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DOUBLE INDEPENDENT SUBSETS OF A GRAPH

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§1 Introduction

It was pointed out by Amari [1] that the set of edges of a graph can be divided into two distinct subsets such that the sum of the rank of one subset and the corank of the other may give a number which is less than both the rank and the corank of the graph. Not long afterwards, this number was formally defined by Ohtsuki, Ishizahi and Watanabe and called the hybrid rank [2]. In the same paper the notion of a minimal hybrid rank taken over all possible partitions of the edge set of a graph was introduced and called the topological degree of freedom. That paper together with the paper of Kishi and Kajitani [3] in which maximally distant pairs of trees and principal partition were introduced provide the foundation of the so-called hybrid approach in graph theory. Since 1967 many papers were published in this area, mostly by Japanese authors [8]-[22].

The appearance of new concepts can be a spurious process and it may not at first be clear which are of value and whether the most useful definitions have been made. In this paper we introduce a concept, that of **double independent subsets** of a graph.This concept, although it has never been given a specific name, has featured from the begining of the hybrid approach in graph theory. In matroid theory it is called matroid intersection [6]. The concept of double independent subset inherently has a hybrid flavour. In a previous paper [7] we introduced a concept, that of a **perfect pair of trees** of a graph which is closely related to the concept of double independent subset. We hope that both concepts provide excellent intuitive insight within this area of study. Throughout this paper we shall be concerned only with 2-connected graphs.

§2 Preliminaries

This section is devoted to some definitions and assertions related to material that follows. We presume that the reader is familiar with the following basic notions in graph theory : graph, edge, circuit and cutset. We take these to be primary notions that need not be defined. However we will define all other notions on the basis of these. Throughout we denote a graph by G and its edge set by E. The terms circuit, cutset, tree, cotree, forest and coforest will be used here to mean a subset of edges of a graph. A forest is a maximal circuitless subset of edges while a coforest is a maximal cutsetless subset of edges. If the graph is connected then a forest is a tree and a coforest is a cotree. In what follows, a tree will be denoted by t and a cotree by t^*. Given a tree t, any edge in the corresponding cotree t^* forms exactly one circuit with edges in t. Such a circuit is called a **fundamental circuit** of G with respect to t. Similarly, any edge of the tree t defines exactly one cutset with the edges in the corresponding cotree t^*. Such a cutset is called a **fundamental cutset** of G with respect to t^*. If E' is a subset of E then the **rank** of E' is the cardinality of the largest circuitless subset of E', the **co-rank** of E' is the cardinality of the largest cutsetless subset of E'and the **complement** of E is the set difference E\E'denoted by E^*. By $|E'|$ we denote the number of elements in (that is, the cardinality of) the subset E'.

The **distance** [4] between two trees t_1 and t_2 of a graph, written $|t_1\backslash t_2|$, is the number of edges which are in t_1 but not in t_2. A tree t_2 is said to be **maximally distant from** another tree t_1 [5] if $|t_1\backslash t_2| \geq |t_1\backslash t|$ for every tree t of G. A pair of trees (t_1,t_2) is defined to be a **perfect pair** of trees [7] if both t_2 is maximally distant from t_1 and t_1 is maximally distant from t_2.

Assertion 1 [5]
Given a tree t_0 of a graph G, $(\forall t)\ |t_0\backslash t| \leq \text{rank } t_0^*$

Assertion 2 [7]

The following five statements are equivalent:

i) t_2 is maximally distant from t_1

ii) the fundamental circuit with respect to t_2 defined by an edge in $t_1^* \cap t_2^*$ contains no edges in $t_1 \cap t_2$.

iii) the fundamental cutset with respect to t_1^* defined by an edge in $t_1 \cap t_2$ contains no edges in $t_1^* \cap t_2^*$.

iv) $|t_1\backslash t_2| = \text{rank } t_1^*$.

v) the number of edges in $t_1 \cap t_2$ is equal to the maximal number of independent cutsets of the graph that belong entirely to the tree t_1.

Assertion 3 (theorem 1 of [7])

The following five statements are equivalent:

i) (t_1,t_2) is a perfect pair

ii) fundamental circuits with respect to t_1 and t_2 defined by edges in $t_1^* \cap t_2^*$ contains no edges in $t_1 \cap t_2$

iii) fundamental cutsets with respect to t_1^* and t_2^* defined by edges in $t_1 \cap t_2$ contains no edges in $t_1^* \cap t_2^*$

iv) rank $t_1^* = |t_1 \backslash t_2| = |t_2 \backslash t_1| =$ rank t_2^*

v) the following three numbers, associated with the pair of trees (t_1,t_2) are equal:

- the maximal number of independent cutsets of the graph that belong to t_1
- the maximal number of independent cutsets of the graph, that belong to t_2
- the number of common edges in t_1 and t_2.

§3 Double independent subsets

A subset of edges of a graph G is said to be a **double independent** subset if it contains no circuits and no cutsets of the graph G

According to the preceding definition, we can consider a double independent subset to be a subset of a tree that does not contain cutsets of the graph or (in dual fashion) as a subset of a cotree that does not contain circuits of the graph.

Remark 1

Because a double independent subset does not contain cutsets, removing all the edges of a double independent subset from a graph the rank of the graph remains the same.

Assertion 4

A subset of edges of a graph is a double independent subset iff it can be represented as a set difference of a pair of trees of G.

Proof

⇒ Let d be a double independent subset of a graph G and let t_1 be a tree that contains a double independent subset d. The subgraph G', obtained by removing all edges of d from G, has the same rank as G (Remark 1). Hence any tree t_2 of the subgraph G' is a tree of G. Therefore $d= t_1 \backslash t_2$.

⇐ Let (t_1,t_2) be a pair of trees of a graph G. Then, $t_1 \backslash t_2$ is a subset of both t_1 and t_2^*. Therefore,

$t_1 \backslash t_2$ is a double independent subset of G. □

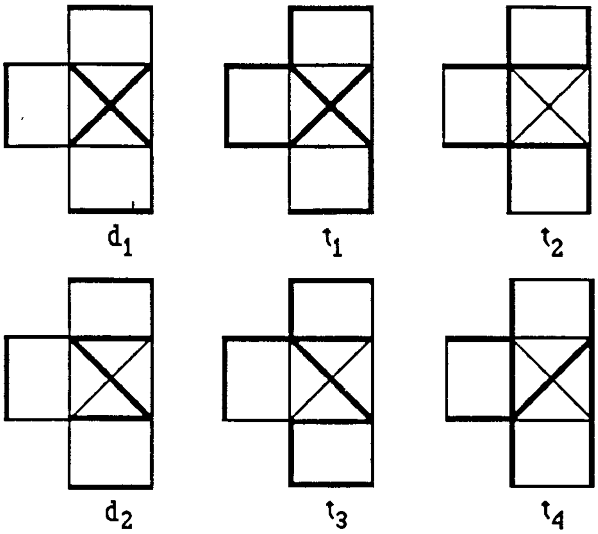

Figure 1

A double independent subset d of edges of a graph G is a **maximally double independent** if, for an arbitrary edge e in the complement of d, $d \cup \{e\}$ is not a double independent subset of G.

Figure 1 shows six copies of a graph and for each copy a different subset of edges is indicated by the use of bold edges. The subset of edges d_1 is a double independent subset and so is d_2. However, d_1 is not a maximally double independent whereas d_2 is. Notice also that $d_1 = t_1 \backslash t_2$ and that $d_2 = t_3 \backslash t_4$, wnere t_1, t_2, t_3 and t_4 are all trees of the graph G.

Assertion 5

A double independent subset d of edges of a graph is a maximally double independent iff every edge in the complement of d form a circuit or/and a cutset with the edges in d only.

Proof

$\Rightarrow$ Given a maximally double independent subset d of G, suppose that there exists an edge e in the complement of d such that for every circuit C_e that contains e, $C_e \backslash (d \cup \{e\})$ is nonempty, and for every cutset S_e that contains e, $S_e \backslash (d \cup \{e\})$ is nonempty. Conseqently, $d \cup \{e\}$ is also a maximally double independent subset which contradict the assumption that d is a maximally double independent subset of G.

$\Leftarrow$ Suppose that, given a double independent subset d of edges of a graph, every edge in the complement of d forms a circuit or/and a cutset with edges in d only. Then, for every edge e in the complement of d, $d \cup \{e\}$ is not double independent due to the fact that it contains a circuit or a cutset. □

According to Assertion 4, for any double independent subset d of a graph, there always exists a pair of trees (t_1, t_2) such that $d = t_1 \backslash t_2$. The next two assertions provide a link between a maximally double independent subset and a perfect pair of trees.

Assertion 6

Let (t_1,t_2) be a pair of trees of a graph G. If $t_1 \backslash t_2$ is a maximal double independent subset of G, then (t_1,t_2) is a perfect pair of trees.

Proof

If $t_1 \backslash t_2$ is a maximal double independent subset of G then, according to Assertion 5, each edge from its complement (including the edges in $t_1^* \cap t_2^*$) makes a circuit or/and a cutset with the elements of $t_1 \backslash t_2$ only. But $t_1 \backslash t_2$ together with $t_1^* \cap t_2^*$ belongs to t_2^* and hence the edges in $t_1^* \cap t_2$ cannot make cutsets with the edges $t_1 \backslash t_2$ only. Therefore the edges in $t_1^* \cap t_2^*$ make circuits with edges in $t_1 \backslash t_2$ only and consequently rank $t_2^* = |t_1 \backslash t_2|$. That means that tree t_1 is maximally distant from the tree t_2. On the other hand, according to Assertion 2, rank $t_1^* \geq |t_1 \backslash t_2|$. We shall now prove that for the case under consideration, equality must occur. That is, t_2 is also maximally distant from t_1. Suppose that this is not true. Then, according to Assertion 1, there exists an edge $e' \in t_1^* \cap t_2^*$ such that a fundamental circuit with respect to t_2, defined by that edge contains an edge $c \in t_1 \cap t_2$. Consequently, $t'_2=(t_2 \backslash e) \cup \{e'\}$ is again a tree and such that $t_1 \backslash t_2 \subseteq t_1 \backslash t'_2$. But subset $t_1 \backslash t'_2$ is, according to Assertion 4, also a double independent subset that contains as a proper subset the maximal double independent subset $t_1 \backslash t_2$. According to Assertion 5, this is a contradiction.

Thus we have proved that t_1 is maximally distant from t_2 and vice versa. Hence (t_1,t_2) is a perfect pair. □

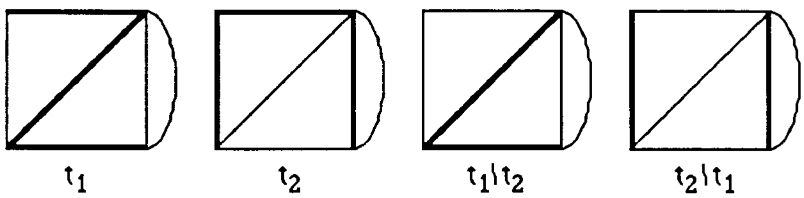

Figure 2

Remark 2

The converse of Assertion 6 is not generally true. That is, if (t_1,t_2) is a perfect pair of trees then their set difference is not necessarily a maximally independent subset. To see this consider figures 2 and 2. Figure 2 shows four copies of the same graph and within each a subset of edges is indicated using bold lines. Now (t_1,t_2) is a perfect pair and (by inspection) $t_1 \backslash t_2$ is a maximal double independent subset while $t_2 \backslash t_1$ is not. Figure 3 shows four copies of the same graph and again various subsets of edges are indicated using bold lines. Again (t_1,t_2) is a perfect pair while neither $t_1 \backslash t_2$ nor $t_2 \backslash t_1$ is a maximal independent subset. The marked edges form neither circuits nor cutsets.

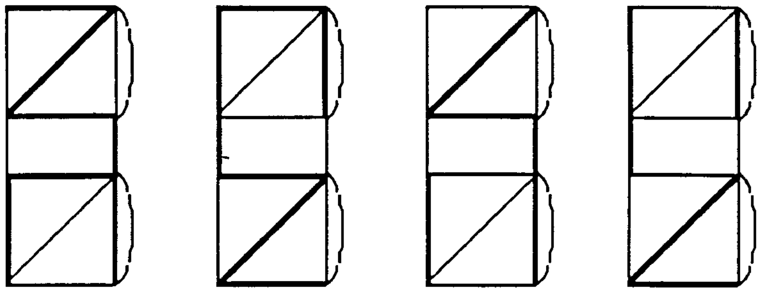

Figure 3

It is obvious that any double independent subset can be embedded in a maximal double independent subset. Also, any subset of a maximal double independent subset is double independent.

Remark 3

Suppose that for a given perfect pair of trees (t_1,t_2), $t_1\setminus t_2$ is not a maximal double independent subset and that we want to enlarge this subset until we obtain a maximal double independent subset. Let (t'_1,t'_2) be another perfect pair such that $t'_1\setminus t'_2$ is a maximal double independent subset and let $t_1\setminus t_2$ be a proper subset of $t'_1\setminus t'_2$. Then $t_2\setminus t_1$ does not belong to $t'_2\setminus t'_1$ as a proper subset. To prove this let us consider a set of edges that have to be added to $t_1\setminus t_2$ in order to obtain $t'_1\setminus t'_2$. Due to properties of perfect pairs (Assertion 3, parts ii) and iii)) we cannot enlarge $t_1\setminus t_2$ with elements of $t_1^*\cap t_2^*$ or $t_1\cap t_2$. So, we have to take some edges from $t_2\setminus t_1$. This means that $t_2\setminus t_1$ partly belongs to $t'_1\setminus t'_2$. But $t'_1\setminus t'_2$ and $t'_2\setminus t'_1$ are disjoint and consequently $t_2\setminus t_1$ only partly belongs to $t'_2\setminus t'_1$ which completes the proof.

To describe more closely the situation when the set difference of a perfect pair of trees is not a maximal double independent subset we establish the following assertion

Assertion 7

Given a perfect pair of trees (t_1,t_2), the following three conditions are equivalent.

(i) $t_1\setminus t_2$ is not a maximal double independent subset.

(ii) There exists an edge in $t_2\setminus t_1$ that belongs to a fundamental circuit with respect to t_2 defined by an edge in $t_1^*\cap t_2^*$ and at the same time forms a fundamental circuit respect to t_1 in which at least one edge is in $t_1\cap t_2$.

(iii) There exists an edge in $t_2\setminus t_1$ that belongs to a fundamental cutset with respect to t_1^* defined by an edge in $t_1\cap t_2$ and at the same time forms a fundamental cutset respect to t_2^* in which at least one edge is in $t_1^*\cap t_2^*$.

Proof

(i)⇐(ii)

Suppose that condition (ii) holds. That is, there exists an edge $e \in t_2 \backslash t_1$ that forms a fundamental circuit with respect to t_1 in which at least one edge is in $t_1 \cap t_2$ (call this conclusion 1). On the other hand, this edge belongs to the fundamental circuit with respect to t_2 defined by an edge in $a \in t_1^* \cap t_2^*$. Because the pair (t_1,t_2) is a perfect pair, each edge from $t_1^* \cap t_2^*$ forms fundamental circuits with respect to t_2 only with edges in $t_2 \backslash t_1$. So, the fundamental circuit defined by a contains only edges from $t_2 \backslash t_1$, including the edge e. As is well known from general graph theory, the intersection of a cutset and a circuit always contains an even number of edges. Therefore, any cutset that includes the edge e, includes at least one more edge from $t_2 \backslash t_1$. Thus, we conclude that edge e does not form a cutset with edges in $t_1 \backslash t_2$ only (call this conclusion 2). According to Assertion 5, conclusions 1 and 2 imply that $t_1 \backslash t_2$ is not a maximal double independent subset.

(i)⇒(ii)

Suppose that condition (ii) is not true. That is, suppose that each edge in $t_2 \backslash t_1$ that belongs to a fundamental circuit with respect to t_2, defined by an edge in $t_1^* \cap t_2^*$ defines a fundamental circuit with respect to t_1 with edges in $t_1 \backslash t_2$ only (call this conclusion 3). The remaining edges in $t_2 \backslash t_1$ that do not belong to fundamental circuits with respect to t_2 defined by an edge in $t_1^* \cap t_2^*$ necessarily form cutsets with edges in $t_1 \backslash t_2$ only (call this conclusion 4). From conclusions 3 and 4 we see that all edges in $t_2 \backslash t_1$ form circuits or cutsets with respect to t_1 with edges only in $t_1 \backslash t_2$. On the other hand, for each perfect pair we have that all edges in $t_1^* \cap t_2^*$ form circuits with edges in $t_1 \backslash t_2$ only and all edges in $t_1 \cap t_2$ form cutsets with edges in $t_1 \backslash t_2$ only. According to Assertion 5, $t_1 \backslash t_2$ is a maximal double independent subset. Using reductio ad absurdum we conclude that (i)⇒(ii).

(ii)⇔(iii)

This is evident from the following well known statement: two edges belong to a circuit iff they both belong to a same cutset.((ii) and (iii) are dual statements) Note also that $t_2 \backslash t_1 = t_1^* \backslash t_2^*$. □

As an immediate consequence of Assertions 6 and 7, we have the following theorem.

Theorem 1

A subset of edges d (of a graph G) is a maximal double independent subset iff the conjunction of the following two statements hold.

(i) There exists a perfect pair (t_1,t_2) such that $d = t_1 \backslash t_2$.

(ii) Each edge in $t_2 \backslash t_1$ that belongs to a fundamental circuit with respect to t_2, defined by an edge in $t_1^* \cap t_2^*$, defines a fundamental circuit with respect to t_1 with edges in $t_1 \backslash t_2$ only.

Assertion 8

If (t_1,t_2) is a maximally distant pair of trees then both $t_1\backslash t_2$ and $t_2\backslash t_1$ are maximal double independent subsets.

Proof

Suppose that one of the subsets $t_1\backslash t_2$ or $t_2\backslash t_1$ is not maximal double independent, for example the subset $t_1\backslash t_2$. Then there exists a maximal double independent subset d that contains $t_1\backslash t_2$ as a proper subset. According to Assertion 6 there is a perfect pair of trees (t'_1,t'_2) such that $t'_1\backslash t'_2=d$. Because $t_1\backslash t_2 \subset d = t'_1\backslash t'_2$, we conclude that $|t_1\backslash t_2| \subset |d| = |t'_1\backslash t'_2|$ which contradicts the assumption that (t_1,t_2) is a maximally distant pair of trees. □

Remark 4

The converse of Assertion 8 is not generally true. That is, if $t_1\backslash t_2$ and $t_2\backslash t_1$ are both maximal double independent subsets, then (t_1,t_2) is not necessarily a maximally distant pair of trees. In order to see this consider figure 4. This figure shows four copies of the same graph with different subsets of edges indicated with bold lines. Now $t_1\backslash t_2$ and $t_2\backslash t_1$ are both maximal double independent subsets but (t_1,t_2) is not a maximally distant pair of trees.

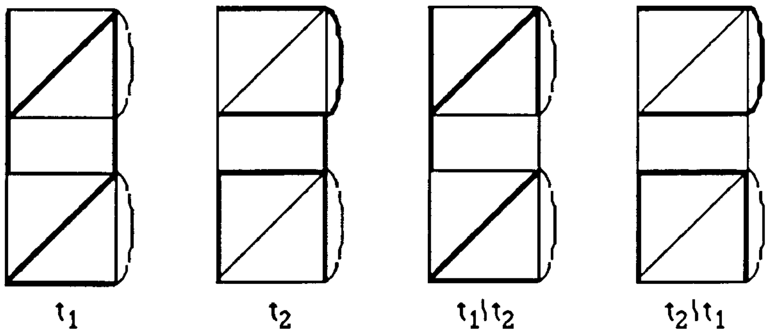

Figure 4

Conclusion

In this paper the notion called maximally double independent subset is considered and related to the concept of perfect pair of trees. Several assertions were stated in order to closely characterise its properties. Also, several examples were included in order to help the reader gain intuitive insight.

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