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# Frobenius-like groups as groups of automorphisms

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**Abstract:** A finite group $FH$ is said to be Frobenius-like if it has a nontrivial nilpotent normal subgroup $F$ with a nontrivial complement $H$ such that $FH/[F,F]$ is a Frobenius group with Frobenius kernel $F/[F,F]$. Such subgroups and sections are abundant in any nonnilpotent finite group. We discuss several recent results about the properties of a finite group $G$ admitting a Frobenius-like group of automorphisms $FH$ aiming at restrictions on $G$ in terms of $C_G(H)$ and focusing mainly on bounds for the Fitting height and related parameters. Earlier such results were obtained for Frobenius groups of automorphisms; new theorems for Frobenius-like groups are based on new representation-theoretic results. Apart from a brief survey, the paper contains the new theorem on almost nilpotency of a finite group admitting a Frobenius-like group of automorphisms with fixed-point-free almost extraspecial kernel.

**Key words:** Frobenius group, Frobenius-like group, fixed points, Fitting height, nilpotency class, derived length, rank, order

## 1. Introduction

Every nonnilpotent finite group contains nilpotent subgroups that are normalized but not centralized by elements of coprime order. Therefore, there are sections of the form $1 \neq [N,g]\langle g\rangle$, where $N$ is a nilpotent $p'$-subgroup and $g$ has prime order $p$. Such a section is a special case of a so-called Frobenius-like group, the formal definition of which is given below. This observation leads us to say that "there is an abundance of Frobenius-like groups around".

**Definition 1.1** *A finite group $G$ is said to be **Frobenius-like** if it contains a nontrivial nilpotent normal subgroup $F$, which is called the **kernel** of $G$; and a nontrivial complement $H$ to $F$ in $G$, which is called the **complement** in $G$ such that*

$$[F,h] = F \text{ for all nonidentity elements } h \in H.$$

**Remark 1.2** *Every Frobenius group is a Frobenius-like group. Conversely, if $FH$ is a Frobenius-like group with kernel $F$ and complement $H$, then $FH/[F,F]$ is a Frobenius group with kernel $F/[F,F]$ and complement $[F,F]H/[F,F]$ isomorphic to $H$. Since $\pi(F)=\pi(F/[F,F])$ we see that $(|F|,|H|)=1$ and $H$ has the structure of a Frobenius complement. In particular (see [9, Chapter 6]),*

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1. *$|H|$ divides $(|F/[F,F]|-1)$,*
2. *all abelian subgroups of $H$ are cyclic, and Sylow subgroups of $H$ are either cyclic or generalized quaternion,*
3. *if all Sylow subgroups of $H$ are cyclic, then $[H,H]$ and $H/[H,H]$ are both cyclic and have coprime orders, $[H,H] \leqslant F(H)$, $F(H)$ is cyclic and $\pi(F(H)) = \pi(H)$.*

The purpose of this paper is to discuss some recent results concerning the structure of a finite solvable group $G$ on which a certain Frobenius-like group $FH$, with kernel $F$ and complement $H$, acts by automorphisms. Earlier similar results, prompted by Mazurov's problem 17.72 in the Kourovka Notebook [18], were obtained in the case of $FH$ being a Frobenius group. In this case, Khukhro, Makarenko, and Shumyatsky in [10–17] obtained restrictions on various parameters of $G$ such as Fitting height, nilpotency class, and exponent, in terms of the fixed-point subgroup $C_G(H)$ of $H$. It is a natural and important problem to extend these results to more general situations, both from the viewpoint of relaxing the strong conditions on the action of the kernel and relaxing the conditions on the structure of the group $FH$ itself. Focusing on the Fitting height and related parameters, Ercan and Güloğlu introduced the concept of a Frobenius-like group and obtained the results presented in [2,3], and together with Khukhro the results in [4].

The paper is structured as follows. The results for $FH$ being a Frobenius group are described in Section 2. Section 3 contains a brief discussion of Frobenius-like groups and the recent results on the structure of groups acted on by them. In Section 4 we obtain a new theorem on almost nilpotency of a finite group admitting a Frobenius-like group of automorphisms with fixed-point-free almost extraspecial kernel, which generalizes Theorem 2.1 in [13] and Proposition C in [3].

## 2. Frobenius groups

We devote this section to the relevant work by Khukhro, Makarenko, and Shumyatsky and assume throughout that the following hypothesis is satisfied.

**Hypothesis 1** *$FH$ is a Frobenius group with kernel $F$ and complement $H$ and $FH$ acts on the finite group $G$ by automorphisms.*

The investigation of the properties and parameters of the group $G$ under Hypothesis I was motivated by Mazurov's problem 17.72 stated in 2010 in "Kourovka Notebook" [18]. He supposes additionally that $GF$ is a Frobenius group with kernel $G$ and complement $F$ (then the group $GFH$ is called a 2-Frobenius group) and asks whether (a) the nilpotency class of $G$ is bounded in terms of the order of $H$ and the nilpotency class of $C_G(H)$, and also whether (b) the exponent of $G$ is bounded in terms of $|H|$ and the exponent of $C_G(H)$.

The question (a) on the nilpotency class was answered affirmatively by Makarenko and Shumyatsky in [16] using also some ideas of Khukhro's. Subsequently it was observed that in order to get very precise structural results about $G$ it suffices to assume that $F$ acts fixed-point-freely on $G$ and not necessarily semiregularly. Therefore, the condition that $C_G(x) = 1$ for all nonidentity elements $x \in F$ was replaced by $C_G(F) = 1$. By a theorem of Belyaev and Hartley [7] based on the classification then $G$ is solvable. Khukhro, Makarenko, and Shumyatsky investigated extensively this case and proved the following theorems over a sequence of papers, namely [10], [11], and [15]. Here $F_i(G)$ denote terms of the Fitting series.

**Theorem 2.1** *Assume that Hypothesis I and the condition $C_G(F) = 1$ are satisfied. Then $G$ is solvable and*

1. *$F_n(G) \cap C_G(H) = F_n(C_G(H))$ for any positive integer $n$,*

2. *the Fitting height of $G$ is equal to the Fitting height of $C_G(H)$,*

3. *the $\pi$-length of $G$ is equal to the $\pi$-length of $C_G(H)$,*

4. *$|G|$ is bounded in terms of $|H|$ and $|C_G(H)|$,*

5. *the rank of $G$ is bounded in terms of $|H|$ and the rank of $C_G(H)$.*

The main ingredient of the proof of Theorem 2.1 is Clifford's theorem, by which any $kFH$-module $V$ on which $F$ acts fixed-point-freely is a free $kH$-module (often also called a regular $kH$-module).

**Theorem 2.2** *Assume that Hypothesis I and the condition $C_G(F) = 1$ are satisfied. If in addition $FH$ is metacyclic and $C_G(H)$ is nilpotent, then $G$ is nilpotent and the nilpotency class of $G$ is bounded in terms of $|H|$ and the nilpotency class of $C_G(H)$.*

Part (b) of Mazurov's question so far has only been answered partially.

**Theorem 2.3** *Assume that Hypothesis I and the condition $C_G(F) = 1$ are satisfied. If in addition $FH$ is metacyclic, then the exponent of $G$ is bounded in terms of $|F|$ and the exponent of $C_G(H)$.*

Theorems 2.2 and 2.3 are proved by reducing each of them to a problem about Lie rings followed by a delicate analysis of the corresponding parameters in the environment of Lie rings.

Although Theorem 2.1 might lead the reader to the expectation that 'all' the parameters of $G$ and $C_G(H)$ must be the same, this is not true for the nilpotency class and exponent, as shown by an example in [1]. It must be mentioned, however, that there are only a few examples of this kind, which cannot support the conjecture that both the nilpotency class and the exponent of $G$ can be arbitrarily larger than those of $C_G(H)$ — of course, with larger complements $H$. It is also worth mentioning that the additional condition of $FH$ being metacyclic is essential in Theorem 2.2, as shown by examples. It is conjectured that this condition can be dropped in Theorem 2.3, but so far a corresponding result was only proved for $|FH| = 12$ by Shumyatsky [17]. It is also conjectured that in Theorem 2.3 the dependence on $|F|$ can be replaced by dependence on $|H|$.

It is now natural to ask what can be said without the assumption that $C_G(F) = 1$. In this direction Khukhro obtained upper bounds for some parameters of the group $G$ in terms of $|H|$ and those of $C_G(H)$ in [12]. Namely, he proved the following theorem, in which $\mathbf{r}(G)$ denotes the rank of a group $G$, that is, the least number $r$ such that every subgroup of $G$ can be generated by $r$ elements.

**Theorem 2.4** *Assuming Hypothesis I and that $(|G|, |FH|) = 1$ we have*

1. *$|G| \leqslant |C_G(F)| \cdot f(|H|, |C_G(H)|)$ and*

2. *$\mathbf{r}(G) \leqslant \mathbf{r}(C_G(F)) + g(|H|, \mathbf{r}(C_G(H)))$,*

*for some functions $f$ and $g$.*

In view of these positive results one can also ask whether it could be possible to prove parts (1) and (2) of Theorem 2.1 under the weaker assumption that $[G, F] = G$. However, the answer is negative as the following example due to Khukhro shows.

**Example 2.5** *Let $FH$ be the Frobenius group of order 6, $K = LM$ be the Frobenius group of order 55, and $T$ be the elementary abelian group of order $7^2$. We can define actions of $FH$ on $K$ and $T$ by automorphisms so that the following hold: $F$ acts trivially on $K$ and fixed-point-freely on $T$; and $H$ acts trivially on $M$ and fixed-point-freely on $L$; and on $T$ by transposing a basis of $T$ so that $|C_T(H)| = 7$. We now define an action of $FH$ on the wreath product of $K$ and $T$, by defining the action of $TFH$ as in "non-commutative induced representation". The base subgroup of the wreath product is $B = K^{t_1} \times K^{t_2} \times \cdots \times K^{t_{49}}$, where $1 = t_1$ and $\{t_i \mid i = 1, 2, \ldots, 49\} = T$. We define the action as $(k^t)^a = (k^a)^{t^a}$ for any $k \in K, t \in T, a \in FH$.*

*Let $U = BT$. Clearly $FH$ acts on $U$. We let $G = [U, F]$. The subgroup $C_U(H)$ contains the Sylow 5-subgroup $M^t$ of $K^t$ for every $t \in C_T(H)$. These $M^t, t \in C_T(H)$ are in $F(C_U(H))$, since for $t \in C_T(H)$, the group $H$ normalizes $L^t$ without fixed-points and $M^t$ centralizes $K^s$ for any $t \neq s \in T$. Clearly $T = [T, F], T < G$ and $G \lhd U$. Therefore, $G$ contains $[K^t, s]$ for any $t, s \in T$. Taking $s, t \in C_T(H)$ and $1 \neq m \in M$ we obtain in $G$ the element $[m^t, s] = (m^{-1})^t m^{ts}$ of order 5 in $M^t \times M^{ts}$, which lies in $F(C_G(H))$. However, for $1 \neq x \in L$, the element $(m^{-1})^t m^{ts}$ acts nontrivially on $\langle x^{-t} x^{ts} \rangle \leqslant [L^t, s] \leqslant G$, and hence $(m^{-1})^t m^{ts}$ is not in $F(G)$. Thus, $F(C_G(H)) \nleqslant F(G)$*

Therefore, the following result of [6] seems to be interesting.

**Theorem 2.6** *Assume Hypothesis I. If $(|G|, |FH|) = 1, [G, F] = G$, and $C_G(F)H$ is a Frobenius group with kernel $C_G(F)$ and complement $H$, then the Fitting height of $G$ is equal to the Fitting height of $C_G(H)$.*

Here the condition that $C_G(F)H$ is a Frobenius group with kernel $C_G(F)$ and complement $H$ implies, of course, that not necessarily $F$ but $FH$ acts fixed-point-freely on $G$. One can ask further whether the same conclusion is true under the assumption that $FH$ acts fixed-point-freely on $G$, and whether the coprimeness condition $(|G|, |FH|) = 1$ could be dropped.

Other recent results on the structure of groups admitting the action of a Frobenius group with a not necessarily fixed-point-free kernel are the following theorems due to Khukhro and Makarenko [13,14].

**Theorem 2.7** *Assume Hypothesis I, assume that $C_G(H)$ is nilpotent of class $c$ and $(|G|, |FH|) = 1$.*

*(a) Then $G$ has a nilpotent characteristic subgroup of index bounded in terms of $|C_G(F)|$ and $|F|$.*

*(b) If in addition $F$ is cyclic, then this subgroup can be chosen to be of index bounded in terms of $c$, $|C_G(F)|$, and $|F|$ and to have nilpotency class bounded in terms of $c$ and $|H|$ only.*

As already mentioned above, the additional condition of $F$ being cyclic cannot be dropped in part (b), even in the case of a fixed-point-free kernel.

**Theorem 2.8** *Suppose that a finite $p$-group $P$ admits a Frobenius group $FH$ of automorphisms with cyclic kernel $F$ of order $p^k$. Let $c$ be the nilpotency class of the fixed-point subgroup $C_P(H)$ of the complement. Then*

*(a) $P$ has a characteristic subgroup $P_1$ of index bounded in terms of $c$, $|F|$, and $|C_P(F)|$ whose nilpotency class is bounded in terms of $c$ and $|H|$ only.*

*(b) $P$ has a characteristic subgroup $P_2$ of index bounded in terms of $|F|$ and $|C_P(F)|$ such that*

*(i) $|P_2| \leqslant |C_P(H)|^{|H|}$;*

*(ii) $\mathbf{r}(P_2) \leqslant |H| \cdot \mathbf{r}(C_P(H))$;*

*(iii) the exponent of $P_2$ is at most $p^{2e}$, where $p^e$ is the exponent of $C_P(H)$.*

## 3. Frobenius-like groups

It is a natural and important problem to extend the results on Frobenius groups of automorphisms to more general situations, both from the viewpoint of (a) relaxing the strong conditions on the action of the kernel and (b) relaxing the conditions on the structure of the group $FH$ itself. As for (a), we saw theorems in Section 2 for a Frobenius group of automorphisms $FH$ under various weaker assumptions. In this section we consider part (b) of this program.

As explained in the Introduction, the concept of a Frobenius-like group was defined during some efforts to understand the real relation between the hypotheses on the acting group $FH$ and its conclusions presented in Section 2. Weakening the condition that $FH$ is a Frobenius group to assuming only that $FH$ is a Frobenius-like group seems to be a very significant generalization, because Frobenius-like groups are much more probable to be encountered in practice. Even if one cannot make use of the full generality of being Frobenius-like, but understands only the case where $F$ is a special group or even an extraspecial group, one gains an important amount of information and methods in analyzing the structure of finite solvable groups with a prescribed subgroup of the group of automorphisms. Indeed, reduction arguments applied while studying the structure of minimal counterexamples often lead us to extraspecial groups $F$ on which a group $H$ acts in such a way that $H$ centralizes $Z(F)$ and acts semiregularly on the Frattini quotient group of $F$, so that $FH$ becomes a Frobenius-like group.

It is worth mentioning that the first difficulty arising in this context when $FH$ is not a Frobenius group is the fact that a $kFH$-module $V$ on which $F$ acts fixed-point-freely no longer must be a free $kH$-module. However, the work by Ercan and Güloğlu in [2, Theorem A] shows that it is not very far from being free, at least for certain Frobenius-like groups, in the sense that it contains a regular $kH$-module that guarantees that $C_V(H)$ is nontrivial. Namely, Theorem A in [2] is proved by reducing the structure of a minimal counterexample to a very restricted configuration and deducing a contradiction by proving the following theorem [2, Proposition C] on representations of some specific groups having a normal extraspecial subgroup, which is also of independent interest.

**Theorem 3.1** *Let $H$ be a group in which each Sylow subgroup is cyclic. Assume that $H/F(H)$ is not a nontrivial 2-group. Let $P$ be an extraspecial group of order $p^{2m+1}$ for some prime $p$ not dividing $|H|$. Suppose that $H$ acts on $P$ in such a way that $H$ centralizes $Z(P)$, and $[P,h]=P$ for any nonidentity element $h \in H$. Let $k$ be an algebraically closed field of characteristic not dividing the order of $G = PH$ and let $V$ be a $kG$-module on which $Z(P)$ acts nontrivially and $P$ acts irreducibly. Let $\chi$ be the character of $G$ afforded by $V$. Then $|H|$ divides $p^m - \delta$ and $\chi_H = \frac{p^m-\delta}{|H|}\rho + \delta\mu$, where $\rho$ is the regular character of $H$, $\mu$ is a linear character of $H$, and $\delta \in \{-1, 1\}$.*

This theorem can be regarded as a generalization of the classical result in [8, Satz V.17.13], and is proved along the same lines as in its proof due to Dade. As an immediate consequence of Theorem 3.1 we have the following.

**Corollary 3.2** *Under the hypotheses and notation of Theorem 3.1, the module $V_H$ contains a regular $kH$-submodule as a direct summand if and only if $|H| \neq p^m + 1$. In particular, if $FH$ is of odd order, then $V_H$ contains a regular $kH$-submodule.*

We now consider the following complicated-looking hypothesis introduced in [4]. It is formulated to avoid the so-called exceptional cases, which possibly occur in Hall–Higman type arguments, and is slightly more general than assuming that $FH$ is of odd order as in the hypothesis of Theorem A in [2].

**Hypothesis 2** *$FH$ is a Frobenius-like group with kernel $F$ and complement $H$ such that a Sylow 2-subgroup of $H$ is cyclic and normal, and $F$ has no extraspecial sections of order $p^{2m+1}$ such that $p^m + 1 = |H_1|$ for some subgroup $H_1 \leqslant H$.*

One can prove the following theorem by repeating word-for-word the proof of [2, Theorem A] (where $|FH|$ was odd).

**Theorem 3.3** *Let $V$ be a nonzero vector space over an algebraically closed field $k$ and let $FH$ be a Frobenius-like group satisfying Hypothesis II and acting on $V$ as a group of linear transformations such that $char(k)$ does not divide the order of $H$. Then $V_H$ has an $H$-regular direct summand if one of the following holds:*

1. $C_V(F) = 0$,
2. $[V, F] \neq 0$ *and $char(k)$ does not divide the order of $F$.*

The upshot for the action of a Frobenius-like group satisfying Hypothesis II on a finite solvable group $G$ is the following.

**Corollary 3.4** *Let $G$ be a finite solvable group acted on coprimely by a Frobenius-like group $FH$ satisfying Hypothesis II so that $[G, F] \neq 1$. Then $C_G(H) \neq 1$.*

This corollary is used in the proof of the following main result of [3].

**Theorem 3.5** *Let $G$ be a finite group admitting a Frobenius-like group of automorphisms $FH$ satisfying Hypothesis II such that $[F, F]$ is of prime order and $[[F, F], H] = 1$. Assume further that $(|G|, |H|) = 1$ and $C_G(F) = 1$. Then*

1. *the Fitting series of $C_G(H)$ coincides with the intersections of $C_G(H)$ with the Fitting series of $G$;*
2. *the Fitting height of $G$ is equal to the Fitting height of $C_G(H)$.*

Exactly as in [11] one can deduce the corresponding theorem about $\pi$-series. Here $O_\pi(G)$ is the largest normal $\pi$-subgroup of a group $G$, for some set of primes $\pi$.

**Theorem 3.6** *Let $G$ be a finite group admitting a Frobenius-like group of automorphisms $FH$ satisfying Hypothesis II such that $[F, F]$ is of prime order and $[[F, F], H] = 1$. Assume further that $(|G|, |H|) = 1$ and $C_G(F) = 1$. Then we have*

1. $O_\pi(C_G(H)) = O_\pi(G) \cap C_G(H)$ *for any set of primes $\pi$,*

2. *the* $\pi$*-length of* $G$ *is equal to the* $\pi$*-length of* $C_G(H)$*,*

3. $O_{\pi_1,\pi_2,\ldots,\pi_k}(C_G(H)) = O_{\pi_1,\pi_2,\ldots,\pi_k}(G) \cap C_G(H)$*, where* $\pi_i$ *is a set of primes for each* $i = 1, \ldots, k$*.*

As the example in [3] shows, the fixed-point-freeness of $F$ on $G$ in the hypothesis of Theorem 3.5 seems to be essential to conclude that $F(G) \cap C_G(H) = F(C_G(H))$, and one cannot even replace the condition $C_G(F) = 1$ by the condition that $C_{C_G(F)}(h) = 1$ for all nonidentity elements $h \in H$, in contrast to Theorem 2.5.

One can obtain similar bounds for some parameters of the group $G$ as in the case where $FH$ is a Frobenius group. Namely we have the following result obtained in [4].

**Theorem 3.7** *Let* $FH$ *be a Frobenius-like group with kernel* $F$ *and complement* $H$ *satisfying Hypothesis II. Let* $P$ *be a finite* $p$*-group admitting* $FH$ *as a group of automorphisms of coprime order so that* $[P, F] = P$*. Then*

1. *the nilpotency class of* $P$ *is at most* $2\log_p |C_P(H)|$*,*

2. $|P|$ *is bounded in terms of* $|H|$ *and* $|C_P(H)|$*,*

3. *the rank of* $P$ *is bounded in terms of* $|H|$ *and the rank of* $C_P(H)$*.*

Recall that the rank of a group $K$ denoted by $\mathbf{r}(K)$ is the smallest integer $s$ such that every subgroup of $K$ can be generated by $s$ elements. With this notation the above theorem leads to an analogue of Theorem 2.6 for Frobenius-like groups; namely we have the following result obtained in [4].

**Theorem 3.8** *Let* $FH$ *be a Frobenius-like group with kernel* $F$ *and complement* $H$ *satisfying Hypothesis II. If a finite group* $G$ *admits* $FH$ *as a group of automorphisms of coprime order, then*

1. $|G| \leqslant |C_G(F)| \cdot f(|H|, |C_G(H)|)$ *and*

2. $\mathbf{r}(G) \leqslant \mathbf{r}(C_G(F)) + g(|H|, \mathbf{r}(C_G(H)))$ *for some functions* $f$ *and* $g$*.*

We present below a result of different nature that is the most recent theorem in this context and appears as the main theorem in [5].

**Theorem 3.9** *Let* $FH$ *be a Frobenius-like group satisfying Hypothesis II acting faithfully by linear transformations on a vector space* $V$ *over a field* $k$ *of characteristic that does not divide* $|FH|$*. Then* $F$ *is solvable of derived length at most* $\log_2 m + 2$*, where* $m = \dim_k C_V(H)$*.*

Here the function $\log_2 m + 2$ is well defined due to the fact that $m \neq 0$ by Corollary 3.4. Notice also that the bound for the derived length is independent of $H$. Finally, it should be noted that additional conditions like Hypothesis II cannot be dropped as shown in Remark 2.4 in [5].

## 4. Frobenius-like group of automorphisms with fixed-point-free almost extraspecial kernel

In this section we prove a new theorem on almost nilpotency of a finite group admitting a Frobenius-like group of automorphisms with fixed-point-free almost extraspecial kernel, which generalizes Theorem 2.7(a). The proof relies on the following generalization of a basic proposition that is essentially used in proving parts (1), (2) of Theorem 2.1 and Theorems 2.6, 2.7, 3.5, 3.6 stated in the previous sections.

**Proposition 4.1** *Let $FH$ be a Frobenius-like group satisfying Hypothesis II such that $[F,F]$ is of prime order and $[[F,F],H]=1$. Suppose that $FH$ acts on a $q$-group $Q$ of class at most 2 for some odd prime $q$ coprime to the order of $FH$. Let $V$ be a $kQFH$-module where $k$ is a field of characteristic not dividing $|QFH|$. Suppose further that $C_V(F)=1$. Then we have $Ker(C_{[Q,F]}(H)\, on\, C_V(H)) = Ker(C_{[Q,F]}(H)\, on\, V)$.*

Here we use alternative notation for the kernel of an action of a group $A$ by automorphisms on a group $B$ denoting $Ker(A\, on\, B) := C_A(B)$ in order to avoid cumbersome subscripts.

**Proof** Suppose the proposition is false and choose a counterexample with minimum $\dim_k V + |QFH|$. To ease the notation we set $K = Ker(C_{[Q,F]}(H)\, on\, C_V(H))$. We proceed in several steps. □

(1) *We may assume that $k$ is a splitting field for all subgroups of $QFH$.*

**Proof** We consider the $QFH$-module $\bar{V} = V \otimes_k \bar{k}$, where $\bar{k}$ is the algebraic closure of $k$. Notice that $dim_k V = dim_{\bar{k}} \bar{V}$ and $C_{\bar{V}}(H) = C_V(H) \otimes_k \bar{k}$. Therefore, once the proposition has been proven for the group $QFH$ on $\bar{V}$, it becomes true for $QFH$ on $V$ also. □

(2) *We have $Q=[Q,F]$ and hence $C_Q(F) \leqslant Q' \leqslant Z(Q)$.*

**Proof** We may assume that $[Q,F]$ acts nontrivially on $V$. If $[Q,F] \neq Q$, then the proposition holds by induction for the group $[Q,F]FH$ on $V$. Since $[Q,F,F]=[Q,F]$ due to the coprime action of $F$ on $Q$, the conclusion of the proposition is true. This contradiction shows that $[Q,F]=Q$ and hence $C_Q(F) \leqslant Q' \leqslant Z(Q)$. □

(3) *$V$ is an irreducible $QFH$-module on which $Q$ acts faithfully.*

**Proof** As $char$(k) is coprime to the order of $Q$ and $K \neq 1$, there is a $QFH$-composition factor $W$ of $V$ on which $K$ acts nontrivially. If $W \neq V$, then the proposition is true for the group $QFH$ on $W$ by induction. That is,

$$Ker(C_Q(H)\, on\, C_W(H)) = Ker(C_Q(H)\, on\, W)$$

and hence

$$K = Ker(K\, on\, C_W(H)) = Ker(K\, on\, W),$$

which is a contradiction with the assumption that $K$ acts nontrivially on $W$. Hence $V=W$.

We next set $\overline{Q} = Q/Ker(Q$ on $V)$ and consider the action of the group $\overline{Q}FH$ on $V$, assuming $Ker(Q$ on $V) \neq 1$. An induction argument gives

$$Ker(C_{\overline{Q}}(H)\, on\, C_V(H)) = Ker(C_{\overline{Q}}(H)\, on\, V),$$

which leads to a contradiction as $\overline{C_Q(H)} = C_{\overline{Q}}(H)$. Thus we may assume that $Q$ acts faithfully on $V$. □

By Clifford's theorem the restriction of the $QFH$-module $V$ to the normal subgroup $Q$ is a direct sum of $Q$-homogeneous components.

(4) *Let $\Omega$ denote the set of $Q$-homogeneous components of $V$. Then $F$ acts transitively on $\Omega$ and $H$ fixes an element of $\Omega$.*

**Proof** Let $\Omega_1$ be an $F$-orbit on $\Omega$ and set $H_1 = Stab_H(\Omega_1)$. Suppose first that $H_1 = 1$. Pick an element $W$ from $\Omega_1$. Clearly, we have $Stab_H(W) \leqslant H_1 = 1$ and hence the sum $X = \sum_{h\in H} W^h$ is direct. It is straightforward to verify that $C_X(H) = \left\{\sum_{h\in H} v^h : v \in W\right\}$. By definition, $K$ acts trivially on $C_X(H)$. Note also that $K$ normalizes each $W^h$ as $K \leqslant Q$. It follows now that $K$ is trivial on $X$. Notice that the action of $H$ on the set of $F$-orbits on $\Omega$ is transitive, and $K \leqslant C_Q(H)$. Hence $K$ is trivial on the whole of $V$ contrary to (3). Thus $H_1 \neq 1$.

The group $H$ acts transitively on $\{\Omega_i : i = 1, 2, \ldots, s\}$, the collection of $F$-orbits on $\Omega$. Let now $V_i = \bigoplus_{W\in\Omega_i} W$ for $i = 1, 2, \ldots, s$. Suppose that $H_1$ is a proper subgroup of $H$, equivalently, $s > 1$. By induction the proposition holds for the group $QFH_1$ on $V_1$, that is,

$$Ker(C_Q(H_1)\,on\,C_{V_1}(H_1)) = Ker(C_Q(H_1)\,on\,V_1).$$

In particular, we have

$$Ker(C_Q(H)\,on\,C_{V_1}(H_1)) = Ker(C_Q(H)\,on\,V_1).$$

On the other hand, we observe that

$$C_V(H) = \{u^{x_1} + u^{x_2} + \cdots + u^{x_s} : u \in C_{V_1}(H_1)\},$$

where $x_1, \ldots, x_s$ is a complete set of right coset representatives of $H_1$ in $H$. By definition, $K$ acts trivially on $C_V(H)$ and normalizes each $V_i$. Then $K$ is trivial on $C_{V_1}(H_1)$ and hence on $V_1$. As $K$ is normalized by $H$ we see that $K$ is trivial on each $V_i$ and hence on $V$ contrary to (3). Therefore, $H_1 = H$ and $F$ acts transitively on $\Omega$ so that $\Omega = \Omega_1$ as desired.

Let now $S = Stab_{FH}(W)$ and $F_1 = F\cap S$. Then $|F : F_1| = |\Omega| = |FH : S|$ and so $|S : F_1| = |H|$. Notice next that as $(|F_1|, |H|) = 1$ there exists a complement, say $S_1$, of $F_1$ in $S$ with $|H| = |S_1|$ by Schur–Zassenhaus theorem. Therefore by passing, if necessary, to a conjugate of $W$ in $\Omega$, we may assume that $S = F_1H$, that is, $W$ is $H$-invariant. This establishes the claim. □

From now on $W$ will denote an $H$-invariant element in $\Omega$ the existence of which is established by (4). It should be noted that the group $Z(Q/C_Q(W))$ acts by scalars on the homogeneous $Q$-module $W$, and so $[Z(Q), H] \leqslant C_Q(W)$ as $W$ is stabilized by $H$. Set $L = K \cap Z(C_Q(H))$. Since $1 \neq K \trianglelefteq C_Q(H)$, the group $L$ is nontrivial. To simplify the notation we set $F_0 = [F, F]$.

(5) *Set $U = \sum_{x\in F_0} W^x$ and $F_2 = Stab_F(U)$. Then $[L, Q] \leqslant C_Q(U)$.*

**Proof** Note that $Z_2(Q) = Q$ by the hypothesis and $Q = [Q, H]C_Q(H)$ as $(|Q|, |H|) = 1$. We have $[Q, L, H] \leqslant [Z(Q), H] \leqslant C_Q(W)$. We also have $[L, H, Q] = 1$ as $[L, H] = 1$. It follows now by the 3-subgroup lemma that $[H, Q, L] \leqslant C_Q(W)$. On the other hand, $[C_Q(H), L] = 1$ by the definition of $L$. Thus $[L, Q] \leqslant C_Q(W)$. Since the group $[L, Q]$ is $F_0$ -invariant as $[F_0, H] = 1$, we conclude that $[L, Q] \leqslant C_Q(U)$. □

(6) *$F_2 = F_1F_0$ is a proper subgroup of $F$, and $K^x$ acts trivially on $U$ for every $x \in F - F_2$. Moreover, $C_V(H) \neq 0$.*

**Proof** For $F_2 = Stab_F(U)$, clearly we have $F_0 \leqslant F_2$ and $F_1 = Stab_F(W) \leqslant F_2$. Assume that $F = F_2$. This forces the equality $V = U$ as $F$ is transitive on $\Omega$ by (4). In fact we have $F = F_1 = F_2$ and so $V = W = U$ as

$F_0 \leqslant \Phi(F)$. Then $[L^{F_2}, Q] \leqslant C_Q(V) = 1$ by (5) and hence $L^{F_2} \leqslant Z(Q)$. Now $Z(Q/C_Q(W))$ and hence $L$ acts by scalars on the homogeneous $Q$-module $V$. Notice that $C_V(H) \neq 0$ by Theorem 3.3 applied to the action of $FH$ on $V$. Since $L$ acts faithfully and by scalars on $V$, we get $L = 1$, which is not the case. Consequently, in any case $F \neq F_2$.

Pick $x \in F - F_2$ and suppose that there exists $1 \neq h \in H$ such that $(U^x)^h = U^x$ holds. Then $[h, x^{-1}] \in F_2$ and so $F_2 x = F_2 x^h = (F_2 x)^h$, implying the existence of an element $g \in F_2 x \cap C_F(h)$ by [[8], Kapitel I, 18.6] by coprimeness. The Frobenius action of $H$ on $F/F_2$ gives that $x \in F_2$, a contradiction. That is, for each $x \in F - F_2$, $Stab_H(U^x) = 1$. In particular, $H-$orbit of $U^x$ is regular and hence we conclude that $C_V(H) \neq 0$.

Set now $U_1 = U^x$ for some $x \in F - F_2$. The sum $Y = \sum_{h \in H} {U_1}^h$ is direct by the preceding paragraph. It is straightforward to verify that $C_Y(H) = \left\{ \sum_{h \in H} v^h : v \in U_1 \right\}$. By definition, $K$ acts trivially on $C_Y(H)$. Note also that $K$ normalizes each ${U_1}^h$ for every $h \in H$ as $K \leqslant Q$. It follows now that $K$ is trivial on $Y$ and hence trivial on $U^x$ for every $x \in F - F_2$, which is equivalent to $K^x$ acting trivially on $U$ for all $x \in F - F_2$ as desired. □

(7) $L \leqslant Z(Q)$ *and hence the group* $LC_Q(W)/C_Q(W)$ *acts by scalars on W.*

**Proof** Recall that $[L, Q] \leqslant C_Q(U)$ by (5). This gives $[L^{F_2}, Q] \leqslant C_Q(U)$. On the other hand, $[L^x, Q] \leqslant [C_Q(U), Q] \leqslant C_Q(U)$ for any $x \in F - F_2$ by (6). Then we have $[L^F, Q] \leqslant C_Q(U)$. It follows that $[L^F, Q] = 1$, that is $L^F \leqslant Z(Q)$. □

(8) $C_U(H) = 0, [U, [F_2, H]] = 0,$ *and hence* $[Q, [F_2, H]] \leqslant C_Q(U)$.

**Proof** It should be noted that the group $[F_2, H]H$ is Frobenius-like. If $[U, [F_2, H]] \neq 0$ then Theorem 3.3 applied to the action of $[F_2, H]H$ on $U$ gives that $C_U(H) \neq 0$. This forces that $C_W(H) \neq 0$ and hence $L$ acts trivially on $W$, which is a contradiction. Therefore, we have $C_U(H) = 0$ and $[U, [F_2, H]] = 0$. As a consequence, $[U, [F_2, H], Q] = 0 = [Q, U, [F_2, H]]$. It follows by the 3-subgroup lemma that $[Q, [F_2, H]] \leqslant C_Q(U)$. □

(9) $[F_2, H] = [F_1, H]$ *and* $[F_1, H] \cap F_0 = 1$

**Proof** By (8), $[F_1, H] \cap F_0 \leqslant C_{Z(F)}(W)$ and hence trivial. □

(10) *If* $F_1 \neq F_2$ *then the theorem follows.*

**Proof** Suppose that $F_1 \neq F_2 = F_1 F_0$. Since $F_0$ is of prime order, $F_0 \cap F_1 = 1$ and hence $F_1 = [F_1, H]$. By (8), $[W, F_1] = 0$. However, $C_W(F_1) = 0$ as $C_V(F) = 0$. This contradiction establishes the claim. □

(11) $[Q, F_1] = 1$

**Proof** Assume the contrary. Note that $F_1 = F_2 = [F_1, H]F_0$. In the case $C_W(F_0) \neq 0$ we apply Lemma 1.3 in [15] to the action of the Frobenius group $(F_2/F_0)H$ on $C_W(F_0)$ and see that $C_W(F_0)|_H$ is free. Since $C_W(H) = 0$ by (8) we must have $C_W(F_0) = 0$. Suppose now that $[Q, F_0]$ is not contained in $C_Q(W)$. Then the group $[Q, F_0]F_0$ is Frobenius-like and it satisfies Hypothesis II as $q$ is odd. This forces by Theorem 3.3 that $C_W(F_0) \neq 0$. This contradiction shows that $[Q, F_0] \leqslant C_Q(W)$ and hence $[Q, F_0] = 1$. By (8) $[Q, F_1] \leqslant C_Q(W)$. As $F_1 \lhd F$ we get $[Q, F_1] \leqslant C_Q(V) = 1$. □

(12) *Final contradiction.*

**Proof** By (7), $L^F \leqslant Z(Q)$. Suppose that $[L^F, F]$ is not contained in $C_Q(W)$ and let $z \in [L^F, F] - C_Q(W)$. It follows now that $\prod_{f \in F} z^f$ is a well defined element of $Q$ which lies in $C_{[L^F, F]}(F) = 1$. Thus, by (7), we have

$$1 = \prod_{f\in F} z^f = (\prod_{f\in F_1} z^f)(\prod_{f\in F-F_1} z^f) \in (\prod_{f\in F_1} z^f)C_Q(W).$$

On the other hand, we have $[Q, F_1] = 1$ by (11). That is $(\prod_{f\in F_1} z^f)C_Q(W) = z^{|F_1|}C_Q(W)$ and so $z \in C_Q(W)$ as $|F_1|$ is coprime to $|z|$. This contradiction shows that $[L^F, F] \leqslant C_Q(W)$, in fact $[L^F, F] = 1$. As a consequence, $L \leqslant Z(QFH)$ and so $C_V(L)$ is $QFH$-invariant. This leads to the contradiction that $[V, L] = 0$ as $0 \neq C_V(H) \leqslant C_V(L)$. □

We can now obtain an analogue of Proposition 2.11 in [13].

**Proposition 4.2** *Let $G$ be a finite solvable group admitting a Frobenius-like group $FH$ of automorphisms of coprime order satisfying Hypothesis II with kernel $F$ and complement $H$ such that $[F, F]$ is of prime order and $[[F, F], H] = 1$. Assume that $V = F(G) = O_p(G)$ is an elementary abelian $p$-group and $C_G(H)$ is nilpotent of odd order. If $C_V(F) = 1$, then $G = VC_G(F)$.*

**Proof** The group $\bar{G} = G/V$ acts faithfully on $V$. Assume that $F$ acts nontrivially on $F(\bar{G}) = S/V$. Then we see by a Hall–Higman type reduction that there exists an $FH$-invariant nontrivial $q$-subgroup $Q$ of $S$ of class at most 2 with $[Q, F] = Q$. It follows by Corollary 3.4 applied to the action of $FH$ on $Q$ that $C_Q(H) \neq 0$. The same corollary applied to the action of $FH$ on $V$ gives $C_V(H) \neq 0$, too. Since $C_G(H)$ is nilpotent we conclude that $C_Q(H)$ centralizes $C_V(H)$, contrary to Theorem 4.1. Thus $F$ is trivial on $F(\bar{G})$. Then $[F, F(\bar{G}), \bar{G}] = \bar{1} = [F(\bar{G}), \bar{G}, F]$. It follows now by the 3-subgroup lemma that $[\bar{G}, F] \leqslant C_{\bar{G}}(F(\bar{G})) \leqslant F(\bar{G})$. Hence $[\bar{G}, F] = \bar{1}$ as $[\bar{G}, F] = [\bar{G}, F, F]$ by coprimeness. □

**Theorem 4.3** *Let $G$ be a finite group admitting a Frobenius-like group $FH$ of automorphisms of coprime order satisfying Hypothesis II with kernel $F$ and complement $H$ such that $[F, F]$ is of prime order and $[[F, F], H] = 1$. Suppose that the fixed-point subgroup $C_G(H)$ of the complement is nilpotent of odd order. Then the index of the Fitting subgroup $F(G)$ is bounded in terms of $|C_G(F)|$ and $|F|$.*

**Proof** This can be proven as in Theorem 2.1 in [13] by the replacement of Proposition 2.11 in [13] by Proposition 4.2 above. □

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