \to K^0(SO(m))\;) \cong R(Z_2) \otimes_{R(Spin(m))} Z$$

and (a) <u>$m = 2n+1$:</u>

$$v(\rho_1),\ldots,v(\rho_{n-1}) \;;\; (\; 2+x\;).v(\Delta_{2n+1}) \quad \text{in } K^{-1}(SO(2n+1))$$

(b) <u>$m = 2n$:</u>

$$v(\rho_1),\ldots,v(\rho_{n-2}) \;;\; v(\Delta^+_{2n}) - v(\Delta^-_{2n}) \;,$$
$$(\; 2+x\;).v(\Delta^+_{2n}) \quad \text{in } K^{-1}(SO(2n)) \;.$$

We now consider the algebra structure of $K^*(SO(m))$, for this we need the following lemma .

<u>Lemma 5.1</u>: Let $j: K^{-1}(SO(m)) \to K^{-1}(Spin(m))$ be the homomorphism induced by the covering map .

There exist elements, $z_m \in K^{-1}(SO(m))$, such that

$$j(z_{2n+1}) = 2.v(\Delta_{2n+1}) \;,$$
$$j(z^+_{2n}) = 2.v(\Delta^+_{2n})$$

and $0 = x.z_m \in K^{-1}(SO(m))$.

<u>Proof</u>: Let $m = 2n+1$, $x+1 = y$ and let V be the 2^n-dimensional representation of $Spin(2n+1)$ which restricts to $2^n.y$, (see (9)p.54 ; (15)p.187) .

We construct an element of $K^{-1}_{Z_2}(Spin(2n+1))$ in a similar way to that used in § 3 .

Let $Spin(2n+1)\times Spin(2n+1)$ have a $Z_2\times Spin(2n+1)$ - action

$$(\; h,g\;).(g_1,g_2) = (\; h,g_1.g^{-1},g_2.g^{-1}\;) \;,$$

($h \in Z_2$; $g,g_1,g_2 \in Spin(2n+1)$)

We have two isomorphisms of Z_2-vector bundles over

$$Spin(2n+1)\times Spin(2n+1)/Spin(2n+1) \cong Spin(2n+1) \quad ,$$

$$\alpha_1 : Spin(2n+1)\times Spin(2n+1)\times_{Spin(2n+1)} V \rightarrow Spin(2n+1)\times 2^n y$$

$$\alpha_2 : Spin(2n+1)\times Spin(2n+1)\times_{Spin(2n+1)} V \rightarrow Spin(2n+1)\times 2^n \quad ,$$

(c.f. §3 ; 2^n is the trivial representation of this dimension).

Since $y^2 = 1$, as a Z_2-representation we obtain two isomorphisms

$$\alpha_1 , \alpha_2 : (Spin\times Spin \times_{Spin} V) \otimes (1+y) \rightarrow Spin \times 2^n(1+y) \quad .$$

This produces a difference element , z_m , which is represented as a Z_2-automorphism , F , (see (4)§2.8) of the product bundle

$$Spin(2n+1)\times V(1+y) \quad .$$

If we write V_1 for $V\otimes 1$, and V_2 for $V\otimes y$, F is given by $F(g,v_1,v_2) = (g,g.v_2,g.v_1)$,

($v_i \in V_i$, $i = 1,2$; $g \in Spin(2n+1)$, here ($g._$) is the multiplication on 2^n give by the Spin-representation .).

The change of factors is necessitated by the requirement that F be an isomorphism of Z_2-vector bundles .

If we forget the Z_2-action then it is clear that the difference element represented by $F: Spin(2n+1)\times 2V \rightarrow Spin(2n+1)\times 2V$ is, (up to sign) , twice the element constructed in §3 .

Now consider the Z_2-vector bundle automorphism

$$F\otimes 1 \quad \text{of} \quad Spin(2n+1)\times(V_1 + V_2)\otimes y$$

which represents $y.z_{2n+1} \in K^{-1}_{Z_2}(Spin(2n+1)$, under the identification of Z_2-vector spaces

$$(V_1 + V_2)\otimes y \xrightarrow{\cong} V_1 + V_2 \ , \ (v_1,v_2) \mapsto (v_2,v_1) \ ,$$

$F\otimes 1$ is just F again.

Hence we have $y.z_{2n+1} = z_{2n+1}$ and $x.z_{2n+1} = 0$.

Similarly we can construct z^+_{2n} and z^-_{2n}.

From ((14) §3), we know that any difference element in $K^{-1}_{Z_2}(Spin(m)) \cong K^{-1}_{Z_2}(Spin(m)\hat{)}$ can be induced from an element in the K_{Z_2}-theory of a suitable conjugate bundle over a point, hence these difference elements square to zero.

Since, from the spectral sequence there can be no other relations than the one given in Lemma 5.1 we have proved :

<u>Lemma 5.2</u>: There exist isomorphisms of Z_2-graded algebras

$$K^*(SO(2n+1)) \cong E_{R(Z_2)\otimes R(Spin(2n+1))^Z}\frac{(v(\rho_1),\ldots,v(\rho_{n-1}),z_{2n+1})}{(x.z_{2n+1}=0)}$$

$$K^*(SO(2n)) \cong E_{R(Z_2)\otimes R(Spin(2n))^Z}\frac{(v(\rho_1),\ldots,v(\Delta^+-\Delta^-),z^+_{2n})}{(x.z^+_{2n}=0)}$$

(i.e. $Tor^*_{R(Spin(m))}(R(Z_2) , Z) \cong K^*(SO(m))$).

Consider the restriction homomorphisms

$$i_m : K^*(SO(m)) \rightarrow K^*(SO(m-1)) .$$

It is clear that $i_{2n+1}(z_{2n+1}) = 2.z^+_{2n} - (2+x).v(\Delta^+-\Delta^-)$

and $i_{2n}(z^+_{2n}) = z_{2n-1}$.

Also we can choose the ρ_1 such that , (see (15)p.187) ,

$i_{2n+1}(\ v(\rho_i)\ ) = v(\rho_i)$, $(\ i = 1,\ldots,n-2\ )$

$i_{2n+1}(\ v(\rho_{n-1})) = 0$

and $i_{2n}(\ v(\rho_i)\ ) = v(\rho_i)$, $(\ i = 1,\ldots,n-2\ )$.

Hence the ring of operations from complex to real K-theory

$$\lim_{\leftarrow m} K^*(SO(m))$$

is equal to $Z[[x]]\hat{\otimes}E_Z(\ v(\rho_1),\ldots,v(\rho_p),\ldots)$, the completed tensor product of a power series ring on x and an exterior algebra on $\{v(\rho_1),\ldots,v(\rho_p),\ldots\}$, where the filtrations are with respect to weights in x and the $v(\rho_i)$.

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