INFORMATION SPREAD IN NETWORKS: GAMES, OPTIMAL CONTROL, AND STABILIZATION

BY

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DISSERTATION

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ABSTRACT

This thesis focuses on designing efficient mechanisms for controlling information spread in networks. We consider two models for information spread. The first one is the well-known distributed averaging dynamics. The second model is a nonlinear one that describes virus spread in computer and biological networks. We seek to design optimal, robust, and stabilizing controllers under practical constraints.

For distributed averaging networks, we study the interaction between a network designer and an adversary. We consider two types of attacks on the network. In ATTACK-I, the adversary strategically disconnects a set of links to prevent the nodes from reaching consensus. Meanwhile, the network designer assists the nodes in reaching consensus by changing the weights of a limited number of links in the network. We formulate two problems to describe this competition where the order in which the players act is reversed in the two problems. Although the canonical equations provided by the Pontryagin's Maximum Principle (MP) seem to be intractable, we provide an alternative characterization for the optimal strategies that makes connection to potential theory. Further, we provide a sufficient condition for the existence of a saddle-point equilibrium (SPE) for the underlying zero-sum game.

In ATTACK-II, the designer and the adversary are both capable of altering the measurements of all nodes in the network by injecting *global* signals. We impose two constraints on both players: a power constraint and an energy constraint. We assume that the available energy to each player is *not sufficient* to operate at maximum power throughout the horizon of the game. We show the existence of an SPE and derive the optimal strategies in closed form for this attack scenario.

As an alternative to the "network designer vs. adversary" framework, we investigate the possibility of stabilizing unknown network diffusion processes using a distributed mechanism, where the uncertainty is due to an attack on the network. To this end, we propose a distributed version of the classical logic-based supervisory control scheme. Given a network of agents whose dynamics contain unknown parameters, the distributed supervisory control scheme is used to assist the agents to converge to a certain set-point without requiring them to have explicit knowledge of that set-point. Unlike the classical supervisory control scheme where a centralized supervisor makes switching decisions among the candidate controllers, in our scheme, each agent is equipped with a local supervisor that switches among the available controllers. The switching decisions made at a certain agent depend only on the information from its neighboring agents. We provide sufficient conditions for stabilization and apply our framework to the distributed averaging problem in the presence of large modeling uncertainty.

For infected networks, we study the stability properties of a susceptible-infected-susceptible (SIS) diffusion model, so-called the n-intertwined Markov model, over arbitrary network topologies. Similar to the majority of infection spread dynamics, this model exhibits a threshold phenomenon. When the curing rates in the network are high, the all-healthy state is the unique equilibrium over the network. Otherwise, an endemic equilibrium state emerges, where some infection remains within the network. Using notions from positive systems theory, we provide conditions for the global asymptotic stability of the equilibrium points in both cases over strongly and weakly connected directed networks based on the value of the basic reproduction number, a fundamental quantity in the study of epidemics.

Furthermore, we demonstrate that the n-intertwined Markov model can be viewed as a best-response dynamical system of a concave game among the nodes. This characterization allows us to cast new infection spread dynamics; additionally, we provide a sufficient condition, for the global convergence to the all-healthy state, that can be checked in a distributed fashion. Moreover, we investigate the problem of stabilizing the network when the curing rates of a limited number of nodes can be controlled. In particular, we characterize the number of controllers required for a class of undirected graphs. We also design optimal controllers capable of minimizing the total infection in the network at minimum cost. Finally, we outline a set of open problems in the area of information spread control.

بِسْمِ ٱللّٰهِ ٱلرَّحْمٰنِ ٱلرَّحِيمْ

In the name of Allah, the Beneficent, the Merciful

Verily, knowledge is a lock and its key is the question.

Imam Ja’far ibn Muḥammad al-Ṣādiq (a.s.)

To my Wife, Son, Daughter, Parents, and Siblings

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LIST OF ABBREVIATIONS

CLF	Control Lyapunov Function
DAG	Directed Acyclic Graph
GAS	Globally Asymptotically Stable
ISS	Input-to-State Stable
MP	Pontryagin's Maximum Principle
ODE	Ordinary Differential Equation
OFDMA	Orthogonal Frequency Division Multiple Access
SCC	Strongly Connected Component
SIR	Susceptible-Infected-Removed
SIS	Susceptible-Infected-Susceptible
SPE	Saddle-Point Equilibrium

NOTATIONS

$\mathbb{R}$	The set of real numbers
$\mathbb{R}_{\geq 0}$	The set of nonnegative real numbers
$\mathbb{R}_{>0}$	The set of positive real numbers
$\mathbb{Z}$	The set of integers
$C^1[a,b]$	The space of continuously differentiable functions over the interval $[a,b] \subseteq \mathbb{R}$
$[n]$	The set $\{1,\ldots,n\}$, $n \in \mathbb{Z}_{\geq 1}$
$\sigma(X)$	The set of eigenvalues of a square matrix X
$\rho(X)$	The eigenvalue with the largest absolute value in $\sigma(X)$
$\mu(X)$	The eigenvalue with the largest real part in $\sigma(X)$
$x \gg y$	$x_i > y_i$ for all $i \in [n]$, $x, y \in \mathbb{R}^n$
$x \succ y$	$x_i \geq y_i$ for all $i \in [n]$ but $x \neq y$, $x, y \in \mathbb{R}^n$
$x \succeq y$	$x_i \geq y_i$ for all $i \in [n]$, $x, y \in \mathbb{R}^n$
$x_{\min}$	The smallest entry in a vector x
$x_{\max}$	The largest entry in a vector x
$\lvert . \rvert$	The absolute value of a scalar, or the cardinality of a set
$\Vert . \Vert_p$	The p-norm of a vector, $p \in [0, \infty]$
$\Vert . \Vert_{L_p}$	The L_p-norm of a vector, $p \in [0, \infty]$
$\Vert f \Vert_{C^1}$	$\Vert f \Vert_{L_\infty} + \left\Vert \frac{d}{dx} f \right\Vert_{L_\infty}$
$\mathcal{L}^2$	The space of measurable functions whose L_2-norm is bounded
$\Vert X \Vert_2$	The induced 2-norm of a square matrix X

diag(X)	A column vector that contains the diagonal entries of a square matrix X
diag(x)	A diagonal matrix with $X_{ii} = x_i$, $i \in [n]$. The same definition holds for diag$(x_1, \dots, x_n)$
X^{-1}	The inverse of a real matrix X
X^T	The transpose of matrix X
$\lambda_1(X)$	The largest eigenvalue of a matrix X with real spectra
$\lambda_n(X)$	The smallest eigenvalue of a matrix X with real spectra
$\sum_{j>i}(.)$	$\sum_{j=2}^{n}\sum_{i=1}^{j-1}(.)$, for some $n \in \mathbb{Z}_{\geq 2}$
I	The identity matrix
$\mathbf{1}$	The all-ones vector

CHAPTER 1

INTRODUCTION

This chapter provides a motivation for the questions studied in the thesis. We provide examples of how information spread in networks could have drastic economical and financial impact on society. We motivate the idea of *control intervention* in networks to control diffusion processes, describe existing attempts, and outline our approach and control design methodology.

1.1 Why Study Information Spread?

Various global patterns in computer, social, and biological networks stem from local interactions among nodes. Examples include birds flying in formation, propagation of rumors and computer viruses, and epidemics. A common ingredient among these examples is the exchange of information in the network, where “information” may refer to ideas, products, or viruses. Studying the propagation of information in networks is important in and relevant to many fields including control, communications, signal processing, and social sciences. Depending on the type of information, the objective of the nodes or the network designer can be either accelerating or decelerating the spread of information across the network. For example, while a network designer would be interested in containing a rumor in an infected network, he would attempt to increase the adoption of a new product in a viral marketing scenario. A large body of literature is dedicated to modeling information diffusion processes in networks; however, controlling such processes is a relatively new area that presents many open problems. Controlling information spread is challenging primarily due to the dependence of the information spread dynamics on the underlying network structure, and the networks we are interested in studying tend to contain a large number of nodes. A typical communication or online social network in today’s world comprises millions of users with numerous connections. While high connectivity provides an unprecedented source of data, controlling such networks is a tall order.

The purpose of this thesis is to demonstrate that tools from control and game theories can be utilized to tackle the problem of information spread control in networks. Before we delve into the details of the problems we study, we provide practical examples that highlight the magnitude of the impact that information spread can have on society.

1.2 Societal Impact of Information Spread

Below we list a few recent events which motivated our research, and occurred due to the propagation of information in networks.

Spread of Rumors and Misinformation over Networks Online social networks provide a medium for the rapid spread of misinformation and rumors. A recent example of how detrimental rumor spread can be occurred in April 2013 when the Twitter account of the Associated Press (AP) was hacked, and a false message claiming that the White House was attacked was sent to the many followers of the account. The message was reportedly retweeted more than 3000 times within a few minutes. The security breach was quickly detected by the AP; nonetheless, this rumor led to a sharp 143 points drop in the Dow Jones industrial average [1] as shown in Fig. 1.1.

Virus Spread in Biological and Computer Networks The spread of viruses among humans is largely dependent on one-to-one interactions. Advances in ground and air transportation systems enabled humans to cover larger distances in shorter times; however, travelers also carry infections with them across borders which may lead to global epidemic outbreaks [3]. Figure 1.2 illustrates disease spread along travel routes and demonstrates the possibility of the emergence of global pandemics.

Computer networks are also prone to virus propagation. The last decade has witnessed many examples of security breaches resulting from virus spread across networks. Perhaps the most notable of this is the recent outbreak of Stuxnet, which is a computer worm designed to attack control machinery in various systems such as assembly lines, power plants, and nuclear plants [5].

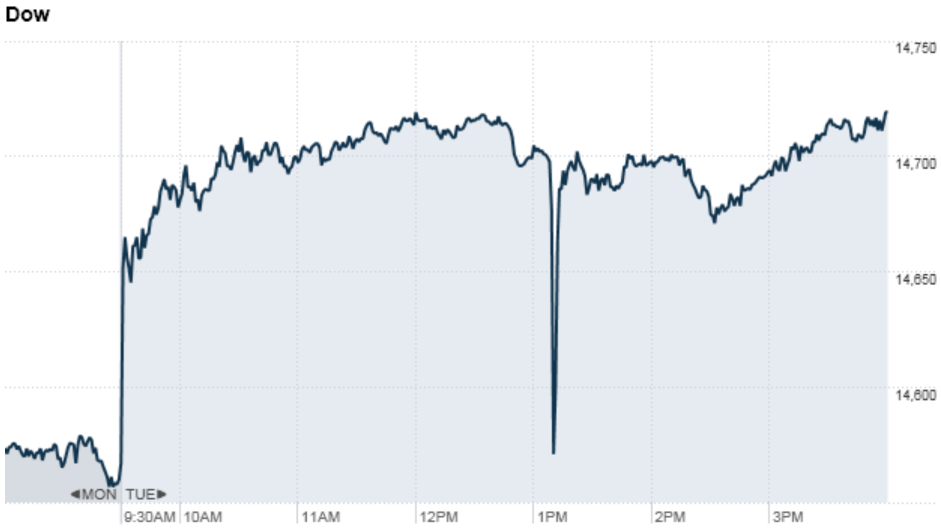


Figure 1.1: Effect of a rumor that was broadcast via the AP Twitter account on the Dow Jones industrial average [2].

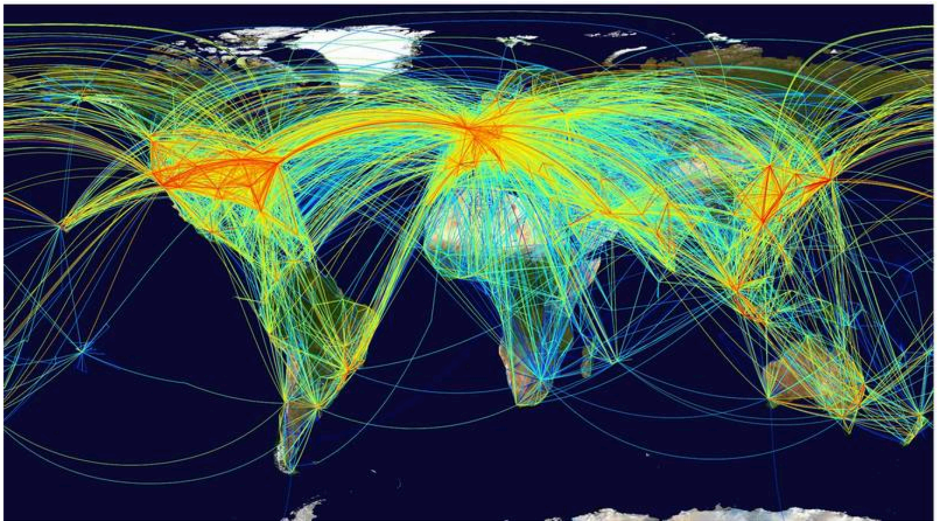


Figure 1.2: Disease spread via mobility [4].

Stuxnet has the ability to spread over computer networks, and it was reported that it was successful in compromising control mechanisms in Iranian nuclear facilities [6].

Delay Propagation in Transportation Networks Another example of information diffusion arises in transportation networks in the form of delay propagation. In the US, it has been estimated that flight delays cost an estimated $40 billion per year according to the 2008 Report of the Congress Joint Economic Committee [7]. Researchers have found that crew and passenger connections are a major source of delays in flight schedules. Further congestion in an airport was shown to propagate to surrounding airports and beyond [8] as demonstrated in Fig. 1.3.

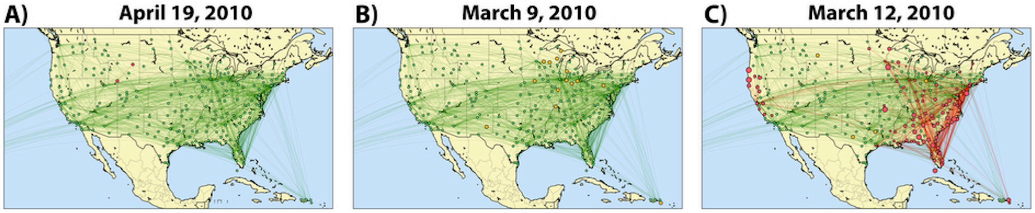


Figure 1.3: Delay propagation across US airports [7].

Whether it is a rumor spreading in a social network or delays propagating across airports, diffusion processes across networks have the ability to replicate local behaviors over extremely large networks in a very short time. These examples emphasize the importance of designing control mechanisms that are capable of effectively responding to such *cascading* effects.

1.3 Dynamical Models for Information Diffusion

The literature is rich with dynamical models that describe information diffusion for various types of networks. Earlier models did not depend on the network structure explicitly. Examples include the Bass model [9] that describes the adoption of a new product in a population, a game-theoretic model to describe the evolution of conventions by Young [10], and a spread model for gonorrhea by Lajmanovich and Yorke [11].

The availability of data in recent years makes capturing the network effect on the diffusion of information a viable direction to pursue. In fact, the examples we listed in Section 1.2, which emanate from different fields, all have clear dependence on the underlying network structure. A wide range of models that depend explicitly on the network structure have been proposed to describe different phenomena occurring in biological and social networks, and we will describe several of them next.

A popular information spread model is the distributed averaging dynamics. In this model, an agent updates its value as a linear combination of the values of its neighbors. Averaging dynamics is the basic building block in many multi-agent systems, and they are widely used whenever an application requires multiple agents, who are graphically constrained, to synchronize their measurements. Examples include formation control, coverage, distributed estimation and optimization, and flocking [12–14]. Besides engineering, linear averaging finds applications in other fields as well. For instance, social scientists use averaging to describe the evolution of opinions in networks [15].

The Hegselmann-Krause dynamics [16] are also used to describe the evolution of opinions. Unlike distributed averaging, the Hegselmann-Krause dynamics allow the underlying graph to change with time as the nodes update their values. Many threshold based models, such as Granovetter's model [17], where agents adopt a certain behavior based on the choices of their neighbors, have been previously proposed; see [18] for an extensive review of such models. These models have applications in voting, riot behavior, and rumor diffusion. Recently proposed models have also incorporated stubborn agents who may represent religious or political leaders. Examples include the voter model and opinion dynamics in the presence of stubborn agents [19–21]. For epidemics, the susceptible-infected-removed (SIR) [22] and susceptible-infected-susceptible (SIS) models are commonly used to describe the spread of viruses in networks [23, 24]. In SIS models, a node is always susceptible to infection, even if it has been infected and cured previously. On the other hand, a cured node becomes immune to infection in SIR models, and hence it is called a "removed" node. Many variants and extensions of the SIS and SIR models are also available in the literature [23, 25–27].

1.4 Control Intervention

As mentioned earlier, the main goal of this thesis is to demonstrate the effectiveness of control intervention in networks. In this section, we provide an overview of previous approaches to information spread control. We also describe the main properties of the control strategies we design in this thesis, and we identify ways in which our designs complement the current literature.

1.4.1 Existing Work

When examining the literature on control of diffusion dynamics, one can observe that the majority of approaches can be classified under four main categories. We identify these categories below along with relevant examples.

Static Approaches Goyal and Vigier investigate the construction of network topologies that facilitate the exchange of goods and information in [28]. The objective of the network designer is to make the network robust to adversarial attacks. Kempe *et al.* study the problem of finding the optimal set of nodes to maximize the spread of influence in a social network [29]. They propose a polynomial-time algorithm based on submodular functions that finds a near-optimal solution. A competition between two opposing campaigns to influence the largest set of nodes was studied in [30], where a greedy algorithm was proposed to find the best set of nodes for one campaign to limit the influence of the other. In the context of epidemics control, Borgs *et al.* propose a static curing rate allocation mechanism in order to cure the network from infection [31]. Omic *et al.* study a similar problem, and they adopt a static game-theoretic approach to perform the curing rates allocation across the network [32].

The controlled parameters in all the above problems are chosen at the initial time and are left static onward. These designs, therefore, cannot handle dynamically changing networks or the presence of other strategic players in the network.

Randomized Algorithms To cure computer networks and populations from viruses, Cohen *et al.* propose the so-called acquaintance immunization strategy in which random acquaintances of a randomly selected set of nodes are immunized [33]. This approach was shown to dramatically reduce the required number of immunized nodes in order to cure the network. Genetic algorithms and random mutation hill climbing were used in [34] to find optimal vaccine distributions in order to minimize the number of illnesses in the event of pandemic influenza. In order to achieve fast information spread, a hybrid algorithm was proposed in [35], which alternates between randomized and deterministic neighbor selection in order to maximize the speed of information spread.

While the above approaches are computationally efficient, they are neither robust to failures in the network nor to adversarial interventions. Networks are susceptible to attacks, and immunization techniques must be robust to such security breaches.

Controller at Each Node A common theme in current research is to assume that the network designer can control all the nodes in the network in order to limit the infection's spread. For instance, Preciado *et al.* have developed a convex framework for optimizing the curing rates across the network, where it is assumed that the curing rate of each node can be controlled [36,37]. Similar optimization problems were also studied in [32, 38].

In reality, such freedom in placing controllers may not be possible. As networks grow in size to include millions of nodes, reducing the number of controllers required to counter the infection's spread will result in vast cost reductions.

Adoption Rate Control In networks described using SIS or SIR models, a large body of literature focuses on controlling the rate at which the numbers of infected, susceptible, or removed nodes increase; see [39, 40] and the references therein. In such approaches, controllers are not explicitly allocated to the nodes, and the graph structure is not exploited. An alternative approach would be to control node-specific parameters such as the curing or infection rates, which will in turn control the adoption rate.

1.4.2 A New Approach

In view of the current state of the literature, we take an alternative approach to the problem of information spread control in networks, where we focus on designing control mechanisms that are dynamic, robust, constrained, and capable of exploiting the underlying network structure. In addition to these properties, we rely on graph theoretical modeling, which allows our results to be applicable in a rich class of networks including computer, social, and biological networks. Below, we highlight the main features of the controllers we design throughout the thesis.

Dynamic By relying on optimal control design, we construct strategies that are capable of responding to dynamical changes in the network.

Robust Using the framework of differential game theory, we propose zero-sum games to construct controllers suitable for competitive dynamic environments, which are robust also to adversarial intervention.

Constrained and Limited As opposed to controlling each and every node in the network, we use tools from nonlinear control to propose a framework for achieving certain control objectives using a limited number of controllers.

Network Structure Dependent By controlling node-specific parameters, we construct controllers that exploit the underlying network structure. Such designs are motivated by the availability of data, e.g., connectivity information of users, which allows for designing more intelligent controllers.

To demonstrate how such controllers can be designed, we consider two models of information spread: one is linear and the other one is nonlinear. We will refer to networks described by linear (nonlinear) information diffusion dynamics as *linear (nonlinear) dynamical networks*. We now briefly describe the problems we study under each type of dynamics.

1.4.3 Linear Dynamical Networks

In Chapters 2, 3 and 4, we study the problem of *robust* information spread control over linear dynamical networks. In Chapters 2 and 3, we design robust strategies by formulating a zero-sum game that describes the interaction between an adversary and a network designer who compete to control a network of nodes performing distributed averaging. The adversary can launch two network-wide attacks which we study separately. In Attack-I, the adversary is capable of disconnecting certain links in the network, while the designer can change the weights of certain links. In Attack-II, the players are capable of injecting global signals to alter the states of the nodes. Both the adversary and the designer are constrained by their physical capabilities, e.g., battery life and communication range. To capture such constraints, we allow the adversary and the designer to affect only a fixed number of links in Attack-I. As for Attack-II, we impose power and energy constraints on both players. However, we assume that the energy constraint does not allow for maximum power operation; this necessitates studying the problem under both constraints as the power constraint does not capture the limited energy budget.

In Chapter 4, we take an alternative approach to this problem, where we model the adversary as a large modeling uncertainty, and focus on designing *distributed* defense mechanisms, as opposed to having a centralized network designer. In particular, we study the problem of distributed stabilization of linear dynamical networks in the presence of uncertainties, where we extend the classical adaptive supervisory control framework to a distributed setting and investigate the conditions required for stability.

1.4.4 Nonlinear Dynamical Networks

The nonlinear model we study here is the so-called n-intertwined Markov model [41], which belongs to the SIS class of epidemiological models. Similar to the majority of virus spread models, the n-intertwined Markov model exhibits a threshold phenomenon. When the curing rate is high, the all-healthy state is the unique equilibrium. When the curing rates are low, however, a strictly positive equilibrium point arises, and a residual infection could persist in the network. We are interested in constructing constrained and dynamic mechanisms that control the virus propagation, while satisfying certain design objectives.

As a first step, we perform stability analysis for this model in Chapter 5, where we employ notions from positive systems theory to thoroughly study the stability properties of both equilibrium points over arbitrary network topologies. Further, we introduce a generic infection diffusion model that is motivated by theory of noncooperative games and show that this model subsumes existing virus spread models.

In Chapter 6, we shift our attention to control design questions. In particular, we identify sufficient conditions for stabilizing the network by controlling the curing rates of a limited number of nodes. We also formulate and solve multiple optimal control problems which aid a network designer in minimizing control cost while reducing infection levels across the network.

1.5 Thesis Organization

The rest of the thesis is organized as follows. In Chapter 2, we formulate the problems describing Attack-I and Attack-II and derive the worst-case adversarial attacks in the absence of the network designer. We introduce a network designer in Chapter 3 and study its interaction with the adversary under Attack-I and Attack-II using tools from differential game theory. The distributed supervisory control framework is introduced in Chapter 4. In Chapter 5, we review the n-intertwined Markov model and discuss its stability properties. We design stabilizing and optimal controllers for infected networks in Chapter 6. We outline open problems in information spread control in Chapter 7 and collect our concluding remarks in Chapter 8.

1.6 Mathematical Preliminaries

We start with some terminology and notational conventions. We use the words “nodes” and “agents” interchangeably. All the matrices and vectors in this thesis are real valued. For a set of $n \in \mathbb{Z}_{\geq 1}$ elements, we use the combinatorial notation $[n]$ to denote $\{1, \dots, n\}$. Unless otherwise mentioned, the (i, j)-th entry of a matrix $X \in \mathbb{R}^{n \times m}$, $n, m \in \mathbb{Z}_{\geq 1}$ is denoted by X_{ij}, and the i-th entry of a vector $x \in \mathbb{R}^n$, $n \in \mathbb{Z}_{\geq 1}$, is denoted by x_i. For two real vectors $x, y \in \mathbb{R}^n$, we write $x \gg y$ if $x_i > y_i$ for all $i \in [n]$, $x \succ y$ if $x_i \geq y_i$ for all $i \in [n]$ but $x \neq y$, and $x \succeq y$ if $x_i \geq y_i$ for all $i \in [n]$. We say a vector $x \in \mathbb{R}^n$ is strictly positive if $x \gg 0$. For any vector $x \in \mathbb{R}^n$, we define

$$x_{\min} := \min_{i \in [n]} x_i, \quad x_{\max} := \max_{i \in [n]} x_i.$$

The absolute value of a scalar variable is denoted by $|.|$. We also denote the cardinality of a finite set by $|.|$, and the purpose this operator is being used for will be clear from the context. The set of eigenvalues of a matrix X is denoted by $\sigma(X)$. The spectral radius of a matrix $X \in \mathbb{R}^{n\times n}$ is given by

$$\rho(X) = \max_{\lambda\in\sigma(X)} |\lambda|,$$

and its abscissa is given by

$$\mu(X) = \max_{\lambda\in\sigma(X)} \mathrm{Re}(\lambda).$$

When the eigenvalues of a matrix X are real, we denote the largest eigenvalue by $\lambda_1(X)$ and the smallest eigenvalue by $\lambda_n(X)$. The Euclidean norm of a vector is denoted by $\|.\|_2$, the ℓ_1-norm is denoted by $\|.\|_1$, and the ℓ_∞-norm is denoted by $\|.\|_\infty$. The induced 2-norm of a matrix $X \in \mathbb{R}^{n\times n}$ is given by

$$\|X\|_2 = \max_{\substack{y\in\mathbb{R}^n \\ \|y\|_2=1}} \|Xy\|_2 = \sqrt{\lambda_1\left(X^T X\right)}.$$

The L_2-norm of a function f defined over a vector space $\mathcal{X}$ is given by

$$\|f\|_{L_2} = \left(\int_{\mathcal{X}} \|f(x)\|_2^2 dx\right)^{\frac{1}{2}},$$

and its L_∞-norm is given by

$$\|f\|_{L_\infty} = \sup_{x\in\mathcal{X}} \|f(x)\|_\infty.$$

If f is differentiable, we can define the C^1-norm of f as follows:

$$\|f\|_{C^1} = \|f\|_{L_\infty} + \left\|\frac{d}{dx} f\right\|_{L_\infty}.$$

The vector space $\mathcal{L}^2$ is the space of all measurable functions for which $\|f\|_{L_2}$ is bounded. Given a time interval $[0,T] \subset \mathbb{R}$, we denote the space of continuously differentiable functions over this interval by $C^1[0,T]$. We recall that $C^1[0,T]$ is a Banach space when endowed with the C^1-norm.

We use the operator diag(.) for two purposes. When applied to a square matrix $X \in \mathbb{R}^{n\times n}$, diag$(X)$ returns a column vector that contains the diagonal entries of X. For a vector $x \in \mathbb{R}^n$, $X = \text{diag}(x)$, or $X = \text{diag}(x_1, \ldots, x_n)$, is a diagonal matrix with $X_{ii} = x_i$, $i \in [n]$. When a diagonal matrix has positive diagonal entries, we call it a positive diagonal matrix. The identity matrix is denoted by I, and the all-ones vector is denoted by $\mathbf{1}$. We assume both I and $\mathbf{1}$ have the appropriate dimensions whenever they are used. We use $[.]^{-1}$ to denote the inverse of a square matrix and $[.]^T$ to denote the transpose of a vector or a matrix. We use the game theoretic notation x_{-i} to refer to the vector comprised of the decision variables of all players except that of player i, where the dimension of x_{-i} will be defined once a game is formally introduced.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function that defines a dynamical system $\dot{x} = f(x)$, and let $\overline{x}$ be an equilibrium point of this system, i.e., $f(\overline{x}) = 0$. The Jacobian matrix of f, $J(x) \in \mathbb{R}^{n \times n}$, is given by $J(x) = \frac{\partial}{\partial x} f(x)$. Let $D \subset \mathbb{R}^{n \times n}$ be a compact domain where the trajectories of the dynamical system $\dot{x} = f(x)$ lie. A continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ is a Lyapunov function if, $V(\overline{x}) = 0$ and $V(x) > 0$ for all $x \in \mathcal{D} \setminus \{\overline{x}\}$. The Lie derivative of V along f is given by

$$\mathcal{L}_f V(x) := \frac{d}{dx} V(x)^T f(x).$$

Matrix Theory

We call two matrices $X, Y \in \mathbb{R}^{n \times n}$ *similar* if there exists a nonsingular matrix $T \in \mathbb{R}^{n \times n}$ such that $Y = T^{-1} X T$. An important property of similar matrices is that they share the same set of eigenvalues [42]. Some of our results rely on properties of Metzler matrices. A real square matrix X is called Metzler if its off-diagonal entries are nonnegative. We say that a matrix $X \in \mathbb{R}^{n \times n}$ is reducible if there exists a permutation matrix T such that

$$T^{-1} X T = \begin{bmatrix} Y & Z \\ 0 & W \end{bmatrix},$$

where Y and W are square matrices, or if $n = 1$ and $X = 0$ [43]. A real square matrix is called irreducible if it is not reducible. A survey on Metzler matrices and their stability properties can be found in [43–45]. Hurwitz Metzler matrices have the following equivalent characterizations.

Proposition 1.1 ([46])**.** *For a Metzler matrix $X \in \mathbb{R}^{n \times n}$, the following statements are equivalent:*

(i) The matrix X is Hurwitz.

(ii) There exists a vector $\xi \gg 0$ such that $X\xi \ll 0$.

(iii) There exists a vector $\nu \gg 0$ such that $\nu^T X \ll 0$.

(iv) There exists a positive diagonal matrix Q such that

$$X^T Q + QX = -K,$$

where K is a positive definite matrix.

The last characterization is often referred to as *diagonal stability* [43, 47].

The Perron-Frobenius (PF) theorem is a fundamental result in spectral graph theory that characterizes some of the properties of the spectra of Metzler and nonnegative matrices, i.e., matrices whose entries are all nonnegative. We first state the PF theorem for irreducible Metzler matrices [44, Theorem 17].

Theorem 1.1 (PF – Irreducible Metzler Case)**.** *Let $X \in \mathbb{R}^{n \times n}$ be an irreducible Metzler matrix. Then*

(i) $\mu(X)$ is an algebraically simple eigenvalue of X.

(ii) Let v_F be such that $Xv_F = \mu(X)v_F$. Then v_F is unique (up to scalar multiple) and $v_F \gg 0$.

(iii) If $v \succ 0$ is an eigenvector of X, then $Xv = \mu(X)v$, and, hence, v is a scalar multiple of v_F.

For irreducible nonnegative matrices, the following version of the PF theorem applies [42, Theorem 8.2.11].

Theorem 1.2 (PF – Irreducible Nonnegative Case)**.** *Let $X \in \mathbb{R}^{n \times n}$ be an irreducible nonnegative matrix. Then*

(i) $\rho(X) > 0$.

(ii) $\rho(X)$ is an algebraically simple eigenvalue of X.

(iii) If $Xv = \rho(X)v$, then $v \gg 0$.

Graph Theory

A *directed graph*, or *digraph*, is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. Given $\mathcal{G}$, we denote an edge from node $i \in \mathcal{V}$ to node $j \in \mathcal{V}$ by (i, j). We say node $i \in \mathcal{V}$ is a neighbor of node $j \in \mathcal{V}$ if and only if $(i, j) \in \mathcal{E}$. When $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$, we call the graph *undirected*. For a graph with $n \in \mathbb{Z}_{\geq 1}$ nodes, we associate an adjacency matrix $A \in \mathbb{R}^{n\times n}$ with entries $a_{ij} \in \mathbb{R}_{\geq 0}$, where $a_{ij} = 0$ if and only if $(i, j) \notin \mathcal{E}$. For undirected graphs, the adjacency matrix is symmetric, i.e., $A = A^T$.

In a digraph, a directed path is a collection of nodes $\{i_1, \ldots, i_\ell\} \subseteq \mathcal{V}$, $\ell \in \mathbb{Z}_{>1}$, such that $(i_k, i_{k+1}) \in \mathcal{E}$ for all $k \in [\ell - 1]$. A digraph is *strongly connected* if there exists a directed path between any two nodes in $\mathcal{V}$. A strongly connected component (SCC) of a graph is a subgraph which itself is strongly connected. When a nonnegative square matrix X is viewed as an adjacency matrix of a digraph, then X is irreducible if and only if its corresponding digraph is strongly connected [43]. A path in an undirected graph is defined in a similar manner. We call an undirected graph *connected* if it contains a path between any two nodes in $\mathcal{V}$. A digraph is called *weakly connected* if when every edge in $\mathcal{E}$ is viewed as an undirected edge, the resulting graph is a connected undirected graph. We call a graph, whether it is directed or undirected, *disconnected* if it contains at least two isolated subgraphs. Throughout the thesis, when the graph $\mathcal{G}$ is directed, we assume that it is either strongly or weakly connected. When $\mathcal{G}$ is undirected, we assume that it is connected.

A directed acyclic graph (DAG) is a digraph with no directed cycles. A node $i \in \mathcal{V}$ is called a source node if $\sum_{j\neq i} \mathbb{1}_{\{a_{ji}\neq 0\}} = 0$, and it is called a sink node if $\sum_{j\neq i} \mathbb{1}_{\{a_{ij}\neq 0\}} = 0$, where $\mathbb{1}_{\{a_{ij}\neq 0\}} = 1$ if and only if $a_{ij} \neq 0$, and is zero otherwise. A DAG can have multiple sources and multiple sinks. For a given graph $\mathcal{G}$, let $\mathcal{S}_{\text{source}}$ denote the set of source nodes, and let $\mathcal{S}_{\text{N-source}}$ be the set of all nodes i in $\mathcal{G}$ such that $a_{ji} \neq 0$ for some $j \in \mathcal{S}_{\text{source}}$.

CHAPTER 2

WORST-CASE ATTACKS ON CONSENSUS NETWORKS

2.1 Background

In practice, communication among agents performing averaging is prone to different types of non-idealities which can affect the convergence properties of the associated distributed algorithms. Transmission delays [48], noisy links [49, 50], and quantization [51] are some examples of non-idealities that are due to the physical nature of the application. In addition to physical restrictions, researchers have also studied averaging dynamics in the presence of malicious nodes in the network [52, 53]. Various algorithms that guarantee resilience against node failures have been proposed in the literature [54].

Here, we study the problem of continuous-time distributed averaging in the presence of an intelligent adversary. We consider two network-wide attacks launched by an adversary attempting to hinder the convergence of the nodes to consensus. The adversarial attacks we explore here differ from the ones studied by [52], [55], and [53], who consider the effect of malicious and compromised agents who could update their values arbitrarily. In the first scenario (called Attack-I) we consider, the adversary can break a set of edges in the network at each time instant. In practice, the adversary would be limited in its resources; we translate this practical limitation to a hard constraint on the total number of links the adversary can compromise at each time instant. In the second case (called Attack-II), the adversary can corrupt the measurements of the nodes by injecting a signal under a maximum power constraint. Our goal is to study the optimal behavior of the adversary in each case, given the imposed constraints.

For both attacks, we formulate the problem of the adversary as a finite horizon maximization problem in which the adversary seeks to maximize the Euclidean distance between the nodes' state and the consensus line. We completely characterize the optimal strategy of the adversary under both attacks; for each case we obtain a closed-form solution, providing also a potential-theoretic interpretation of the adversary's optimal strategy in Attack-I.

Our model is different from the models in the current literature in two ways: (i) the adversary interacts with a dynamical network. This is different from the problems studied in the computer science and economics communities where the network is usually static [28]; (ii) the adversary in our model is constrained and does not have an infinite budget. This enables us to model practical scenarios more closely rather than allowing the malicious behavior to be unrestricted as in [52, 55, 56], where it is assumed that the network contains nodes that are misbehaving. In addition, those papers focus on finding necessary and sufficient conditions for the network to reach consensus in the presence of malicious nodes, and observability theory is the main tool used to study such problems. Here, we assume that all the nodes are normal, and we focus on identifying the links that are of importance to the adversary. This requires us to borrow tools from optimal control theory.

2.2 Main Results

The contributions of this chapter are as follows. For Attack-I, we model the behavior of the adversary using an optimal control problem. We study the existence of solutions and the structure of the solution using Pontryagin's maximum principle (MP). We provide a method to compute the optimal attack strategy without requiring the adjoint equations to be solved. This method provides a new characterization for the optimal strategies in terms of potential-theoretic quantities. For Attack-II, we derive the optimal attack strategy in closed form using a fixed-point argument.

Organization

In Section 2.3, we formulate and provide the preliminaries of Attack-I. We show the existence of solutions, study the problem using the MP, and derive the optimal attack strategy. Attack-II is formulated and studied in Section 2.4. Numerical examples are provided in Section 2.5. We summarize the main results of the chapter in Section 2.6. Section 2.7 contains a technical result that is used in proving one of the main results.

Terminology and Notation

We will often use x to refer to a function or its value at a given time instant; the context should make the distinction clear. We will use the words "strategy" and "action" interchangeably; since we are seeking optimal open-loop strategies in this chapter, both terms are equivalent. We will use $\sum_{j>i}(.)$ to mean $\sum_{j=2}^{n}\sum_{i=1}^{j-1}(.)$, for some $n \in \mathbb{Z}_{\geq 2}$. Given an undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of edges, we will use e_{ij} as a shorthand notation for an edge from node $i \in \mathcal{N}$ to node $j \in \mathcal{N}$, i.e., $e_{ij} := (i, j)$. We define the projection operator $\Phi : \mathcal{E} \times \mathbb{R} \to \mathcal{E}$ such that $\Phi((e, r)) = e$, for some $(e, r) \in \mathcal{E} \times \mathbb{R}$. When applied to a set $S \subset \mathcal{E} \times \mathbb{R}$, the mapping Φ is defined as follows:

$$\Phi(S) \;=\; \begin{cases} \bigcup\limits_{(e,r)\in S} \Phi((e,r)), & S \neq \emptyset \\ 0, & S = \emptyset \end{cases} .$$

Given $S \subset \mathcal{E} \times \mathbb{R}$, with $|S| = k$, let $\pi(S) = \{(e_1, r_1), \ldots, (e_k, r_k)\}$, where $r_i \in \mathbb{R}$ and $e_i \in \mathcal{E}$ for all $i \in [k]$, be an ordering of the elements of S such that $r_1 \leq \ldots \leq r_k$. Then, given $\ell \in \mathbb{Z}_{\geq 0}$, we define the set operator $\Phi_\ell : \mathcal{E} \times \mathbb{R} \to \mathcal{E}$ as:

$$\Phi_\ell(S) \;=\; \begin{cases} \Phi(S), & \ell > k \\ \{e_1, \ldots, e_\ell\}, & 0 < \ell \leq k \\ 0, & \ell = 0 \text{ or } k = 0 \end{cases} .$$

Throughout this chapter, we will be dealing with undirected graphs. Although both e_{ij}, e_{ji} belong to the set of edges $\mathcal{E}$ in such graphs, we do not distinguish between the two edges, and we treat them as a single edge. As a result, in any set defined over $\mathcal{E} \times \mathbb{R}$, we include a *single* tuple (e_{ij}, r_{ij}), $r_{ij} \in \mathbb{R}$, to represent both edges.

2.3 Attack-I: Single-Player Case

Consider a connected network of n nodes and m links described by a weighted undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with vertex set $\mathcal{N}$, $|\mathcal{N}| = n$, and edge set $\mathcal{E}$, $|\mathcal{E}| = m$. The value, or state, of the nodes at time instant $t \in [0, \infty)$ is given by $x(t) = [x_1(t), ..., x_n(t)]^T$. The nodes start with an initial value $x(0) = x_0$, and they are interested in computing the average of their initial measurements, $x_{\text{avg}} = \frac{1}{n}\sum_{i=1}^{n} x_i(0)$, via local averaging. We consider the continuous-time averaging dynamics given by

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \tag{2.1}$$

where the matrix A, $A_{ij} = a_{ij}$, has the following properties:

$$A = A^T, \quad A\mathbf{1} = 0, \tag{2.2}$$

$$A_{ij} \geq 0, \quad A_{ij} = 0 \iff e_{ij} \notin \mathcal{E}, \quad i \neq j. \tag{2.3}$$

Define $\bar{x} = \mathbf{1}x_{\text{avg}}$ and let $M = \frac{\mathbf{1}\mathbf{1}^T}{n}$. A well-known result states that, under the above assumptions, the nodes will reach consensus as $t \to \infty$, i.e., $\lim_{t\to\infty} x(t) = \bar{x}$ [12]. To achieve his objectives, the adversary controls the elements of A as we describe next. This will render the matrix A to be time-varying.

The adversary attempts to slow down convergence by breaking at most $\ell \leq m$ links at each time t. Let $u_{ij}(t) \in \{0, 1\}$ be the weight the adversary assigns to link e_{ij} at time t. He breaks link e_{ij} when $u_{ij}(t) = 1$. Define $r := \binom{n}{2}$. The action set of the adversary can then be written as

$$\begin{aligned} U \quad = \quad & \{w \in \mathbb{R}^r \mid w = [w_{12}, ..., w_{1n}, w_{23}, ..., w_{(n-1)n}]^T, w_{ij} \in \{0, 1\}, \\ & w_{ij} = 0 \text{ if } e_{ij} \notin \mathcal{E}, \|w\|_1 \leq \ell\}. \end{aligned}$$

The set of admissible controls consists of all functions that are piecewise continuous in time and whose range is U. Given a time interval $[0, T]$, we can formally write

$$\mathcal{U} = \{u : [0, T] \to U \mid u \text{ is a piecewise continuous function of } t\}.$$

Given the above definitions, we can write down the (i, j)-th element of the matrix $A(u(t))$ as

$$A_{ij}(u(t)) = a_{ij}(1 - u_{ij}(t)).$$

Note that using the structure of A, we can re-write the dynamics (2.1) as follows:

$$\dot{x} = G(x(t))(\mathbf{1} - u(t)), \tag{2.4}$$

where $G_{ij} = a_{ij}(x_j - x_i)$. This demonstrates that the system we are considering is affine in the adversary's control u.

Define the functional:

$$J(u) = \frac{1}{2}\int_0^T k(t)\,\|x(t) - \bar{x}\|_2^2\,dt,$$

where the weighting factor k is positive and integrable over $[0, T]$, which can, for example, be viewed as a discounting factor, such as $k(t) = e^{-\alpha t}$ for some $\alpha > 0$. This constitutes the utility function of the adversary. The adversary's problem can now be formally written as

$$\begin{array}{ll} \sup\limits_{u \in \mathcal{U}} & J(u) \\ \text{subject to} & \dot{x}(t) = A(u(t))x(t), \quad x(0) = x_0. \end{array}$$

We make the following assumption:

Assumption 2.1. *The initial matrix $A(0)$, the time interval $[0, T]$, the value ℓ, and the initial state x_0 are known to the adversary.*

Based on the above formulation, the adversary is capable of changing the system matrix. This renders the system we are studying as a switched one. The optimal controllers for such systems can exhibit Zeno behavior, i.e., they may switch infinitely many times over a finite interval. Extensive simulation results show that the optimal controllers derived below switch a few times only. In order to explicitly eliminate the possibility of infinite switching, we make the following assumption in the remainder of this chapter.

Assumption 2.2. *Let $u \in \mathcal{U}$ be an arbitrary controller with switching times $0 \leq r_1 < \ldots < r_{K_u} \leq T$. We assume that $K_u \in \mathbb{Z}_{\geq 0}$ is finite, and that there exists a globally minimum dwell time $\tau > 0$ such that*

$$\tau \leq \min_{i \in [K_u]} r_{i+1} - r_i \tag{2.5}$$

over which the system matrix $A(u)$ is time-invariant.

Note that this assumption is well motivated for practical reasons. Consider, for example, a communication network where an adversary is a jammer injecting an interfering signal at some links. If the adversary chooses to change the set of links it is jamming, there must be some delay for it to change its configuration. This shows that the above assumption is not restrictive. Note that we do not require the computation of τ; we just need τ to be nonzero. Since simulation results show that Zeno behavior does not occur, we strongly believe that this assumption is not required.

Under the above assumption, we can restrict our development to piecewise continuous controllers. Hence, the right hand side of the ordinary differential equation (ODE) is piecewise continuous in t, continuous in u, and Lipschitz in x (uniformly over u). Hence, the ODE admits a unique solution over $[0, T]$, and the optimal control problem is well-posed by Filippov's Theorem (see the next subsection).

To arrive at the optimal strategy of the adversary, we employ the maximum principle. In what follows, we will often drop the time index and other arguments for notational simplicity.

2.3.1 Existence of Optimal Control

The MP provides a necessary condition for optimality, and before one can apply it, it is important to show that an optimal solution indeed exists for the given problem. Recall Filippov's Existence Theorem.

Theorem 2.1 ([57, 58])**.** *Consider the following optimal control problem:*

$$\sup_{u\in\mathcal{U}} \int_0^T L(x,u,t)dt \quad \textit{subject to} \quad \dot{x} = f(x,u,t), \quad x(0) = x_0.$$

Assume that the solutions of the ODE exist over $[0, T]$ for all $u : [0, T] \to U$ and that for every pair (t, x), the set U is compact, and the set

$$Q(x,t) = \{(z^0, z) \mid z^0 \geq L(x,u,t), z = f(x,w,t) \textit{ for some } w \in U\}$$

is convex. Then, an optimal control exists for the above problem.

In order to apply Filippov's Theorem, we first need to convexity the action set U as follows:

$$\begin{aligned}U_c &= \{w \in \mathbb{R}^r \mid w = [w_{12}, ..., w_{1n}, w_{23}, ..., w_{(n-1)n}]^T, w_{ij} \in [0,1], \\ &\quad w_{ij} = 0 \text{ if } e_{ij} \notin \mathcal{E}, \|w\|_1 \leq \ell\}.\end{aligned}$$

Consider the *convexified* problem:

$$\begin{aligned}\sup_{u \in \mathcal{U}_c} \quad & J(u) \\ \text{subject to} \quad & \dot{x} = A(u)x, \quad x(0) = x_0,\end{aligned}$$

where $\mathcal{U}_c$ is defined in a similar manner to $\mathcal{U}$ with U replaced with U_c in its definition.

In the remainder of this subsection, we will work with the convexified problem. We will show that the optimal solution of the convexified problems takes values at the boundaries of the set $\mathcal{U}_c$. Hence, this convexification does not change the optimal solution of the original problem. We are now ready to prove the existence of optimal controls for the convexified problem.

Lemma 2.1. *The convexified problem admits an optimal solution.*

Proof. For each fixed pair (x,t), the set

$$Q_u(x,t) = \left\{(z^0, z) \;\middle|\; z^0 \geq \frac{k}{2}\|x - \bar{x}\|_2^2, z = A(w)x \text{ for some } w \in U_c\right\}$$

is convex and compact. Indeed, let $\lambda \in [0,1]$, and let $(z_1^0, z_1), (z_2^0, z_2) \in Q_u(x,t)$. We then have $\lambda z_1^0 + (1-\lambda)z_2^0 \geq \frac{k}{2}\|x - \bar{x}\|_2^2$. Recalling the affine representation in (2.4), we can write

$$\begin{aligned}\lambda z_1 + (1-\lambda)z_2 &= \lambda G(x)(\mathbf{1} - w_1) + (1-\lambda)G(x)(\mathbf{1} - w_2) \\ &= G(x)(\lambda(\mathbf{1} - w_1) + (1-\lambda)(\mathbf{1} - w_2)) \\ &= A(\tilde{w})x, \quad \tilde{w} = \lambda w_1 + (1-\lambda)w_2 \in U_c.\end{aligned}$$ □

Having shown that optimal solutions exists, we can now replace "sup" by "max" in what follows.

2.3.2 Solution via the MP

The Hamiltonian associated with the above problem is:

$$H(x,p,u) = \frac{1}{2}k(t)\left\|x(t)-\bar{x}\right\|_2^2 + p^T(t)A(u(t))x(t).$$

The first-order necessary conditions for optimality are (noting that $A^T = A$) [58]:

$$\begin{aligned}
\dot{p} &= -\frac{\partial}{\partial x}H \\
&= -k(x-\bar{x}) - Ap, \quad p(T)=0 && (2.6) \\
\dot{x} &= Ax, \quad x(0)=x_0 && (2.7) \\
u^\star &= \arg\max_{U_c} H(x,p,u).
\end{aligned}$$

To find the optimal strategies, let us first write

$$\begin{aligned}
p^TAx &= \sum_{i=1}^{n} p_i\left(\sum_{j=1}^{n} A_{ij}x_j\right) \\
&= \sum_{i=1}^{n} p_i\left(-\sum_{j=1,j\neq i}^{n} A_{ij}x_i + \sum_{j=1,j\neq i}^{n} A_{ij}x_j\right) \\
&= \sum_{j>i} a_{ij}(1-u_{ij})(p_j-p_i)(x_i-x_j).
\end{aligned}$$

Define the function

$$f_{ij} = (p_j - p_i)(x_i - x_j),$$

and write

$$\max_{u\in U_c} H = \frac{1}{2}k\left\|x-\bar{x}\right\|_2^2 + \max_{u\in U_c}\sum_{j>i} A_{ij}f_{ij}. \tag{2.8}$$

Note that we cannot decouple the maximization into $\binom{n}{2}$ maximization problems, each corresponding to a link or a single term inside the double summation. This is due to the constraint on the number of links that can be targeted by the adversary. To find the optimal strategy, let $\mathcal{D}_\ell \subseteq \mathcal{E}$ be the set containing the ℓ links with the lowest negative f_{ij} values, if such links exist. Formally, define the set $S = \{(e_{ij}, a_{ij}f_{ij}) \mid e_{ij} \in \mathcal{E}, f_{ij} < 0\} \subset \mathcal{E} \times \mathbb{R}$. We can then write $\mathcal{D}_\ell = \Phi_\ell(S)$. Note that the definition of Φ_ℓ allows us to account for the case when $|S| < \ell$. Given this definition and Eq. (2.8), we conclude that the optimal control of the adversary in the convexified maximization problem should be of the following form:

$$u^{\star}_{ij}(t) = \begin{cases} 1, & e_{ij} \in \mathcal{D}_\ell \\ 0, & e_{ij} \notin \mathcal{D}_\ell \text{ or } f_{ij} > 0 \\ \{0, 1\}, & \text{otherwise} \end{cases} . \tag{2.9}$$

Since the optimal control takes values at the boundaries U_c, it constitutes a solution for the original nonconvex maximization problem.

2.3.3 Solution via Potential Theory

The optimal strategy is defined in terms f_{ij}'s, for $e_{ij} \in \mathcal{E}$, which depend on the state x and the costate p. However, we have not derived the optimal trajectories that satisfy the canonical equations given by the MP in (2.6) and (2.7), and hence in that sense the solution is incomplete. Since the system is linear time-varying, the solutions will be given in terms of a state transition matrix. Also, the functions f_{ij} depend on both the state and the costate, which in turn are defined in terms of the control. This makes working with f_{ij} intractable.

Under Assumption 2.2, the following theorem provides a procedure to arrive at the optimal solutions without the need to compute p. We will be using the term "connected component" to refer to a set of connected nodes which have the same values. Define $\nu_{ij} = -(x_i - x_j)^2$, $e_{ij} \in \mathcal{E}$.

Theorem 2.2. *Under Assumptions 2.1 and 2.2, the rankings performed as part of the optimal strategies of the maximization problem can be carried out by replacing $f_{ij}(t)$ by $\nu_{ij}(t)$, for all $e_{ij} \in \mathcal{E}$.*

Furthermore, it is optimal for the adversary to modify a total of ℓ links. If the adversary has an optimal strategy of modifying less than ℓ links, then either $\mathcal{G}$ has a cut of size less than ℓ or the nodes have reached consensus at time t. In either of the cases, modifying ℓ links is also optimal.

Proof. We will show that it is optimal for the adversary to rank the links based on their $w_{ij} := a_{ij}\nu_{ij}$ values instead of the $a_{ij}f_{ij}$'s. The main complication in solving the adjoint equations is that the system is time-varying. However, under Assumption 2.2, the functions x, p become piecewise continuous.

Hence, the function f_{ij}, for all $e_{ij} \in \mathcal{G}$, is also piecewise continuous and its value cannot change abruptly over a finite interval. As a result, we can regard the system as a time-invariant one over a small interval $[t_0, t_0 + \delta] \subset [0, T]$, where $0 < \delta \leq \tau$, and τ was defined in (2.5). The proof consists of three steps.

(i) Show that it is optimal for the adversary to change ℓ links.

(ii) Show that, over a small interval $[t_0, t_0 + \delta]$, it is optimal for the adversary to switch from a strategy $u \in \mathcal{U}$ to another strategy $u^\star \in \mathcal{U}$, where $u^\star$ entails ranking the links based on their w_{ij} values.

(iii) Show that allowing $u^\star$ to mimic u for the remaining time of the problem preserves the gain obtained over $[t_0, t_0 + \delta]$.

Over a small interval, u and $u^\star$ induce certain system matrices. Let the system matrix corresponding to u over $[t_0, t_0 + \delta]$ be $A(u) = A$, and let $\|u\|_1 < \ell$ over this interval. Since the control strategy of the adversary is fixed over this interval, the state trajectory is given by

$$x(t) = e^{A(t-t_0)}x(t_0), \quad t \in [t_0, t_0 + \delta].$$

Let $P(t) := e^{At}$. Due to the structure of A, $P(t)$ is a doubly stochastic matrix for $t \geq 0$; see [59, p. 63].

Note that we can write $x(t_0) = \tilde{P}x_0$, where $\tilde{P}$ is some doubly stochastic matrix. Indeed, assume that either or both controls had switched once at some time $\tilde{t}_0 \in [0, t_0)$, and that the system matrix over $[0, \tilde{t}_0)$ was $\tilde{A}_1$, and the system matrix corresponding to $[\tilde{t}_0, t_0)$ was $\tilde{A}_2$. Then $x(t_0) = e^{\tilde{A}_2(t_0 - \tilde{t_0})}e^{\tilde{A}_1\tilde{t_0}}x_0$. Since both $e^{\tilde{A}_1 t}$, $e^{\tilde{A}_2 t}$ are doubly stochastic matrices, their product is also doubly stochastic. We can readily generalize this result to any number of switches in the interval $[0, t_0)$. With this observation, we can write

$$x(t) - \bar{x} = P(t - t_0)\tilde{P}x_0 - Mx_0 = (P(t - t_0) - M)x(t_0),$$

where the last equality follows from the fact that

$$\tilde{P}M = M\tilde{P} = M, \ \tilde{P} \text{ is doubly stochastic.} \tag{2.10}$$

We want to show that switching to strategy $u^\star$ at some time $t^\star \in [t_0, t_0 + \delta]$ can improve the utility of the adversary. To this end, we assume that the matrix induced by $u^\star$ over $[t_0, t^\star)$ is A, while the system matrix corresponding to $u^\star$ over $[t^\star, t_0+\delta]$ is B. Define the doubly stochastic matrix $Q(t) := e^{Bt}$, $t \geq 0$. Over $[t^\star, t_0+\delta]$, the strategies u and $u^\star$ are identical except at link $e_{ij} \in \mathcal{E}$, where $u_{ij} = 0$ and $u^\star_{ij} = 1$. It follows that:

$$A_{ij} > B_{ij} = 0, \quad A_{kl} = B_{kl} \quad \forall e_{kl} \neq e_{ij}. \tag{2.11}$$

Formally, we want to prove the following inequality:

$$\begin{aligned}\int_{t_0}^{t_0+\delta} k(t) \left\|(P(t-t_0) - M)x(t_0)\right\|_2^2 dt \\ < \int_{t_0}^{t^\star} k(t) \left\|(P(t-t_0) - M)x(t_0)\right\|_2^2 dt \\ + \int_{t^\star}^{t_0+\delta} k(t) \left\|(Q(t-t^\star) - M)P(t^\star - t_0)x(t_0)\right\|_2^2 dt,\end{aligned}$$

or equivalently

$$\begin{aligned}\int_{t^\star}^{t_0+\delta} k(t) \cdot \Big[\left\|(Q(t-t^\star) - M)P(t^\star - t_0)x(t_0)\right\|_2^2 \\ - \left\|(P(t-t_0) - M)x(t_0)\right\|_2^2\Big] dt > 0.\end{aligned} \tag{2.12}$$

Using (2.10) and the semi-group property, Eq. (2.12) simplifies to

$$\int_{t^\star}^{t_0+\delta} k(t) \cdot x(t_0)^T \Lambda(t, t^\star) x(t_0) dt > 0, \tag{2.13}$$

where $\Lambda(t, t^\star) = P(t^\star - t_0)Q(2(t - t^\star))P(t^\star - t_0) - P(2(t - t_0))$. A sufficient condition for (2.13) to hold is

$$h(t, x(t_0)) = x(t_0)^T \Lambda(t, t^\star) x(t_0) > 0, \text{ for } t > t^\star.$$

As $\delta \downarrow 0$, we can write $P(t) = I + tA + \mathcal{O}(\delta^2)$, where $\mathcal{O}(\delta^2)/\delta \leq L$ for sufficiently small δ and some finite constant L. We therefore have

$$\begin{aligned}\Lambda(t, t^*) &= \left(I + (t^\star - t_0)A + \mathcal{O}\left(\delta^2\right)\right)\left(I + 2(t - t^\star)B + \mathcal{O}\left(\delta^2\right)\right) \\ &\quad \left(I + (t^\star - t_0)A + \mathcal{O}\left(\delta^2\right)\right) - \left(I + 2(t - t_0)A + \mathcal{O}\left(\delta^2\right)\right) \\ &= 2(t - t^\star)B + 2(t^\star - t_0)A - 2(t - t_0)A + \mathcal{O}\left(\delta^2\right)\end{aligned}$$

$$= 2(t - t^\star)(B - A) + \mathcal{O}\left(\delta^2\right).$$

For sufficiently small δ, the first term dominates the second term. Recall that the quadratic form of a Laplacian matrix L exhibits the following form: $x^T L x = \sum_{l>k} L_{kl}(x_l - x_k)^2$, for any $x \in \mathbb{R}^n$. Note that $B - A$ is in fact a negative Laplacian. Using (2.11), we can then write

$$\begin{aligned} h(t, x(t_0)) &= 2(t - t^\star) \sum_{r>s} (A_{sr} - B_{sr}) \left(x_r(t_0) - x_s(t_0)\right)^2 + \mathcal{O}\left(\delta^2\right) \\ &= 2(t - t^\star) A_{ij} \left(x_j(t_0) - x_i(t_0)\right)^2 + \mathcal{O}\left(\delta^2\right). \end{aligned} \tag{2.14}$$

For small enough δ, the higher order terms are dominated by the first term. Hence, if there is a link e_{ij} such that $x_i(t_0) \neq x_j(t_0)$, there exists $t^\star$ such that $h(t, x(t_0)) > 0$ for $t \in (t^\star, t_0 + \delta]$. Since t_0 was arbitrary, we conclude that the optimal strategy must satisfy $\|u^\star(t)\|_1 = \ell$ for all t, given that each of the ℓ links connects two nodes having different values.

If no link such that $x_i(t_0) \neq x_j(t_0)$ exists at a given time t_0, the adversary does not need to break additional links, although breaking more links does not affect optimality because $h(t, x(t_0)) = 0$ in such a case. There are two cases under which the adversary cannot find a link to make $h(t, x(t_0)) > 0$: (i) The graph at time t_0 is one connected component. In this case, the nodes have already reached consensus and $\|u^\star\|_1 < \ell$. This is a *losing strategy* for the adversary as it failed in preventing nodes from reaching agreement; (ii) The graph at time t_0 has multiple connected components, and the number of links connecting the components is less than ℓ. The adversary here possesses a *winning strategy* with $\|u^\star\|_1 < \ell$, as it can disconnect $\mathcal{G}$ into multiple components and prevent consensus.

The second step is to show that the adversary will modify the ℓ links with the lowest $w_{ij} = a_{ij}\nu_{ij}$ values, $e_{ij} \in \mathcal{E}$. Let us again restrict our attention to the interval $[t_0, t_0 + \delta]$ where the adversary applies strategy u. Assume (to the contrary) that the links the adversary breaks over this interval are not the ones with the lowest $w_{ij}(t)$ values. In particular, assume that the adversary chooses to break link $e_{kl} \in \mathcal{E}$, while there is a link $e_{ij} \in \mathcal{E}$ such that $w_{ij} < w_{kl}$. Assume that the adversary switches at time $t^\star \in [t_0, t_0 + \delta]$ to strategy $u^\star$ by *breaking* link e_{ij} and *unbreaking* link e_{kl}. Then, (2.14) becomes

$$h(t, x(t_0)) = 2(t - t^*) \left(w_{kl}(t_0) - w_{ij}(t_0)\right) + \mathcal{O}\left(\delta^2\right).$$

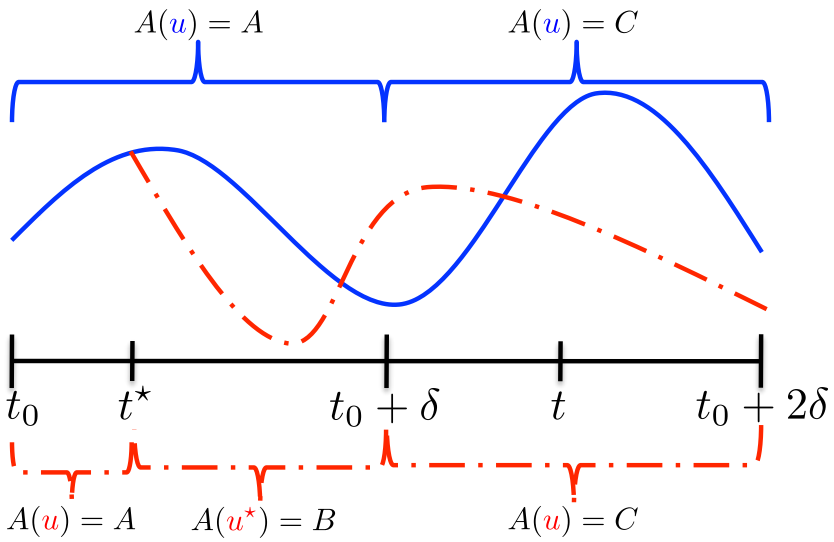


Figure 2.1: A demonstration of the technique used in the third step of the proof. The blue solid trajectory corresponds to u while the red dashed trajectory corresponds to $u^\star$.

Hence, by following the same arguments as above, we can conclude that breaking e_{kl} is not optimal. This proves that the optimal strategy for the adversary is to break the links with the lowest w_{ij} values.

The final step of the proof is to show that switching to strategy $u^\star$ guarantees an improved utility for the adversary *regardless of how the original trajectory corresponding to u changes beyond time $t_0 + \delta$*. To this end, we will assume that from time $t_0 + \delta$ onward, strategy $u^\star$ will *mimic* strategy u. Assume that strategy u switches from matrix A to matrix C over the interval $[t_0 + \delta, t_0 + 2\delta]$, and define $R(t) := e^{Ct}$. Hence, strategy $u^\star$ will also switch from the system matrix B to matrix C. However, the trajectories corresponding to u and $u^\star$ will have different initial conditions at time $t_0 + \delta$, due to the switch that strategy $u^\star$ made at time $t^\star$. Figure 2.1 illustrates this idea. Consider the behavior of the system over the interval $[t_0 + \delta, t_0 + 2\delta]$ where we can assume that the system is time-invariant. To show that the gain obtained over $[t_0, t_0 + \delta]$ by the switch made by $u^\star$ is maintained over $[t_0 + \delta, t_0 + 2\delta]$, we must prove the following inequality:

$$\int_{t_0+\delta}^{t_0+2\delta} k(t) \cdot \Bigg[\underbrace{\|(R(t-(t_0+\delta))-M)Q(t_0+\delta-t^\star)P(t^\star-t_0)x(t_0)\|_2^2}_{:=L_1} - \underbrace{\|(R(t-(t_0+\delta))-M)P(t_0+\delta-t_0)x(t_0)\|_2^2}_{:=L_2} \Bigg] dt > 0.(2.15)$$

As before, it suffices to prove that the integrant $L_1 - L_2$ is positive. Let us now expand both L_1 and L_2.

$$\begin{aligned} L_1 &= x(t_0)^T P(t^\star - t_0)Q(t_0+\delta-t^\star)(R(t-(t_0+\delta))-M) \\ &\quad (R(t-(t_0+\delta))-M)Q(t_0+\delta-t^\star)P(t^\star-s)x(t_0) \\ &= x(t_0)^T P(t^\star-t_0)Q(t_0+\delta-t^\star)(R(2(t-(t_0+\delta)))-M) \\ &\quad Q(t_0+\delta-t^\star)P(t^\star-t_0)x(t_0) \\ &= x(t_0)^T(P(t^\star-t_0)Q(t_0+\delta-t^\star)R(2(t-(t_0+\delta))) \\ &\quad Q(t_0+\delta-t^\star)P(t^\star-t_0)-M)x(t_0). \end{aligned}$$

Similarly,

$$L_2 = x(t_0)^T(P(\delta)R(2(t-(t_0+\delta)))P(\delta)-M)x(t_0).$$

We can then write

$$\begin{aligned} L_1 - L_2 &= x(t_0)^T(P(t^\star-t_0)Q(t_0+\delta-t^\star)R(2(t-(t_0+\delta)))Q(t_0+\delta-t^\star) \\ &\quad P(t^\star-t_0)-P(\delta)R(2(t-(t_0+\delta)))P(\delta))x(t_0) \\ &:= x(t_0)^T(F_1-F_2)x(t_0). \end{aligned}$$

Before we perform a first-order Taylor expansion to the above terms, let us define the following quantities:

$$\tau_1 = t^\star - t_0, \quad \tau_2 = (t_0+\delta)-t^\star, \quad \tau_3 = t-(t_0+\delta),$$

where $t^\star \in [t_0, t_0+\delta]$ and $t \in [t_0+\delta, t_0+2\delta]$.

Using Proposition 2.1 (see Section 2.7), we can now expand F_1 and F_2 as follows:

$$\begin{aligned}
F_1 &= \left(I + \tau_1 A + \mathcal{O}\left(\tau_1^2\right)\right)\left(I + \tau_2 B + \mathcal{O}\left(\tau_2^2\right)\right)\left(I + 2\tau_3 C + \mathcal{O}\left(\tau_3^2\right)\right) \\
&\quad \left(I + \tau_2 B + \mathcal{O}\left(\tau_2^2\right)\right)\left(I + \tau_1 A + \mathcal{O}\left(\tau_1^2\right)\right) \\
&= \left(I + \tau_1 A + \tau_2 B + \mathcal{O}\left(\delta^2\right)\right)\left(I + 2\tau_3 C + \mathcal{O}\left(\delta^2\right)\right) \\
&\quad \left(I + \tau_1 A + \tau_2 B + \mathcal{O}\left(\delta^2\right)\right) \\
&= \left(I + \tau_1 A + \tau_2 B + 2\tau_3 C + \mathcal{O}\left(\delta^2\right)\right)\left(I + \tau_1 A + \tau_2 B + \mathcal{O}\left(\delta^2\right)\right) \\
&= I + 2\tau_1 A + 2\tau_2 B + 2\tau_3 C + \mathcal{O}\left(\delta^2\right). \\
F_2 &= \left(I + \delta A + \mathcal{O}\left(\delta^2\right)\right)\left(I + 2\tau_3 C + \mathcal{O}\left(\tau_3^2\right)\right)\left(I + \delta A + \mathcal{O}\left(\delta^2\right)\right) \\
&= \left(I + \delta A + 2\tau_3 C + \mathcal{O}\left(\delta^2\right)\right)\left(I + \delta A + \mathcal{O}\left(\delta^2\right)\right) \\
&= I + 2\delta A + 2\tau_3 C + \mathcal{O}\left(\delta^2\right).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
F_1 - F_2 &= 2\left(\tau_1 - \delta\right) A + 2\tau_2 B + \mathcal{O}\left(\delta^2\right), \\
&= 2\tau_2\left(B - A\right) + \mathcal{O}\left(\delta^2\right) \\
&= 2\left(\left(t_0 + \delta\right) - t^\star\right)\left(B - A\right) + \mathcal{O}\left(\delta^2\right),
\end{aligned}$$

and thereby we obtain

$$\begin{aligned}
L_1 - L_2 &= 2\left(\left(t_0 + \delta\right) - t^\star\right)\sum_{r>s}\left(A_{sr} - B_{sr}\right)\left(x_r(t_0) - x_s(t_0)\right)^2 + \mathcal{O}\left(\delta^2\right) \\
&= 2\left(t_0 + \delta - t^\star\right) A_{ij}\left(x_j(t_0) - x_i(t_0)\right)^2 + \mathcal{O}\left(\delta^2\right).
\end{aligned}$$

Thus, for small enough δ, we conclude that $L_1 - L_2 > 0$, which implies that (2.15) is satisfied, and the gain obtained by switching to system matrix B at $t^\star \in [t_0, t_0 + \delta]$ is maintained over $[t_0 + \delta, t_0 + 2\delta]$. Note that the effect of switching to matrix C is cancelled in $F_1 - F_2$, and hence $L_1 - L_2$, since the strategy $u^\star$ is mimicking strategy u. Hence, by dividing the interval $(t_0 + 2\delta, T]$ into small intervals of length δ and repeating the above analysis, we conclude that the gain due to the switch at time $t^\star$ is preserved over the remaining time of the problem. The proof is therefore complete. □

Remark 2.1. *(Potential-Theoretic Analogy) When the graph is viewed as an electrical network, a_{ij} can be viewed as the conductance of link $e_{ij} \in \mathcal{E}$ and $x_i - x_j$ as the potential difference across the link. Therefore, the optimal* *strategy of the adversary involves breaking the links with highest power dissipation given by $a_{ij}(x_i - x_j)^2$. These links correspond to the edges with the highest information flow; therefore, for the purpose of delaying or preventing consensus, attacking these links is optimal.* •

2.4 Attack II: Single Player Case

Assume now that the adversary is capable of adding a noise signal to all the nodes in the network in order to slow down convergence. The dynamics in this case are:

$$\dot{x}(t) = Ax(t) + u(t), \quad x(0) = x_0. \tag{2.16}$$

We assume that the instantaneous power $\|u(t)\|_2^2$ that the adversary can expend cannot exceed a fixed value $P_{\max}$. We also assume that the adversary has sufficient energy $E_{\max}$ to allow it to operate at maximum instantaneous power. Accordingly, the action set of the adversary is

$$U = \{w \in \mathbb{R}^n \mid \|w\|_2^2 \leq P_{\max}\}.$$

The set of admissible controls consists of all functions that are continuously differentiable in time and whose range is U. Given a time interval $[0, T]$, we can formally write

$$\mathcal{U} = \left\{u : [0, T] \to U \mid u \in C^1[0, T]\right\}.$$

The adversary's problem is given by

$$\begin{aligned} \max_{u \in \mathcal{U}} \quad & J(u) & (2.17)\\ \text{subject to} \quad & \dot{x}(t) = Ax(t) + u(t), \quad x(0) = x_0, & (2.18) \end{aligned}$$

where A satisfies the properties given in (2.2) and (2.3). The Hamiltonian in this case is given by

$$\begin{aligned} H(x, p, u) \quad = \quad & k(t)\left\|x(t) - \bar{x}\right\|_2^2 + p(t)^T\left(Ax(t) + u(t)\right) \\ & + \lambda(t)\left(\|u(t)\|_2^2 - P_{\max}\right), \end{aligned}$$

where λ is a continuously differentiable scalar Lagrange multiplier associated with the power constraint. As before, we let $x, p \in C^1[0,T]$. Here, λ must satisfy

$$\lambda(t) \leq 0, \quad \lambda(t)\left(\|u(t)\|_2^2 - P_{\max}\right) = 0, \quad \forall t \in [0,T].$$

The first-order necessary conditions for optimality are:

$$\begin{aligned} \dot{p}(t) &= -\frac{\partial}{\partial x} H(x,p,u) \\ &= -2k(t)(x(t) - \bar{x}) - Ap(t), \quad p(T) = 0 \\ \dot{x}(t) &= Ax(t) + u(t), \quad x(0) = x_0 \\ \frac{\partial}{\partial u} H(x,p,u) &= 2\lambda(t)u(t) + p(t) = 0. \end{aligned} \tag{2.19}$$

To find $u^\star$, consider the following cases:

Case 1: ($\lambda(t_0) < 0 \implies \|u(t_0)\|_2^2 = P_{\max}$*, for some* $t_0 \in [0,T]$) Using (2.19), we obtain $\lambda(t_0)\|u(t_0)\|_2^2 = -\frac{1}{2}u(t_0)^T p(t_0)$; hence,

$$\lambda(t_0) = -\frac{1}{2P_{\max}} u(t_0)^T p(t_0), \tag{2.20}$$

which we can then use to solve for the optimal control:

$$u^\star(t_0) = P_{\max}\frac{p(t_0)}{u(t_0)^T p(t_0)} = \frac{E_{\max}}{T} \cdot \frac{p(t_0)}{u(t_0)^T p(t_0)}.$$

Remark 2.2. *The optimal strategy $u^\star(t_0)$ is the vector of maximum power that it is aligned with $p(t_0)$. To see this, note that (2.20) implies that $u(t_0)^T p(t_0) > 0$, because $\lambda(t_0) < 0$. Hence, the vectors $u^\star(t_0)$ and $p(t_0)$ are aligned. Define the unit vector $\bar{p}(t_0) = p(t_0)/\|p(t_0)\|_2$. Then, we can further write*

$$u^\star(t_0) = \frac{E_{\max}/T}{\|u(t_0)\|_2^2} \cdot \bar{p}(t_0) = \sqrt{P_{\max}} \cdot \bar{p}(t_0). \tag{2.21}$$

Hence, the adversary's optimal solution in this case is to operate at the maximum power available. •

Case 2: ($\|u(t_0)\|_2^2 < P_{\max} \implies \lambda(t_0) = 0$*, for some* $t_0 \in [0,T]$) Using (2.19), we obtain $p(t_0) = 0$. In this case the control is singular, since it does not appear in $\frac{\partial}{\partial u}H = 0$. By continuity of the costate p, there exists an interval $\Delta t = [t_0, t_0 + \delta]$, $\delta > 0$, such that $p(t) = 0$, for all $t \in \Delta t$. This implies that all the time derivatives of p must also be zero:

$$\frac{d}{dt}\frac{\partial}{\partial u}H = \dot{p}(t) = -2k(t)\left(x(t)-\bar{x}\right) - A^T p(t) = 0, \quad \forall t \in \Delta t,$$

which implies that $x(t) = \bar{x}$, for all $t \in \Delta t$. The conditions obtained by taking the time derivatives are also necessary conditions that must be satisfied at the optimal trajectory. However, having $x(t) = \bar{x}$, for all $t \in \Delta t$, could violate the initial condition when $t_0 = 0$. In order to resolve this inconsistency, we set the control at $t = 0$ to be an impulse, $u_i(t) = c \cdot \delta(t)$, for all $i \in \mathcal{V}$, in order to make $x(0) = \bar{x}$, where $c \in \mathbb{R}$ is chosen to guarantee $\|u(t)\|_2^2 < P_{\max}$. Note that we still have not recovered the control, and therefore we need to differentiate again:

$$\frac{d^2}{dt^2}\frac{\partial}{\partial u}H = \dot{x}(t) = Ax(t) + u(t) = 0, \quad \forall t \in \Delta t,$$

which implies that $u(t) = -Ax(t) = -A\bar{x} = 0$, for all $t \in \Delta t$.

Remark 2.3. *Note that $x(t) = \bar{x}$ leads to having $u(t) = 0$. This result matches intuition; when the nodes reach consensus, $J(u) = 0$ for all $u \in \mathcal{U}$. Hence, no matter what the control is, the utility of the adversary will always be zero. Thus, expending power becomes sub-optimal, and the optimal strategy is to do nothing.* •

Since the adversary attempts to increase the Euclidean distance between x and $\bar{x}$, we can readily see that $u \equiv 0$ cannot be optimal, unless $x(0) = \bar{x}$. The following lemma proves this formally.

Lemma 2.2. *The solution of the problem (2.17)-(2.18) satisfies $\|u(t)\|_2^2 = P_{\max}$.*

Proof. Let $u_1 \in \mathcal{U}$ be such that $\|u_1(t)\|_2^2 < P_{\max}$, for all $t \in [0, T]$. Then, it follows from Case 2 that $J(u_1) = 0$. Consider another solution, $u_2 \in \mathcal{U}$, which satisfies the power constraint with equality. Namely, let $u_2(t) = \sqrt{\frac{P_{\max}}{n}}\mathbf{1}$, for all $t \in [0, T]$. Using the solution to (2.16), and by defining the doubly stochastic matrix $P(t) = e^{At}$ we can write

$$x(t) = P(t)x_0 + \sqrt{\frac{P_{\max}}{n}}\mathbf{1}t, \quad t \in [0, T].$$

In this case, for all $t \in [0,T]$, we have

$$\begin{aligned}
\|x(t)-\bar{x}\|_2^2 &= x_0^T(P(t)-M)^T(P(t)-M)x_0 \\
&\quad + P_{\max}t^2 + 2\sqrt{\frac{P_{\max}}{n}}x_0^T(P(t)-M)\mathbf{1}t \\
&= x_0^T(P(t)^2 - 2MP(t) - M^2)x_0 + P_{\max}t^2 \qquad (2.22) \\
&= x_0^T P(2t)(I-M)x_0 + P_{\max}t^2, \qquad (2.23)
\end{aligned}$$

where (2.22) follows because $P(t), t \in [0,T]$, and M are stochastic matrices, and (2.23) follows from (2.10) and the semi-group property. Being a stochastic matrix, $P(2t)$ is positive semidefinite, for $t \in [0,T]$. Also, $I-M$ is a Laplacian matrix; therefore, it is also positive semidefinite. Further, note that

$$P(2t)(I-M) = P(2t) - MP(2t) = (I-M)P(2t), \quad t \in [0,T].$$

Hence, $P(2t)(I-M)$ is also positive semidefinite, and therefore $x_0^T P(2t)(I-M)x_0 \geq 0$, for all $t \in [0,T]$. This in turn implies

$$\begin{aligned}
J(u_2) &= \int_0^T k(t)\left[x_0^T P(2t)(I-M)x_0 + P_{\max}t^2\right]dt \\
&\geq \frac{P_{\max}}{3}T^3 > J(u_1) = 0.
\end{aligned}$$

We conclude that not utilizing the power budget available yields a lower utility for the adversary. □

From the above analysis, and with Lemma 2.2 at hand, we conclude that the optimal control of the adversary must be given by (2.21), for all $t_0 \in [0,T]$. Hence, it remains to determine the costate vector in order to completely characterize $u^\star$. To do so, we will invoke Banach's fixed-point theorem. To this end, we will work with the scaled utility $\tilde{J}(u) = \nu J(u)$, $\nu > 0$, without loss of generality. Note that $u^\star$ in (2.21) is also the solution to the maximization problem of $\tilde{J}(u)$. The costate trajectory is given by

$$p(t) = 2\nu \int_t^T k(\tau)P(\tau - t)(x(\tau)-\bar{x})d\tau. \qquad (2.24)$$

Substituting (2.21) and the solution to (2.16) into (2.24) yields

$$p(t) = g(t) + 2\nu\sqrt{P_{\max}} \int_t^T \int_0^\tau k(\tau)P(2\tau - (t+s))\bar{p}(s)dsd\tau,$$

where $g(t) = 2\nu \int_t^T P(\tau - t)k(\tau)(P(\tau)x_0 - \bar{x})d\tau$. Note that $2\tau - (t+s) \geq 0$ for $0 \leq s \leq \tau$, $t \leq \tau \leq T$, and hence $P(.)$ is a well-defined doubly stochastic matrix over the region of integration. We define the mapping $\mathcal{T}(p)(t) := p(t)$. By its structure, it is readily seen that $\mathcal{T}(p)(t) : C^1[0,T] \to C^1[0,T]$. The following lemma aids in obtaining the costate vector.

Lemma 2.3. *Let $\tilde{\mathcal{T}}(x)(t) := k(t)\int_0^t P(s)x(s)ds$, $t \in [0,T]$, where $P(t)$ is a doubly stochastic matrix, and fix $x(t) \in C^1[0,T]$. Then*

$$\left\|\tilde{\mathcal{T}}(x)\right\|_{L_\infty} \leq \sup_{0\leq t\leq T} tk(t) \cdot \|x\|_{L_\infty}.$$

Proof. We have:

$$\begin{aligned}
\left\|\tilde{\mathcal{T}}(x)\right\|_{L_\infty} &= \sup_{0\leq t\leq T} \left\|k(t)\int_0^t P(s)x(s)ds\right\|_\infty \\
&= \sup_{0\leq t\leq T} k(t) \sup_{1\leq i\leq n} \left|\int_0^t \sum_{j=1}^n P_{ij}(s)x_j(s)ds\right| \\
&\leq \sup_{0\leq t\leq T} k(t) \sup_{1\leq i\leq n} \int_0^t \sum_{j=1}^n P_{ij}(s)\left|x_j(s)\right|ds \\
&\leq \sup_{0\leq t\leq T} k(t) \sup_{1\leq i\leq n} \int_0^t \left(\sum_{j=1}^n P_{ij}(s)\right) \sup_{1\leq j\leq n} \left|x_j(s)\right|ds \\
&= \sup_{0\leq t\leq T} k(t) \int_0^t \sup_{1\leq j\leq n} \left|x_j(s)\right|ds \\
&\leq \sup_{0\leq t\leq T} k(t) \int_0^t \sup_{0\leq s\leq T} \sup_{1\leq j\leq n} \left|x_j(s)\right|ds \\
&= \sup_{0\leq t\leq T} tk(t) \cdot \|x\|_{L_\infty},
\end{aligned}$$

where the second inequality follows from Hölder's inequality. □

Theorem 2.3. *By choosing $\nu < \frac{1}{2\sqrt{P_{\max}}(\check{k}+\hat{k})}$, where $\check{k} = \sup_{0\leq t\leq T} tk(t)$ and $\hat{k} = \sup_{0\leq t\leq T}\int_t^T \tau k(\tau)d\tau$, the mapping $\mathcal{T}(p)(t) : C^1[0,T] \to C^1[0,T]$ has a unique fixed point $p^\star \in C^1[0,T]$ that can be obtained by any sequence gener-*

ated by the iteration $p_{k+1}(t) = \mathcal{T}(p_k)(t)$, $t \in [0,T]$, starting from an arbitrary vector $p_0 \in C^1[0,T]$.

Proof. The theorem will follow if for this choice of ν, the mapping $\mathcal{T}$ is a contraction. Consider two vectors $y, z \in C^1[0,T]$ and let $\bar{y}, \bar{z}$ be the corresponding normalized unit norm vectors. Let $\bar{w} = \bar{y} - \bar{z}$. Then

$$\begin{aligned}
&\frac{1}{2\nu\sqrt{P_{\max}}} \left\| \mathcal{T}(y) - \mathcal{T}(z) \right\|_{C^1} = \\
&\qquad \sup_{0\le t\le T} k(t) \sup_{1\le i\le n} \left| \int_0^t \sum_{j=1}^n P_{ij}(t-s)\bar{w}_j(s)ds \right| \\
&\qquad + \sup_{0\le t\le T} \sup_{1\le i\le n} \left| \int_t^T k(\tau) \int_0^\tau \sum_{j=1}^n P_{ij}(2\tau - (t+s))\bar{w}_j(s)dsd\tau \right| \\
&\qquad \le \sup_{0\le t\le T} tk(t) \left\| \bar{w} \right\|_{L_\infty} + \sup_{0\le t\le T} \sup_{1\le i\le n} \int_t^T k(\tau) \\
&\qquad \cdot \int_0^\tau \sum_{j=1}^n P_{ij}(2\tau - (t+s)) \left| \bar{w}_j(s) \right| dsd\tau,
\end{aligned}$$

where the last inequality follows from Lemma 2.3. Using arguments similar to those used in proving Lemma 2.3, we have:

$$\begin{aligned}
&\frac{1}{2\nu\sqrt{P_{\max}}} \left\| \mathcal{T}(y) - \mathcal{T}(z) \right\|_{C^1} \\
&\qquad \le \left(\sup_{0\le t\le T} tk(t) + \sup_{0\le t\le T} \int_t^T \tau k(\tau) d\tau \right) \left\| \bar{w} \right\|_{L_\infty} \\
&\qquad \le (\check{k} + \hat{k}) \left\| y - z \right\|_{L_\infty} \le 2\nu\sqrt{P_{\max}}(\check{k} + \hat{k}) \left\| y - z \right\|_{C^1},
\end{aligned}$$

where the second inequality follows from the properties of similar triangles. We readily see that by selecting $\nu < \frac{1}{2\sqrt{P_{\max}}(\check{k}+\hat{k})}$, the last inequality implies that $\mathcal{T}(p)(t)$ is a contraction mapping. Since $C^1[0,T]$ endowed with $\|.\|_{C^1}$ is a Banach space, Banach's contraction mapping principle guarantees the existence of a unique fixed point $p^\star \in C^1[0,T]$ which can be obtained from the iteration $p_{k+1}(t) = \mathcal{T}(p_k)(t)$ as $k \to \infty$, $t \in [0,T]$, for any initial point. □

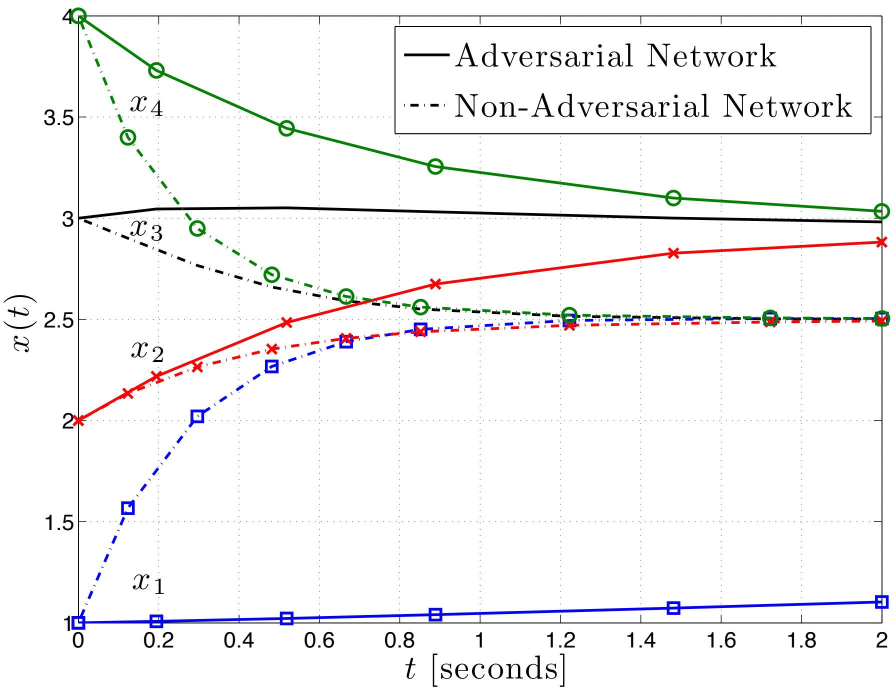


Figure 2.2: Effect of ATTACK-I on the convergence to consensus. $T = 2$, $n = 4$, $\ell = 2$, and $x_0 = [1, 2, 3, 4]$.

2.5 Numerical Studies

In this section, we provide numerical examples for ATTACK-I and ATTACK-II. We consider the complete graph with $n = 4$. The matrix $A(0)$ was generated. We let $T = 2$ and $x_0 = [1, 2, 3, 4]^T$—hence, $x_{\text{avg}} = 2.5$. We simulated the network using MATLAB's BVP SOLVER.

For ATTACK-I, we fixed $\ell = 2$, and computed the optimal control using (2.9), which was found to be $u^\star(t) = [1, 0, 1, 0, 1, 1]^T$ for $t \in [0, 2]$. Indeed, at $t = 0$, the highest w_{ij} values are $w_{13}(0) = 2.2101$ and $w_{14}(0) = 13.8979$ which confirms the conclusion of Theorem 2.2. In this particular example, w_{13}, w_{14} remain dominant throughout the problem's horizon, and hence the control is stationary. Figure 2.2 simulates the network at hand with and without the presence of the adversary. Note that the adversary was successful in delaying convergence. Since both links the adversary broke emanate from node 1, x_1 is far from consensus.

For ATTACK-II, we fixed $P_{\max} = 2$, and Fig. 2.3 demonstrates the network with and without the presence of the adversary. Since the adversary in this

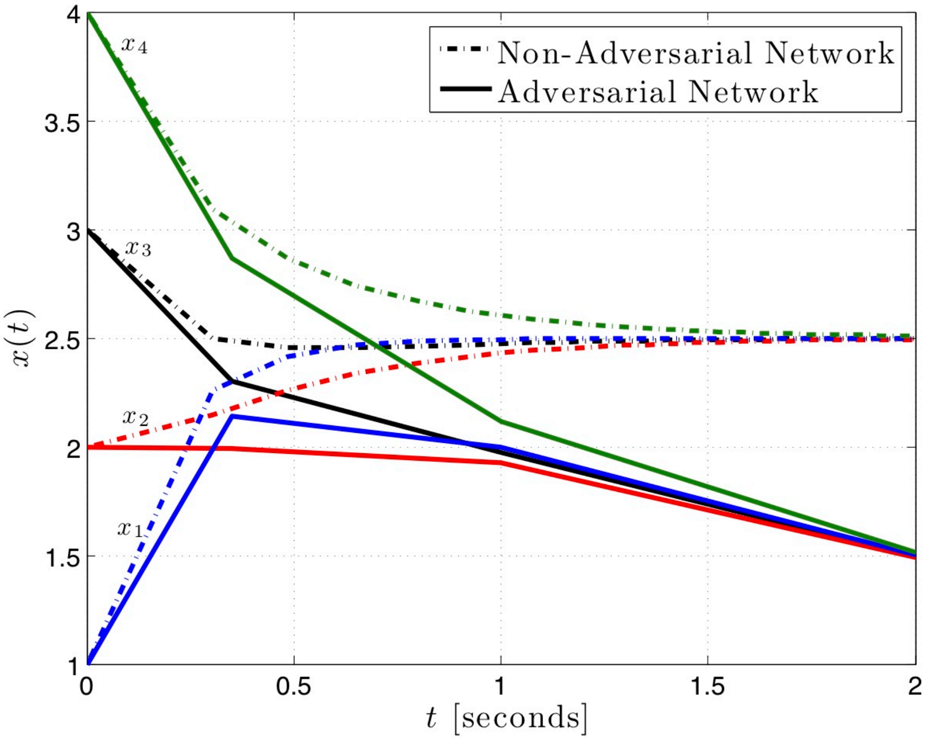


Figure 2.3: Effect of Attack-II on the convergence to consensus. $T = 2$, $n = 4$, and $x_0 = [1, 2, 3, 4]$.

attack is capable of targeting nodes, he was capable of diverting the values of all the nodes away from x_{avg}.

2.6 Summary

We have considered two types of adversarial attacks on a network of agents performing distributed averaging. Both attacks have the common objective of slowing down the convergence of the nodes to the global average. Attack-I involves an adversary that is capable of compromising links, with a constraint on the number of links it can break. Despite the interdependence of the state, costate, and control, we were able to find the optimal strategy. We also presented a potential-theoretic interpretation of the solution. In Attack-II, a finite power adversary attempts to corrupt the values of the nodes by injecting a signal of bounded power. We assumed that the adversary has sufficient energy $E_{\max}$ to operate at maximum instantaneous power and derived the corresponding optimal strategy.

2.7 Additional Proof

The following proposition is used in proof Theorem 2.2. It will also be used in proving some of the main results of Chapter 3.

Proposition 2.1. *Given τ_1, τ_2, τ_3, which were defined in terms of $\delta > 0$ in the proof of Theorem 2.2, let f be a real-valued function. Then, if $f(\delta) = \mathcal{O}\left(\tau_i^2\right)$ as $\delta \to 0$, we have $f(\delta) = \mathcal{O}\left(\delta^2\right), i \in [3]$. Also, if $f(\delta) = \tau_i \mathcal{O}\left(\tau_j^2\right)$ as $\delta \to 0$, then $f(\delta) = \mathcal{O}\left(\delta^3\right), i, j \in [3]$.*

Proof. Recall that we write $f(x) = \mathcal{O}\left(g(x)\right)$, for some real-valued function g, as $x \to a$ if there exist constants M, γ such that $|f(x)| \leq M|g(x)|$, for all x satisfying $|x - a| < \gamma$. Since $f(\delta) = \mathcal{O}\left(\tau_i^2\right)$ as $\delta \to 0$, and recalling that by definition we have $\tau_i \leq \delta$ for $i \in [3]$, we can write $f(\delta) \leq M\tau_i^2 \leq M\delta^2$. Hence, $f(\delta) = \mathcal{O}\left(\delta^2\right)$. To prove the second statement, recall that $h(x)\mathcal{O}\left(g(x)\right) = \mathcal{O}\left(h(x)g(x)\right)$, for any two real-valued functions h, g. Hence, as $\delta \to 0$, we have $f(\delta) = \tau_i \mathcal{O}\left(\tau_j^2\right) = \mathcal{O}\left(\tau_i \tau_j^2\right)$. Therefore, $f(\delta) \leq M\tau_i\tau_j^2 \leq M\delta^3$ and $f(\delta) = \mathcal{O}\left(\delta^3\right)$. □

CHAPTER 3

A COMPETITION OVER CONSENSUS NETWORKS

3.1 Background

Having studied the worst-case attacks on consensus networks in the previous chapter, we now introduce a network designer and study its interaction with the adversary. We consider a setting similar to that of the previous chapter: the network consists of nodes performing continuous-time distributed averaging, and the adversary strategically attempts to prevent the nodes from reaching consensus by launching either Attack-I or Attack-II. By modeling the adversary as a strategic player and deriving optimal defense strategies, we guarantee robustness against worst-case attacks, unlike existing approaches in which attacks on links were modeled as random failures [60].

For Attack-I, the adversary strategically disconnects a set of links to prevent the nodes from reaching consensus. Meanwhile, the network designer assists the nodes in reaching consensus by changing the weights of a limited number of links in the network. We formulate two Stackelberg games to describe this competition where the order in which the players act is reversed in the two problems. Although the canonical equations provided by the Pontryagin's maximum principle seem to be intractable, we provide an alternative characterization for the optimal strategies that makes connection to potential theory. Finally, we provide a sufficient condition for the existence of a saddle-point equilibrium (SPE) for the underlying zero-sum game.

In Attack-II, the designer and the adversary are both capable of altering the measurements of all nodes in the network by injecting *global* signals. We impose two constraints on both players: a power constraint and an energy constraint. We assume that the available energy to each player is *not sufficient* to operate at the maximum power throughout the horizon of the game. We show the existence of an SPE and derive the optimal strategies in closed form for this attack scenario.

Such an interaction between a network designer and an adversary can occur in various practical applications. For example, in a wireless network, the adversary can be a jammer who is capable of breaking links by injecting high noise signals that disrupt the communication among nodes. The link weights in such a network represent the capacities of the corresponding links. The designer can modify the capacity of a certain link using various communication techniques such as introducing parallel channels between two nodes as in orthogonal frequency division multiple access (OFDMA) networks [61]. In OFDMA networks, the number of parallel links between two nodes is usually limited [62]. To capture this limitation, we limit the amount by which the designer can increase the capacity of a given link. The adversary can be a jammer who is capable of breaking links by injecting high noise signals that disrupt the communication among nodes. The adversary is assumed to have sufficient transmit power to disrupt the communication over any link, no matter what the number of parallel channels is.

3.2 Main Results

For Attack-I, we capture the interaction between the designer and the adversary by formulating two separate problems. In the min–max problem, the designer declares a strategy first to which the adversary reacts by its optimal response. The second problem is a max–min one, where the order of play is reversed. Assuming that the controllers do not switch infinitely many times over a finite interval among the available actions, we derive the optimal strategies for both problems in terms of potential-theoretic quantities by working directly with the utility functional. Furthermore, we demonstrate that the derived strategies satisfy the necessary conditions provided by the MP. Further, we derive a sufficient condition guaranteeing the existence of an SPE. For Attack-II, we show that an SPE always exists and derive the optimal strategies in closed-form.

Organization

The rest of this chapter is organized as follows. In Section 3.3, we provide the preliminaries of Attack-I and formulate the min–max and max–min problems. In Section 3.4, we derive the Stackelberg strategies and show that they satisfy the MP. We provide a sufficient condition for the existence of an SPE in Section 3.5. Attack-II is introduced in Section 3.6, where the optimal strategies for both players are derived in closed form. We end the chapter with the concluding remarks of Section 3.7. Section 3.8 includes a proof of one of the theorems and a technical result.

Terminology and Notation

We will adopt the same notation and terminology outlined in Chapter 2.

3.3 Attack-I: Adversary vs. Network Designer

Consider a connected network of n nodes and m links described by a weighted undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$. The value, or state, of the nodes at time instant $t \in \mathbb{R}_{\geq 0}$ is given by $x(t) = [x_1(t), ..., x_n(t)]^T$. The nodes start with an initial value $x(0) = x_0$, and they are interested in computing the average of their initial values, $x_{\text{avg}} = \frac{1}{n}\sum_{i=1}^{n} x_i(0)$, via local averaging. We consider the continuous-time averaging dynamics given by

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \tag{3.1}$$

where the matrix A, $A_{ij} = a_{ij} \in \mathbb{R}$, has the following properties:

$$A = A^T, \quad A\mathbf{1} = 0, \tag{3.2}$$

$$A_{ij} \geq 0, \quad A_{ij} = 0 \iff e_{ij} \notin \mathcal{E}, \quad i \neq j. \tag{3.3}$$

Define $\bar{x} = \mathbf{1}x_{\text{avg}} \in \mathbb{R}^n$ and let $M = \frac{1}{n}\mathbf{1}\mathbf{1}^T$. A well-known result states that, under the above assumptions, the nodes will reach consensus as $t \to \infty$, i.e., $\lim_{t\to\infty} x(t) = \bar{x}$ [12]. To achieve their respective objectives, the designer and the adversary control the elements of A as we describe next. This will render the matrix A to be time-varying.

The adversary attempts to slow down convergence by breaking at most $\ell \leq m$ links at each time t. Let $u_{ij}(t) \in \{0,1\}$ be the weight the adversary assigns to link $e_{ij} \in \mathcal{E}$ at time $t \in \mathbb{R}_{\geq 0}$. He breaks link e_{ij} when $u_{ij}(t) = 1$. Define $r := \binom{n}{2}$. The action set of the adversary is then

$$\begin{aligned} U &= \{w \in \mathbb{R}^r \mid w = [w_{12}, ..., w_{1n}, w_{23}, ..., w_{(n-1)n}]^T, w_{ij} \in \{0,1\}, \\ &\quad w_{ij} = 0 \text{ if } e_{ij} \notin \mathcal{E}, \|w\|_1 \leq \ell\}. \end{aligned}$$

The set of admissible controls, $\mathcal{U}$, consists of all functions that are piecewise continuous in time and whose range is U. Given a time interval $[0,T]$, we can formally write

$$\mathcal{U} = \{u : [0,T] \to U \mid \ u \text{ is a piecewise continuous function of } t\}.$$

We introduce a network designer who attempts to accelerate convergence by controlling the weights of the edges. The designer can change the weight of a given link by adding $v_{ij}(t)$ to its weight a_{ij}. We assume that $v_{ij}(t) \in \{0,b\}$ and that the number of links the designer modifies is at most $\ell \leq m$. Given the above definitions, we can write down the (i,j)-th element, $i \neq j$, of the matrix $A(u(t), v(t))$ as

$$A_{ij}(u(t), v(t)) = (a_{ij} + v_{ij}(t))(1 - u_{ij}(t)), \quad \text{for all } e_{ij} \in \mathcal{E}. \tag{3.4}$$

We require that the resulting matrix is a negative Laplacian of the graph; hence, we must have $A_{ii}(u(t), v(t)) = -\sum_{j \neq i} A_{ij}(u(t), v(t))$, for all $i \in \mathcal{V}$.

Given a time interval $[0,T]$, define the following functional:

$$J(u,v) = \frac{1}{2}\int_0^T k(t) \left\|x(t) - \bar{x}\right\|_2^2 dt,$$

where the weighting factor $k(t)$ is positive and integrable over $[0,T]$. This constitutes the utility function of the adversary, and that of the designer is $-J(u,v)$. We will study two problems. In the first one, the adversary acts first by selecting the links he is interested in breaking. Then, the network designer optimizes his choices over the resulting graph, which we denote by $\mathcal{G}(u(t)) = (\mathcal{N}, \mathcal{E}(u(t)))$, where $\mathcal{E}(u(t)) = \mathcal{E} \setminus \{e_{ij} \in \mathcal{E} : u_{ij}(t) = 1\}$. In this case, the action set of the designer can be written as

$$\begin{aligned} V(u(t)) \ &= \ \big\{w \in \mathbb{R}^r \mid w = [w_{12}, ..., w_{1n}, w_{23}, ..., w_{(n-1)n}]^T, w_{ij} \in \{0, b\}, \\ &\qquad w_{ij} = 0 \text{ if } e_{ij} \notin \mathcal{E}(u(t)), \|w\|_1 \le b\ell\big\} \,. \end{aligned}$$

The set of admissible controls for the designer, $\mathcal{V}(u)$, consists of all piecewise continuous functions whose range is $V(u)$. Formally, we define

$$\mathcal{V}(u) = \{v : [0, T] \to V(u(t)) \mid \ v \text{ is a piecewise continuous function of } t\} \,.$$

The max–min problem can now be formally written as[1]

$$\begin{aligned} &\sup_{u\in\mathcal{U}} \inf_{v\in\mathcal{V}(u)} \quad J(u, v) \\ &\text{subject to} \quad \dot{x}(t) = A(u(t), v(t))x(t), \quad x(0) = x_0. \end{aligned}$$

In the second problem, the order is reversed. Since the designer acts first in this problem, he can optimize over the entire graph $\mathcal{G}$. Thus, the action set of the designer in this problem is $V := V(0)$ and the set of its admissible controls is $\mathcal{V} := \mathcal{V}(0)$; the sets of actions and admissible controls of the adversary remain the same. We can then write

$$\begin{aligned} &\inf_{v\in\mathcal{V}} \sup_{u\in\mathcal{U}} \quad J(u, v) \\ &\text{subject to} \quad \dot{x}(t) = A(u(t), v(t))x(t), \quad x(0) = x_0. \end{aligned}$$

In a computer network, the max–min problem allows the network designer (who is the maximizer here) to architect networks that are robust against strategic virus diffusion. The min–max problem finds applications in army combat situations where the designer (the minimizer) attempts to counter the attacks of the enemy intending to disrupt the network communication.

Given the nature of the players' possible modifications of the network, as described by (3.4), we can view the actions of the players as switches among the possible Laplacian matrices resulting from modifying the links. Moreover, the capability of the designer and the adversary to change the system matrix

[1] Even though existence of a maximum and a minimum has not yet been shown at this stage, we will still call this the "max–min" problem in anticipation of such an existence result later in the chapter. The formal definition below is still in terms of sup and inf. The same argument applies to the min–max problem to be introduced shortly.

renders it as a "switched" one. The optimal controllers for such systems can exhibit Zeno effect, i.e., they may switch infinitely many times over a finite interval. In order to explicitly eliminate the possibility of infinite switching, we make the following assumption in the remainder of this chapter.

Assumption 3.1. *Let $0 \leq r_1 < \ldots < r_{K_u} \leq T$ be the switching times of some $u \in \mathcal{U}$ and $0 \leq s_1 < \ldots < s_{K_v} \leq T$ be those of some $v \in \mathcal{V}$. We assume that $K_u, K_v \in \mathbb{Z}_{\geq 0}$ are finite, and that there exists a globally minimum dwell time $\tau > 0$ such that*

$$\tau \leq \min \left\{ r_{i+1} - r_i, s_{i+1} - s_i, |r_i - s_j| \,\middle|\, i \in [K_u], j \in [K_v] \right\}, \tag{3.5}$$

over which the system matrix $A(u, v)$ is time-invariant.

Note that this assumption is well motivated for practical reasons. Consider, for example, a communication network where an adversary is a jammer injecting an interfering signal at some links. If the adversary chooses to change the set of links it is jamming, there must be some delay for the adversary to change its configuration. Now, we make the following assumption for both problems:

Assumption 3.2. *The initial matrix $A(0, 0)$, the time interval $[0, T]$, the values ℓ and b, and the initial state x_0 are common information to both players.*

We recall the definition of an SPE.

Definition 3.1 (Saddle-Point Equilibrium (SPE) [63])**.** *The pair $(u^\star, v^\star)$ constitutes an SPE if it satisfies the following pair of inequalities*

$$J(u, v^\star) \leq J(u^\star, v^\star) \leq J(u^\star, v), \tag{3.6}$$

for $u \in \mathcal{U}$, $v \in \mathcal{V}$.

The following remarks are now in order.

Remark 3.1. *(Non-Rectangular Strategy Sets and Existence of SPE) When the strategy sets are rectangular, i.e., the strategy of one player does not restrict the strategy space of the other, the following relationship holds:*

$$\underline{V} = \sup_{u \in \mathcal{U}} \inf_{v \in \mathcal{V}} J(u, v) \leq \inf_{v \in \mathcal{V}} \sup_{u \in \mathcal{U}} J(u, v) = \overline{V}, \tag{3.7}$$

where $\underline{V}, \overline{V}$ are called, respectively, the lower and upper values of the game. When the strategy sets are non-rectangular, however, the order in (3.7) may not hold. Moreover, one should not expect the pair of inequalities (3.6) to hold, and hence an SPE may not exist. In the max–min problem in this chapter, the strategy sets of the players are non-rectangular as the adversary's action, removing links from $\mathcal{G}$, could restrict the actions available to the designer. •

Remark 3.2. *(Problem Complexity) Let us consider the problem of the adversary for a given strategy of the designer. Assume that the adversary can act at $K_u \in \mathbb{Z}_{\geq 0}$ given time instances over the interval $[0, T]$. Then, for $\ell \leq m$, assuming that $\|u(t)\|_1 = \ell$ for all $t \in \mathbb{R}_{\geq 0}$, the total number of links that need to be tested in a brute-force approach is*

$$\binom{m}{\ell}^{K_u} \geq \left(\frac{m}{\ell}\right)^{\ell K_u}. \tag{3.8}$$

Clearly, the brute-force approach leads to an exponential number of computations as a function of K_u. The same argument applies to the problem faced by the network designer. •

3.4 Optimal Strategies

We will now present the solutions to the two problems introduced above. In [64], we have shown that the canonical equations provided by the MP are intractable due to the interdependence between the state, costate, and the optimal controls; therefore, it may not be possible to obtain the optimal strategies in closed form using the MP. Here, we take an alternative route to arrive at the optimal strategies of the players by working directly with the objective functional. In what follows, we will often drop the time index and other arguments for notational simplicity. We will be using the term "connected component" to refer to a set of connected nodes which have the same values. The following quantities, which we associate with each $e_{ij} \in \mathcal{E}$, will be central to the derivation of the optimal strategies:

$$\nu_{ij} := -(x_i - x_j)^2, \quad w_{ij} := (a_{ij} + v_{ij})\nu_{ij}. \tag{3.9}$$

3.4.1 The Min–Max Problem

The following theorem presents the optimal strategy of the adversary in the min–max problem. Define the set

$$\mathcal{L}_\ell(v) = \Phi_\ell\left(\{(e_{ij}, (a_{ij} + v_{ij})\nu_{ij}) \mid e_{ij} \in \mathcal{E}\}\right) \subseteq \mathcal{E}. \tag{3.10}$$

Theorem 3.1. *Under Assumptions 3.1 and 3.2, and for a fixed strategy v of the designer, the optimal strategy of the adversary in the min–max problem is*

$$u_{ij}^\star(v) \quad = \quad \begin{cases} 1, & e_{ij} \in \mathcal{L}_\ell(v) \\ 0, & e_{ij} \notin \mathcal{L}_\ell(v) \end{cases} .$$

If the adversary has an optimal strategy of breaking fewer than ℓ links, then either $\mathcal{G}$ has a cut of size less than ℓ or the nodes have reached consensus by time t. In either of these cases, breaking ℓ links is also optimal.

Proof. For a fixed strategy of the designer $v \in \mathcal{V}$, we will show that it is optimal for the maximizer to rank the links based on their w_{ij} values, where w_{ij} was defined in (3.9). Under Assumption 3.1, the function x becomes piecewise continuous. Hence, the function w_{ij}, for all $e_{ij} \in \mathcal{E}$, is also piecewise continuous and its value cannot change abruptly over a finite interval. As a result, we can regard the system as a time-invariant one over a small interval $[t_0, t_0 + \delta] \subset [0, T]$, where $0 < \delta \leq \tau$, and τ was defined in (3.5). The proof consists of two steps.

(i) Showing that, over a small interval $[t_0, t_0 + \delta]$, it is optimal for the adversary to switch from a strategy $u \in \mathcal{U}$ to another strategy $u^\star \in \mathcal{U}$, where $u^\star$ entails breaking the ℓ links with the lowest w_{ij} values.

(ii) Showing that allowing $u^\star$ to mimic u for the remaining time of the problem preserves the gain obtained over $[t_0, t_0 + \delta]$.

Over a small interval, u and $u^\star$ induce certain system matrices. Let the system matrix corresponding to u over $[t_0, t_0 + \delta]$ be $A(u, v) = A$, and let $\|u\|_1 < \ell$ over this interval. Since the control strategies of both players are time-invariant over this interval, we have

$$x(t) = e^{A(t-t_0)}x(t_0), \quad t \in [t_0, t_0 + \delta]. \tag{3.11}$$

Let $P(t) := e^{At}$. Due to the structure of A, $P(t)$ is a doubly stochastic matrix for $t \geq 0$ [59, p. 63]. Note that we can write $x(t_0) = \tilde{P}x_0$, where $\tilde{P}$ is some doubly stochastic matrix. Indeed, assume that either or both controls had switched once at some time $\tilde{t_0} \in [0, t_0)$, and that the system matrix over $[0, \tilde{t_0})$ was $\tilde{A}_1$, and the system matrix corresponding to $[\tilde{t_0}, t_0)$ was $\tilde{A}_2$. Then $x(t_0) = e^{\tilde{A}_2(t_0-\tilde{t_0})}e^{\tilde{A}_1\tilde{t_0}}x_0$. Since both $e^{\tilde{A}_1 t}$, $e^{\tilde{A}_2 t}$ are doubly stochastic matrices, their product is also doubly stochastic. We can readily generalize this result to any number of switches in the interval $[0, t_0)$. With this observation, we can write

$$x(t) - \bar{x} = P(t-t_0)\tilde{P}x_0 - Mx_0 = (P(t-t_0) - M)x(t_0),$$

where the last equality follows from the fact that

$$\tilde{P}M = M\tilde{P} = M, \;\; \tilde{P} \text{ is doubly stochastic.} \tag{3.12}$$

We want to show that switching from strategy u to strategy $u^\star$ at some time $t^\star \in [t_0, t_0 + \delta]$, can improve the utility of the adversary. To this end, we assume that the matrix induced by $u^\star$ over $[t_0, t^\star)$ is A, while the system matrix corresponding to $u^\star$ over $[t^\star, t_0 + \delta]$ is B. Define the doubly stochastic matrix $Q(t) := e^{Bt}$, $t \geq 0$. Over $[t^\star, t_0 + \delta]$, the strategies u and $u^\star$ are identical except at link $e_{ij} \in \mathcal{E}$, where $u_{ij} = 0$ and $u^\star_{ij} = 1$, i.e., $\|u\|_1 < \|u^\star\|_1$ over this sub-interval. It follows that:

$$A_{ij} > B_{ij} = 0, \quad A_{kl} = B_{kl} \quad \forall e_{kl} \neq e_{ij}. \tag{3.13}$$

Formally, we want to prove the following inequality:

$$\begin{aligned}\int_{t_0}^{t_0+\delta} & k(t)\, \|(P(t-t_0) - M)x(t_0)\|_2^2 \, dt \\ & < \int_{t_0}^{t^\star} k(t)\, \|(P(t-t_0) - M)x(t_0)\|_2^2 \, dt \\ & + \int_{t^\star}^{t_0+\delta} k(t)\, \|(Q(t-t^\star) - M)P(t^\star - t_0)x(t_0)\|_2^2 \, dt,\end{aligned}$$

or equivalently

$$\int_{t^\star}^{t_0+\delta} k(t) \cdot \left[\|(Q(t-t^\star) - M)P(t^\star - t_0)x(t_0)\|_2^2 \right. \\ \left. - \|(P(t-t_0) - M)x(t_0)\|_2^2\right] dt > 0. \tag{3.14}$$

Using (3.12) and the semi-group property, (3.14) simplifies to

$$\int_{t^\star}^{t_0+\delta} k(t) \cdot x(t_0)^T \Lambda(t, t^\star) x(t_0) dt > 0, \tag{3.15}$$

where $\Lambda(t, t^\star) = P(t^\star - t_0)Q(2(t - t^\star))P(t^\star - t_0) - P(2(t - t_0))$. A sufficient condition for (3.15) to hold is

$$h(t, x(t_0)) = x(t_0)^T \Lambda(t, t^\star) x(t_0) > 0, \text{ for } t > t^\star.$$

As $\delta \downarrow 0$, we can write $P(t) = I + tA + \mathcal{O}(\delta^2)$, where $\mathcal{O}(\delta^2)/\delta \leq L$ for sufficiently small δ and some finite constant L. We therefore have

$$\begin{aligned}\Lambda(t, t^*) &= \left(I + (t^\star - t_0)A + \mathcal{O}\left(\delta^2\right)\right)\left(I + 2(t - t^\star)B + \mathcal{O}\left(\delta^2\right)\right) \\ &\quad \left(I + (t^\star - t_0)A + \mathcal{O}\left(\delta^2\right)\right) - \left(I + 2(t - t_0)A + \mathcal{O}\left(\delta^2\right)\right) \\ &= 2(t - t^\star)B + 2(t^\star - t_0)A - 2(t - t_0)A + \mathcal{O}\left(\delta^2\right) \\ &= 2(t - t^\star)(B - A) + \mathcal{O}\left(\delta^2\right). \end{aligned} \tag{3.16}$$

For sufficiently small δ, the first term dominates the second term. Recall that the quadratic form of a Laplacian matrix L exhibits the following form: $x^T L x = \sum_{l=1}^{n} \sum_{k=1}^{l-1} L_{kl}(x_l - x_k)^2$, for any $x \in \mathbb{R}^n$. Note that $B - A$ is in fact a negative Laplacian. Using (3.13), we can then write

$$\begin{aligned} h(t, x(t_0)) &= 2(t - t^\star) \sum_{r>s} (A_{sr} - B_{sr})\left(x_r(t_0) - x_s(t_0)\right)^2 + \mathcal{O}\left(\delta^2\right) \\ &= 2(t - t^\star) A_{ij} \left(x_j(t_0) - x_i(t_0)\right)^2 + \mathcal{O}\left(\delta^2\right). \end{aligned} \tag{3.17}$$

For small enough δ, the higher order terms are dominated by the first term. Hence, if there is a link e_{ij} such that $x_i(t_0) \neq x_j(t_0)$, there exists $t^\star$ such that $h(t, x(t_0)) > 0$ for $t \in (t^\star, t_0 + \delta]$. Since t_0 was arbitrary, we conclude that the optimal strategy must satisfy $\|u^\star(t)\|_1 = \ell$ for all t, given that each of the ℓ links connects two nodes having different values.

If no link such that $x_i(t_0) \neq x_j(t_0)$ exists at a given time t_0, the adversary does not need to break additional links, although breaking more links does not affect optimality because $h(t, x(t_0)) = 0$ in such a case. There are two cases under which the adversary cannot find a link to make $h(t, x(t_0)) > 0$: (i) The graph at time t_0 is one connected component. In this case, the nodes have already reached consensus and $\|u^\star(t)\|_1 < \ell$. This is a *losing strategy* for the adversary as he has failed in preventing nodes from reaching agreement; (ii) The graph at time t_0 has multiple connected components, and the number of links connecting the components is less than ℓ. The adversary here possesses a *winning strategy* with $\|u^\star(t)\|_1 < \ell$, as he can disconnect $\mathcal{G}$ into multiple components and prevent consensus.

Next, we need to show that the adversary will modify the ℓ links with the lowest w_{ij} values. Let us again restrict our attention to the interval $[t_0, t_0 + \delta]$ where the adversary applies strategy u. Assume (to the contrary) that the links the adversary breaks over this interval are not the ones with the lowest w_{ij} values. In particular, assume that the adversary chooses to break link e_{kl}, while there is a link e_{ij} such that $w_{ij} < w_{kl}$. Assume that the adversary switches at time $t^\star \in [t_0, t_0 + \delta]$ to strategy $u^\star$ by *breaking* link e_{ij} and *unbreaking* link e_{kl}. Then, (3.17) becomes

$$h(t, x(t_0)) = 2(t - t^*) \left(w_{kl}(t_0) - w_{ij}(t_0)\right) + \mathcal{O}\left(\delta^2\right).$$

Hence, by following the same arguments as above, we can conclude that breaking e_{kl} is not optimal.

The second step of the proof is to show that switching to strategy $u^\star$ guarantees an improved utility for the adversary *regardless of how the original trajectory corresponding to u changes beyond time $t_0 + \delta$*. To this end, we will assume that from time $t_0 + \delta$ onward, strategy $u^\star$ will *mimic* strategy u. Assume that strategy u switches from matrix A to matrix C over the interval $[t_0+\delta, t_0+2\delta]$, and define $R(t) := e^{Ct}$. Hence, strategy $u^\star$ will also switch from the system matrix B to matrix C. However, the trajectories corresponding to u and $u^\star$ will have different initial conditions at time $t_0 + \delta$, due to the switch that strategy $u^\star$ made at time $t^\star$. Figure 3.1 illustrates this idea. Recall that according to A, we have $\|u\|_1 < \ell$ and $u_{ij} = 0$. Here, the system matrix B can differ from the matrix A in two ways: either (i) B dictates breaking one additional link compared to A, or (ii) B dictates breaking link e_{ij} and

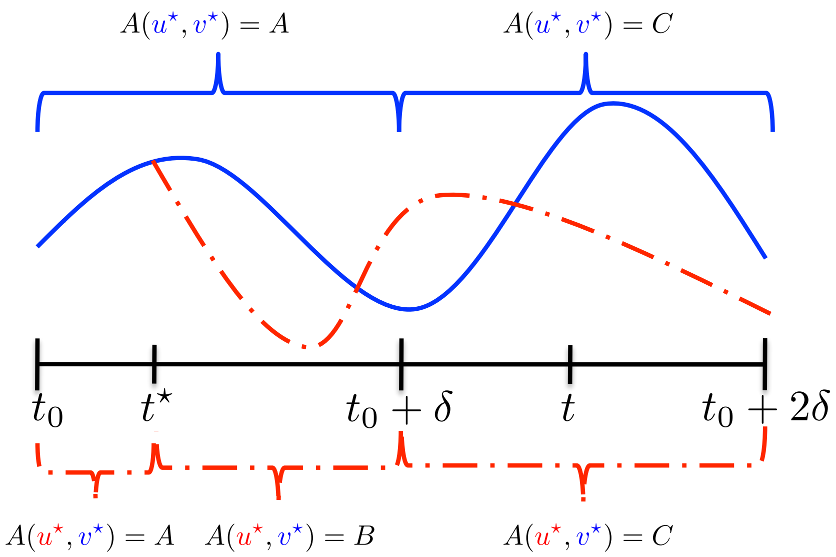

$A(u^\star, v^\star) = A$
$A(u^\star, v^\star) = C$
t_0
$t^\star$
$t_0 + \delta$
t
$t_0 + 2\delta$
$A(u^\star, v^\star) = A$
$A(u^\star, v^\star) = B$
$A(u^\star, v^\star) = C$

Figure 3.1: A demonstration of the technique used in the proof. The blue solid trajectory corresponds to u while the red dashed trajectory corresponds to $u^\star$.

unbreaking link e_{kl} where $w_{ij} < w_{kl}$. Consider Case (i) first and let us study the behavior of the system over the interval $[t_0 + \delta, t_0 + 2\delta]$ where we can assume that the system is time-invariant. To show that the gain obtained over $[t_0, t_0 + \delta]$ by the switch made by $u^\star$ is maintained over $[t_0 + \delta, t_0 + 2\delta]$, we must prove the following inequality:

$$\int_{t_0+\delta}^{t_0+2\delta} k(t) \cdot [L_1 - L_2]\, dt > 0, \tag{3.18}$$

where

$$\begin{aligned} L_1 &:= \left\| (R(t - (t_0 + \delta)) - M) Q(t_0 + \delta - t^\star) P(t^\star - t_0) x(t_0) \right\|_2^2, \\ L_2 &:= \left\| (R(t - (t_0 + \delta)) - M) P(t_0 + \delta - t_0) x(t_0) \right\|_2^2. \end{aligned}$$

As before, it suffices to prove that the integrand $L_1 - L_2$ is positive. Let us now expand both L_1 and L_2.

$$\begin{aligned} L_1 &= x(t_0)^T P(t^\star - t_0) Q(t_0 + \delta - t^\star)(R(t - (t_0 + \delta)) - M) \\ &\quad (R(t - (t_0 + \delta)) - M) Q(t_0 + \delta - t^\star) P(t^\star - s) x(t_0) \\ &= x(t_0)^T P(t^\star - t_0) Q(t_0 + \delta - t^\star)(R(2(t - (t_0 + \delta))) - M) Q(t_0 + \delta - t^\star) \\ &\quad P(t^\star - t_0) x(t_0) \end{aligned}$$

$$\begin{aligned} &= x(t_0)^T(P(t^\star - t_0)Q(t_0+\delta-t^\star)R(2(t-(t_0+\delta)))Q(t_0+\delta-t^\star) \\ &\quad P(t^\star-t_0)-M)x(t_0). \end{aligned}$$

Similarly,

$$L_2 = x(t_0)^T(P(\delta)R(2(t-(t_0+\delta)))P(\delta)-M)x(t_0).$$

We can then write

$$\begin{aligned} L_1 - L_2 &= x(t_0)^T(P(t^\star - t_0)Q(t_0+\delta-t^\star)R(2(t-(t_0+\delta)))Q(t_0+\delta-t^\star) \\ &\quad P(t^\star-t_0)-P(\delta)R(2(t-(t_0+\delta)))P(\delta))x(t_0) \\ &:= x(t_0)^T(F_1-F_2)x(t_0). \end{aligned}$$

Before we perform a first-order Taylor expansion to the above terms, let us define the following quantities: $\tau_1 = t^\star - t_0$, $\tau_2 = (t_0+\delta)-t^\star$, and $\tau_3 = t-(t_0+\delta)$, where $t^\star \in [t_0, t_0+\delta]$ and $t \in [t_0+\delta, t_0+2\delta]$. Using Proposition 2.1 in Section 2.7, we can now expand F_1 and F_2 as follows:

$$\begin{aligned} F_1 &= \left(I+\tau_1 A+\mathcal{O}\left(\tau_1^2\right)\right)\left(I+\tau_2 B+\mathcal{O}\left(\tau_2^2\right)\right)\left(I+2\tau_3 C+\mathcal{O}\left(\tau_3^2\right)\right) \\ &\quad \left(I+\tau_2 B+\mathcal{O}\left(\tau_2^2\right)\right)\left(I+\tau_1 A+\mathcal{O}\left(\tau_1^2\right)\right) \\ &= \left(I+\tau_1 A+\tau_2 B+\mathcal{O}\left(\delta^2\right)\right)\left(I+2\tau_3 C+\mathcal{O}\left(\delta^2\right)\right) \\ &\quad \left(I+\tau_1 A+\tau_2 B+\mathcal{O}\left(\delta^2\right)\right) \\ &= \left(I+\tau_1 A+\tau_2 B+2\tau_3 C+\mathcal{O}\left(\delta^2\right)\right)\left(I+\tau_1 A+\tau_2 B+\mathcal{O}\left(\delta^2\right)\right) \\ &= I+2\tau_1 A+2\tau_2 B+2\tau_3 C+\mathcal{O}\left(\delta^2\right) \\ F_2 &= \left(I+\delta A+\mathcal{O}\left(\delta^2\right)\right)\left(I+2\tau_3 C+\mathcal{O}\left(\tau_3^2\right)\right)\left(I+\delta A+\mathcal{O}\left(\delta^2\right)\right) \\ &= \left(I+\delta A+2\tau_3 C+\mathcal{O}\left(\delta^2\right)\right)\left(I+\delta A+\mathcal{O}\left(\delta^2\right)\right) \\ &= I+2\delta A+2\tau_3 C+\mathcal{O}\left(\delta^2\right). \end{aligned}$$

Hence, we have

$$\begin{aligned} F_1 - F_2 &= 2\left(\tau_1-\delta\right)A+2\tau_2 B+\mathcal{O}\left(\delta^2\right) \\ &= 2\tau_2\left(B-A\right)+\mathcal{O}\left(\delta^2\right) \\ &= 2\left(\left(t_0+\delta\right)-t^\star\right)\left(B-A\right)+\mathcal{O}\left(\delta^2\right), \end{aligned}$$

and thereby we obtain

$$\begin{aligned} L_1 - L_2 &= 2\left((t_0+\delta) - t^\star\right) \sum_{r>s} (A_{sr} - B_{sr}) \left(x_r(t_0) - x_s(t_0)\right)^2 + \mathcal{O}\left(\delta^2\right) \\ &= 2\left(t_0 + \delta - t^\star\right) A_{ij} \left(x_j(t_0) - x_i(t_0)\right)^2 + \mathcal{O}\left(\delta^2\right). \end{aligned}$$

If instead the matrix B dictates breaking link e_{ij} and unbreaking link e_{kl}, where $w_{ij} < w_{kl}$, the difference in the utilities would be

$$L_1 - L_2 = 2\left(t_0 + \delta - t^\star\right)\left(w_{kl}(t_0) - w_{ij}(t_0)\right) + \mathcal{O}\left(\delta^2\right).$$

Hence, in both cases, for small enough δ, we conclude that $L_1 - L_2 > 0$, which implies that (3.18) is satisfied, and the gain obtained by switching to system matrix B at $t^\star \in [t_0, t_0 + \delta]$ is maintained over $[t_0 + \delta, t_0 + 2\delta]$. Note that the effect of switching to matrix C is cancelled out in $F_1 - F_2$, and hence $L_1 - L_2$, since the strategy $u^\star$ is mimicking strategy u. Hence, by partitioning the interval $(t_0 + 2\delta, T]$ into small sub-intervals of length δ and repeating the above analysis, we conclude that the gain due to the switch at time $t^\star$ is preserved over the remaining time of the problem. □

We can now derive the optimal strategy of the designer in the min–max problem. Recall the set $\mathcal{L}_\ell(v) \subseteq \mathcal{E}$ defined in (3.10). Let $\mathcal{L}_{\ell,k}(v) \in \mathcal{E}$ denote the k-th link of $\mathcal{L}_\ell(v)$, $k \in [\ell]$. Also, define $\mathcal{L}_{\ell,k}^{-1}(v) \in \mathbb{R}$ as the value such that $\Phi(\mathcal{L}_{\ell,k}(v), \mathcal{L}_{\ell,k}^{-1}(v)) = \mathcal{L}_{\ell,k}(v)$. We assume that $\mathcal{L}_{\ell,1}^{-1}(v) \geq \ldots \geq \mathcal{L}_{\ell,\ell}^{-1}(v)$. Further, define the sets $\mathcal{P}(v) = \{(e_{ij}, a_{ij}\nu_{ij}) \mid e_{ij} \notin \mathcal{L}_\ell(v)\} \subset \mathcal{E} \times \mathbb{R}$ and $\overline{\mathcal{P}}(v) = \{(e_{ij}, \nu_{ij}) \mid e_{ij} \notin \mathcal{L}_\ell(v)\} \subset \mathcal{E} \times \mathbb{R}$. We also define

$$[v_{\mathcal{S}}(b)]_{ij} = \begin{cases} b, & e_{ij} \in \mathcal{S} \\ 0, & e_{ij} \notin \mathcal{S} \end{cases}.$$

Theorem 3.2. *In the min–max problem, and under Assumptions 3.1 and 3.2, the optimal strategy of the designer is to run Algorithm 3.1 and set $v_{ij}^\star \in \{0, b\}$ if $\nu_{ij} = 0$. Further, it is optimal for the designer to modify ℓ links.*

Proof. By Theorem 3.1, we deduce that $\|v^\star(t)\|_1 = b\ell$, because the designer would be at a disadvantage if he modifies fewer links than the adversary.

Algorithm 3.1: Computing the optimal strategy for the minimizer in the min–max problem.

```
input: a strategy $v$ with $\|v\|_1 = 0$
 for $i = \ell \downarrow 1$
     if $\exists \mathcal{S} \subseteq \Phi(\mathcal{P}(0)), |\mathcal{S}| = i, \mathcal{L}_{\ell,i}(0) \notin \mathcal{L}_\ell(v_{\mathcal{S}}(b))$
         Set $v^\star_{ij} = b$, $\forall e_{ij} \in \mathcal{S} \cup \Phi_{\ell-i}\left(\overline{\mathcal{P}}(v_{\mathcal{S}}(b))\right)$.
         Exit for loop.
     end
 end
 if $\|v\|_1 = 0$
     Set $v^\star_{ij} = b$ for all $e_{ij} \in \Phi_\ell\left(\overline{\mathcal{P}}(0)\right)$.
 end
```

We first consider the designer's strategy over a fixed small interval $[t_0, t_0+\delta]$ over which both u and v are fixed. Using similar steps as those leading to (3.15), and after applying a first-order Taylor expansion, we can write the designer's utility over $[t_0, t_0 + \delta]$ as

$$\int_{t_0}^{t_0+\delta} k(t) \cdot 2(t - t_0) \sum_{j>i} (a_{ij} + v_{ij})(1 - u_{ij})(x_i(t_0) - x_j(t_0))^2 dt + \mathcal{O}\left(\delta^2\right). \quad (3.19)$$

According to Theorem 3.1, and in the absence of the designer, it is optimal for the adversary to break the links in $\mathcal{L}_\ell(0)$. Therefore, the designer must attempt to modify the ranking of the links such that the links (or a subset of them) in $\mathcal{L}_\ell(0)$ are not in $\mathcal{L}_\ell(v^\star)$. In essence, this is what Algorithm 3.1 attempts to achieve. Being of the lowest negative value, and hence the link both the adversary and the designer are interested in, let us explore how the designer can push $\mathcal{L}_{\ell,\ell}(0)$ higher in the ranking of the link values. The designer can achieve this if under some strategy $v \in \mathcal{V}$, the value $\mathcal{L}^{-1}_{\ell,\ell}(0)$ is no longer among the lowest ℓ negative values; in other words, the designer can alter the ranking if there is a set $\mathcal{S} \subset \mathcal{P}(0)$, $|\mathcal{S}| = \ell$, such that when he sets $v_{ij} = b$ for all links in $\mathcal{S}$, there will be ℓ values that are smaller than $\mathcal{L}^{-1}_{\ell,\ell}(0)$ (steps 2 and 3 in Algorithm 3.1). The adversary will then break the links in $\mathcal{S}$ and will spare the link corresponding to $\mathcal{L}^{-1}_{\ell,\ell}(0)$ as required. To see why this is optimal, consider the following two cases, covering the types of links that can be in $\mathcal{S}$.

Case 1: If a link in $\mathcal{S}$ is also in $\mathcal{L}_\ell(0)$, then this is optimal due to the fact that the adversary will disconnect that link since it is in $\mathcal{L}_\ell(0)$. Hence, if the designer can utilize this link to modify the ranking and protect a link whose associated value is more negative ($\mathcal{L}_{\ell,\ell}(0)$ in this case), then this can only improve his utility. The same reasoning applies if more than one of the links in $\mathcal{S}$ are also in $\mathcal{L}_\ell(0)$.

Case 2: If none of the links in $\mathcal{S}$ is in $\mathcal{L}_\ell(0)$, then necessarily some of the links in $\mathcal{L}_\ell(0)$ will also be protected along with the link corresponding to $\mathcal{L}^{-1}_{\ell,\ell}(0)$. This is because $|\mathcal{S}| = \ell$, and the adversary can break at most ℓ links. Hence, this scenario is more favorable to the designer than the previous one and can therefore only improve his utility.

If such an $\mathcal{S}$ exists, then the designer would have exhausted all possible moves, since $|\mathcal{S}| = \ell$, and the algorithm terminates (step 4 of the algorithm). Otherwise, if no such set exists in $\mathcal{P}(0)$, then the designer should try to protect the next most negative link whose value is precisely $\mathcal{L}^{-1}_{\ell,\ell-1}(0)$ by finding a set $\mathcal{S}$ of size $\ell-1$. Since $\mathcal{L}^{-1}_{\ell,\ell-1}(0) \geq \mathcal{L}^{-1}_{\ell,\ell}(0)$, the link corresponding to $\mathcal{L}^{-1}_{\ell,\ell}(0)$ along with $\mathcal{S}$ will constitute the set of ℓ links that the adversary will break. Then, the designer should set $v_{ij} = b$ for all the links in $\mathcal{S}$, and for the remaining action the designer should select the link with the most negative ν_{ij} that is *not* in $\mathcal{L}_\ell(v_\mathcal{S}(b))$; this is precisely the set $\Phi_1\left(\overline{\mathcal{P}}(v_\mathcal{S}(b))\right)$ (step 3 of the algorithm). The reason behind searching in $\overline{\mathcal{P}}(v_\mathcal{S}(b))$ and not in $\mathcal{P}(v_\mathcal{S}(b))$ after finding $\mathcal{S}$ is that the a_{ij}'s only affect the utility of the designer when he attempts to alter the ranking.

This procedure then repeats until the designer has tried to protect all the links in $\mathcal{L}_\ell(0)$. If the designer fails in protecting *all* the links in $\mathcal{L}_\ell(0)$, then we must have $\|v\|_1 = 0$, i.e., the input strategy was not altered. Then, the optimal strategy is to set $v_{ij} = b$ for the links with most negative ν_{ij}'s in $\overline{\mathcal{P}}(0)$ (steps 7 and 8 in Algorithm 3.1).

The final step of the proof is to show that applying Algorithm 3.1 over $[0, T]$ is optimal for the designer. To this end, it suffices to show that modifying links with lower ν_{ij} values is more beneficial to the designer, as Algorithm 3.1 attempts to protect these links. Given the links $e_{ij}, e_{kl} \in \mathcal{E}$, assume that $\nu_{ij} < \nu_{kl}$. Consider the two system matrices A and B, and let $v_{ij} = 0$, $v_{kl} = b$ and $v^\star_{ij} = b$, $v^\star_{kl} = 0$. Assume that a strategy v dictates applying matrix A over $[t_0, t_0 + \delta]$ and applying the matrix C over $[t_0 + \delta, t_0 + 2\delta]$. Also, assume that according to $v^\star$, the designer applies A over $[t_0 + \delta, t^\star)$, B over $(t^\star, t_0 + \delta]$, and C over $[t_0+\delta, t_0+2\delta]$. Following the steps presented in step 2 of the proof of Theorem 3.1, we conclude that, for δ small enough, the quantity $-2\left(t_0+\delta-t^\star\right) b(\nu_{kl}-\nu_{ij}) + \mathcal{O}\left(\delta^2\right)$ is negative. It then follows that the gain obtained by switching to system matrix B at $t^\star \in [t_0, t_0+\delta]$ is maintained over $[t_0+\delta, t_0+2\delta]$. Hence, by partitioning the interval $(t_0+2\delta, T]$ into small sub-intervals of length δ and repeating the above analysis, we conclude that Algorithm 3.1 is optimal over $[0, T]$. □

3.4.2 The Max–Min Problem

The following theorem specifies the optimal strategies of the adversary and the designer in the max–min problem. Let $\mathcal{F}_\ell(u) = \Phi_\ell(\{(e_{ij}, \nu_{ij}) \mid e_{ij} \in \mathcal{E}(u)\}) \subset \mathcal{E}(u)$, where we recall that $\mathcal{E}(u) = \mathcal{E} \setminus \{e_{ij} \in \mathcal{E} \mid u_{ij}(t) = 1\}$, for some $u \in \mathcal{U}$. If $m < 2\ell$, the sets $\mathcal{E}(u), \mathcal{F}_\ell(u)$ could contain fewer than ℓ links. For simplicity, we assume that $m \geq \ell$ in the following proof, which guarantees that $|\mathcal{F}_\ell(u)| = \ell$. However, the result of the theorem applies regardless of this assumption, and the modification of the proof is straightforward.

Theorem 3.3. *Under Assumptions 3.1 and 3.2, and for a fixed strategy u of the adversary, the optimal strategy of the network designer in the max–min problem is given by*

$$v_{ij}^\star(u) \;=\; \begin{cases} b, & e_{ij} \in \mathcal{F}_\ell(u) \\ 0, & e_{ij} \notin \mathcal{F}_\ell(u) \end{cases}.$$

If the designer has an optimal strategy of modifying fewer than ℓ links, then either $\mathcal{G}$ has a cut of size less than ℓ or the nodes have reached consensus by time t. In either of these cases, breaking ℓ links is also optimal.

Proof. The proof follows the same two steps used to prove Theorem 3.1. For a fixed strategy of the adversary u, we will show that it is optimal for the minimizer to rank the links based on their ν_{ij} values. Under Assumption 3.1, the function x becomes piecewise continuous. Hence, the function ν_{ij}, for all $e_{ij} \in \mathcal{E}(u)$, is also piecewise continuous and its value cannot change abruptly over a finite interval. As a result, we can regard the system as a time-invariant one over a small interval $[t_0, t_0+\delta] \subset [0, T]$, where $0 < \delta \leq \tau$, and τ was defined in (3.5).

Let v be an arbitrary strategy of the designer with $\|v\|_1 < b\ell$. Over a small interval, v and $v^\star$ induce certain system matrices. Let the system matrix corresponding to v over $[t_0, t_0+\delta]$ be $A(u,v) = A$. Since the control strategies of both players are time-invariant over this interval, the state trajectory is given by (3.11). We want to show that switching from strategy v to strategy $v^\star$ at some time $t^\star \in [t_0, t_0+\delta]$ can improve the utility of the designer. To this end, we assume that the matrix induced by $v^\star$ over $[t_0, t^\star)$ is A, while the system matrix corresponding to $v^\star$ over $[t^\star, t_0+\delta]$ is B. Assume that $e_{ij} \in \mathcal{E}(u)$, i.e., $u_{ij} = 0$. Over $[t^\star, t_0+\delta]$, the strategies v and $v^\star$ are identical except at link e_{ij}, where $v_{ij} = 0$ and $v^\star_{ij} = b$, i.e., $\|v\|_1 < \|v^\star\|_1$ over this sub-interval. It follows that:

$$B_{ij} = a_{ij} + b > A_{ij} = a_{ij}, \quad A_{kl} = B_{kl}, \quad \forall e_{kl} \neq e_{ij}. \tag{3.20}$$

Following similar steps to those in the proof of Theorem 3.1, we conclude that it suffices to prove

$$h(t, x(t_0)) = x(t_0)^T \Lambda(t, t^\star) x(t_0) < 0, \text{ for } t > t^\star,$$

where $\Lambda(t, t^\star)$ was defined in the proof of Theorem 3.1. For sufficiently small δ, we can arrive at the expansion in (3.16). Using (3.20) and properties of Laplacian matrices, we can then write

$$\begin{aligned} h(t, x(t_0)) &= 2(t - t^\star) \sum_{r>s} (A_{sr} - B_{sr}) \left(x_r(t_0) - x_s(t_0)\right)^2 + \mathcal{O}\left(\delta^2\right) \\ &= -2(t - t^\star) b \left(x_j(t_0) - x_i(t_0)\right)^2 + \mathcal{O}\left(\delta^2\right). \end{aligned} \tag{3.21}$$

For small enough δ, the higher order terms are dominated by the first term. Hence, if there is a link e_{ij} such that $x_i(t_0) \neq x_j(t_0)$, there exists $t^\star$ such that $h(t, x(t_0)) < 0$ for $t \in (t^\star, t_0+\delta]$. Since t_0 was arbitrary, we conclude that the optimal strategy must satisfy $\|v^\star(t)\|_1 = b\ell$ for all t, given that each of the ℓ links connects two nodes having different values.

If no link such that $x_i(t_0) \neq x_j(t_0)$ exists at a given time t_0, the designer does not need to break additional links, although breaking more links does not affect optimality because $h(t, x(t_0)) = 0$ in such a case. There are two cases where the designer cannot find a link to make $h(t, x(t_0)) < 0$, and they were presented in the proof of Theorem 3.1 in the case of the adversary.

However, unlike the case of the adversary, Case (i) presents a winning strategy for the designer as the nodes are in agreement. Case (ii) is not necessarily a winning or a losing strategy for the designer.

Next, we need to show that the designer will modify the ℓ links in $\mathcal{E}(u)$ with the lowest ν_{ij} values. Let us again restrict our attention to the interval $[t_0, t_0+\delta]$ where the designer applies strategy v. Assume (to the contrary) that the links the designer modifies over this interval are not the ones with the lowest ν_{ij} values. In particular, assume that the designer chooses to modify link $e_{kl} \in \mathcal{E}(u)$, while there is a link $e_{ij} \in \mathcal{E}(u)$ such that $\nu_{ij} < \nu_{kl}$. Assume that the designer switches at time $t^\star \in [t_0, t_0+\delta]$ to strategy $v^\star$ by modifying link e_{ij} instead of link e_{kl}. Then, (3.21) becomes

$$h(t, x(t_0)) = -2(t-t^*)b\left(\nu_{kl}(t_0) - \nu_{ij}(t_0)\right) + \mathcal{O}\left(\delta^2\right).$$

Hence, by following the same arguments as above, we can conclude that modifying e_{kl} is not optimal.

The second step of the proof is to show that switching to strategy $v^\star$ guarantees an improved utility for the designer regardless of how the original trajectory corresponding to v changes beyond time $t_0+\delta$. To this end, we will assume that from time $t_0+\delta$ onward, strategy $v^\star$ will mimic strategy v. Assume that strategy v switches from matrix A to matrix C over the interval $[t_0+\delta, t_0+2\delta]$. Hence, strategy $v^\star$ will also switch from the system matrix B to matrix C. However, the trajectories corresponding to v and $v^\star$ will have different initial conditions at time $t_0+\delta$, due to the switch that strategy $v^\star$ made at time $t^\star$. Recall that according to A, we have $\|v\|_1 < b\ell$ and $v_{ij} = 0$. Here, the system matrix B can differ from the matrix A in two ways: either (i) B dictates modifying one additional link compared to A, or (ii) B dictates modifying link e_{ij} instead of link e_{kl} where $\nu_{ij} < \nu_{kl}$. Consider the behavior of the system over the interval $[t_0+\delta, t_0+2\delta]$ where we can assume that the system is time-invariant. To show that the gain obtained over $[t_0, t_0+\delta]$ by the switch made by $v^\star$ is maintained over $[t_0+\delta, t_0+2\delta]$, it suffices to prove that the integrant $L_1 - L_2$ is negative, where L_1 and L_2 were defined in the proof of Theorem 3.1. For Case (i), by following the steps presented in the proof of Theorem 3.1, we can write

$$L_1 - L_2 \;\; = \;\; -2\left(t_0+\delta-t^\star\right) b\left(x_j(t_0) - x_i(t_0)\right)^2 + \mathcal{O}\left(\delta^2\right).$$

For Case (ii), the difference in utilities would be

$$L_1 - L_2 = 2\left(t_0 + \delta - t^\star\right)\left(w_{kl}(t_0) - w_{ij}(t_0)\right) + \mathcal{O}\left(\delta^2\right).$$

Hence, for small enough δ, we conclude that $L_1 - L_2 < 0$. By partitioning the interval $(t_0 + 2\delta, T]$ into small sub-intervals of length δ and repeating the above analysis, we conclude that the gain due to the switch at time $t^\star$ is preserved over the remaining time of the problem. This concludes the proof. □

Next, we present the optimal strategy of the adversary. To this end, define the set

$$\mathcal{D}_\ell = \Phi_\ell(\{(e_{ij,}a_{ij}\nu_{ij}) \mid e_{ij} \in \mathcal{E}\} \cup \{(e_{ij}, (a_{ij} + b)\nu_{ij}) : e_{ij} \in \mathcal{E}\}).$$

Theorem 3.4. *In the max–min problem, and under Assumptions 3.1 and 3.2, the optimal strategy of the adversary is given by*

$$u^\star_{ij}(t) \;=\; \begin{cases} 1, & e_{ij} \in \mathcal{D}_\ell \\ 0, & e_{ij} \notin \mathcal{D}_\ell \end{cases}.$$

Further, it is optimal for the adversary to break ℓ links.

Proof. By Theorem 3.3, we deduce that $\|u^\star(t)\|_1 = \ell$, because the adversary would be at a disadvantage if he breaks fewer links than the designer. We first consider the adversary's strategy over a fixed small interval $[t_0, t_0 + \delta]$ over which both u and v are fixed. Using a first-order Taylor expansion, the adversary's utility over $[t_0, t_0 + \delta]$ is given by (3.19).

In this problem, the adversary has the first-mover-advantage and needs to dispose of the links that can reduce his utility. The adversary knows that, according to $v^\star(u)$, the designer attempts to make the ν_{ij}'s smaller by adding b to the corresponding edge weights. However, we cannot rule out the possibility that $(a_{lk} + b)\nu_{lk} > a_{ij}\nu_{ij}$, for some links e_{kl} and e_{ij}. Hence, the adversary is not only interested in finding the smallest negative $(a_{ij} + b)\nu_{ij}$'s, but also needs to consider the $a_{ij}\nu_{ij}$'s themselves. It follows that the adversary needs to find the terms that can become very small (negative) and set $u_{ij} = 1$ to the corresponding links. But those links are exactly the ones included in $\mathcal{D}_\ell$. Formally, we can write

$$-\sum_{\substack{j>i\\ e_{ij}\in\mathcal{D}_\ell}} (a_{ij}+v_{ij})\nu_{ij} \leq -\sum_{\substack{j>i\\ e_{ij}\notin\mathcal{D}_\ell}} (a_{ij}+v_{ij})\nu_{ij},$$

This confirms that, over the interval $[t_0, t_0+\delta]$, $u^\star$ is as claimed.

The final step of the proof is to show that switching from a strategy u to strategy $u^\star$ guarantees an improved utility for the designer over $[0, T]$. To this end, it suffices to show that modifying links with lower w_{ij} values is more beneficial to the adversary. For the links $e_{ij}, e_{kl} \in \mathcal{E}$, assume that $w_{ij} < w_{kl}$. Consider the two system matrices A and B, and let $u_{ij} = 0$, $u_{kl} = 1$ and $u^\star_{ij} = 1$, $u^\star_{kl} = 0$. Assume that the strategy u dictates applying matrix A over $[t_0, t_0+\delta]$ and applying the matrix C over $[t_0+\delta, t_0+2\delta]$. On the other hand, we assume that according to $u^\star$, the adversary applies A over $[t_0+\delta, t^\star)$, B over $(t^\star, t_0+\delta]$, and C over $[t_0+\delta, t_0+2\delta]$. Following the steps presented in step 2 of the proof of Theorem 3.1, we conclude that, for δ small enough, the quantity $2\left(t_0+\delta-t^\star\right)\left(w_{kl}-w_{ij}\right) + \mathcal{O}\left(\delta^2\right)$ is positive, which implies that the gain obtained by switching to system matrix B at $t^\star \in [t_0, t_0+\delta]$ is maintained over $[t_0+\delta, t_0+2\delta]$. Hence, by partitioning the interval $(t_0+2\delta, T]$ into small sub-intervals of length δ and repeating the above analysis, we conclude that $u^\star$ is optimal over $[0, T]$. □

Remark 3.3. *(Potential-Theoretic Analogy) When the graph is viewed as an electrical network, $a_{ij}+v_{ij}$ can be viewed as the conductance of link $e_{ij} \in \mathcal{E}$, and $x_i - x_j$ as the potential difference across the link. Therefore, according to Theorems 3.2 and 3.3, the optimal strategy of the designer in both problems involves finding the links with the highest potential difference (or the lowest ν_{ij}'s) and increasing the conductance of those links by setting $v_{ij} = b$. This leads to increasing the power dissipation across those links, which translates to increasing the information flow across the network and results in faster convergence. The optimal strategy of the adversary should therefore involve breaking the links with the highest power dissipation. But power dissipation is given by $(a_{ij}+v_{ij})(x_i-x_j)^2$, and this is exactly what the adversary targets according to Theorems 3.1 and 3.4.* •

3.4.3 From Potential Theory to the Maximum Principle

In this section, we show that the strategies derived in the above theorems satisfy the first-order necessary conditions for optimality given by the MP. We will address here the min–max problem; a theorem similar to the one presented below can be obtained also for the max–min problem. In [64], we showed that the optimal strategies provided by the MP for the min–max problem are the same as those derived in Theorems 3.1 and 3.2, with the ranking of the links performed after replacing the quantity ν_{ij} with the quantity $(p_j - p_i)(x_i - x_j)$, where p is the costate vector. The next theorem states that the potential-theoretic strategies satisfy the MP if the controllers do note switch infinitely many times over $[0, T]$.

Theorem 3.5. *Under Assumptions 3.1 and 3.2, the optimal strategies in Theorems 3.1 and 3.2 satisfy the canonical equations of the MP.*

Proof. See Section 3.8. □

3.4.4 Complexity of the Optimal Strategies

We next study the complexity of the optimal strategies. We first start with the max–min problem. Assuming, as in Remark 3.2, that the players switch their strategies a total of K times over $[0, T]$, we conclude that the worst-case complexity of the strategy of either player is $\mathcal{O}(K \cdot m \log m)$ as their strategies involve merely the ranking of sets of size at most $2m$. As for the min–max problem, the complexity of the adversary's strategy is $\mathcal{O}(K \cdot m \log m)$. The main bottleneck in the strategy of the designer is step 2 in Algorithm 3.1. The size of the set $\mathcal{P}(0)$ is at most $m - \ell$; thus, the worst-case complexity for the designer is $K \cdot \sum_{i=1}^{m-\ell} \binom{m-\ell}{i} \approx K \cdot \sum_{i=1}^{\ell} (m - \ell)^i$. By comparison with (3.8), we conclude that the optimal strategies achieve vast complexity reductions.

3.4.5 An Illustrative Example

The goal of this example is twofold: (i) to show how the players execute their strategies; and (ii) to serve as a counterexample showing that an SPE may not exist and to provide some guidelines as to when one would exist. We will study the interaction between the designer and the adversary for the case when $T = \tau$, and τ is very small. By Assumption 3.1, we conclude that the players cannot change the actions they choose at time $t = 0$. Assume that $\mathcal{G}$ is a complete graph with three nodes with the following weights:

$$A(0,0) = \begin{bmatrix} -4 & 3 & 1 \\ 3 & -5 & 2 \\ 1 & 2 & -3 \end{bmatrix}.$$

Define $e_1 = (1,2)$, $e_2 = (2,3)$, $e_3 = (1,3)$. Let $\nu_{12} = -1$, $\nu_{23} = -2$, and $\nu_{13} = -5$. Let $x(0) = [1,2,3]^T$ and $\ell = 1$. Consider the following two cases:
Case 1: *($b = 1$)* Let us first consider the max–min problem. We have

$$\mathcal{D}_1 = \Phi_1(\{(e_1,-3),(e_1,-4),(e_2,-4),(e_3,-5),(e_2,-6),(e_3,-10)\}) = \{e_3\}.$$

Hence, according to Theorem 3.4, the adversary breaks e_3, and we have that $\mathcal{E}(u^\star) = \mathcal{E} \setminus e_3$. We also have $\mathcal{F}_1(u^\star) = \{e_2\}$, which means that $v^\star = [0,1,0]^T$ and $u^\star = [0,0,1]^T$. Hence, using (3.19), we can write

$$\begin{aligned} \underline{V} &= \int_0^T k(t) \cdot 2t[3(x_1(0) - x_2(0))^2 + 3(x_2(0) - x_3(0))^2]dt + \mathcal{O}\left(\delta^2\right) \\ &= \int_0^T k(t) \cdot 12t dt + \mathcal{O}\left(\delta^2\right). \end{aligned}$$

For the min–max problem, Algorithm 3.1 uses the following sets $\mathcal{L}_1(0) = \{e_3\}$ and $\mathcal{P}(0) = \{(e_1,-3),(e_2,-4)\}$. Let $\mathcal{S} = \{e_2\}$, and note that $\mathcal{S} \in \Phi(\mathcal{P}(0)) = \{e_1, e_2\}$. We then have $v_{\mathcal{S}}(1) = [0,1,0]^T$ and $\mathcal{L}_1(v_{\mathcal{S}}(1)) = \{e_2\}$. Note that $\mathcal{L}_1(0) \notin \mathcal{L}_1(v_{\mathcal{S}}(1))$. Hence, the condition in step 2 of the algorithm is satisfied with this choice of $\mathcal{S}$, and we have $v^\star = v_{\mathcal{S}}(1)$. Then, Theorem 3.2 says that the designer will increase the weight of e_2, and Theorem 3.1 says that the adversary will break the same link, i.e., $v^\star = [0,1,0]^T$ and $u^\star = [0,1,0]^T$. We thus have

$$\overline{V} = \int_0^T k(t) \cdot 14t dt + \mathcal{O}\left(\delta^2\right).$$

We conclude that in this case $\overline{V} > \underline{V}$, and an SPE does not exist.
Case 2: *($b = 0.4$)* By repeating the above steps, we conclude that in the max–min problem we have $v^\star = [0, 0.4, 0]^T$ and $u^\star = [0,0,1]^T$, and we can

write

$$\underline{V} = \int_0^T k(t) \cdot 10.8t dt + \mathcal{O}\left(\delta^2\right).$$

For the min–max problem, one cannot find a set $\mathcal{S}$ satisfying the conditions of step 2 in Algorithm 3.1. To execute step 8 of the algorithm, note that $\mathcal{L}_1(0) = \{e_3\}$, and hence $\Phi_1(\overline{\mathcal{P}}(0)) = \{e_2\}$. We therefore have $v^\star = [0, 0.4, 0]^T$ and $u^\star = [0, 0, 1]^T$, and hence

$$\overline{V} = \int_0^T k(t) \cdot 10.8t dt + \mathcal{O}\left(\delta^2\right).$$

In this case, the pair of inequalities (3.6) are satisfied and an SPE exists. The main difference between the two cases was that the designer was able to find a set $\mathcal{S}$ that allows him to alter the ranking and deceive the adversary when $b = 1$. This made the adversary break e_3 in the max–min problem and break e_2 in the min–max problem which led to having $\overline{V} \neq \underline{V}$. When such a set does not exit, the strategy of the adversary is unchanged in both problems, and hence the upper and lower values would agree. Hence, for an SPE to exist, one needs a behavior similar to Case 2 to occur throughout the problem horizon $[0, T]$. This of course depends on the value of b and the weights a_{ij}. Section 3.5 explores the question of existence of an SPE further.

3.5 A Sufficient Condition for the Existence of an SPE

Thus far, we have solved the min–max and max–min problems separately and showed that the derived optimal strategies achieve the upper and lower values. Hence, to prove the existence of an SPE, it remains to verify whether the pair of inequalities (3.6) can be satisfied under some assumptions, even though the action sets of the players are non-rectangular in the max–min problem. Besides the issue of non-rectangular action sets, the main reason that the upper and lower values are different is mainly due to the ability of the minimizer to *deceive* the maximizer by altering the ranking of the most negative values. If we remove this ability from the network designer, we should expect that an SPE would exist. The following theorem makes this argument formal. Define $\gamma := \frac{4\|x_0\|_\infty^2}{\epsilon^2}$, $\epsilon > 0$. We assume that ϵ is chosen to guarantee $\gamma > 1$.

Theorem 3.6. *Given $\epsilon > 0$, assume that T is small enough such that (3.36) in Section 3.8 holds. Then, under Assumptions 3.1 and 3.2, a sufficient condition for the existence of an SPE for the underlying zero-sum game between the designer and the adversary is to select b such that*

$$0 \leq b \leq \min_{e_{ij}, e_{kl} \in \mathcal{E}} |\gamma a_{ij} - a_{kl}| , \tag{3.22}$$

given that $a_{ij} \neq a_{kl}$ and $a_{ij} > \gamma a_{kl}$ whenever $a_{ij} > a_{kl}$, for all $e_{ij}, e_{kl} \in \mathcal{E}$.

Proof. It suffices to show that $\mathcal{L}_\ell(v^\star) = \mathcal{L}_\ell(0) = \mathcal{D}_\ell$ as this would imply that the adversary would break the same links whether he acts first or second, and as a result the strategy of the minimizer in both problems will be the same. This will guarantee that (3.6) is satisfied. This would occur if the minimizer cannot protect any of the links in $\mathcal{L}_\ell(0)$. In other words, this will happen if the minimizer cannot satisfy the condition in step 2 of Algorithm 3.1 for any $i \in [\ell]$. A sufficient condition for $\mathcal{L}_\ell(v^\star) = \mathcal{L}_\ell(0) = \mathcal{D}_\ell$ to hold is to require

$$\min_{e_{ij} \in \Phi(\mathcal{P}(0))} (a_{ij} + b)\nu_{ij} \quad > \quad \max_{e_{ij} \in \mathcal{L}_\ell(0)} a_{ij}\nu_{ij}.$$

This implies that no matter how the designer changes the weights of the links in $\Phi(\mathcal{P}(0))$, he cannot make those links more negative than the links in $\mathcal{L}_\ell(0)$. To satisfy this inequality, we will establish that whenever $a_{ij}\nu_{ij} > a_{kl}\nu_{kl}$, we must have $(a_{ij} + b)\nu_{ij} > a_{kl}\nu_{kl}$, for all $e_{ij}, e_{kl} \in \mathcal{E}$. We can then re-write the condition on b as

$$b \leq \frac{a_{ij}\nu_{ij} - a_{kl}\nu_{kl}}{-\nu_{ij}} = a_{kl}\frac{|\nu_{kl}|}{|\nu_{ij}|} - a_{ij}, \quad \forall e_{ij}, e_{kl} \in \mathcal{E} \tag{3.23}$$

Consider the following two cases. If $\nu_{kl} \geq \nu_{ij}$, then we must have $a_{kl} > a_{ij}$. Then, by assumption we have that $a_{kl} > \gamma a_{ij}$. By Lemma 3.1 in Section 3.8, we can write

$$a_{kl}\frac{|\nu_{kl}|}{|\nu_{ij}|} - a_{ij} \geq \frac{1}{\gamma}a_{kl} - a_{ij} > 0. \tag{3.24}$$

Next, consider the case when $\nu_{ij} > \nu_{kl}$. In this case, a_{ij} can be larger or smaller than a_{kl}. However, if $a_{ij} > a_{kl}$, and recalling that $a_{ij}\nu_{ij} > a_{kl}\nu_{kl}$, then

$$\gamma a_{kl} < a_{ij} < a_{kl}\frac{|\nu_{kl}|}{|\nu_{ij}|} \leq \gamma a_{kl},$$

which is a contradiction. The case $a_{kl} = a_{ij}$ is excluded by assumption. Hence, in this case, we must have $a_{ij} < a_{kl}$, and the inequality in (3.24) applies. Thus, by choosing b as in (3.22), we obtain the condition we are seeking. Note that we do not need to consider the case when $a_{ij}\nu_{ij} = a_{kl}\nu_{kl}$ since the players will be indifferent as to which link to choose. □

Remark 3.4. *The condition derived in the Theorem 3.6 requires the network to be "sufficiently diverse" in the sense that the weights of the links have to be not only different from each other, but also a factor γ apart. This is due to the fact that we were seeking uniform bounds on the ν_{ij}'s, for all $e_{ij} \in \mathcal{E}$. If we allow b to vary with time, then one can find less restrictive conditions to ensure the existence of an SPE. However, this would require (3.23) to be verified at each time instant. Further, the bound derived in (3.24) is loose, because it was obtained by bounding $|v_{kl}|$ and $|v_{ij}|$ independently. Tighter bounds could be obtained by studying the dynamics of $|\nu_{kl}|/|\nu_{ij}|$. However, studying the time derivative of this ratio is not tractable.* •

Remark 3.5. *This result highlights the fact that, in general, Stackelberg games are more natural to study security problems than zero-sum games. In fact, the leader-follower formulation fits many real-world security scenarios; see [65] and the references therein. However, the sufficient condition we derive here is a step in the right direction for establishing the existence of an SPE for the zero-sum game between the designer and the adversary. We are currently investigating whether this condition is also necessary.* •

3.6 Attack-II: Adversary vs. Network Designer

Assume now that both the adversary and the designer are capable of adding signals to all the nodes in the network in order to carry out their respective objectives. The dynamics in this case are given by:

$$\dot{x}(t) = Ax(t) + v(t) + u(t), \quad x(0) = x_0, \tag{3.25}$$

where $x(t)$ is the state of the network. Also, $u(t)$ is the signal to be added by the adversary and that controlled by the designer is $v(t)$. The system matrix A is time-invariant in this setting and it satisfies the properties in (3.2) and (3.3), with $A_{ij} = a_{ij}$. To capture physical constraints, we assume that both signals must satisfy power and energy constraints. Formally, the action spaces of the players in this case are

$$\begin{aligned}\mathcal{U} &= \left\{u \in C^1[0,T] \mid \|u(t)\|_2^2 \leq P_{\max}, \forall t \in [0,T],\right.\\ &\qquad \left.\|u\|_{L_2}^2 \leq E_{\max} < TP_{\max}\right\},\\ \mathcal{V} &= \left\{v \in C^1[0,T] \mid \|v(t)\|_2^2 \leq P_{\max}, \forall t \in [0,T],\right.\\ &\qquad \left.\|v\|_{L_2}^2 \leq E_{\max} < TP_{\max}\right\}.\end{aligned}$$

It will be evident from the structure of the Hamiltonian that if we allow $E_{\max} \geq TP_{\max}$, it is straightforward to show that both $v^\star$, $u^\star$ will have magnitude $\sqrt{P_{\max}}$ throughout $[0,T]$, and the energy constraint will be satisfied; see Section 2.4 for a similar case. We thus consider the more interesting case where the players do not have enough energy to operate at maximum power throughout $[0,T]$. Also, the case where the power and budgets are different for the players can be readily obtained from the results we demonstrate below.

As in the previous section, we are interested in studying the interaction between the designer, who attempts to minimize $J(u,v)$, and the adversary who is interested in maximizing $J(u,v)$. We make the following assumption.

Assumption 3.3. The matrix A, the time interval $[0,T]$, the values $P_{\max}$ and $E_{\max}$, and the initial state x_0 are known to both players.

Unlike Attack-I, the zero-sum game played by the designer and the adversary admits a pure-strategy SPE as the following theorem proves. To derive the feedback SPE, we will invoke Theorem 8.1 in [63, p. 427] which provides a necessary and sufficient condition for the existence of a feedback SPE. We will also derive the open-loop SPE using Theorem 8.2 in [63, p. 428]. Let $V(t,x)$ be the value function we seek. Define the vector $V_x := \frac{\partial}{\partial x}V$. The Hamiltonian associated with the game can then be written as:

$$\begin{aligned}H(x, V_x, u, v) = \frac{k(t)}{2}\|x-\bar{x}\|_2^2 &+ V_x^T(Ax+u+v)\\ &+ \lambda_1(t)(\|u\|_2^2 - P_{\max}) + \nu_1(\|u\|_2^2 - E_{\max})\\ &+ \lambda_2(t)(\|v\|_2^2 - P_{\max}) + \nu_2(\|v\|_2^2 - E_{\max}),\end{aligned}$$

where $\lambda_1, \lambda_2 \in C^1[0,T]$ and ν_1, ν_2 are constant Lagrange multipliers. Recall

Isaacs condition [63]:

$$\min_{\|v\|_2^2 \le P_{\max}} \max_{\|u\|_2^2 \le P_{\max}} H(x, V_x, u, v) = \max_{\|u\|_2^2 \le P_{\max}} \min_{\|v\|_2^2 \le P_{\max}} H(x, V_x, u, v), \quad (3.26)$$

and note that it is satisfied here because the Hamiltonian is separable in u and v.

Theorem 3.7. *Under Assumption 3.3, the value function $V(t, x) = x^T X x$ satisfies Isaacs condition (3.26), where $X \ge 0$ is a symmetric matrix that satisfies the following Riccati differential equation:*

$$\dot{X} = XA + A^T X + \frac{k}{2} M - \frac{k}{2} I, \quad X(T) = 0.$$

Further, the pair

$$v^\star(t) = -\sqrt{E_{\max}/T} \frac{Xx}{\|Xx\|_2}, \quad u^\star(t) = \sqrt{E_{\max}/T} \frac{Xx}{\|Xx\|_2},$$

constitutes a feedback SPE for the zero-sum game between the network designer and the adversary.

Proof. Note that

$$\begin{aligned}
\arg\max_{\|u\|_2^2 \le P_{\max}} H(x, V_x, u, v^\star) &= \arg\max_{\|u\|_2^2 \le P_{\max}} V_x^T u + (\lambda_1(t) + \nu_1)\|u\|_2^2 \\
&\le \arg\max_{\|u\|_2^2 \le P_{\max}} \|u\|_2 \|V_x\|_2 + (\lambda_1(t) + \nu_1)\|u\|_2^2, \\
\arg\min_{\|v\|_2^2 \le P_{\max}} H(x, V_x, u^\star, v) &= \arg\min_{\|v\|_2^2 \le P_{\max}} V_x^T v + (\lambda_2(t) + \nu_2)\|v\|_2^2 \\
&\ge \arg\min_{\|v\|_2^2 \le P_{\max}} -\|v\|_2 \|V_x\|_2 + (\lambda_2(t) + \nu_2)\|v\|_2^2,
\end{aligned}$$

where the inequalities follow from the Cauchy–Schwarz inequality. We therefore conclude that $u^\star$ must be *aligned* with V_x, whereas $v^\star$ and V_x must have opposite directions.

To obtain $u^\star$, we differentiate the Hamiltonian:

$$\frac{\partial}{\partial u} H = 2(\lambda_1(t) + \nu_1)u + V_x = 0. \quad (3.27)$$

Note that optimality requires that both players spend all the energy available to them. Hence, the energy constraints are, in fact, equality ones. We therefore have that $\lambda_1 \leq 0$ and $\nu_1 \neq 0$. By complementary slackness, we have the following two cases.

Case 1: ($\lambda_1(t) < 0 \implies \|u(t)\|_2^2 = P_{\max}$, *for some* $t \in [0, T]$) Using (3.27), we obtain

$$-\frac{1}{2(\lambda_1 + \nu_1)} = \frac{\|u\|_2^2}{u^T V_x}. \tag{3.28}$$

We thus have

$$u_1^\star = \frac{\|u\|_2^2}{\|u\|_2 \|V_x\|_2} V_x = \sqrt{P_{\max}} \frac{V_x}{\|V_x\|_2}.$$

Case 2: ($\|u(t)\|_2^2 < P_{\max} \implies \lambda_1(t) = 0$, *for some* $t \in [0, T]$) Using (3.27), we obtain

$$\frac{-1}{2\nu_1} = \frac{u^T V_x}{\|V_x\|_2^2}, \tag{3.29}$$

and therefore

$$u = \frac{u^T V_x}{\|V_x\|_2^2} V_x,$$

This enables us to write

$$E_{\max} = \int_0^T \|u\|_2^2 dt = \int_0^T \frac{(u^T V_x)^2}{\|V_x\|_2^2} dt,$$

which is satisfied by the control

$$u_2^\star = \sqrt{E_{\max}/T} \frac{V_x}{\|V_x\|_2}.$$

It is not clear whether it is optimal to apply $u_1^\star$ or $u_2^\star$ for some $t \in [0, T]$ or throughout $[0, T]$. We need to solve for the Lagrange multipliers in order to characterize the optimal control. By (3.29), we can obtain

$$\nu_1 = -\frac{1}{2} \frac{\|V_x\|_2}{\sqrt{E_{\max}/T}}.$$

Since ν_1 is time-invariant, it must maintain this value even when $\lambda(t) < 0$, i.e., in Case 1. Hence, we can substitute this value in (3.28) to obtain

$$\lambda_1 = \frac{1}{2} \left(\frac{1}{\sqrt{E_{\max}/T}} - \frac{1}{\sqrt{P_{\max}}} \right) \|V_x\|_2.$$

However, $E_{\max} < TP_{\max}$ by assumption, which implies that $\lambda_1(t) > 0$. This contradicts the assumption in Case 1. It therefore follows that either $\lambda_1(t) < 0$ for all $t \in [0,T]$ *or* $\|u\|_2^2 < P_{\max}$ for all $t \in [0,T]$. The former is impossible because $\int_0^T \|u_1^\star\|_2^2 dt = TP_{\max} > E_{\max}$. Hence, $u^\star = u_2^\star$ as claimed. By following the same steps as above, and noting that $\lambda_2 \geq 0$ and $\nu_2 \neq 0$, it is straightforward to show that $v^\star = -u^\star$.

It remains to find V_x to completely characterize $u^\star$ and $v^\star$. To this end, we can now write [63, (8.9) on p. 426]:

$$-\frac{\partial}{\partial t}V = \frac{k}{2}\|x-\bar{x}\|_2^2 + V_x^T Ax.$$

Since the cost functional is quadratic in the state, we make the guess that $V(t,x) = x^T X(t) x$, $X \geq 0$. Recalling that $M = \frac{1}{n}\mathbf{1}\mathbf{1}^T$, we can write

$$\begin{aligned} -x^T\dot{X}x &= \frac{k}{2}\|x-\bar{x}\|_2^2 + x^T XAx + x^T A^T Xx \\ &= \frac{k}{2}(x^T x - 2x^T\bar{x} + \bar{x}^T\bar{x}) + x^T XAx + x^T A^T Xx \\ &= \frac{k}{2}(x^T x - 2x^T Mx + x^T Mx) + x^T XAx + x^T A^T Xx. \end{aligned}$$

We therefore conclude that X must satisfy the following Riccati differential equation:

$$\dot{X} = XA + A^T X + \frac{k}{2}M - \frac{k}{2}I, \quad X(T) = 0,$$

as claimed. The proof of the theorem is thus complete. □

The following theorem derives the open-loop SPE for the zero-sum game.

Theorem 3.8. *Under Assumption 3.3, the pair*

$$v^\star(t) = -\sqrt{E_{\max}/T}\frac{p}{\|p\|_2}, \quad u^\star(t) = \sqrt{E_{\max}/T}\frac{p}{\|p\|_2}, \tag{3.30}$$

constitutes an open-loop SPE for the zero-sum game between the network designer and the adversary, where p is the costate vector:

$$p(t) = \int_t^T e^{A(2\tau - t)}(x_0 - \bar{x})d\tau,$$

which can be computed offline, and locally, by both players.

Proof. The proof follows the same steps as the proof of Theorem 3.7. The difference here is that p is the costate vector (p plays the role of V_x which appears in the previous theorem) which must satisfy the costate equation [63, (8.14) on p. 429]:

$$\dot{p} = -\frac{\partial}{\partial x} H = -Ap - 2(x - \bar{x}), \quad p(T) = 0.$$

It follows that

$$p(t) = 2 \int_t^T e^{A(\tau - t)} (x(\tau) - \bar{x}) d\tau,$$

and by substituting the optimal controllers (3.30) in the ODE (3.25), we can find the optimal trajectory x and write

$$p(t) = \int_t^T e^{A(2\tau - t)} (x_0 - \bar{x}) d\tau,$$

as claimed. □

Remark 3.6. *The above two theorems imply that if the graph is connected, and T is large enough, the nodes would reach consensus even though an adversary is present. This is because the designer is able to cancel the signal of the adversary completely. Note that this is due to the assumption that the designer has complete knowledge about A and x. An interesting future direction is to study this problem in the case where the players have varying knowledge about the network's topology and state. This will cast the game into an asymmetric information setting. We suspect that the designer would not always be successful in annihilating the adversary's effect in this case. This invites one to establish an analogy with communication networks where an extensive body of research has been devoted to the study of interference cancelation under different types of uncertainties.* •

3.7 Summary

In this chapter, we have considered two types of adversarial attacks on a network of agents performing distributed averaging. Both attacks have the common objective of slowing down the convergence of the computation at the nodes to the global average. We introduced a network designer whose objective is to assist the nodes in reaching consensus by countering the attacks of the adversary. Attack-I involves an adversary and a network designer who are capable of targeting links. We have formulated and solved two problems that capture the competition between the two players. We considered practical models for the players by constraining their actions along the problem horizon. The derived strategies were shown to exhibit a low worst-case complexity. When Zeno behavior is excluded, we showed that the optimal strategies admit a potential-theoretic analogy. Finally, we showed that when the link weights are sufficiently diverse, an SPE exists for the zero-sum game between the designer and the adversary. ATTACK-II, on the other hand, involves an adversary and a network designer who are able to modify the values of the nodes by injecting signals of bounded power and energy. We utilized the maximum principle to completely characterize the optimal strategies of the players and showed that an SPE exists in this case.

3.8 Additional Proofs

In this section, we provide a proof of Theorem 3.5. We also provide a technical result that is instrumental in proving Theorem 3.6.

Proof for Theorem 3.5 . For a fixed strategy v of the designer, it was shown in [64] that the adversary's strategy derived using the MP requires finding the lowest $f_{ij} = (a_{ij} + v_{ij})(p_i - p_j)(x_j - x_i)$ values, for all $e_{ij} \in \mathcal{E}$. However, Theorem 3.1 requires finding the lowest w_{ij}'s. The designer's strategy relies on finding the lowest $(p_i - p_j)(x_j - x_i)$ values according to the MP, and it requires finding the lowest ν_{ij}'s according to Theorem 3.2. In order to prove the theorem, and since $w_{ij} = (a_{ij} + v_{ij})\nu_{ij}$, $a_{ij} + v_{ij} \geq 0$, it is sufficient to show that $w_{ij} \leq w_{kl}$ implies that $f_{ij} \leq f_{kl}$, for all e_{ij}, $e_{kl} \in \mathcal{E}$. Without loss of generality, we will assume that $v_{ij} = v_{kl} = 0$.

The Hamiltonian associated with the min–max problem is:

$$H(x, p, u, v) = \frac{1}{2}k(t) \left\| x(t) - \bar{x} \right\|_2^2 + p(t)^T A(u(t), v(t))x(t),$$

where $p(t)$ is the costate vector, whose existence is guaranteed by the MP because an optimal solution for the min–max exists. The first-order necessary conditions for optimality are (noting that $A^T = A$ and recalling that $V =$

$V(0)$) [66]:

$$\begin{aligned}
\dot{p} &= -\frac{\partial}{\partial x} H \\
&= -k(x-\bar{x}) - Ap, \quad p(T) = 0 && (3.31) \\
\dot{x} &= Ax, \quad x(0) = x_0 && (3.32) \\
u^\star(v) &= \arg\max_U H(x,p,u,v), \quad v^\star = \arg\max_V H(x,p,u^\star(v),v).
\end{aligned}$$

To prove the theorem, we will rely on approximating the state and costate up to first-order using Taylor expansion. To this end, we partition the problem's horizon into $L > K$ small sub-intervals of length $0 < \delta \leq \tau$, where τ was defined in (3.5), over which the system is time-invariant. More formally, define the times $0 = t_1 < t_2 < \ldots < t_L < t_{L+1} = T$. Let A_i be the system matrix corresponding to the interval $[t_i, t_{i+1}]$, $i \in [L]$. We will denote the i-th row of matrix A_k by $A_{k,i}$ and its (i,j)-th element by a^k_{ij}. The proof comprises two steps:

(i) We establish the claim of the theorem over $[t_L, t_{L+1}]$.

(ii) We generalize the argument to hold over $[0,T]$.

We start by considering the interval $[t_L, t_{L+1}]$. The solutions to ODEs (3.31) and (3.32) over this interval are:

$$\begin{aligned}
x_{A_L}(t) &= e^{A_L(t-t_L)} x_{A_L}(t_L) \\
p_{A_L}(t) &= \int_t^T e^{-A_L(t-\tau)}(x_{A_L}(\tau) - \bar{x}) d\tau.
\end{aligned}$$

Let $P_i(t) := e^{A_i t} = I + tA_i + \mathcal{O}(\delta^2)$. We can then re-write the above expressions as

$$\begin{aligned}
x_{A_L}(t) &= P_L(t-t_L) x_{A_L}(t_L) \\
&= (I + (t-t_L)A) x_{A_L}(t_L) + \mathcal{O}\left(\delta^2\right) \\
p_{A_L}(t) &= \int_t^T P_L(\tau - t)[P_L(\tau - t_L) - M] x_{A_L}(t_L) d\tau \\
&= \int_t^T [P_L(2\tau - t - t_L) - M] x_{A_L}(t_L) d\tau \\
&= \int_t^T [I + (2\tau - t - s)A_L - M] x_{A_L}(t_L) d\tau + \mathcal{O}\left(\delta^2\right)
\end{aligned}$$

$$
\begin{aligned}
&= [(T-t)I + (T-t)(T-t_L)A_L - (T-t)M]x_{A_L}(t_L) + \mathcal{O}\left(\delta^2\right) \\
&= (T-t)(I-M)x_{A_L}(t_L) + \mathcal{O}\left(\delta^2\right),
\end{aligned}
$$

where the last equality follows because $(T-t)(T-t_L)A_L = \mathcal{O}(\delta^2)$. Define $\xi(\alpha,\beta) := \alpha - \beta$, $\alpha, \beta \in \mathbb{R}$, and write

$$
\begin{aligned}
x_{A_L}(t) &= (I + \xi(t,t_L)A_L)\, x_{A_L}(t_L) + \mathcal{O}\left(\delta^2\right) \\
p_{A_L}(t) &= \xi(T,t)(I-M)x_{A_L}(t_L) + \mathcal{O}\left(\delta^2\right). \qquad (3.33)
\end{aligned}
$$

Further, define the matrices

$$
G := I + \xi(t,t_L)A_L, \quad R := \xi(T,t)(I-M),
$$

and write

$$
\begin{aligned}
w_{ij} &= a^L_{ij}(x_{A_L,i} - x_{A_L,j})(x_{A_L,j} - x_{A_L,i}) \\
&= a^L_{ij} x_{A_L}(t_L)^T (G_i - G_j)(G_j - G_i)^T x_{A_L}(t_L) + \mathcal{O}\left(\delta^2\right) \\
f_{ij} &= a^L_{ij} x_{A_L}(t_L)^T (R_i - R_j)(G_j - G_i)^T x_{A_L}(t_L) + \mathcal{O}\left(\delta^2\right),
\end{aligned}
$$

where R_i^T, R_i^T are the i-th rows of G and R, respectively. Using the definitions of G and R, we obtain

$$
\begin{aligned}
(G_i - G_j)(G_j - G_i)^T = -(I_i - I_j)(I_i - I_j)^T &- \xi(t,t_L)((I_i - I_j)(A_{L,i} - A_{L,j})^T \\
&+ (A_{L,i} - A_{L,j})(I_i - I_j)^T) \\
&- \xi(t,t_L)^2 (A_{L,i} - A_{L,j})(A_{L,i} - A_{L,j})^T.
\end{aligned}
$$

The last term is quadratic, and thus we can absorb it in $\mathcal{O}(\delta^2)$. We then have

$$
\begin{aligned}
&a^L_{ij}(G_i - G_j)(G_j - G_i)^T - a^L_{kl}(G_k - G_l)(G_l - G_k)^T = a^L_{kl}(I_k - I_l)(I_k - I_l)^T \\
&\quad - a^L_{ij}(I_i - I_j)(I_i - I_j)^T + (a^L_{kl}(I_k - I_l)(A_{L,k} - A_{L,l})^T \\
&\quad - a^L_{ij}(I_i - I_j)(A_{L,i} - A_{L,j})^T)\xi(t,t_L) + (a^L_{kl}(A_{L,k} - A_{L,l})(I_k - I_l)^T \\
&\quad - a^L_{ij}(A_{L,i} - A_{L,j})(I_i - I_j)^T)\xi(t,t_L) + \mathcal{O}\left(\delta^2\right).
\end{aligned}
$$

Similarly, we have

$$a_{ij}^L(R_i - R_j)(G_j - G_i)^T - a_{kl}^L(R_k - R_l)(G_l - G_k)^T = (a_{kl}^L(I_k - I_l)(I_k - I_l)^T \\ - a_{ij}^L(I_i - I_j)(I_i - I_j)^T)\xi(T,t) + \mathcal{O}\left(\delta^2\right).$$

Let $\Gamma_1 = a_{kl}^L(I_k - I_l)(I_k - I_l)^T - a_{ij}^L(I_i - I_j)(I_i - I_j)^T$ and $\Gamma_2 = a_{kl}^L(I_k - I_l)(A_{L,k} - A_{L,l})^T - a_{ij}^L(I_i - I_j)(A_{L,i} - A_{L,j})^T$. We now have

$$\begin{aligned} w_{ij} - w_{kl} &= x_{A_L}(t_L)^T(\Gamma_1 + \xi(t,t_L)\Gamma_2 + \xi(t,t_L)\Gamma_2^T)x_{A_L}(t_L) + \mathcal{O}\left(\delta^2\right) \\ f_{ij} - f_{kl} &= \xi(T,t)x_{A_L}(t_L)^T\Gamma_1 x_{A_L}(t_L) + \mathcal{O}\left(\delta^2\right). \end{aligned}$$

If $w_{ij} - w_{kl} \leq 0$, since $\xi(T,t) \geq 0$, we can write

$$\begin{aligned} \xi(T,t)(w_{ij} - w_{kl}) &= x_{A_L}(t_L)^T(\xi(T,t)\Gamma_1 + \xi(T,t)\xi(t,t_L)\Gamma_2 \\ &\quad +\xi(T,t)\xi(t,t_L)\Gamma_2^T)x_{A_L}(t_L) + \mathcal{O}\left(\delta^2\right) \\ &\leq 0, \end{aligned}$$

or

$$\xi(T,t)x_{A_L}(t_L)^T\Gamma_1 x_{A_L}(t_L) + \mathcal{O}\left(\delta^2\right) \leq 0,$$

but the left hand side is $f_{ij} - f_{kl}$; hence, $w_{ij} \leq w_{kl}$ implies that $f_{ij} \leq f_{kl}$ as required.

So far, we have verified the claim of the theorem over the interval $[t_L, T]$ only. We are now in a position to generalize the statement of the theorem to the interval $[0, T]$. The only complication that arises when studying this interval is that the terminal condition, i.e. $p_{L-1}(t_L)$, is not forced to be zero as in $[t_L, T]$.

Over the interval $[t_{L-1}, t_L]$, the state and costate are

$$\begin{aligned} x_{L-1}(t) &= e^{A_{L-1}(t-t_{L-1})}x_{A_{L-1}}(t_{L-1}) \\ p_{A_{L-1}}(t) &= e^{-A_{L-1}(t-t_{L-1})}p_{A_{L-1}}(t_{L-1}) - \int_{t_{L-1}}^{t} e^{-A_{L-1}(t-\tau)}(x_{A_{L-1}}(\tau) - \bar{x})d\tau. \end{aligned}$$

Solving for $p_{A_{L-1}}(t_{L-1})$ in terms of $p_{A_{L-1}}(t_L)$ and substituting back, we can write $p_{A_{L-1}}(t)$ in terms of $p_{A_{L-1}}(t_L)$ as follows:

$$p_{A_{L-1}}(t) = e^{-A_{L-1}(t-t_L)}p_{A_{L-1}}(t_L) + \int_{t}^{t_L} e^{-A_{L-1}(t-\tau)}(x_{A_{L-1}}(\tau) - \bar{x})d\tau. \quad (3.34)$$

By continuity of the state and costate functions, it follows that $x_{A_{L-1}}(t_L) = x_{A_L}(t_L)$, $p_{A_{L-1}}(t_L) = p_{A_L}(t_L)$. Using a first-order Taylor expansion and (3.33), we can write

$$\begin{aligned} p_{A_{L-1}}(t) &= (I + \xi(t_L, t)A_{L-1})p_{A_L}(t_L) + \xi(t_L, t)(I - M)x_{A_{L-1}}(t_{L-1}) + \mathcal{O}\left(\delta^2\right) \\ &= (I + \xi(t_L, t)A_{L-1})(\xi(t_L, t)(I - M)x_{A_L}(t_L)) \\ &\quad + \xi(t_L, t)(I - M)x_{A_{L-1}}(t_{L-1}) + \mathcal{O}\left(\delta^2\right) \\ &= \xi(t_L, t)(I - M)x_{A_{L-1}}(t_L) + \xi(t_L, t)(I - M)x_{A_{L-1}}(t_{L-1}) + \mathcal{O}\left(\delta^2\right). \end{aligned}$$

We can further simplify this expression using $x_{A_{L-1}}(t)$ as follows:

$$\begin{aligned} \xi(t_L, t)(I - M)x_{A_{L-1}}(t_L) &= \xi(t_L, t)(I - M)e^{A_{L-1}(t_L - t_{L-1})}x_{A_{L-1}}(t_{L-1}) \\ &= \xi(t_L, t)(I - M)(I + \xi(t_L, t_{L-1})A_{L-1})x_{A_{L-1}}(t_{L-1}) \\ &\quad + \mathcal{O}\left(\delta^2\right) \\ &= \xi(t_L, t)(I - M)x_{A_{L-1}}(t_{L-1}) + \mathcal{O}\left(\delta^2\right), \end{aligned}$$

and therefore we have

$$p_{A_{L-1}}(t) = 2\xi(t_L, t)(I - M)x_{A_{L-1}}(t_{L-1}) + \mathcal{O}\left(\delta^2\right). \tag{3.35}$$

Comparing (3.33) and (3.35), we conclude that the argument used to prove the claim over the interval $[t_L, T]$ applies over $[t_{L-1}, t_L]$. Hence, $w_{ij} - w_{kl} \leq 0$ implies that $f_{ij} - f_{kl} \leq 0$ over $[t_{L-1}, t_L]$.

Note that we can generalize (3.34) to any interval $[t_i, t_{i+1}]$, $i \in [L]$, as follows:

$$p_{A_i}(t) = e^{-A_i(t - t_{i+1})}p_{A_i}(t_{i+1}) + \int_t^{t_{i+1}} e^{-A_i(t-\tau)}(x_{A_i}(\tau) - \bar{x})d\tau.$$

Following similar steps to the above, we can arrive at

$$p_{A_i}(t) = \frac{T - t_i}{\delta}\xi(t_{i+1}, t)(I - M)x_{A_i}(t_i) + \mathcal{O}\left(\delta^2\right), \quad t \in [t_i, t_{i+1}],$$

which maintains the same structure as in (3.35), and the claim therefore holds for the interval $[t_i, t_{i+1}]$, $i \in [L]$, and the theorem is proved. □

Lemma 3.1. *Given $\epsilon > 0$ and $\delta \leq \tau$, τ defined in (3.5), one can select the* *problem horizon T small enough such that*

$$\epsilon \leq |x_i(t) - x_j(t)| \leq 2\left\|x_0\right\|_\infty, \quad \forall e_{ij} \in \mathcal{E}, \tag{3.36}$$

for all $t \in [0, T]$.

Proof. By the structure of the system matrix in (3.1), we can deduce that $|x_i - x_j|$ cannot increase as $t \to T$. Thus

$$\begin{aligned} |x_i(t) - x_j(t)| &\leq \max_{i,j \in [n]} |x_i(0) - x_j(0)| \\ &\leq 2 \max_{i \in [n]} |x_i(0)| = 2\left\|x_0\right\|_\infty. \end{aligned}$$

This provides the uniform upper bound. In order to obtain a uniform lower bound, we need to ensure that $|x_i(t) - x_j(t)|$ does not approach zero as $t \to T$. We are seeking a time $t^\star$ such that for a given $\epsilon > 0$, we have $|x_i(t) - x_j(t)| \geq \epsilon$ for all $t < t^\star$ and all $e_{ij} \in \mathcal{E}$. We can then fix $T < t^\star$ to ensure the existence of a uniform lower bound on $|x_i(t) - x_j(t)|$. Let us again restrict our attention to a small interval $[t_0, t_0 + \delta]$ where the system is time-invariant, and let the system matrix over this interval be A. We require that the system did not reach equilibrium over this interval, i.e., $x(t_0 + \delta) \neq \bar{x}$. Without loss of generality, we assume that $x_1(t_0) > \ldots > x_n(t_0)$[2]. Define the following dynamics:

$$\begin{aligned} \frac{d}{dt}(\overline{y}_i - x_1(t_0)) &= \sum_{j \neq i} A_{ij}(x_1(t_0) - \overline{y}_i) \\ \frac{d}{dt}(\underline{y}_i - x_n(t_0)) &= \sum_{j \neq i} A_{ij}(x_n(t_0) - \underline{y}_i), \end{aligned}$$

with initial conditions $\overline{y}_i(t_0) = 2x_1(t_0)$, $\underline{y}_i(t_0) = 2x_n(t_0)$. Note that $\dot{x}_i = \sum_{j \neq i} A_{ij}(x_j - x_i)$. It follows that $\dot{\underline{y}}_i \leq \dot{x}_i \leq \dot{\overline{y}}_i$. By the comparison principle, we conclude that $\underline{y}_i - x_n(t_0) \leq x_i \leq \overline{y}_i - x_1(t_0)$, for $i \in \mathcal{N}$. Note that we can readily find the solution trajectories for $\overline{y}$ and $\underline{y}$. By defining $a_i = \sum_{j \neq i} A_{ij}$, we can then write

$$\overline{y}_i - x_1(t_0) = e^{-a_i(t - t_0)} x_1(t_0), \quad \underline{y}_i - x_n(t_0) = e^{-a_i(t - t_0)} x_n(t_0).$$

[2] We are making the implicit assumption that $x_1(0) > \ldots > x_n(0)$.

By solving the equation $\overline{y}_{i-i} - x_1(t_0) = \underline{y}_i - x_n(t_0)$, we can find a time $t_i^\star$ when x_{i-1} can potentially meet x_i:

$$t_i^\star = \frac{1}{a_{i-1} - a_i} \ln\left(\frac{x_1(t_0)}{x_n(t_0)}\right) + t_0.$$

If $t_i^\star > t_0 + \delta$, for all $i \in \mathcal{N}$, then we need to propagate the solution forward, and keeping in mind that the system matrix could change, until we find a time $t_i^\star$ in some interval $[\tilde{t}, \tilde{t} + \delta]$ where $\overline{y}_{i-i} = \underline{y}_i$ for some $i \in \mathcal{N}$. Then, for a given $\epsilon > 0$, we can select $T < t_i^\star$ such that $|x_i - x_{i-1}| \geq |\underline{y}_i - \overline{y}_{i-1}| \geq \epsilon$; hence, we conclude that for this choice of T we can guarantee that $|x_i - x_j| \geq \epsilon > 0$ for all $e_{ij} \in \mathcal{E}$. $\square$

CHAPTER 4

STABILIZATION IN THE PRESENCE OF MODELING UNCERTAINTY

4.1 Background

In the previous chapters, we have taken a differential game-theoretic approach toward designing robust strategies to control spread of information. In this chapter, we take a different approach, assuming that the adversary has already acted on the network, and that his intervention has led to a large modeling uncertainty in the system. Instead of having a *centralized* network designer as in the previous chapters, our alternative approach here investigates the ability of the nodes to stabilize the network, in the presence of a large modeling uncertainty, using a *distributed* control law.

Logic-based switching supervisory control has been proposed as a method to overcome limitations of adaptive control schemes [67]. A fundamental difference between the two approaches is that while adaptive control requires continuous tuning of parameters, supervisory control relies on logic-based switching among a collection of candidate controllers. Continuous tuning suffers from well-known issues such as loss of stabilizability. In the classical supervisory control scheme, a centralized supervisor estimates the state of the plant, and based on the history of estimation errors, it activates a certain candidate controller. For a more detailed study of supervisory control, see Chapter 6 of [68].

Supervisory control has been used in various problems and applications [69–77]. In [69, 70], the set-point control problem has been studied using a supervisory control framework. It has also been utilized in path-following problems for underactuated systems with large modeling uncertainties [72]. Recently, supervisory control has been extended to addresses the problem of stabilizing uncertain systems with quantized outputs [77].

In this chapter, motivated by its attractive properties, we extend the supervisory control framework to a distributed setting. A distributed version of supervisory control can have wide applications in stabilization and tracking problems over networked systems in the presence of large modeling uncertainties.

4.2 Main Results

The main contribution of this chapter is extending the centralized supervisory control framework to a distributed setting. We first provide a detailed description of the main components in this scheme. We prove that when the set in which the unknown parameters take values is finite, the switching stops in finite time at each node. Further, we provide sufficient conditions for achieving set-point tracking using this framework without requiring the individual agents to have explicit knowledge of the desired set-point. Finally, we apply this scheme to the distributed averaging problem in the presence of unknown parameters.

Organization

In Section 4.3, we introduce the system model and present the problem formulation. The main components of the distributed supervisory control scheme are provided in Section 4.4. Section 4.5 contains the stability analysis of the proposed scheme. An application to the distributed averaging problem is presented in Section 4.6. We conclude the chapter in Section 4.7.

Notation

We denote the i-th row of a matrix $X \in \mathbb{R}^{n\times m}$ by $[X]_i \in \mathbb{R}^m$, and the (i, j)-th entry of that matrix by $[X]_{ij} \in \mathbb{R}$. Similarly, we denote the i-th entry of a vector $x \in \mathbb{R}^n$ by $[x]_i \in \mathbb{R}$.

4.3 System Model

Consider a network with n nodes, and let $x \in \mathbb{R}^n$ be the state of the network, where $[x]_i \in \mathbb{R}$ is the state of node i. It is possible to extend this setting to the case where the state of the i-th agent is k_i-dimensional, where $k_1+\ldots+k_n = n$; however, in this chapter, we restrict our attention to the case where the state of each node is scalar for simplicity. Let $u \in \mathbb{R}^n$ be a vector consisting of the inputs to all the nodes with $[u]_i \in \mathbb{R}$ being a scalar input to node i. Further, let $y \in \mathbb{R}^n$ be a vector consisting of the outputs of all the nodes with $[y]_i \in \mathbb{R}$ being a scalar output of node i. Similar to the state variables, it is possible to allow the nodes to take multiple inputs and produce multiple outputs, and the restriction to the single-input single-output set-up is for purpose of clarity in presentation.

The network is described by a graph whose topology is unknown, i.e., the interconnections among the n nodes are not known. Let $\mathcal{P} = [r]$ be a finite index set. To each $p \in \mathcal{P}$, we associate a graph $\mathcal{G}_p = (\mathcal{V}_p, \mathcal{E}_p)$, where $\mathcal{V}_p$ is the set of vertices, and $\mathcal{E}_p \subseteq \mathcal{V}_p \times \mathcal{V}_p$ is the set of edges. The index $p^\star \in \mathcal{P}$ is unknown to the nodes, and its corresponding graph, $\mathcal{G}_{p^\star}$, describes the actual network under study. The graphs $\mathcal{G}_p$, $p \neq p^\star$, are different possibilities of what $\mathcal{G}_{p^\star}$ might be. To each graph $\mathcal{G}_p$, there corresponds a linear dynamical system represented by a triple (A_p, B_p, C_p), where $A_p, B_p, C_p \in \mathbb{R}^{n\times n}$. Each triple represents a different possibility of the actual system $(A_{p^\star}, B_{p^\star}, C_{p^\star})$ that governs the dynamics of the network. In particular, we assume that the nodes operate according to the following linear dynamics:

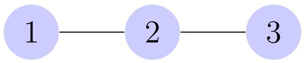
$$\begin{aligned} \dot{x} &= A_{p^\star}x + B_{p^\star}u, \quad x(0) = x_0, \\ y &= C_{p^\star}x. \end{aligned}$$

We define the neighborhood of node i in the graph $\mathcal{G}_p$ as

$$N_p(i) = \{j \in \mathcal{V}_p \mid (j, i) \in \mathcal{E}_p\}.$$

Note that we have not explicitly included i in $N_p(i)$ to allow for applications where node i is not able to measure its own state, for example.

In order to capture the underlying network topology, we must have that the state x_i, control u_i, and measurement y_i of node i can only depend on the



$$A_p = \begin{bmatrix} 0 & * & 0 \\ * & 0 & * \\ 0 & * & 0 \end{bmatrix}$$

Figure 4.1: A path graph with 3 nodes and its corresponding A_p.

states, control inputs, and measurements of the nodes in $N_p(i)$. To this end, we impose the following *sparsity* constraint on the matrices $\{A_p, B_p, C_p \mid p \in \mathcal{P}\}$:

$$j \notin N_p(i) \implies [A_p]_{ij} = [B_p]_{ij} = [C_p]_{ij} = 0, \quad p \in \mathcal{P}. \tag{4.1}$$

Under this constraint, the matrices A_p, B_p, C_p can be seen as an *encoding* of the topology of the graph $\mathcal{G}_p$. To demonstrate the sparsity constraint, consider the 3-node path graph shown in Fig. 4.1. For this graph, the matrix A_p must have the shown structure, where " $*$ " can be any nonzero real number.

Further, in order to be able to design decentralized controllers, we must restrict the knowledge of node i about the graph $\mathcal{G}_p$. In particular, we assume that the knowledge of node i about the topology of $\mathcal{G}_p$ is only local; this can be captured by restricting the knowledge of node i to the set $\{[A_p]_i, [B_p]_i, [C_p]_i\}$. Formally, we make the following assumption.

Assumption 4.1. The set $\mathcal{P}$ is finite, and the set $\{[A_p]_i, [B_p]_i, [C_p]_i \mid p \in \mathcal{P}\}$ is known to node i.

Our goal is to design decentralized control inputs $[u]_i$, via an extension of the classical supervisory control scheme, in order to track the following stable linear reference model:

$$\begin{aligned} \dot{x}_m &= A_m x_m, \quad x_m(0) = x_m^0, \\ y_m &= C_m x_m, \end{aligned} \tag{4.2}$$

where $x_m \in \mathbb{R}^n$ and $A_m, C_m \in \mathbb{R}^{n \times n}$. Define the tracking error, e_T, as follows:

$$e_T = y_m - y.$$

The problem we are solving here is not the general tracking problem, because there is no external reference signal. The reason behind introducing the reference model is motivated by applications where the agents attempt to converge to a certain set-point *without the explicit knowledge of that point.* An example of such a scenario is the distributed averaging problem where nodes attempt to compute the average of their initial values, x_0, without knowing the value of the average a priori. We will apply our framework to the distributed averaging problem in Section 4.6. Moreover, the standard stabilization problem, i.e., regulating the state x to the origin, is a special case of the problem we are solving and can be achieved by removing the reference model, i.e., setting $x_m \equiv 0$.

In the following section, we will introduce the distributed supervisory control scheme, and explain the functions of its main components in detail.

4.4 Distributed Supervisory Control Architecture

Figure 4.2 illustrates the general architecture of the distributed supervisory control scheme. In this scheme, each node has access to a bank of candidate controllers that take as input the outputs of the nodes in its neighborhood as well as the tracking error. The "Sparse Filters" block in the figure emphasizes that the local dynamics and controllers of node i can only use information from neighboring nodes. It should be noted that there is no *centralized* sparse filter implemented, and this block is introduced for the sake of demonstration only. In this section, we will precisely explain how the information from the neighboring nodes affect the dynamics and control inputs of node i. Each node has a *local supervisor*: a dynamical system that takes as input the outputs and control inputs of the neighboring nodes and produces a *switching signal.* The switching signal provided by the supervisor activates one of the available controllers. The choice of a given control input is intended to minimize the tracking error. We will study the supervisor in more detail next.

4.4.1 The Distributed Supervisor

We will refer to the collection of the local supervisors by the *distributed supervisor.* As illustrated in Fig. 4.3, the distributed supervisor has three main blocks: a multi-estimator, a monitoring signal generator, and a switching logic component. As in the centralized supervisory control case, there are cer-

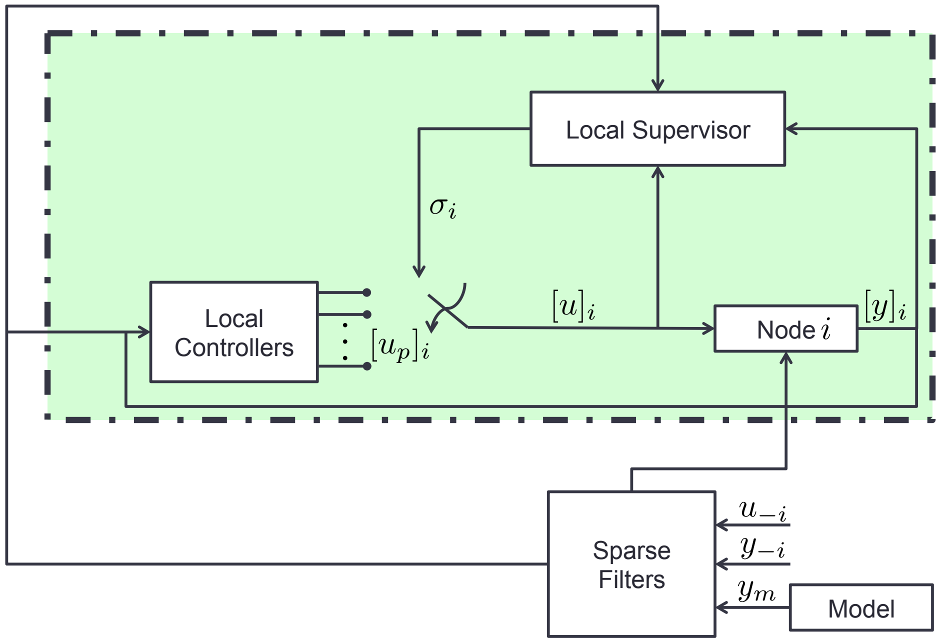


Figure 4.2: Distributed supervisory control architecture.

tain properties we require from the individual blocks of the local supervisors which are crucial for achieving tracking. In particular, the multi-estimators must guarantee that at least one estimation error e_p is small. This will guarantee that switching halts in finite time. As for the candidate controllers, they must ensure that the closed loop system is detectable with respect to the estimation error. The switching logic must ensure that the estimation error is bounded, while avoiding fast switching. Here, we will work with a specific choice of these three blocks.

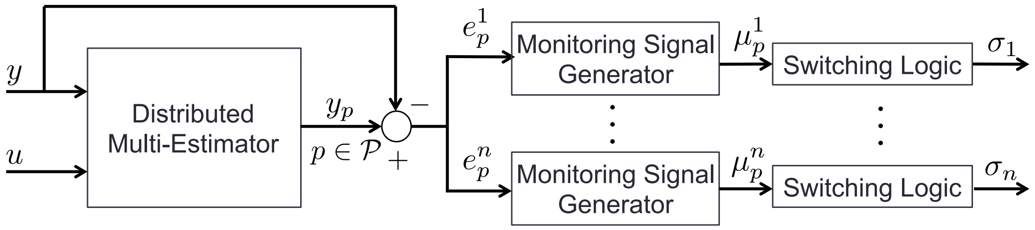


Figure 4.3: The distributed supervisor.

Distributed Multi-Estimator and Candidate Controllers

For now, we assume that the control input u is given. We will explain how to select the control below. The distributed multi-estimator is a collection of local multi-estimators that are implemented at the nodes. At node i, the local multi-estimator is a dynamical system that takes as input the outputs and control inputs of the neighboring nodes, and it produces an estimate $[y_p]_i$, $p \in \mathcal{P}$. At each node, we adopt the standard Luenberger observer to design the multi-estimator. Let the matrix L_p be sparse:

$$j \notin N_p(i) \implies [L_p]_{ij} = 0. \tag{4.3}$$

The estimator equations at node i can then be written as

$$\begin{aligned} [\dot{x}_p]_i &= \sum_{j \in N(i)} [A_p]_{ij}[x_p]_j + [B_p]_{ij}[u]_j + [L_p]_{ij}[y_p - y]_j, \\ [y_p]_i &= \sum_{j \in N(i)} [C_p]_{ij}[x_p]_j, \end{aligned}$$

with arbitrary initial values $[x_p(0)]_i$. To write the estimator equations more compactly, let $x_p = [[x_p]_1, \ldots, [x_p]_n]^T$ and $y_p = [[y_p]_1, \ldots, [y_p]_n]^T$, for all $p \in \mathcal{P}$. Recalling that the matrices A_p, B_p, L_p, C_p are sparse, we can now write

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p u + L_p(y_p - y), \quad x_p(0) = x_p^0, \\ y_p &= C_p x_p, \end{aligned}$$

where $x_p^0 = [[x_p(0)]_1, \ldots, [x_p(0)]_n]^T$. It is important to note that x_p, y_p are not stored at any node in the network, since they are centralized quantities, and are introduced merely for notational simplicity.

We define the estimation error as $e_p = y_p - y$, $p \in \mathcal{P}$. We denote the estimation error at the i-th node by

$$e_p^i = [y_p - y]_i, \quad p \in \mathcal{P}.$$

As for the candidate control inputs at node i, we assume they are linear and given by

$$[u_p]_i = \sum_{j \in N(i)} [K_p]_{ij}[x_p]_j + [F_p]_{ij}[e_T]_j, \quad p \in \mathcal{P},$$

where the gain matrices K_p and F_p must be sparse to guarantee that the controllers are decentralized. Formally, we have the following constraint on the gain matrices:

$$j \notin N_p^i \implies [K_p]_{ij} = [F_p]_{ij} = 0, \quad p \in \mathcal{P}. \tag{4.4}$$

Similar to the estimators, for each $p \in \mathcal{P}$, we collect the control inputs of the nodes into the vector $u_p = [[u_p]_1, \ldots, [u_p]_n]^T$. We can then write

$$u_p = K_p x_p + F_p e_T, \quad p \in \mathcal{P}.$$

In general, the number of candidate control inputs need not be equal to $|\mathcal{P}| = r$. However, we will assume in this chapter, for simplicity, that each node has access to r controllers.

Monitoring Signal Generators

Each node implements a monitoring signal generator which keeps track of the history of the estimation errors. This allows the switching decisions (to be explained next) to be based on the history of errors instead of the instantaneous estimation error values. The monitoring signals can be defined as any norm of the estimation error. Here, we define the monitoring signal at the i-th node as the square of the L_2 norm of e_p^i. Formally, we write

$$\mu_p^i(t) = \int_0^t \|e_p^i(s)\|_2^2 ds. \tag{4.5}$$

It is more convenient for implementation purposes to express the monitoring signal as an ODE:

$$\dot{\mu}_p^i = \|e_p^i\|_2^2, \quad \mu_p^i(0) = 0, \quad p \in \mathcal{P}.$$

Switching Logic

The switching logic at each node takes the monitoring signals μ_p^i, $p \in \mathcal{P}$, as inputs and produces a switching signal $\sigma_i : [0, +\infty) \to \mathcal{P}$ which determines the controller to be applied at each time instant. In particular, we have $[u]_i = [u_{\sigma_i}]_i$, $i \in [n]$. The chosen controller should correspond to the monitoring signal that has the lowest value. However, if we set $\sigma_i = \min_{p \in \mathcal{P}} \mu_p^i$, we run into the risk of fast switching, which could be detrimental for the stability of the system [68]. To this end, we will employ hysteresis switching logic at each node with hysteresis constant $h_i > 0$. The hysteresis constant is introduced in order to prevent σ_i from switching its value too quickly. At each node, we first initialize the switching signal as follows:

$$\sigma_i(0) = \min_{p \in \mathcal{P}} \mu_p^i(0).$$

Let $\hat{p}_i(t) := \arg\min_{p \in \mathcal{P}} \mu_p^i(t)$. The signal σ_i switches its value at time t if $\mu_{\hat{p}_i}^i + h_i \leq \mu_{\sigma_i}^i$. Figure 4.4 illustrates the hysteresis based logic at node i.

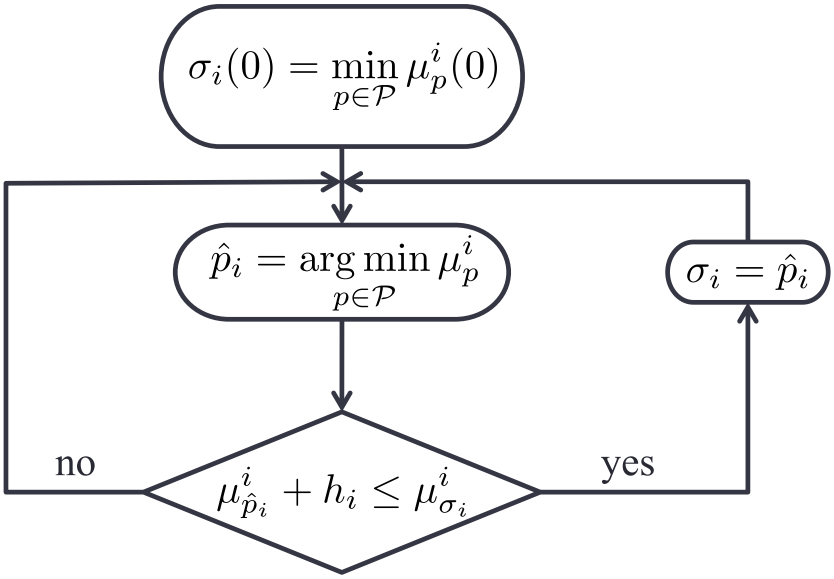

$\sigma_i(0) = \min_{p \in \mathcal{P}} \mu_p^i(0)$
$\hat{p}_i = \arg\min_{p \in \mathcal{P}} \mu_p^i$
$\sigma_i = \hat{p}_i$
no
yes
$\mu_{\hat{p}_i}^i + h_i \leq \mu_{\sigma_i}^i$

Figure 4.4: Hysteresis based switching logic.

4.5 Stability Analysis

In this section, we will obtain sufficient conditions for driving the tracking error to zero. Our approach will consist of two main steps. First, we will show that switching at all the nodes will halt in finite time. Then, assuming that the switching has stopped at all the nodes, we will study the detectability properties of the closed-loop system.

In order to prove that switching terminates in finite time, it is instrumental to show that $e_{p^\star}$ converges to zero exponentially fast. When $p = p^\star$, we have

$$\dot{x}_{p^\star} - \dot{x} = (A_{p^\star} + L_{p^\star} C_{p^\star})(x_{p^\star} - x).$$

To guarantee that $x_{p^\star}$ converges exponentially fast to x, we need to impose the following condition.

Condition 4.1. The matrix $A_{p^\star} + L_{p^\star} C_{p^\star}$ is Hurwitz with $L_{p^\star}, C_{p^\star}$ satisfying (4.1) and (4.3), respectively.

Remark 4.1. *This condition can be viewed as a* distributed *version of detectability for the plant. In the case when $C_p = I$, for all $p \in \mathcal{P}$, this condition can be satisfied via diagonal dominance. Diagonal dominance can be achieved by choosing*

$$[L_p]_{ii} < -[A_p]_{ii} - \max_{p \in \mathcal{P}} \sum_{\jmath \neq i} |[A_p]_{ij}|.$$

Note that the maximization can be carried out locally at each node because of Assumption 4.1. To guarantee that L_p is sparse, we can select it to be a diagonal matrix. With such choice of L, the matrix $A_{p^\star} + L_{p^\star}$ becomes diagonally dominant with negative diagonal entries, and by Gershgorin's circle theorem, it follows that the matrix is Hurwitz. •

Under Condition 4.1, $x_{p^\star}$ converges exponentially fast to x, and consequently $e_p^\star = C_{p^\star}(x_{p^\star} - x)$ converges to zero exponentially fast regardless of the applied control u. We now have the following proposition, which is an immediate extension of its counterpart in the centralized architecture [68,78].

Proposition 4.1. *For all $i \in [n]$, there exists a time $T_i^\star$ and an index $q_i^\star \in \mathcal{P}$ such that $\sigma_i(t) = q_i^\star$, for all $t \geq T_i^\star$. Moreover, $e^i_{q_i^\star} \in \mathcal{L}^2$, for all $i \in [n]$.*

Proof. Since $e_p^\star$ converges to zero exponentially fast, it follows from (4.5) that $\mu^i_{p^\star}$ is bounded. Let $K_i \in \mathbb{N}$ be such that $\mu^i_{p^\star} \leq K_i$. By definition, μ^i_p is a nondecreasing function, for all $p \in \mathcal{P}$. Hence, each μ^i_p must have a limit. Since $\mathcal{P}$ is finite, there exists a time T_i such that either $\mu^i_p \geq K_i$ or $\mu^i_p(t_2) - \mu^i_p(t_1) < h_i$ for all $t_2 > t_1 \geq T_i$; therefore, at most one more switch can occur for $t \geq T_i$. This in turn implies that there exists a time $T_i^\star$ such that $\sigma_i(t) = q_i^\star$, $q_i^\star \in \mathcal{P}$, for $t \geq T_i^\star$. Since $\mu^i_{p^\star}$ is bounded, $\mu^i_{q_i^\star}$ must also be bounded. By (4.5), it then follows that $e^i_{q_i^\star} \in \mathcal{L}^2$. □

Note that after the switching stops, the estimate of node i, $q_i^\star$, might not match that of another node j, $q_j^\star$. In other words, the perception of node i about the underlying graph will in general be different than that of node j. This leads to new analysis challenges that were not present in the centralized structure.

In order to study the stability of the system following termination of switching, we first define

$$\begin{aligned}\hat{x}_{q^\star} &:= [[x_{q_1^\star}]_1, \ldots, [x_{q_n^\star}]_n]^T, \\ q^\star &:= [q_1^\star, \ldots, q_n^\star]^T.\end{aligned}$$

Further, we need to construct the following matrices:

$$\hat{A}_{q^\star} := \begin{bmatrix} [A_{q_1^\star}]_1 \\ \vdots \\ [A_{q_n^\star}]_n \end{bmatrix}, \hat{B}_{q^\star} := \begin{bmatrix} [B_{q_1^\star}]_1 \\ \vdots \\ [B_{q_n^\star}]_n \end{bmatrix}, \hat{C}_{q^\star} := \begin{bmatrix} [C_{q_1^\star}]_1 \\ \vdots \\ [C_{q_n^\star}]_n \end{bmatrix},$$

$$\hat{K}_{q^\star} := \begin{bmatrix} [K_{q_1^\star}]_1 \\ \vdots \\ [K_{q_n^\star}]_n \end{bmatrix}, \hat{F}_{q^\star} := \begin{bmatrix} [F_{q_1^\star}]_1 \\ \vdots \\ [F_{q_n^\star}]_n \end{bmatrix}, \hat{L}_{q^\star} := \begin{bmatrix} [L_{q_1^\star}]_1 \\ \vdots \\ [L_{q_n^\star}]_n \end{bmatrix}.$$

With these definitions, we can write the control law u after the switching stops as

$$u = \hat{K}_{q^\star}\hat{x}_{q^\star} + \hat{F}_{q^\star} e_T.$$

Define $\overline{x} := [x^T, \hat{x}_{q^\star}^T]^T$. After the switching stops, the closed-loop system becomes:

$$\begin{aligned}\dot{\overline{x}} &= \overline{A}\overline{x} + \overline{D}x_m \\ \hat{e}_{q^\star} &= \overline{C}\overline{x},\end{aligned}$$

where

$$\overline{A} = \begin{bmatrix} A_{p^\star} - B_{p^\star}\hat{F}_{q^\star}C_{p^\star} & B_{p^\star}\hat{K}_{q^\star} \\ -(\hat{B}_{q^\star}\hat{F}_{q^\star} + \hat{L}_{q^\star})C_{p^\star} & \hat{A}_{q^\star} + \hat{B}_{q^\star}\hat{K}_{q^\star} + \hat{L}_{q^\star}\hat{C}_{q^\star} \end{bmatrix},$$

$$\overline{D} = \begin{bmatrix} B_{p^\star}\hat{F}_{q^\star}C_m \\ \hat{B}_{q^\star}\hat{F}_{q^\star}C_m \end{bmatrix},$$

$$\overline{C} = \begin{bmatrix} -C_{p^\star} & \hat{C}_{q^\star} \end{bmatrix}.$$

Consider now the matrix

$$\overline{\Gamma} = \begin{bmatrix} B_{p^\star}\hat{F}_{q^\star} + L_{p^\star} \\ \hat{B}_{q^\star}\hat{F}_{q^\star} + \hat{L}_{q^\star} \end{bmatrix},$$

and note that

$$\overline{A} - \overline{\Gamma}\,\overline{C} = \begin{bmatrix} A_{p^\star} + L_{p^\star}C_{p^\star} & B_{p^\star}(\hat{K}_{q^\star} - \hat{F}_{q^\star}\hat{C}_{q^\star}) - L_{p^\star}\hat{C}_{q^\star} \\ 0 & \hat{A}_{q^\star} + \hat{B}_{q^\star}(\hat{K}_{q^\star} - \hat{F}_{q^\star}\hat{C}_{q^\star}) \end{bmatrix}.$$

Using output injection, we can write

$$\dot{\overline{x}} = (\overline{A} - \overline{\Gamma}\,\overline{C})\overline{x} + \overline{\Gamma}\,\hat{e}_{q^\star} + \overline{D}x_m. \tag{4.6}$$

To achieve tracking, the matrix $\overline{A} - \overline{\Gamma}\,\overline{C}$ must be Hurwitz. Hence, in addition to Condition 4.1, we need to impose the following condition.

Condition 4.2. The matrix $\hat{A}_{q^\star} + \hat{B}_{q^\star}(\hat{K}_{q^\star} - \hat{F}_{q^\star}\hat{C}_{q^\star})$ is Hurwitz for all $q^\star = [q_1^\star, \ldots, q_n^\star]^T$ with $\{q_1^\star, \ldots, q_n^\star\} \subset \mathcal{P}$, while satisfying (4.1) and (4.4).

Remark 4.2. *Assume that $B_p = C_p = I$, for all $p \in \mathcal{P}$, and let us select $[K_p]_i = -[A_p]_i$, for all i and p. Note that such selection for $[K_p]_i$ is made possible by Assumption 4.1. In this case, Condition 4.2 simplifies to requiring $-\hat{F}_{q^\star}$ to be sparse and Hurwitz. This can be achieved by selecting $F_p = kI$, where $k \in \mathbb{R}_{>0}$.* ●

We are now ready to state the main result of this section. Denote the state to which the reference model converges by $x_m^\star$.

Proposition 4.2. *Under Conditions 4.1 and 4.2, and assuming that x_m converges asymptotically to $x_m^\star$, the state of the plant x remains bounded, and it asymptotically converges to*

$$x^\star = -(A_{p^\star} + B_{p^\star}(\hat{K}_{q^\star} - \hat{F}_{q^\star}\hat{C}_{q^\star}))^{-1}B_{p^\star}\hat{F}_{q^\star}C_m x_m^\star.$$

Proof. Under Conditions 4.1 and 4.2, the matrix $\overline{A} - \overline{\Gamma}\,\overline{C}$ is Hurwitz. We know from Proposition 4.1 that $e^i_{q_i^\star} \in \mathcal{L}^2$ for all $i \in [n]$. Noting that $\hat{e}_{q^\star} = [e^1_{q_1^\star}, \ldots, e^n_{q_n^\star}]$, we conclude that $\hat{e}_{q^\star}$ converges to zero as $t \to \infty$. Then, because x_m is bounded, we deduce from (4.6) that x must remain bounded. Using the fact that $\hat{e}_{q^\star}$ converges to zero, the steady-state expression follows immediately from (4.6). □

Remark 4.3. *Since the objective of the controller is to enable the plant to track the reference model, we are interested in cases where $x^\star = x_m^\star$. Assuming that B_p, C_p, C_m are all equal to the identity matrix, for all $p \in \mathcal{P}$, the steady-state expression simplifies to*

$$x^\star = -(A_{p^\star} + \hat{K}_{q^\star} - \hat{F}_{q^\star})^{-1}\hat{F}_{q^\star}x_m^\star.$$

Hence, by setting $K_p = -A_p$ for all $p \in \mathcal{P}$, we will have $x^\star = x_m^\star$ if and only if $p^\star = q^\star$, i.e., when all the nodes correctly identify the unknown topology. Otherwise, there will be a discrepancy between x and the reference trajectory x_m. Nonetheless, in certain scenarios, this discrepancy may be negligible as we will demonstrate in Section 4.6. •

Finally, we note that the multi-estimators and controllers we used here are only a specific possibility which we adopted to demonstrate the idea behind distributed supervisory control. One possible variation is to select the control inputs as

$$u_p = K_p y_p + F_p e_T, \quad p \in \mathcal{P}.$$

By following similar steps to the above, one can show that, with this choice of controllers, the matrix that is required to be Hurwitz in Condition 4.2 becomes

$$\hat{A}_{q^\star} + \hat{B}_{q^\star}(\hat{K}_{q^\star} - \hat{F}_{q^\star})\hat{C}_{q^\star}.$$

Hence, different choices of the controllers will provide different conditions on the system parameters to ensure stability. We are currently investigating different design choices that would place less restrictions on the system parameters.

4.6 Application: Tracking Consensus Dynamics

In this section, we apply the distributed supervisory control scheme to the distributed averaging problem [12, 79] in the case where the dynamics of the nodes contain unknown parameters. In distributed averaging networks, the nodes attempt to converge to the average of their initial values, $x(0)$, by performing local averaging. When the dynamics of the nodes contain unknown parameters, adaptive control techniques have been applied to solve this problem in [80]. By performing logic-based switching, our scheme enables convergence to the average without requiring continuous tuning of parameters as in the adaptive control approach. In [12, 81], the problem of achieving consensus when the underlying topologies are switching has been studied. Note that the topology in our case is unknown, but fixed, and the switching is performed at each node to choose the controller that minimizes the tracking error.

To specialize the reference model (4.2) to the distributed averaging dynamics, we assume that A_m is the negative of the weighted Laplacian matrix of a connected undirected graph. In particular, we have

$$\begin{aligned} &A_m = A_m^T, \quad A_m \mathbf{1} = 0, \\ &[A_m]_{ij} \geq 0, \quad [A_m]_{ij} = 0 \iff (i,j) \notin \mathcal{E}, \quad i \neq j, \end{aligned}$$

where the weights $[A_m]_{ij}$, $j \neq i$ are randomly generated. The connectivity of the graph corresponding to A_m is necessary for the convergence to the average [12]. We assume that there is full state observation across the network; we therefore set $C_m = I$ and $C_p = I$, for all $p \in \mathcal{P}$. We also set $B_p = I$, for all $p \in \mathcal{P}$. Since the agents attempt to compute the average of their initial values, we set $x_m^0 = x_0$ and $x_p^0 = x_0$, for all $p \in \mathcal{P}$.

We consider a network of $n = 5$ agents and set $x_0 = [1, \ldots, 5]^T$. The agents will therefore attempt to converge to $\frac{1}{5}\mathbf{1}^T x_0 = 3$. We let $|\mathcal{P}| = 10$, that is, there are 10 possible topologies, and we set $p^\star = 10$. The matrices $\{A_p\}_{p \in \mathcal{P}}$ are generated at random, without any connectivity requirements.

In order to satisfy Condition 4.1, we pick $L_p = -kI$, for all $p \in \mathcal{P}$, where $k \in \mathbb{R}$ is selected as explained in Remark 4.1. In view of Remark 4.2, we set $K_p = -A_p$ and $F_p = 5I$, for all $p \in \mathcal{P}$, in order to satisfy Condition 4.2.

We will run two experiments, where we generate different $\{A_p\}_{p \in \mathcal{P}}$, A_m matrices, each time while respecting the connectivity constraint on A_m. Figure 4.5 demonstrates the trajectories of the state of the network x, the state of the reference model x_m, the switching signals σ_i, and the tracking error e_T for the first experiment. In this case, all the agents correctly converge to the correct topology $\mathcal{G}_{p^\star}$, and, hence, converge to the average value 3. The tracking error therefore converges to zero.

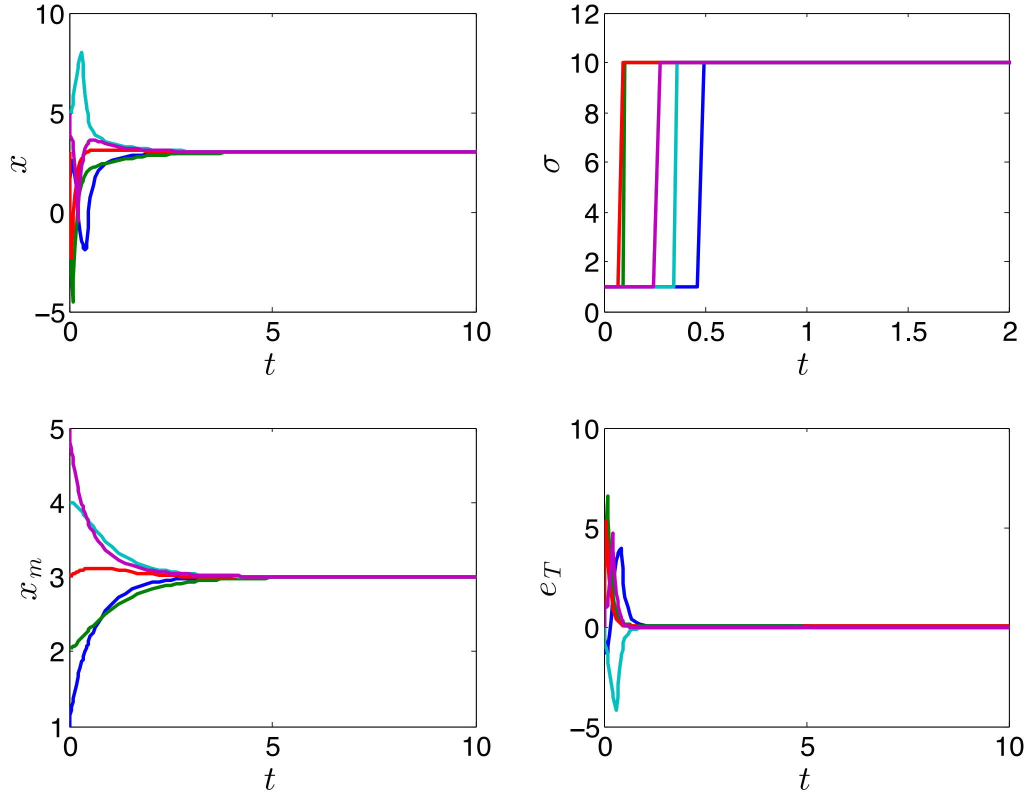


Figure 4.5: All the agents correctly identify the unknown topology.

Figure 4.6 illustrates the same signals for the second experiment. In this case, agent 3 does not select the correct topology, i.e., $q_3^\star \neq 10$. Nonetheless, it converges to 3.09, and the tracking error is very small. The remaining nodes all converge to 3. A potential future research direction is quantifying the tracking error in the event where $q_i^\star \neq p^\star$.

4.7 Summary

We proposed a distributed version of the classical centralized supervisory control scheme. Our scheme is based on logic-based switching among candidate controllers at each node. The switching decisions performed at each node depend only on information from neighboring nodes. The goal of the controllers is to track a set-point, without requiring the agents to have explicit knowledge of this point. The classical stabilization or regularization problem

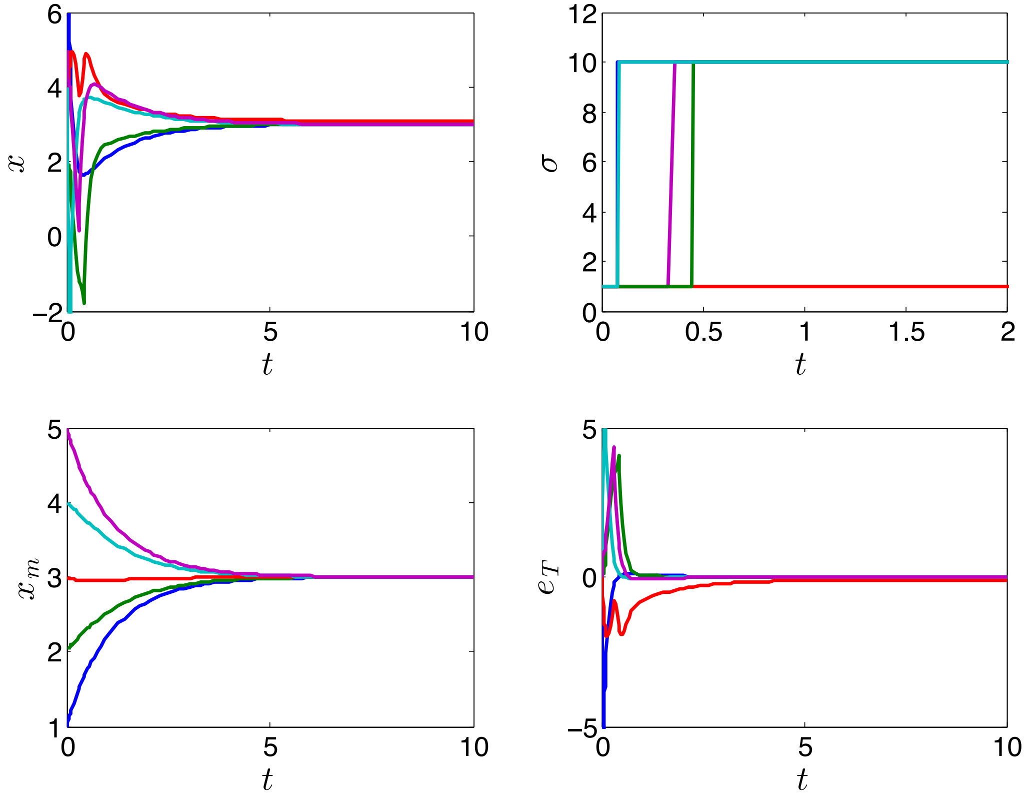


Figure 4.6: One of the agents does not identify the correct topology.

is a special case of this set-point tracking problem. We showed that switching stops in finite time at each node, and we provided sufficient conditions for stability. We applied our scheme to the distributed averaging problem when the dynamics of the agents contain unknown parameters. Simulation results demonstrated the efficacy of our scheme.

CHAPTER 5

VIRUS SPREAD IN NETWORKS: STABILITY ANALYSIS

5.1 Background

In this chapter, and in preparation for studying various control design questions in infected networks in the following chapter, we perform stability analysis for the n-intertwined Markov model [41, 82], which is a virus spread model that belongs to the SIS class. Epidemiological models for disease spread among humans constitute important classes of spread dynamics, as they can potentially provide models for many engineering related phenomena such as the spread of viruses in computer networks [41, 83–85]. There is a vast literature on various aspects of epidemiological models and the study of infection propagation over networks; we refer the reader to [84,86,87] and the references therein. Characterization of the stability properties of such diffusion dynamics is a crucial first step towards designing efficient algorithms for controlling the evolution of such dynamics. Most dynamical epidemiological models, including the n-intertwined Markov model studied here, can possess two equilibrium points, under certain conditions: an *all-healthy* state at which the network is cured, and an *endemic* state at which the infection persists in the network [11,26,88,89]. A threshold called the basic reproduction number, whose value depends on the curing and infection rates across the network as well as the network topology, determines to which equilibrium point the state of the network will converge [88].

For the n-intertwined Markov model, the basic reproduction number, introduced as a critical threshold in [41, 82], characterizes this threshold phenomenon. In particular, when the basic reproduction number is less than or equal to 1, the unique equilibrium is the all-healthy state; otherwise, the endemic state emerges. Our aim in this chapter is to fully characterize the stability properties of this model over networks with directed topologies.

A sufficient condition for the stability of the all-healthy state over strongly connected digraphs has been established in [36]. For compartmental SIS models, a necessary and sufficient condition for the global asymptotic stability of this equilibrium was presented in [26] using a linear Lyapunov function. For the same model, the global asymptotic stability of the endemic state over strongly connected directed graphs has been studied in [26,89,90]—see [89] for a summary of other approaches to establish this result. The results in [26,90] rely on the assumption that the state of the model will evolve in the strictly positive quadrant when the state of the network is initialized away from the origin. The result in [89] was established using a non-quadratic Lyapunov function. In contrast, in this chapter, using the theory of positive systems, we establish the global asymptotic stability of the endemic state using a quadratic Lyapunov function. This allows us to provide novel results for the stability properties of epidemic dynamics over weakly connected topologies; in all the aforementioned results, the underlying graphs were assumed to be strongly connected (or connected when the graph is undirected). Nonetheless, weakly connected directed graphs are common in practice, and characterizing the equilibrium points as well as their stability properties over these graphs present new challenges in studying epidemiological networks.

5.2 Main Results

The main contributions of this chapter are as follows. First, using tools from the theory of positive systems, we fully characterize the stability properties of the all-healthy and endemic state equilibrium points of the n-Intertwined Markov model over strongly connected digraphs. In particular, we show that the all-healthy state is globally asymptotically stable (GAS) if and only if the basic reproduction number is less than or equal to one. When the basic reproduction number is greater than one, we show that the endemic state is locally exponentially stable, and when the network is not initialized at the all-healthy state, we show that the endemic state is GAS. Unlike [26, 90], the proof we present here does not make any assumption on the evolution of the state, and unlike [89], the stability properties are established using a quadratic Lyapunov function. Using this key construction, our next contribution is to study the existence, uniqueness, and stability properties of the all-healthy and endemic states over weakly connected digraphs. By studying the input-to-state stability of the network, we provide conditions for a GAS endemic state to emerge over weakly connected digraphs. Unlike endemic states over strongly connected digraphs, we show that at the endemic states emerging over weakly connected graphs a subset of the nodes could be healthy while the rest become infected.

Finally, we provide a game-theoretic framework that can prescribe more general classes of infection dynamics. Using this model, we show that the n-Intertwined Markov model prescribes the best-response dynamics of a concave game. This allows us to provide a new condition for the stability of the all-healthy state, which can be checked in a distributed way by the nodes.

Organization

In Section 5.3, we recall the n-intertwined Markov model, and discuss a connection with a game-theoretic formulation. Sections 5.4 and 5.5 contain our results on the stability of the n-intertwined Markov model over, respectively, strongly and weakly connected digraphs. Numerical studies are provided in Section 5.6. We collect our conclusions in Section 5.7. Section 5.8 contains technical results that are used in proving some of our main results.

5.3 The n-Intertwined Markov Model

In this section, we recall the heterogeneous n-intertwined Markov model that has recently been proposed [41, 82]. This model is related to the so-called multi-group SIS model that was proposed earlier in [11]; see also [26, 89]. We prescribe the infection model over a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with n nodes, where $\mathcal{V}$ is the set of nodes, and $\mathcal{E}$ is the set of edges. Each node in the network has two states: infected or cured. The curing and infection of a given node $i \in \mathcal{V}$ are described by two independent Poisson processes with rates δ_i and β_i, respectively. Throughout the chapter, we assume that $\delta_i > 0$ and $\beta_i > 0$. The transition rates between the healthy and infected states of a given node's Markov chain depend on its curing rate as well as the infection probabilities among its neighbors. A mean-field approximation is introduced to "average" the effect of infection probabilities of the neighbors on the infection probability of a given node. This approximation yields a dynamical system that describes the evolution of the probability of infection of node $i \in \mathcal{V}$ and is central to our upcoming developments. We briefly review this dynamical system next.

Let $p_i(t) \in [0,1]$ be the infection probability of node $i \in \mathcal{V}$ at time $t \in \mathbb{R}_{\geq 0}$, and let $p(t) = [p_1(t), \dots, p_n(t)]^T$. Also, let $D = \operatorname{diag}(\delta_1, \dots, \delta_n)$, $P(t) = \operatorname{diag}(p(t))$, and $B = \operatorname{diag}(\beta_1, \dots, \beta_n)$. The n-intertwined Markov model is prescribed by the mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, where

$$\begin{aligned} \dot{p}(t) &= \Phi(p(t)) \\ &:= (A^T B - D)p(t) - P(t)A^T B p(t). \end{aligned} \tag{5.1}$$

It can be shown that when $p(0) \in [0,1]^n$, $p(t) \in [0,1]^n$, for all $t \in \mathbb{R}_{>0}$ [41]. Hereinafter, for most parts, we will drop the time index for notational simplicity.

5.3.1 Equilibrium States of the n-Intertwined Markov Model

We next focus on characterizing the set of equilibria of the dynamical system (5.1). We give this characterization using the so-called *basic reproduction number*, denoted by $\mathcal{R}_o$, which is defined as the expected number of infected nodes produced in a completely susceptible population due to the infection of a neighboring node [88]. For the n-intertwined Markov model, the basic reproduction number was found in [82], where it was called the "critical threshold", to be equal to

$$\mathcal{R}_o = \rho(D^{-1}A^T B).$$

For connected undirected graphs, it is shown in [82] that the all-healthy state is the unique equilibrium for the n-intertwined Markov model when $\mathcal{R}_o \leq 1$. When $\mathcal{R}_o > 1$, in addition to the all-healthy equilibrium, an endemic equilibrium, denoted by $p^\star$, emerges. In fact, it is shown that $p^\star \gg 0$. We call a strictly positive endemic state *strong*. When $p^\star \succ 0$, we call it a *weak* endemic state. A recursive expression for the endemic state $p^\star$ is provided in [82], which is shown to depend on the problem parameters only: A, δ_i, β_i,

$i \in \mathcal{V}$. To arrive at this expression, consider the steady-state equation

$$0 = (A^T B - D)p - PA^T Bp. \tag{5.2}$$

Define $\xi_i := \sum_{j \neq i} a_{ji}\beta_j p_j$ and $\xi_i^\star := \sum_{j \neq i} a_{ji}\beta_j p_j^\star$, $i \in \mathcal{V}$. We can then write $p_i^\star$ as

$$p_i^\star = \frac{\xi_i^\star}{\delta_i + \xi_i^\star} = 1 - \frac{\delta_i}{\delta_i + \xi_i^\star}, \quad i \in \mathcal{V}. \tag{5.3}$$

Since we assumed that $\delta_i > 0$, we conclude that $p_i^\star < 1$, for all $i \in \mathcal{V}$. We can then re-write (5.2), evaluated at $p^\star$, in the following form:

$$A^T B p^\star = (I - P^\star)^{-1} D p^\star, \tag{5.4}$$

where $P^\star = \operatorname{diag}(p^\star)$.

5.3.2 The n-Intertwined Markov Model as a Concave Game

In this subsection, we demonstrate that the n-intertwined Markov model can be cast as the best response dynamical system associated with a noncooperative game. An important by-product of this study is the development of a larger class of infection dynamics with reasonable convergence properties. Further, our exposition provides a decision-based interpretation to virus spread models, which are often based on the theory of Markov chains. Although our focus here is the study of virus spread, our model can be applied to other diffusion phenomena such as the spread of spam in computer networks.

To this end, consider a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with n nodes, and let $0 \leq x_i \leq 1$ be the rate with which node i sends messages. We associate an objective function, denoted by $f_i : \mathbb{R}^n \to \mathbb{R}$, to node i that is comprised of a local utility function $U_i : [0,1] \to \mathbb{R}$, and a component that encapsulates the influence of the neighboring nodes. The influence of node j on node i is described via the function $\tilde{g}_{ji} : [0,1] \times [0,1] \to \mathbb{R}$, where $\tilde{g}_{ji} \equiv 0$ if and only if $(j,i) \notin \mathcal{E}$. We can then write the objective function of node i as

$$f_i(x_i, x_{-i}) = U_i(x_i) + \sum_{j \neq i} \tilde{g}_{ji}(x_i, x_j). \tag{5.5}$$

Each node is interested in maximizing its own objective function f_i. Formally, we can write the problem of the i-th agent as

$$\max_{0 \leq x_i \leq 1} f_i(x_i, x_{-i}), \quad \text{for each fixed } x_{-i}. \tag{5.6}$$

When f_i is concave in x_i, and because the objective function of each player depends also on the actions of other players, problem (5.6) describes a concave game [63, 91].

The solution concept we are interested in studying here is the pure-strategy Nash equilibrium (PSNE).

Definition 5.1 ([63]). *The vector $x^\star \in [0,1]^n$ constitutes a PSNE if, for all $i \in \mathcal{V}$, the inequality*

$$f_i(x_i^\star, x_{-i}^\star) \geq f_i(x_i, x_{-i}^\star)$$

is satisfied for all $x_i \in [0,1]$.

Note that under the PSNE, no agent has any incentive to unilaterally deviate from the solution $x^\star$. The next proposition establishes the existence and uniqueness of the PSNE for the game in (5.6), when the game is concave.

Proposition 5.1 ([91]). *For each $i \in \mathcal{V}$, let $f_i(x_i, x_{-i})$ in* (5.5) *be strictly concave in $x_i \in [0,1]$, for every $x_j \in [0,1], j \in \mathcal{V}, j \neq i$. Then the resulting concave game in* (5.6) *admits a unique PSNE under the following diagonal dominance condition:*

$$2\left|\frac{\partial^2}{\partial x_i^2} U_i(x_i)\right| \quad > \quad \sum_{j \neq i} \left|\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\tilde{g}_{ij}(x_i, x_j) + \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\tilde{g}_{ji}(x_j, x_i)\right|. \tag{5.7}$$

The following lemma establishes a relationship between virus spread in networks and concave games. In the virus spread case, the probability of infection p_i plays the role of the transmission rate x_i.

Lemma 5.1. *The dynamics of the n-intertwined Markov model are best-response dynamics of a concave game among the nodes, where the decision variable of node $i \in \mathcal{V}$ is $p_i \in [0,1]$, and its objective function is given by*

$$f_i(p_i, p_{-i}) = -\frac{\delta_i}{2} p_i^2 + p_i \left(1 - \frac{p_i}{2}\right) \sum_{j \neq i} a_{ji} \beta_j p_j. \tag{5.8}$$

Proof. Recall the objective functions defined in (5.5). Let $U_i(p_i) = -\frac{\delta_i}{2}p_i^2$ and $\tilde{g}_{ji}(p_i, p_j) = p_i(1 - \frac{p_i}{2})a_{ji}\beta_j p_j$, $i \in \mathcal{V}$. We then obtain

$$\frac{\partial^2}{\partial p_i^2} f_i(p_i, p_{-i}) = -\delta_i - \sum_{j \neq i} a_{ji}\beta_j p_j < 0, \quad i \in \mathcal{V},$$

which shows that the f_i's are strictly concave in self-variables. It is now not hard to see that the dynamics of the n-intertwined Markov model (5.1) correspond to the gradient flow dynamics when the agents aim at maximizing their own objective functions (5.8). □

5.4 Stability of Epidemic Dynamics over Strongly Connected Graphs

We start by studying the stability properties of the n-intertwined model over directed graphs with strongly connected topologies.

5.4.1 Stability of the All-Healthy State

As a stepping stone, we first provide an alternative proof for the necessary and sufficient condition for the global asymptotic stability of the all-healthy state, see [26,36], using the theory of positive systems. As we will see shortly, the proof strategy provided here is essential in some of our upcoming results.

Proposition 5.2. *Suppose $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a strongly connected digraph. The origin is GAS if and only if $\mathcal{R}_o \leq 1$.*

Proof. Note that the matrix $A^TB - D$ is Metzler, because the entries of A^TB are nonnegative. Using the convergent regular splitting property of Metzler matrices, it can be shown that $\mathcal{R}_o < 1$ if and only if $\mu(A^TB - D) < 0$, and $\mathcal{R}_o = 1$ if and only if $\mu(A^TB - D) = 0$ [43, Theorem 2.3].

As a result, when $\mathcal{R}_o < 1$, the matrix $A^TB - D$ is Hurwitz. Since it is also Metzler, by Proposition 1.1(iv), there exists a positive diagonal matrix R_1 satisfying $(A^TB - D)^TR_1 + R_1(A^TB - D) = -K$, where K is a positive definite matrix. Consider the Lyapunov function $V_1(p) = p^TR_1p$. Using (5.1), we have

$$\begin{aligned}
\mathcal{L}_\Phi V_1(p) &= p^T((A^TB-D)^TR_1+R_1(A^TB-D))p \\
&- 2p^TR_1PA^TBp \\
&\leq p^T((A^TB-D)^TR_1+R_1(A^TB-D))p \\
&= -p^TKp \leq \lambda_1(-K)\|p\|_2^2 < 0, \quad p \neq 0, \qquad (5.9)
\end{aligned}$$

where the first inequality follows because $p^TR_1PA^TBp \geq 0$, for all $p \in [0,1]^n$, and (5.9) follows because K is positive definite. This implies that the all-healthy state is GAS.

When $\mathcal{R}_o = 1$, we have $\mu(A^TB-D) = 0$. Since $\mathcal{G}$ is strongly connected, it follows that A^TB-D is irreducible [43]. Recalling that A^TB-D is also Metzler, we conclude from Lemma 5.2 that there exists a positive diagonal matrix R_2 such that $(A^TB-D)^TR_2+R_2(A^TB-D)$ is negative semidefinite. Using the Lyapunov function $V_2(p) = p^TR_2p$, we can write

$$\begin{aligned}
\mathcal{L}_\Phi V_2(p) &= p^T((A^TB-D)^TR_2+R_2(A^TB-D))p \\
&- 2p^TR_2PA^TBp \\
&\leq -2p^TR_2PA^TBp.
\end{aligned}$$

We next prove that $p^TR_2PA^TBp = 0$ if and only if $p = 0$. Since R_2 is a positive diagonal matrix, we have that $p^TR_2PA^TBp = 0$ if and only if

$$p_i^2 \sum_{j \neq i} a_{ji}\beta_j p_j = 0, \qquad (5.10)$$

for all $i \in \mathcal{V}$. Assume that there is a solution p that satisfies $p^TR_2PA^TBp = 0$ at some time $t_0 \in \mathbb{R}_{\geq 0}$, and let $p_i(t_0) \neq 0$ for some $i \in \mathcal{V}$. Then, by continuity of the state p, there exists an interval $\tau = [t_0, t_0+\delta]$, $\delta > 0$, such that $p_i(t) \neq 0$, for all $t \in \tau$. Using (5.10), we hence conclude that for all $j \in \mathcal{V}$ that are neighbors of i, i.e., $a_{ji} \neq 0$, we must have that $p_j(t) = 0$ and $\dot{p}_j(t) = 0$ for all $t \in \tau$, for all $j \in \mathcal{V}$ with $a_{ji} \neq 0$. Then, for some $j \in \mathcal{V}$ such that $a_{ji} \neq 0$, we have $\dot{p}_j(t) = \sum_{k \neq j} a_{kj}\beta_k p_k(t) = 0$, for all $t \in \tau$. This implies that $p_k(t) = 0$ for all $t \in \tau$ and for all $k \in \mathcal{V}$ such that $a_{kj} \neq 0$. By repeating this argument, we conclude that $p_l(t) = 0$ for all $t \in \tau$ for any node $l \in \mathcal{V}$ from which there is a directed path to node j. Since $\mathcal{G}$ is strongly connected, there is a directed path from node i to node j, and we must then have $p_i(t) = 0$ for all $t \in \tau$, which contradicts our initial hypothesis. It then follows that $p^T R_2 P A^T B p = 0$ if and only if $p \equiv 0$. Hence, the all-healthy state is GAS. This proves the sufficiency part.

We will show necessity by proving the contrapositive. The Jacobian matrix of the vector field in (5.1) evaluated at the origin is given by $J(0) = A^T B - D$. If $\mathcal{R}_o > 1$, we have $\mu(A^T B - D) > 0$, and we conclude by Lyapunov's indirect method that the original nonlinear system is not stable. This proves that $\mathcal{R}_o \leq 1$ is also necessary for the origin to be asymptotically stable. □

It is worth noting that, when $\mathcal{R}_0 < 1$, the proof of the global asymptotic stability of the all-healthy state does not rely on the strong connectivity assumption. This is also true for the instability proof, when $\mathcal{R}_0 > 1$. We only used the strong connectivity of the graph to prove global asymptotic stability when $\mathcal{R}_o = 1$.

5.4.2 Existence and Stability of an Endemic State

In this section, we use notions from positive systems theory to prove the local and global asymptotic stability of an endemic state over strongly connected digraphs. We first note that the existence of a unique endemic state for (5.1) over strongly connected digraphs can be concluded from [26, Section 2.2], as stated next.

Proposition 5.3. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a strongly connected digraph. Then, a unique strong endemic state $p^\star \gg 0$ exists if and only if $\mathcal{R}_o > 1$.*

Next, we compute the Jacobian of Φ, given by (5.1), at $p^\star$. Note that

$$\begin{aligned} J_{ii}(p^\star) &= \frac{\partial}{\partial p_i}\Phi_i(p^\star) = -(\delta_i + \xi_i^\star), \quad i \in \mathcal{V}, \\ J_{ij}(p^\star) &= \frac{\partial}{\partial p_j}\Phi_i(p^\star) = (1 - p_i^\star) a_{ji}\beta_j, \quad j \neq i, j \in \mathcal{V}, \end{aligned}$$

where $\Phi_i(p^\star)$ is i-th entry of $f(p^\star)$. Using the definition of $p^\star$ in (5.3), we realize that $J_{ii}(p^\star) = -\delta_i/(1 - p_i^\star)$, $i \in \mathcal{V}$. As a result, we conclude that

$$J(p^\star) = -(I - P^\star)^{-1} D + (I - P^\star) A^T B.$$

Our first result establishes the local stability of $p^\star$.

Theorem 5.1. *Suppose that $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a strongly connected digraph and that $\mathcal{R}_o > 1$. Then, the strong endemic state $p^\star$ is locally exponentially stable.*

Proof. We invoke Lyapunov's indirect method. Since $\mathcal{G}$ is strongly connected, A is irreducible. From (5.4), we deduce that $Dp^\star = (I - P^\star)A^T Bp^\star$. We can then write

$$\begin{aligned} J(p^\star)p^\star &= -A^T Bp^\star + (I - P^\star)A^T Bp^\star \\ &= -P^\star A^T Bp^\star \ll 0, \end{aligned}$$

where the last strict inequality follows because $p^\star \gg 0$, B is a positive diagonal matrix, and A is irreducible. The matrix $J(p^\star)$ is Metzler, because its off-diagonal entries are nonnegative. Then, using Proposition 1.1(ii), we conclude that $J(p^\star)$ is Hurwitz. □

We are now in a position to state the following result.

Theorem 5.2. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a strongly connected digraph, and assume that $p(0) \neq 0$. If $\mathcal{R}_o > 1$, then the strong endemic state $p^\star$ is GAS.*

Proof. Recall that $p(t) \in [0, 1]^n$ for all $t \in \mathbb{R}_{\geq 0}$. When $\mathcal{R}_o > 1$, Proposition 5.2 implies that the origin is unstable. Therefore, under this condition, the set $W = [0, 1]^n \backslash \{0\}$ is invariant under the evolutions of (5.1).

Next, define the state $\tilde{p} = p - p^\star$. Let $\tilde{P} = \text{diag}(\tilde{p})$. The dynamics of $\tilde{p}$ can then be written as follows:

$$\begin{aligned} \dot{\tilde{p}} &= (A^T B - D)(\tilde{p} + p^\star) - (\tilde{P} + P^\star)A^T B(\tilde{p} + p^\star) \\ &= (-D + (I - P^\star)A^T B)\tilde{p} - \tilde{P}A^T Bp. \end{aligned}$$

Define the matrix $\Lambda(p^\star) := -D + (I - P^\star)A^T B$, and note that the off-diagonal entries of $\Lambda(p^\star)$ are nonnegative; hence, $\Lambda(p^\star)$ is a Metzler matrix. Since $\mathcal{G}$ is strongly connected, the matrix $\Lambda(p^\star)$ is also irreducible. From (5.4), it follows that $\Lambda(p^\star)p^\star = 0$, and since $p^\star$ is strictly positive, it follows from Theorem 1.1 that $\mu(\Lambda(p^\star)) = 0$. Thus, it follows from Lemma 5.2 that there exists a positive diagonal matrix R such that the matrix $\Lambda(p^\star)^T R + R\Lambda(p^\star)$ is negative semidefinite.

Consider the Lyapunov function $V(\tilde{p}) = \tilde{p}^T R \tilde{p}$. We have

$$\begin{aligned}\mathcal{L}_\Phi V(\tilde{p}) &= \tilde{p}^T(\Lambda(p^\star)^T R + R\Lambda(p^\star))\tilde{p} - 2\tilde{p}^T \tilde{P} R A^T B p \\ &\leq -2\tilde{p}^T R \tilde{P} A^T B p = -2\tilde{p}^T \tilde{P} R A^T B p,\end{aligned}$$

where the inequality follows because $\Lambda(p^\star)^T R + R\Lambda(p^\star)$ is negative semidefinite, and the last equality follows because $\tilde{P}$ and R commute, since they are both diagonal matrices.

We next prove that $p^T R P A^T B p = 0$ if and only if $p = p^\star$. Since R is a positive diagonal matrix, we have $\tilde{p}^T \tilde{P} R A^T B p = 0$ if and only if $\tilde{p}_i^2 \sum_{j \neq i} a_{ji}\beta_j p_j = 0$, for all $i \in \mathcal{V}$. Assume that there is a vector p that satisfies $\tilde{p}^T \tilde{P} R A^T B p = 0$ while $p_i \neq p_i^\star$, for some $i \in \mathcal{V}$. We then must have $\sum_{j \neq i} a_{ji}\beta_j p_j = 0$, which implies that $p_j = 0$ for all $j \in \mathcal{V}$ such that $a_{ji} \neq 0$. Then, for some $j \in \mathcal{V}$ for which $a_{ji} \neq 0$, we must also have $\sum_{k \neq j} a_{kj}\beta_k p_k = 0$, because $p_j = 0 < p_j^\star$. By repeating this argument, we conclude that $p_l = 0$ for any node $l \in \mathcal{V}$ from which there is a directed path to node j. Since $\mathcal{G}$ is strongly connected, there is a directed path from node i to node j, and we must have $p_i = 0$. This implies that $p = 0$, which contradicts our initial assumption. Therefore, since the set W is invariant under (5.1), we have that $\dot{V}(\tilde{p}) = 0$ if and only if $p = p^\star$. □

Remark 5.1. *The novelty of our proof lies in its use of notions from positive systems theory, which enables us to construct a quadratic Lyapunov function. A proof for a weaker statement is established in [26, 90], where it is assumed that for $p(0) \neq 0$, there exists a time $T \in \mathbb{R}_{>0}$ such that $p(t) \in (0,1]^n$ for all $t \geq T$. An alternative proof that utilizes a logarithmic Lyapunov function has recently appeared in [89].*

In addition to the useful characteristics of using a quadratic Lyapunov function for studying additional properties such as convergence rates, our proof allows for establishing the stability properties of the equilibrium points over weakly connected digraphs in the next section. •

5.4.3 A Simplified Stability Condition through a Game-Theoretic Perspective

The game-theoretic connection we established in Lemma 5.1 enables us to provide a simplified condition for the global asymptotic stability of the all-healthy state. In particular, by applying the diagonal dominance condition in (5.7) to (5.8), we obtain the following sufficient condition:

$$\frac{1}{2}\sum_{j\neq i} a_{ij}\beta_j < \delta_i, \quad \text{for all } i \in \mathcal{V}. \tag{5.11}$$

Recall that the conditions $\mathcal{R}_0 < 1$ and $\mu(A^T B - D) < 0$ are equivalent. Note the similarities between the conditions $\mu(A^T B - D) < 0$ and (5.11). The two conditions are related by the Gershgorin Circle Theorem. While (5.11) is more restrictive than $\mu(A^T B - D) < 0$, it is linear and easier to compute. More importantly, condition (5.11) can be checked in a distributed fashion, which makes it more suitable for the design of distributed algorithms.

5.5 Stability of Epidemic Dynamics over Weakly Connected Graphs

In this section, we study the stability properties of the n-intertwined Markov model over weakly connected graphs. This class is of great importance, since it is conceivable that in many practical scenarios there exist connected components that collectively serve as an infection source, but are not affected by the rest of the nodes. Such scenarios cannot be captured by strongly connected topologies.

We start by introducing some notations. When the graph $\mathcal{G}$ is weakly connected, its adjacency matrix can be transformed into an upper triangular form using an appropriate labeling of the nodes. Assuming that $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ contains $N \in \mathbb{Z}_{\geq 1}$ strongly connected components, we can write

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1N} \\ 0 & A_{22} & A_{23} & \dots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \dots & 0 & A_{NN} \end{bmatrix},$$

where A_{ii} are irreducible for all $i \in [N]$, and, hence, correspond to SCCs in $\mathcal{G}$ [43]. For notational simplicity, we will use A_i instead of A_{ii}. The matrices A_{ij}, $j \neq i$ are not necessarily irreducible. We denote an SCC of $\mathcal{G}$ by $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i)$, $i \in [N]$, where $\cup_{i=1}^N \mathcal{V}_i = \mathcal{V}$ and $\cup_{i=1}^N \mathcal{E}_i = \mathcal{E}$. For each $i \in [N]$, we introduce the positive diagonal matrices D_i, B_i which contain, respectively, the curing and infection rates of the nodes in $\mathcal{V}_i$ along their diagonals. We introduce the partial order '$\prec$' among SCCs, and we write $\mathcal{G}_i \prec \mathcal{G}_j$, for some $i, j \in [N]$, if there is a directed path from $\mathcal{G}_i$ to $\mathcal{G}_j$ but not vice versa.

For a given $i \in [N]$, we denote the state of the nodes in $\mathcal{G}_i$ by $q_i \in \mathbb{R}^{|\mathcal{V}_i|}$ and the state of the k-th node in $\mathcal{V}_i$ by $q_{i,k} \in \mathbb{R}$. The state, p, of the entire network is given by $p = [q_1^T, \ldots, q_N^T]$. Let $c_i = \sum_{j \neq i} A_{ji}^T B_j q_j \in \mathbb{R}^{|\mathcal{V}_i|}$, $i \in [N]$, be the input infection from the nodes in $\mathcal{G} \backslash \mathcal{G}_i$. We can now write the dynamics of the nodes in $\mathcal{G}_i$, $i \in [N]$, given by the mapping $\tilde{\Phi}_i : \mathbb{R}^{|\mathcal{V}_i|} \times \mathbb{R}^{|\mathcal{V}_i|} \to \mathbb{R}^{|\mathcal{V}_i|}$, as

$$\begin{aligned} \dot{q}_i &= \tilde{\Phi}_i(q_i, c_i) \\ &:= (A_i^T B_i - D_i) q_i - Q_i A_i^T B_i q_i + (I - Q_i) c_i, \end{aligned} \tag{5.12}$$

where $Q_i = \mathrm{diag}(q_i)$. When an SCC comprises a single node, $A_i^T B_i - D_i$ is equal to $-\delta_i$. In what follows, we say $\mathcal{G}_i$ is stable to mean that the dynamics (5.12) are stable. When an endemic state $p^\star$ emerges over the graph $\mathcal{G}$, we call the steady-state of q_i an endemic state of $\mathcal{G}_i$, and we denote it by $q_i^\star$. Hence, the endemic state emerging over the entire network is given by $p^\star = [q_1^{\star T}, \ldots, q_N^{\star T}]^T$.

We first state some results about the special case where the network topology is given by a DAG.

Proposition 5.4. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a DAG and suppose $\delta_i > 0$ for all $i \in \mathcal{V}$. Then the origin is the unique equilibrium. Moreover, this equilibrium is GAS.*

Proof. Let us denote the steady-state of (5.1) by $p(\infty)$. The steady-state equation for the source nodes of the DAG is of the form $0 = -\delta_i p_i(\infty)$, $i \in \mathcal{S}_{\text{source}}$, which implies that $p_i(\infty) = 0$ for all source nodes. For a node $i \in \mathcal{S}_{\text{N-source}}$, its steady-state equation can be written as $0 = -\delta_i p_i(\infty) + (1 - p_i(\infty)) \sum_{j \in \mathcal{S}_{\text{source}}} a_{ij} \beta_j p_j(\infty)$. The sum evaluates to zero, and again we obtain $p_i(\infty) = 0$. By repeating this argument, we conclude that $p_i(\infty) = 0$, for all $i \in \mathcal{S}_{\text{N-source}}$. By propagating this argument all the way to the sink nodes, we conclude that zero is the unique solution of the steady-state equation.

Next, we prove the second statement. In a DAG, the dynamics of the source nodes become $\dot{p}_i = -\delta_i p_i$, $i \in \mathcal{S}_{\text{source}}$. Hence, all source nodes are globally exponentially stable. Let $v_i := \sum_{j \in \mathcal{S}_{\text{source}}} a_{ij}\beta_j p_j$, and define the following linear dynamical system for all $i \in \mathcal{S}_{\text{N-source}}$

$$\dot{\bar{p}}_i = -\delta_i \bar{p}_i + v_i, \quad \bar{p}_i(0) = p_i(0).$$

Then, we have from (5.1) that $\dot{p}_i \leq \dot{\bar{p}}_i$, for all $i \in \mathcal{S}_{\text{N-source}}$. By the comparison lemma, it follows that $p_i \leq \bar{p}_i$, for all t and all $i \in \mathcal{S}_{\text{N-source}}$. It is well-known that if the input of an exponentially stable linear system converges to zero, its state converges to zero. Thus, since v_i converges to zero, $\bar{p}_i$ must also converge to zero, for all $i \in \mathcal{S}_{\text{N-source}}$. Since $p_i \geq 0$, we conclude that p_i converges to zero for all $i \in \mathcal{S}_{\text{N-source}}$. The proposition follows by repeating this argument for the remaining nodes in the graph. □

We begin by studying the existence, uniqueness, and the stability properties of an endemic state over a weakly connected digraph consisting of two SCCs; the generalization to multiple SCCs is straightforward.

Proposition 5.5. *Let $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i)$ be an SCC, $i \in [N]$, and let $q_i^\star$ be its endemic state equilibrium. If $q_{i,i_1}^\star > 0$ for some $i_1 \in \mathcal{V}_i$, then $q_i^\star \gg 0$.*

Proof. Let $i_1 \in \mathcal{V}_i$ be a node with $q_{i,i_1}^\star > 0$. Since $\mathcal{G}_i$ is strongly connected, for any node $i_m \in \mathcal{V}_i$, where m is an integer satisfying $m \leq |\mathcal{V}_i|$, there exists a directed path from node i_1 to node i_m. Let $i_2 \in \mathcal{V}_i$ be a node along this path such that $(i_1, i_2) \in \mathcal{E}_i$. It follows from (5.3), that $q_{i,i_2}^\star > 0$. By the same argument, it follows that $q_{i,i_k}^\star > 0$ for every node $i_k \in \mathcal{V}_i$ along the directed path from i_1 to i_m, including i_m. Since nodes i_1 and i_m were arbitrary, the proof is complete. □

Let $\mathcal{R}_o^i := \rho(D_i^{-1} A_i^T B_i)$ be the basic reproduction number corresponding to $\mathcal{G}_i$. We have the following existence and uniqueness result.

Theorem 5.3. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a weakly connected digraph consisting of two SCCs $\mathcal{G}_1$, $\mathcal{G}_2$ such that $\mathcal{G}_1 \prec \mathcal{G}_2$. Assume that $q_i(0) \neq 0$ for all $i \in [2]$. Then the following statements hold:*

(i) If $\mathcal{R}_o^1 > 1$, and $\mathcal{R}_o^2$ being arbitrary, then $p = 0$ and $p^\star = [q_1^{\star T}, q_2^{\star T}]^T$ are the only possible equilibrium points over $\mathcal{G}$, where $q_1^\star$ and $q_2^\star$ are unique strong endemic equilibrium points over $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.

(ii) If $\mathcal{R}_o^1 \le 1$ and $\mathcal{R}_o^2 > 1$, then $p = 0$ and $p^\star = [0^T, q_2^{\star T}]^T$ are the only possible equilibrium points over $\mathcal{G}$, where $q_2^\star$ is a unique strong endemic equilibrium point over $\mathcal{G}_2$.

(iii) If $\mathcal{R}_o^i \le 1$, $i \in [2]$, then $p = 0$ is the only possible equilibrium over $\mathcal{G}$.

Proof. In all the cases, the fact that $p = 0$ is an equilibrium point follows directly from the structure of the dynamics. Since $\mathcal{G}_1 \prec \mathcal{G}_2$, we have $c_1 = 0$, i.e., the dynamics of the nodes in $\mathcal{G}_1$ are not affected by those in $\mathcal{G}_2$.

We first prove (i). First, consider the case when $\mathcal{R}_o^2 > 1$. Since $\mathcal{R}_o^1 > 1$ and $\mathcal{G}_1$ is an SCC, we conclude by Theorems 5.3 and 5.2 that there exists a strong endemic state $q_1^\star \gg 0$ over $\mathcal{G}_1$, which is GAS, assuming that $q_1(0) \neq 0$. Hence, c_2 converges to $c_2^\star := A_{12}^T B_2 q_1^\star$, which is a nonnegative vector. We can now write the steady-state equation for $\mathcal{G}_2$ as

$$(A_2^T B_2 - D_2) q_2 - Q_2 A_2^T B_2 q_2 + (I - Q_2) c_2^\star = 0, \tag{5.13}$$

or

$$A_2^T B_2 q_2 - \text{diag}(A_2^T B_2 q_2) q_2 - (D_2 + C_2^\star) q_2 + c_2^\star = 0,$$

where $C_2^\star = \text{diag}(c_2^\star)$. Define $G_2 = D_2 + C_2$, and note that this is an invertible diagonal matrix because D_2 is a strictly positive diagonal matrix. We then conclude that

$$G_2^{-1} A_2^T B_2 q_2 - (I + \text{diag}(G_2^{-1} A_2^T B_2 q_2)) q_2 + G_2^{-1} c_2^\star = 0,$$

or

$$q_2 = (I + \text{diag}(G_2^{-1} A_2^T B_2 q_2))^{-1} G_2^{-1} (A_2^T B_2 q_2 + c_2^\star). \tag{5.14}$$

Since $\mathcal{G}_2$ is an SCC, A_2 is irreducible, and therefore $G_2^{-1} A_2^T B_2$ is irreducible as well. Furthermore, we have $G_2^{-1} c_2^\star \ll 1$ by construction. It then follows by Theorem 5.5 in Section 5.8 that there exists a unique strong endemic state $q_2^\star$ over $\mathcal{G}_2$. From (5.4), it follows that the steady-state of any node in $\mathcal{G}_2$ that is connected to a node in $\mathcal{G}_1$ is strictly positive. Then, it follows from Proposition 5.5 that $[q_1^\star, 0]$ cannot be an equilibrium over $\mathcal{G}$, and $[q_1^{\star T}, q_2^{\star T}]^T$ is the unique equilibrium over $\mathcal{G}$ in this case.

When $\mathcal{R}_o^2 \leq 1$, it follows from (5.4) that the steady-state of any node in $\mathcal{G}_2$ that is connected to a node in $\mathcal{G}_1$ is strictly positive. Hence, by Proposition 5.5, there exists a strong endemic state $q_2^\star$ over $\mathcal{G}_2$. Finally, and because the steady-state equation over $\mathcal{G}_2$ is given by (5.14), it follows from Proposition 5.7 in Section 5.8 that $q_2^\star$ must be unique.

For (ii), since $c_1 = 0$ and $\mathcal{R}_o^1 \leq 1$, it follows by Proposition 5.2 and Theorem 5.3 that the only valid equilibrium over $\mathcal{G}_1$ is $q_1 = 0$, which is GAS. Hence, in steady-state, $\mathcal{G}_2$ can be viewed as an isolated irreducible graph, and it follows from Theorems 5.3 and 5.2 that there exists a unique strictly positive equilibrium $q_2^\star$ over $\mathcal{G}_2$.

Finally, for (iii), and similar to (ii), the only possible equilibrium over $\mathcal{G}_1$ is $q_1 = 0$, which is GAS. This in turn leads to having $c_2^\star = 0$, and since $\mathcal{R}_o^2 \leq 1$, the only possible equilibrium over $\mathcal{G}_2$ is $q_2 = 0$. □

From (ii), we conclude that a weak endemic state could emerge over weakly connected graphs. A strong endemic state could emerge in case (i), and the all-healthy state is the only possible equilibrium in case (iii). It is important to note that the endemic state $q_2^\star$ resulting in cases (i) and (ii) are not necessarily the same.

Next, we study the stability properties of weak and strong endemic equilibria.

Theorem 5.4. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a weakly connected digraph consisting of two SCCs $\mathcal{G}_1$, $\mathcal{G}_2$ such that $\mathcal{G}_1 \prec \mathcal{G}_2$. Assume that $q_i(0) \neq 0$ for all $i \in [2]$. Then, $\mathcal{G}_2$ is input-to-state stable (ISS). Further, the equilibrium over $\mathcal{G}$ is GAS.*

Proof. First, note that the dynamics over $\mathcal{G}_1$ are not affected by $\mathcal{G}_2$. Hence, the global asymptotic stability of the equilibrium (all-healthy or strong endemic, depending on the value of $\mathcal{R}_o^1$) over $\mathcal{G}_1$ follows immediately. We will start by proving that $\mathcal{G}_2$ is ISS for different values of $\mathcal{R}_o^1$ and $\mathcal{R}_o^2$. Consider the following cases.

(i) $\mathcal{R}_o^2 < 1$: In this case, we have $\mu(A_2^T B_2 - D_2) < 0$, and therefore the matrix $A_2^T B_2 - D_2$ is Hurwitz. Since it is also Metzler, it follows from Proposition 1.1 that there exists a positive diagonal matrix R which satisfies

$$(A_2^T B_2 - D_2)^T R + R(A_2^T B_2 - D_2) = -K,$$

where K is a positive definite matrix. Similar to the proof of Proposition 5.2, consider the Lyapunov function $V_R(q_2) = q_2^T R q_2$. We have

$$\begin{aligned}
\mathcal{L}_{\tilde{\Phi}_2} V_R(q_2) &= q_2^T((A_2^T B_2 - D_2)^T R + R(A_2^T B_2 - D_2))q_2 \\
&- 2q_2^T R Q_2 A_2^T B_2 q_2 + 2q_2^T R(I - Q_2)c_2 \\
&\leq -q_2^T K q_2 + 2q_2^T R c_2,
\end{aligned}$$

where the inequality follows because $q_2^T R Q_2 A_2^T B_2 q_2 \geq 0$, for all $q_2 \in [0,1]^n$, and $q_2^T R Q_2 c_2 \geq 0$, for all $c_2, q_2 \in [0,1]^n$. Let $0 < \epsilon < 1$. We can then write

$$\mathcal{L}_{\tilde{\Phi}_2} V_R(q_2) \leq -(1-\epsilon)q_2^T K q_2 - \epsilon q_2^T K q_2 + 2q_2^T R c_2.$$

We will prove that there exists a class $\mathcal{K}_\infty$ function, χ, such that $-\epsilon q_2^T K q_2 + 2q_2^T R c_2 \leq 0$ for $\|q_2\|_2 \geq \chi(\|c_2\|_2)$. To this end, note that $q_2^T R c_2 \leq \|R\|_2 \cdot \|q_2\|_2 \cdot \|c_2\|_2$. Also, because K is positive definite, we can write $q_2^T K q_2 \geq \lambda_n(K)\|q\|_2^2 > 0$. Define $\chi(r) := \frac{2\|R\|_2 \cdot r}{\epsilon \lambda_n(K)}$, where $r \in \mathbb{R}$. We then have $-\epsilon q_2^T K q_2 + 2q_2^T R c_2 \leq 0$ for $\|q_2\|_2 \geq \chi(\|c_2\|_2)$, and hence

$$\mathcal{L}_{\tilde{\Phi}_2} V_R(q_2) \leq -(1-\epsilon)q_2^T K q_2, \quad \|q_2\|_2 \geq \chi(\|c_2\|_2).$$

This implies that the system $\mathcal{G}_2$ is ISS when $\mathcal{R}_o^2 < 1$ and $\mathcal{R}_o^1$ is arbitrary.

(ii) $\mathcal{R}_o^2 = 1$*:* Following the same reasoning in the proof of Proposition 5.2, we conclude that there exists a positive diagonal matrix S such that $(A_2^T B_2 - D_2)^T S + S(A_2^T B_2 - D_2)$ is negative semidefinite. Then, using the Lyapunov function $V_S(q_2) = q_2^T S q_2$, we can write

$$\begin{aligned}
\mathcal{L}_{\tilde{\Phi}_2} V_S(q_2) &\leq -2q_2^T Q_2 S A_2^T B_2 q_2 + 2q_2^T S c_2 \\
&\leq -q_2^T Q_2 S A_2^T B_2 q_2 + 2\sqrt{n}\|S\|_2 \cdot \|c_2\|_2,
\end{aligned}$$

where the second inequality follows by using the bound $\|q_2\|_2 \leq \sqrt{|\mathcal{V}_2|} \leq \sqrt{n}$. Define the function $\rho : \mathbb{R} \to \mathbb{R}$ as $\rho(\|c_2\|_2) = 2\sqrt{n}\|S\|_2 \cdot \|c_2\|_2$, and note that $\rho \in \mathcal{K}_\infty$ since it is linear in $\|c\|_2$. Define the function $g : \mathbb{R}_{\geq 0}^n \to \mathbb{R}$ as $g(q_2) = 2q_2^T Q_2 S A_2^T B_2 q_2$. Following similar steps to those in the proof of Proposition 5.2, we can show that $g(q_2) = 0$ if and only if $q_2 = 0$. Note that $g(q_2) > 0$ for all $q_2 \in \mathbb{R}_{\geq 0}^n$ such that $q_2 \neq 0$. Furthermore, the function g is continuous and radially unbounded. Hence, it follows by [92, Lemma 4.3] that there exists a class $\mathcal{K}_\infty$ function $\alpha : \mathbb{R} \to \mathbb{R}$ such that $g(q_2) \geq \alpha(\|q_2\|_2)$. We therefore have

$$\mathcal{L}_{\tilde{\Phi}_2} V_S(q_2) \leq -\alpha(\|q_2\|_2) + \rho(\|c_2\|_2).$$

As a result, it follows from [93, Remark 2.4] that the system $\mathcal{G}_2$ is ISS when $\mathcal{R}_o^2 = 1$ and $\mathcal{R}_o^1$ is arbitrary.

(iii) $\mathcal{R}_o^2 > 1$: Define the state $\tilde{q}_2 = q_2 - q_2^\star$, and the control input $\tilde{c}_2 = c_2 - c_2^\star$, where $c_2^\star$ was defined in the proof of Theorem 5.3 as the steady-state of c_2. Let $\tilde{Q}_2 = \mathrm{diag}(\tilde{q}_2)$, $Q_2^\star = \mathrm{diag}(q_2^\star)$, and $C_2^\star = \mathrm{diag}(c_2^\star)$. The dynamics of $\tilde{q}_2$ can then be written as

$$\begin{aligned}
\dot{\tilde{q}}_2 &= (A_2^T B_2 - D_2)(\tilde{q}_2 + q_2^\star) - (\tilde{Q}_2 + Q_2^\star)A_2^T B_2(\tilde{q}_2 + q_2^\star) \\
&\quad + (I - \tilde{Q}_2 - Q_2^\star)(\tilde{c}_2 + c_2^\star) \\
&= (-D_2 + (I - Q_2^\star)A_2^T B_2)\tilde{q}_2 - \tilde{Q}_2 A_2^T B_2 q_2 \\
&\quad + (I - Q_2)\tilde{c}_2 - \tilde{Q}_2 c_2^\star \qquad (5.15) \\
&= (-D_2 - C_2^\star + (I - Q_2^\star)A_2^T B_2)\tilde{q}_2 - \tilde{Q}_2 A_2^T B_2 q_2 \\
&\quad + (I - Q)\tilde{c}_2, \qquad (5.16)
\end{aligned}$$

where (5.15) follows from the steady-state equation in (5.13) evaluated at $q_2 = q_2^\star$, and (5.16) follows because $\tilde{Q}_2 c_2^\star = C_2^\star \tilde{q}_2$.

Next, define the matrix $\tilde{\Lambda}(q_2^\star) = -D_2 - C_2^\star + (I - Q_2^\star)A_2^T B_2$, which is Metzler since its off-diagonal entries are nonnegative. Since $\mathcal{G}_2$ is an SCC, the matrix $\tilde{\Lambda}(q_2^\star)$ is also irreducible. We wish to study the sign of $\mu\left(\tilde{\Lambda}(q_2^\star)\right)$. Using the steady-state equation in (5.13) evaluated at $q_2 = q_2^\star$, it follows that $\tilde{\Lambda}(q_2^\star)q_2^\star = -c_2^\star$, where we recall that $c_2^\star \succeq 0$. Consider the following two cases.

(iii.a) $\mathcal{R}_o^1 \leq 1$ *and* $\mathcal{R}_o^2 > 1$: In this case, the all-healthy state is GAS over $\mathcal{G}_1$; see Proposition 5.2. Then, $c_2^\star = 0$, and $\tilde{\Lambda}(q_2^\star)q_2^\star = 0$. Since $q_2^\star$ is strictly positive, it follows from Theorem 1.1 that $\mu\left(\tilde{\Lambda}(q_2^\star)\right) = 0$. Thus, it follows from Lemma 5.2 that there exists a positive diagonal matrix R such that the matrix $\tilde{\Lambda}(q_2^\star)^T R + R\tilde{\Lambda}(q_2^\star)$ is negative semidefinite. Consider the Lyapunov

function $V_R(\tilde{p}) = \tilde{p}^T R \tilde{p}$. We have

$$\begin{aligned} \mathcal{L}_{\tilde{\Phi}_2} V_R(\tilde{p}) &= \tilde{q}_2^T(\tilde{\Lambda}(q_2^\star)^T R + R\tilde{\Lambda}(q_2^\star))\tilde{q}_2 - 2\tilde{q}_2^T \tilde{Q}_2 R A_2^T B_2 q_2 \\ &\quad +2\tilde{q}_2^T R(I - Q_2)\tilde{c}_2 \\ &\leq -2\tilde{q}_2^T \tilde{Q}_2 R A_2^T B_2 q_2 + 2\tilde{q}_2^T R(I - Q_2)\tilde{c}_2 \\ &\leq -2\tilde{q}_2^T \tilde{Q}_2 R A_2^T B_2 q_2 + 4\sqrt{n}\|R\|_2 \cdot \|\tilde{c}_2\|_2, \end{aligned} \tag{5.17}$$

where the last inequality follows from $\|\tilde{q}_2\|_2 \leq \|q_2\|_2 + \|q_2^\star\|_2 \leq 2\sqrt{n}$, and the fact that $\|I - Q_2\|_2 \leq 1$. Define the scalar function $\rho(\|\tilde{c}_2\|_2) := 4\sqrt{n}\|R\|_2 \cdot \|\tilde{c}_2\|_2$, and note that $\rho \in \mathcal{K}_\infty$, since it is linear in $\|\tilde{c}_2\|_2$. Following similar steps to those in the proof of Theorem 5.2, one can show that $\tilde{q}_2^T \tilde{Q}_2 R A_2^T B_2 q_2 = 0$ if and only if $\tilde{q}_2 = 0$. Then, using the same reasoning as in the proof of Theorem 5.4, we conclude that there exists a class $\mathcal{K}_\infty$ function $\alpha : \mathbb{R} \to \mathbb{R}$ such that $2\tilde{q}_2^T \tilde{Q}_2 R A_2^T B_2 q_2 \geq \alpha(\|\tilde{q}_2\|_2)$. We therefore have $\mathcal{L}_{\tilde{\Phi}_2} V_R(\tilde{p}) \leq -\alpha(\|\tilde{q}_2\|_2) + \rho(\|\tilde{c}_2\|_2)$, and it follows from [93, Remark 2.4] that the system $\mathcal{G}_2$ is input-to-state-stable when $\mathcal{R}_o^1 \leq 1$ and $\mathcal{R}_o^2 > 1$.

(iii.b) $\mathcal{R}_o^1 > 1$ *and* $\mathcal{R}_o^2 > 1$: In this case, the endemic state is GAS over $\mathcal{G}_1$; see Theorem 5.2. Then, $c_2^\star \succ 0$, and $\tilde{\Lambda}(q_2^\star)q_2^\star \prec 0$. Since $q_2^\star$ is strictly positive, it follows from [26, Theorem 2.4] that $\mu\left(\tilde{\Lambda}(q_2^\star)\right) < 0$; therefore, $\tilde{\Lambda}(q_2^\star)$ is Hurwitz. Thus, it follows from Proposition 1.1(iv) that there exists a positive diagonal matrix S such that the matrix $\tilde{\Lambda}(q_2^\star)^T S + S\tilde{\Lambda}(q_2^\star)$ is negative definite. Hence, using $V_S(\tilde{p}) = \tilde{p}^T S \tilde{p}$, one can derive the same bound as in (5.17), with R replaced with S, and by repeating the same steps as above, one can show that $\mathcal{G}_2$ is input to state stable when $\mathcal{R}_o^1 > 1$ and $\mathcal{R}_o^2 > 1$.

Since $\mathcal{G}_1$ is GAS, and $\mathcal{G}_2$ is ISS, it follows from [92, Lemma 4.7] that the equilibrium of the cascaded system is GAS. In particular, when $\mathcal{R}_o^2 \leq 1$ and $\mathcal{R}_o^1 \leq 1$, it follows from Theorem 5.3(iii) that the all-healthy state is GAS. When $\mathcal{R}_o^2 \leq 1$ and $\mathcal{R}_o^1 > 1$, it follows from Theorem 5.3(i) that the strong endemic equilibrium $[q_1^{\star T}, q_2^{\star T}]^T$ is GAS, assuming that $q_i(0) \neq 0$ for all $i \in [2]$. When $\mathcal{R}_o^2 > 1$ and $\mathcal{R}_o^1 \leq 1$, it follows from Theorem 5.3(ii) that the weak endemic state $[0^T, q_2^{\star T}]^T$ is GAS, assuming that $q_2(0) \neq 0$. Finally, when when $\mathcal{R}_o^2 > 1$ and $\mathcal{R}_o^1 > 1$, it follows from Theorem 5.3(i) that the strong endemic state $[q_1^{\star T}, q_2^{\star T}]^T$ is GAS, assuming that $q_i(0) \neq 0$ for $i \in [2]$. □

The following corollary is an immediate consequence of Theorems 5.3 and 5.4.

Corollary 5.1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a weakly connected digraph consisting of N SCCs ordered as $\mathcal{G}_1 \prec \ldots \prec \mathcal{G}_N$. Assume that $q_i(0) \neq 0$ for all $i \in [n]$.*

(i) If $\mathcal{R}_o^i \leq 1$ for all $i \in [N]$, then the all-healthy state is GAS.

(ii) If $\mathcal{R}_o^k > 1$ for some $k \in [N]$, and $\mathcal{R}_o^i \leq 1$ for $i \in [k-1]$, then the endemic state $p^\star = [0, \ldots, 0, q_k^{\star T}, \ldots, q_N^{\star T}]^T$ is GAS.

5.6 Numerical Studies

We demonstrate the emergence of a weak endemic state over the Pajek GD99c network [94], which is a weakly connected directed network shown in Fig. 5.1. The network consists of 105 nodes and it contains 66 SCCs. The nodes marked "red" in Fig. 5.1 constitute an SCC, which we refer to as $\mathcal{G}_1$.

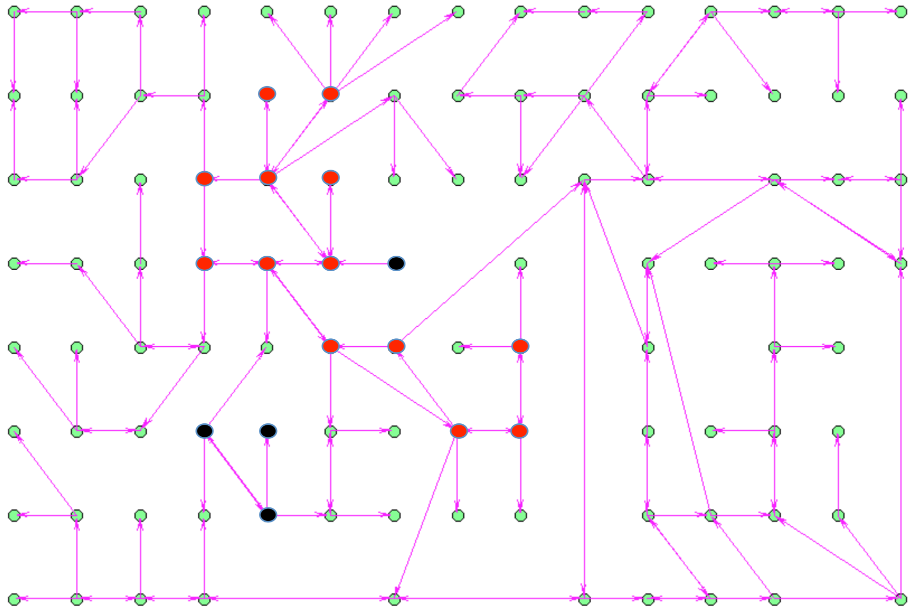


Figure 5.1: The Pajek GD99c network. The "red" nodes belong to $\mathcal{G}_1$ for which $\mathcal{R}_o^1 > 1$. The "black" nodes are the only ones with no direct path from $\mathcal{G}_1$.

We will select the curing rates over $\mathcal{G}_1$ to be low in order to make $\mathcal{R}_o^1 > 1$. For the remaining nodes, we will set $\delta_i = \sum_{j \neq i} a_{ji}\beta_j + 0.5$, which is a sufficient condition to ensure $\mathcal{R}_o^i < 1$ as per (5.11). The infection rates β_i and the weights a_{ij} are all selected to be equal to 1. There are only 4 nodes for which there is no directed path from $\mathcal{G}_1$, and they are marked "black" in Fig. 5.1. The initial infection profile is selected at random.

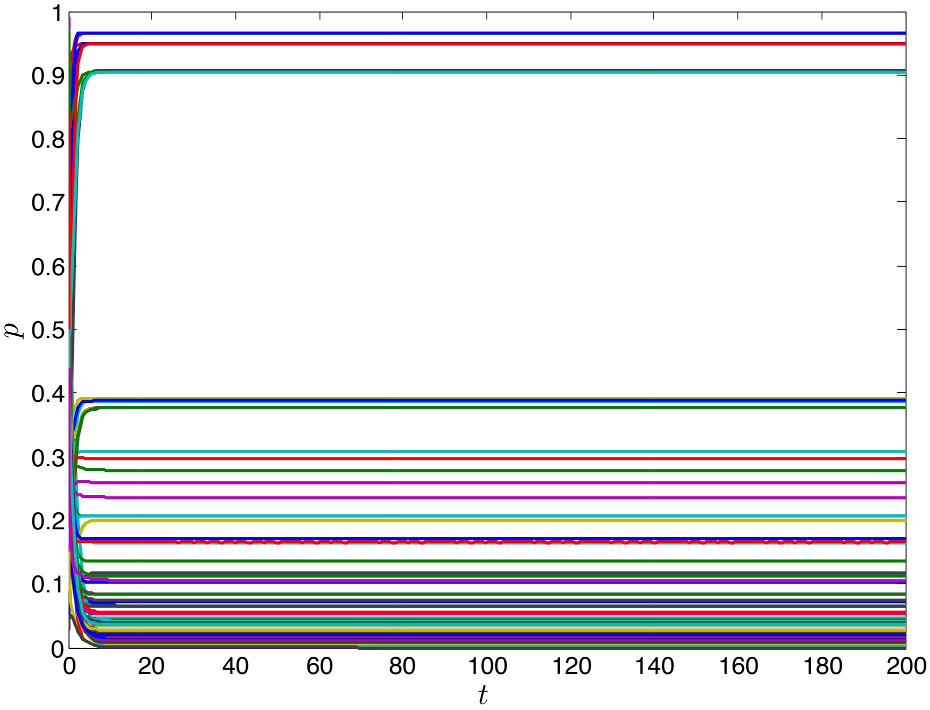


Figure 5.2: Infection probabilities of the nodes.

Figure 5.2 plots the state trajectories. By examining the histogram of the values to which the state converges, which is shown in Fig. 5.3, we notice that there are 13 nodes with high infection probabilities, and those are the nodes comprising $\mathcal{G}_1$. Note that $\mathcal{G}_1$ is asymptotically stable even though it takes input from other SCCs, as shown in the figure, and $\mathcal{R}_o^1 > 1$. There are 4 nodes that become healthy, and those are the "black" nodes which are not reached by a directed path from $\mathcal{G}_1$. The remaining nodes all have positive infection probabilities with varying levels depending on their distance from $\mathcal{G}_1$, with the nodes that are farthest from $\mathcal{G}_1$ enjoying the lowest infection probabilities.

Next, we will demonstrate the global asymptotic stability of $p^\star$ over connected undirected graphs, which follows from Theorem 5.2. The infection rates, the edge weights, and the initial infection profile were generated randomly. The curing rates were selected such that $\mathcal{R}_o > 1$.

Figure 5.4 shows the state of a ring graph with 20 nodes. The figure also plots the Lyapunov function $V(\tilde{p}) = \frac{1}{2}\tilde{p}^T\tilde{p}$. As claimed, the system

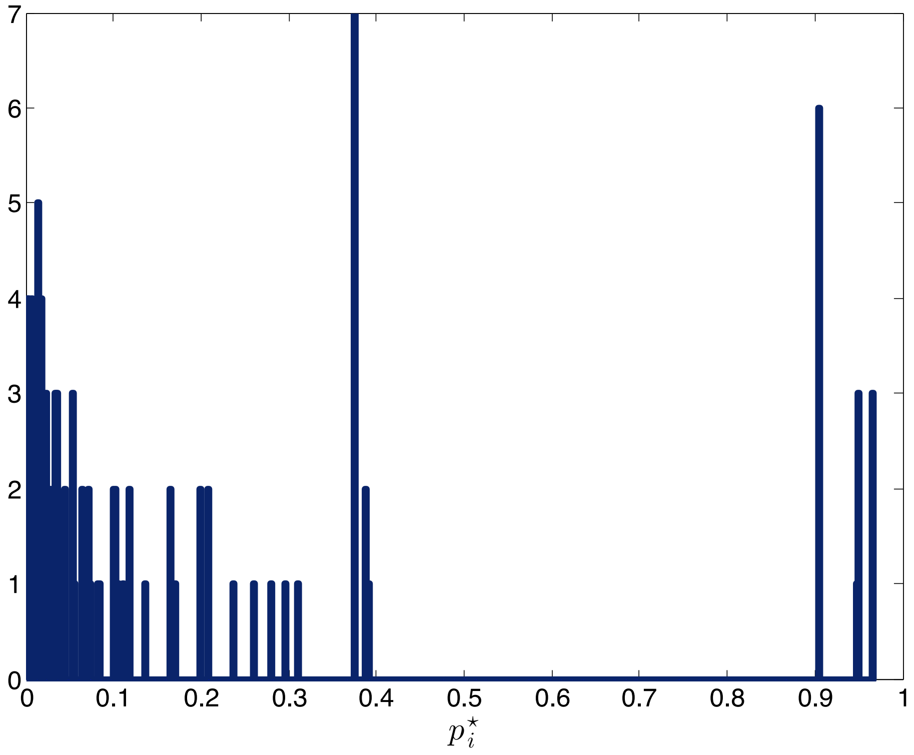


Figure 5.3: A histogram of the endemic state value across the network.

converges to the strictly positive state $p^\star$, and the Lyapunov function decays monotonically to zero.

Figure 5.5 shows the same simulation for a connected undirected random graph with 100 nodes. The probability that an edge occurs in the graph was selected to be $\frac{3}{10}$. The specific graph realization used in this experiment contained 1704 edges. Again, we observe that the state converges to $p^\star$. It is interesting to note that convergence here is faster than the case of the ring graph.

5.7 Summary

We have utilized tools from positive systems theory to establish the stability properties of the n-intertwined Markov model over digraphs. For strongly connected digraphs, we have proved that when the basic reproduction number is less than or equal to 1, the all-healthy state is GAS. When the basic reproduction number is greater than 1, however, we have shown that the endemic state is GAS, and that locally around this equilibrium, the convergence

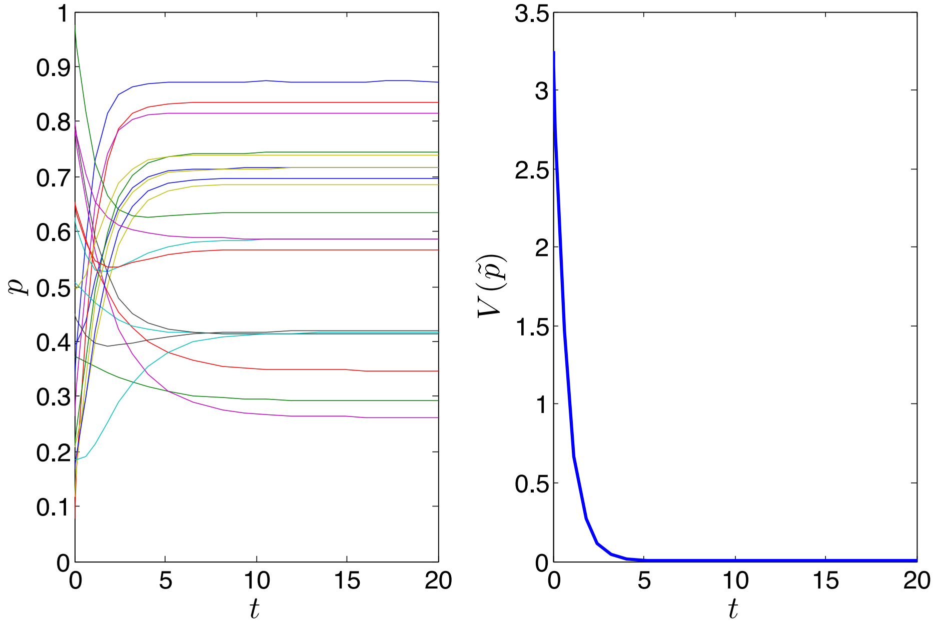


Figure 5.4: Stabilization of a ring graph with 20 nodes.

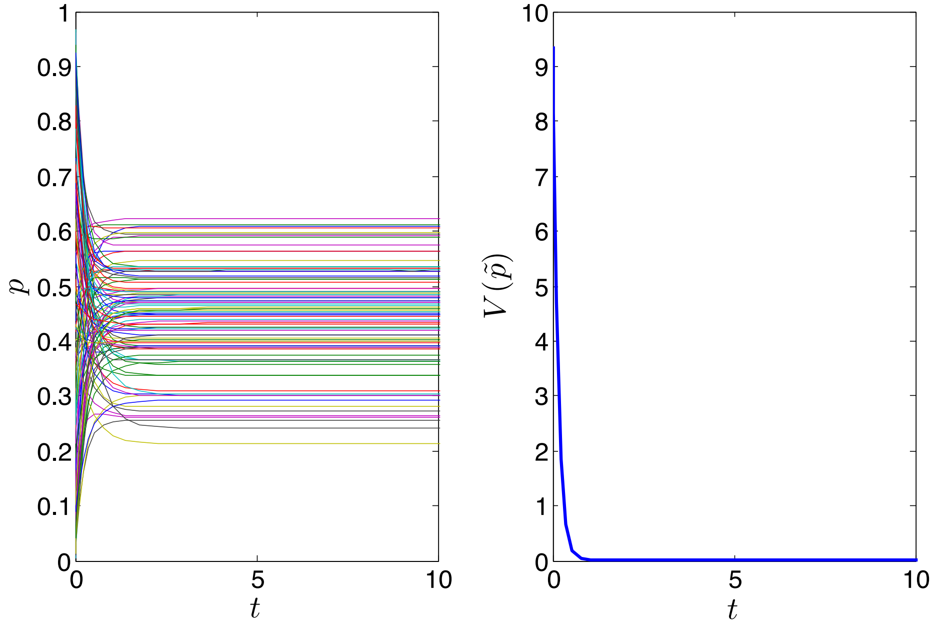


Figure 5.5: Stabilization of a random graph with 100 nodes and 1704 edges.

is exponentially fast. Furthermore, we have studied the stability properties of weakly connected graphs. By viewing an arbitrary weakly connected graph as a cascade of SCCs, we were able to establish the existence and uniqueness of weak and strong endemic states. We have also studied the stability properties of weakly connected graphs using input-to-state stability. Finally, we have proposed a dynamical model that describes the interaction among nodes in an infected network as a concave game and demonstrated that the n-intertwined Markov model is a special case of our model. This alternative description provides a new condition, which can be checked collectively by agents, for the stability of the origin.

5.8 Additional Proofs

In this Section, we collect and prove some results pertinent to the development in he main body of the chapter. We start with the next result, which is key in proving some of the results in Sections 5.4 and 5.5.

Lemma 5.2. *Let $X \in \mathbb{R}^{n\times n}$ be an irreducible Metzler matrix such that $\mu(X) = 0$. Then, there exists a positive diagonal matrix $R \in \mathbb{R}^{n\times n}$ such that the matrix $X^T R + RX$ is negative semidefinite.*

Proof. From Theorem 1.1, it follows that there exists a vector $\nu \in \mathbb{R}^{n\times n}$ such that $\nu \gg 0$ and $X\nu = 0$. Since $\sigma(X) = \sigma(X^T)$, we have $\mu(A^T) = 0$. Using Theorem 1.1 again, we conclude that there exists a vector $\xi \in \mathbb{R}^{n\times n}$ such that $\xi \gg 0$ and $X^T\xi = 0$. Let $R \in \mathbb{R}^{n\times n}$ be a positive diagonal matrix defined with $R_{ii} = \xi_i/\nu_i$, for all $i \in [n]$. Consider now the matrix $X^T R + RX$. The matrix RX is Metzler, since R is a positive diagonal matrix. For the same reason, and because X is irreducible, we conclude that RX is irreducible. By a similar argument, $X^T R$ is also an irreducible Metzler matrix. Since the sum of two Metzler matrices is Metzler, the matrix $X^T R + RX$ is Metzler. Also, because both RX and $X^T R$ are Metzler and irreducible, the matrix $X^T R + RX$ is also irreducible. Further, by construction, we have $(X^T R + RX)\nu = X^T R\nu = X^T\xi = 0$. Since $X^T R + RX$ is symmetric, it has real eigenvalues, and since ν is strictly positive, it follows from Theorem 1.1 that $X^T R + RX$ is negative semidefinite. □

Next, we prove an instrumental result, which can be thought of as a non-homogeneous extension of a result of [26]. We start by providing two key properties of the continuous mapping $T : [0,1]^n \to [0,1]^n$ defined as

$$T(p) := (I + \mathrm{diag}(Xp))^{-1}(Xp + y).$$

Proposition 5.6. *Let $X \in \mathbb{R}^{n\times n}$ be a nonnegative matrix, and let $y \in \mathbb{R}^n$ be a vector satisfying $0 \preceq y \ll \mathbf{1}$. Then, the mapping T is monotonic.*

Proof. Let the vectors $p, q \in \mathbb{R}^n$ be such that $p \preceq q$. For $i \in [n]$, we have

$$\begin{aligned} T_i(p) &= \frac{(Xp)_i + y_i}{1 + (Xp)_i} = 1 - \frac{1 - y_i}{1 + (Xp)_i} \\ &\leq 1 - \frac{1 - y_i}{1 + (Xq)_i} = T_i(q), \end{aligned}$$

where the inequality follows because X is nonnegative. This implies that the mapping T is monotonic. □

Proposition 5.7. *Let $X \in \mathbb{R}^{n\times n}$ be a nonnegative matrix, and let $y \in \mathbb{R}^n$ be a vector satisfying $0 \preceq y \ll \mathbf{1}$. If the mapping T has strictly positive fixed point, then it must be unique.*

Proof. We will prove the claim by contradiction. Assume that there are two fixed points $p^\star, q^\star \in \mathbb{R}^n$, $p^\star \neq q^\star$. We will first show that $p^\star \preceq q^\star$. To this end, define

$$\eta := \max_{i\in[n]} \frac{p_i^\star}{q_i^\star}, \quad k := \arg\max_{i\in[n]} \frac{p_i^\star}{q_i^\star}.$$

Note that $p^\star \preceq \eta q^\star$. For $p^\star \preceq q^\star$ to hold, we must have $\eta \leq 1$; assume that, to the contrary, $\eta > 1$. Then, using Proposition 5.6, we have

$$\begin{aligned} p_k^\star &= T_k(p^\star) \leq T_k(\eta q^\star) = \frac{\eta(Xq^\star)_k + y_k}{1 + \eta(Xq^\star)_k} \\ &< \eta\frac{(Xq^\star)_k + y_k}{1 + (Xq^\star)_k} = \eta T_k(q^\star) = \eta q^\star, \end{aligned}$$

where the strict inequality follows from the assumption that $\eta > 1$, and the last equality follows because $q^\star$ is a fixed point. By definition, we have $p_k^\star = \eta q_k^\star$. Hence, if $\eta > 1$ were true, we would have $p_k^\star < \eta q_k^\star = p_k^\star$, which is a contradiction. Hence, we must have $\eta \leq 1$ and $p^\star \preceq q^\star$. By switching the roles of $p^\star$ and $q^\star$, and repeating the above steps with $\hat{\eta} = \max_{i\in[n]} \frac{q_i^\star}{p_i^\star}$

instead of η, we conclude that $p^\star \succeq q^\star$. Thus, $p^\star = q^\star$, and the fixed point is unique. $\square$

We are now ready to prove the main result.

Theorem 5.5. *Let $X \in \mathbb{R}^{n \times n}$ be a nonnegative irreducible matrix such that $\rho(X) > 1$, and let $y \in \mathbb{R}^n$ be a vector satisfying $0 \preceq y \ll \mathbf{1}$. Then, the mapping $T : [0,1]^n \to [0,1]^n$ has a unique fixed point, which is strictly positive.*

Proof. We will prove that there exists a closed sub-interval of $(0,1)^n$ which is invariant under T. By Theorem 1.2, it follows that X has an eigenvector $v \gg 0$ satisfying $Xv = \rho(X)v$. Without loss of generality, we assume that $v \preceq 1$, which can be achieved by an appropriate scaling of the eigenvector corresponding to $\rho(X)$.

Define $\overline{\kappa} := \sqrt{\frac{\rho(X)+y_{\max}}{1+\rho(X)}}$, and note that $\overline{\kappa} < 1$. Let us choose $\overline{\epsilon} > 0$ such that $\overline{\kappa} \le \overline{\epsilon} v_{\min}$. Note that with such a choice of $\overline{\epsilon}$, we can guarantee, for all $i \in [n]$, that $\overline{\epsilon} v_i < 1$, since $v_i \le 1$ and $\overline{\kappa} < 1$. This choice of $\overline{\epsilon}$ implies that $\overline{\epsilon} v_i \ge \overline{\kappa}$ or $(\overline{\epsilon} v_i)^2 \ge \frac{\rho(X)+y_{\max}}{1+\rho(X)}$, for all $i \in [n]$. This in turn implies, for $i \in [n]$,

$$\overline{\epsilon} v_i \ge \frac{1}{\overline{\epsilon} v_i} \cdot \frac{\rho(X) + y_i}{1 + \rho(X)} > \frac{\overline{\epsilon}\rho(X) v_i + y_i}{1 + \overline{\epsilon} v_i \rho(X)} = T_i(\overline{\epsilon} v_i), \tag{5.18}$$

where the last inequality follows since $\overline{\epsilon} v_i < 1$. We therefore have $T(\overline{\epsilon} v) < \overline{\epsilon} v$.

Define $\underline{\kappa} := \frac{\rho(X)+y_{\min}-1}{1+\rho(X)}$, and note that $\underline{\kappa} < 1$, as $y_{\min} < 1$. Let us choose $\underline{\epsilon} > 0$ such that $0 < \underline{\epsilon} v_{\max} \le \underline{\kappa}$. Then, for all $i \in [n]$, we have

$$\underline{\epsilon} v_i \le \frac{\rho(X) + y_i - 1}{\rho(X) + 1} < \frac{\rho(X) + y_i - 1}{\rho(X)}.$$

We thus have $\underline{\epsilon}\rho(X) v_i + 1 < \rho(X) + y_i$, for all $i \in [n]$. Equivalently, for all $i \in [n]$, we can write

$$\underline{\epsilon} v_i < \underline{\epsilon} v_i \frac{\rho(X) + y_i}{\underline{\epsilon}\rho(X) v_i + 1} < \frac{\underline{\epsilon}\rho(X) v_i + y_i}{\underline{\epsilon}\rho(X) v_i + 1} = T_i(\underline{\epsilon} v), \tag{5.19}$$

where the second strict inequality holds since $\underline{\epsilon} v_i < \kappa < 1$. We therefore have $T(\underline{\epsilon} v) > \underline{\epsilon} v$.

Since $v \gg 0$ and $\underline{\epsilon} > 0$, we have $\underline{\epsilon} v \gg 0$. We also have that $\overline{\epsilon} > \underline{\epsilon}$ because

$$\begin{aligned} \overline{\epsilon} &\geq \frac{\overline{\kappa}}{v_{\min}} > \frac{\overline{\kappa}^2}{v_{\min}} = \frac{\rho(X) + y_{\max}}{v_{\min}(1 + \rho(X))} \\ &> \frac{\rho(X) + y_{\min} - 1}{v_{\max}(1 + \rho(X))} = \frac{\underline{\kappa}}{v_{\max}} \geq \underline{\epsilon}, \end{aligned}$$

where the first strict inequality follows because $\overline{\kappa} < 1$. This implies that $\underline{\epsilon} v \ll \overline{\epsilon} v$. Further, by construction, we have $\overline{\epsilon} v_i < 1$, for all $i \in [n]$, and therefore $\overline{\epsilon} v \ll 1$. To summarize, we have the following bounds: $0 \ll \underline{\epsilon} v \ll \overline{\epsilon} v \ll 1$.

We can now define the closed and bounded set

$$K := \{p \in [0,1]^n \mid \epsilon_1 v \preceq p \preceq \epsilon_2 v\} \subset (0,1)^n.$$

By (5.18) and (5.19), and since T is monotonic as proved in Proposition 5.6, we conclude that $T : K \to K$. Since T is continuous, it follows from Brouwer's fixed-point theorem that there exists a strictly positive fixed point $p^\star \in K$ such that $T(p^\star) = p^\star$. Finally, it follows from Proposition 5.7 that $p^\star$ must be unique. □

CHAPTER 6

VIRUS SPREAD IN NETWORKS: CONTROL DESIGN

6.1 Background

In this chapter, we focus on control design problems for networks whose dynamics are given by the n-intertwined Markov model described in Chapter 5. To this end, we view the curing rates as control inputs, and we investigate the design of stabilizing and optimal control laws.

We have seen in the previous chapter that stabilizing the all-healthy state requires allocating high curing rates across the network. However, for networks that contain a large number of nodes, allocating a high curing rate to each and every node could incur a high cost. Motivated by this challenge, we investigate the possibility of stabilizing the all-healthy state when the curing rates of only a limited number of nodes can be controlled.

A common approach in the literature to stabilize the all-healthy state has been to assign constant curing rates across the network [32, 36–38, 95–98]. This approach seems to be quite wasteful, especially if the infection probability of a given node approaches the healthy state, in which case that node would not need a high rate of curing. Here, using nonlinear control designs, we study the behavior of dynamic controllers that are able to exploit the state of the network. Moreover, we study multiple optimal control problems that are designed to yield controllers capable of minimizing the total infection in the network at a low cost.

6.2 Main Results

The main contributions of this chapter are as follows. When the curing rates of a limited number of nodes can be controlled, we identify conditions under which the network can be stabilized to the origin, and we identify graph classes that can be stabilized using a limited number of controllers. Further, we propose a *dynamic* optimization framework that allows the network designer to design an optimal controller that minimizes the total infection in the network at minimum cost. We show that this controller is of the bang-bang type, and that it may exhibit multiple switches. In addition, we propose two static control laws: one is obtained by optimizing the vaccination levels at time zero, and the other one is based on a second-order approximation. We demonstrate that the optimal dynamic controller and the static control laws exhibit similar performances over *sparse* graphs. Finally, we transform the network dynamics into a form that is linear in controls, and we study an optimal control problem subject to these dynamics. We show that the optimal controller of this problem exhibits at most one switch. By analyzing the switching behavior of the dynamic controllers, we observe that optimal controllers reduce the curing rates of those nodes that are approaching the healthy state, which matches our intuition.

Organization

Section 6.3 contains our results on the design of stabilizing controllers for infected networks. An optimal control problem is formulated and studied in Section 6.4. In Section 6.5, we propose a static optimization framework and compare the performances of dynamic and static control laws. In Section 6.6, we study another optimal control problem subject to a transformed version of the n-intertwined Markov model. The main results of the chapter are summarized in Section 6.7.

6.3 Stabilization

Consider a network of n nodes described by a connected undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of vertices, and $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$ is the set of edges. Recall the n-intertwined Markov model introduced in the previous chapter:

$$\dot{p} = (AB - D)p - PABp, \quad p(0) = p_0, \tag{6.1}$$

where A is the adjacency matrix of $\mathcal{G}$, $P = \text{diag}(p)$, $B = \text{diag}(\beta_1, \ldots, \beta_n)$, and $D = \text{diag}(\delta_1, \ldots, \delta_n)$.

In this section, we will investigate the possibility of reducing the infection by altering the curing rates at a limited number of nodes belonging to a set $S_{\text{control}} \subset \mathcal{V}$, and we define $r := |S_{\text{control}}|$. For this reason, throughout this section, we replace δ_i with $u_i(t)$, where $i \in S_{\text{control}}$. Given the necessary conditions presented in the recent paper [99], we will use the assumption that there exists a small curing rate of α_i at any node in $\mathcal{F} = \mathcal{V} \backslash S_{\text{control}}$. This amount of self-healing may, however, not be enough to stabilize the system to the origin. Recall from Proposition 5.2 that $\mathcal{R}_o \leq 1$ is a necessary and sufficient condition for the all-healthy state to be GAS; we are interested in answering the following question: *When the condition $\mathcal{R}_o \leq 1$ is initially violated, can we stabilize the system to the origin by controlling the nodes in S_{control} only?*

By construction, we have $S_{\text{control}} \cap \mathcal{F} = \emptyset$ and $S_{\text{control}} \cup \mathcal{F} = \mathcal{V}$. Let $U(t)$ be a diagonal matrix such that $U_{ii}(t) = u_i(t)$ if and only if $i \in S_{\text{control}}$, and zero otherwise. Similarly, let Γ be a diagonal matrix such that $\Gamma_{ii} = \alpha_i$ if and only if $i \in \mathcal{F}$, and zero otherwise. The n-intertwined Markov model dynamics introduced in Chapter 5 can then be written as:

$$\dot{p} = (AB - \Gamma - U)p - PABp.$$

Note that this system is affine in controls. To see this, define $h(p) = (AB - \Gamma)p - PABp$ and $g_i(p) := -p_i e_i$, where $\{e_1, \ldots, e_n\}$ is the fundamental basis. We can then write

$$\dot{p} = h(p) + \sum_{i \in S_{\text{control}}} g_i(p) u_i.$$

In what follows we consider two special cases for which a limited number of controllers can stabilize the system.

Lemma 6.1. *In a star graph, the all-healthy state can be stabilized by placing an appropriate controller at the root node and arbitrarily small α-self-loops everywhere else.*

Proof. We will proceed by showing that the function $V(p) = \frac{1}{2}p^T p$ is a control Lyapunov function (CLF). Without loss of generality, let node 1 be the root. The dynamics of all other nodes is then given by $\dot{p}_i = -\alpha_i p_i + (1 - p_i) a_{i1} \beta_1 p_1$.

A necessary and sufficient condition for $V(p)$ to be a CLF is

$$\frac{\partial}{\partial p}V(p)^T g_1(p) = -p_1^2 = 0 \implies \frac{\partial}{\partial p}V(p)^T h(p) < 0, \quad p \neq 0.$$

But when $p_1 = 0$, we have

$$\frac{\partial}{\partial p}V(p)^T h(p) = p^T(AB - \Gamma)p - p^T PABp = -p^T \Gamma p,$$

which is negative. Hence, $V(p)$ is indeed a CLF, and we can stabilize the system using Sontag's universal controller [100]. □

Note that Sontag's controller requires the controlling node to have knowledge of the entire state. In the above, the root node is connected to all the nodes, and hence it has access to the state vector p.

Lemma 6.2. *In an odd (or even) length path graph, a maximum of* $(n-1)/2$ *(or* $n/2$*) controllers are required to stabilize the all-healthy state, provided that all other nodes implement arbitrarily small* α*-self-loops.*

Proof. The proof is similar to the star graph case. We will show that $V(p) = \frac{1}{2}p^T p$ is a CLF. Let us place the controllers at nodes $\{2, 4, \ldots\}$. Then, from the structure of A, it follows that $p^T ABp = 0$ when $\frac{\partial}{\partial p}V(p)^T\,(g_2(p), g_4(p), \ldots) = -\sum_{i \in S_{\text{control}}} p_i^2 = 0$. This implies that $\frac{\partial}{\partial p}V(p)^T h(p) = -p^T \Gamma p$, and $V(p)$ is a CLF. The size of S_{control} follows from the way we have placed the controllers. This concludes the proof. □

Similar results can be obtained for other classes of graphs. The key idea behind the above results is to place the controllers in such a way that no path can be drawn between two nodes in $\mathcal{F}$ without passing through a node in S_{control}. For example, in a tree with an even number of levels, stabilization can be achieved by controlling the nodes in every other level, and placing arbitrarily small α-self-loops everywhere else. The following corollary is immediate.

Corollary 6.1. *In a binary tree with an even number of levels,* ℓ*, it suffices to control* $\frac{1}{3}(2^\ell - 1)$ *nodes to stabilize the all-healthy state.*

The above results characterize the number of controllers that would be sufficient to stabilize the network. For the nodes in S_{control}, there is a variety of choices for the specific control law to be implemented. We will next compare the performance of Sontag's universal controller [100] to that of a constant controller based on the cost of control as given by $\int_0^T u_i(t)dt$. Sontag's universal controller is a state-feedback controller that is used to stabilize systems that are affine in controls. It relies on deriving a control Lyapunov function $V : \mathbb{R}^{n \times n} \to \mathbb{R}$ for the system under study, and it is given by the following universal formula:

$$u_{\text{Sontag}}(p) = \begin{cases} -\frac{\xi + \sqrt{\xi^2 + \|\eta\|_2^4}}{\|\eta\|_2^2} b, & \eta \neq 0 \\ 0, & \eta = 0 \end{cases},$$

$$\xi(p) = \frac{\partial}{\partial p} V(p)^T h(p), \tag{6.2}$$

$$\eta(p) = \left[\frac{\partial}{\partial p} V(p)^T g_1(p), \ldots, \frac{\partial}{\partial p} V(p)^T g_r(p) \right]^T. \tag{6.3}$$

Consider a star graph with 10 nodes. By Lemma 6.1, we know that it suffices to control the root node to stabilize the network. Let node 1 be at the root. Recalling from the proof of Lemma 6.1 that $V = \frac{1}{2} p^T p$ is a CLF for the n-intertwined model over star graphs, and the gains ξ, η in (6.2), (6.3) in this case become

$$\begin{aligned} \xi(p) &= p^T (AB - \Gamma) p - p^T PABp, \\ \eta(p) &= -\left[p_1^2, 0, \ldots, 0 \right]^T. \end{aligned}$$

We assume that the rest of the nodes implement a self-loop $\alpha = 0.1$. The horizon of the simulation, T, is chosen to be 100. Fig. 6.1 depicts the performance of a constant controller $u_1 = 8$, while the performance of Sontag's universal controller is shown in Fig. 6.2. We observe that the stabilization properties of both controllers are similar. However, Sontag's universal controller incurs a lower cost compared to the constant controller; the total cost incurred by the constant controller is 800, while that incurred by Sontag's controller is 738.6.

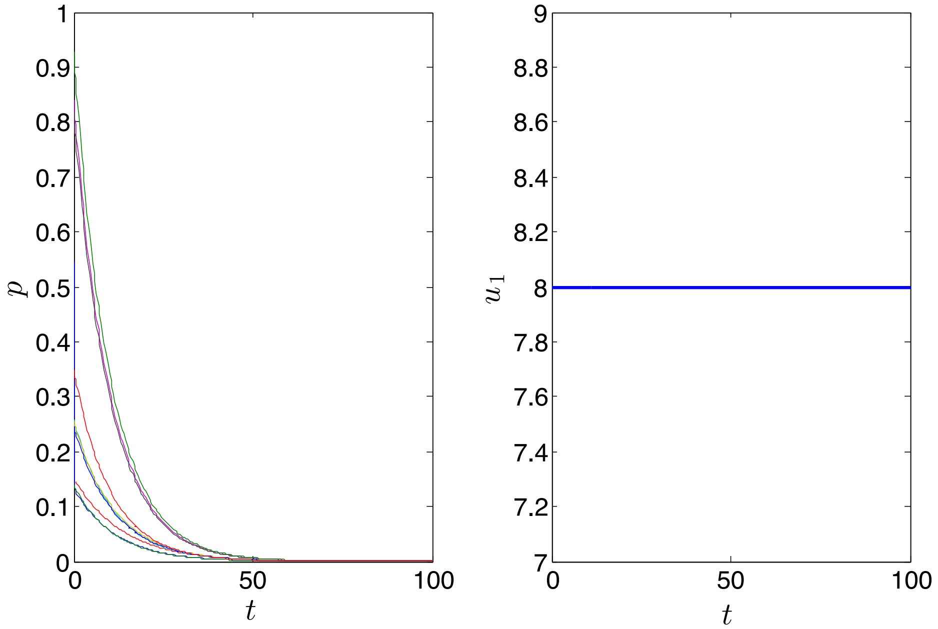


Figure 6.1: A star graph with a constant controller implemented at the root. $n = 10$.

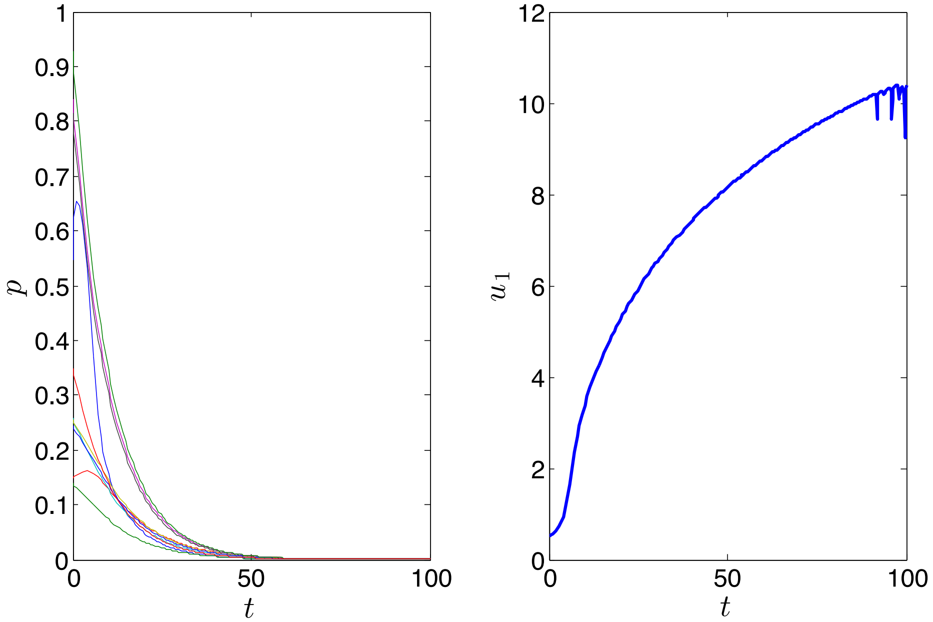


Figure 6.2: A star graph with Sontag's universal controller implemented at the root. $n = 10$.

6.4 Optimal Control

We now focus on designing optimal controllers for infected networks. We assume that the designer can control the curing rates of all nodes. To capture this, we re-label the matrix D in (6.1) as U, where $U = \text{diag}(u_1, \ldots, u_n)$. We assume that there are upper and lower bounds on the curing rates: $\underline{u} \leq u_i(t) \leq \overline{u}$, for all $i \in \mathcal{V}$ and all $t \in \mathbb{R}_{\geq 0}$, where $\underline{u}$ corresponds to the natural immunity of the node, and $\overline{u}$ corresponds to the maximum vaccination level available. The action set of the designer can be written as

$$W = \{w \in \mathbb{R}^n \mid \underline{u} \leq w_i \leq \overline{u}\}.$$

The set of admissible controls, $\mathcal{U}$, consists of all functions that are piecewise continuous in time and whose range is W. Given a time interval $[0, T]$, we can formally write

$$\mathcal{U} = \{u : [0, T] \to W \mid \ u \text{ is a piecewise continuous function of } t\}.$$

The designer aims at reducing the infection probabilities across the network, while minimizing the cost associated with modifying the curing rates. Let $c \in \mathbb{R}^n_{\geq 0}$ be the cost associated with the state, and let $d \in \mathbb{R}^n_{\geq 0}$ be the cost associated with the control. We can then write the cost functional of the designer as follows:

$$J(u) = \int_0^T [c^T p + d^T u] dt.$$

In order to minimize the cost associated with the state, the designer must attempt to stabilize the state to the origin. To this end, we will linearize the dynamics in (5.1) around the origin to obtain $\dot{p} = (AB - U)p$. Noting that $p_i \sum_{j \neq i} a_{ij} \beta_j p_j \geq 0$, for all $i \in \mathcal{V}$ and $p \in [0, 1]^n$, we conclude that

$$(AB - U)p - PABp \leq (AB - U)p.$$

This serves as a confirmation of the fact that the linear part of the dynamics is what is important when the focus is stabilization to the origin. We will therefore work with the linearized dynamics hereinafter.

Consider the following optimal control problem:

$$\begin{array}{rl} \inf\limits_{u \in \mathcal{U}} & J(u) \\ \text{subject to} & \dot{p} = (AB - U)p, \quad p(0) = p_0. \end{array} \tag{6.4}$$

Proposition 6.1. *The optimal dynamic controller for node $i \in \mathcal{V}$ for the above optimal control problem is given by*

$$u_i^\star \;=\; \begin{cases} \overline{u}, & d_i - p_i^\star q_i^\star < 0 \\ \underline{u}, & d_i - p_i^\star q_i^\star > 0 \\ \{\underline{u}, \overline{u}\}, & \textit{otherwise} \end{cases} . \tag{6.5}$$

where $p^\star$ is the optimal trajectory and $q^\star$ is the costate vector provided by the canonical equations of the MP. Further, sufficiently close to the terminal time T, the optimal controller is $u^\star = \underline{u}\mathbf{1}$.

Proof. The existence of optimal control for this problem follows by a straightforward application of Filippov's existence theorem [58]. The Hamiltonian associated with this problem is

$$H(p, q, u) = c^T p + d^T u + q^T (AB - U)p,$$

where q is the costate vector. The MP dictates that there exists a costate vector q satisfying the following canonical equations along the optimal trajectory:

$$\dot{p}^\star \;=\; (AB - U^\star)p^\star, \quad p^\star(0) = p_0, \tag{6.6}$$

$$\dot{q}^\star \;=\; -\frac{\partial}{\partial p} H = -(AB - U^\star)^T q^\star - c, \quad q^\star(T) = 0. \tag{6.7}$$

Further, the optimal controller minimizes the Hamiltonian:

$$u^\star = \arg\min_{u \in W} H(p^\star, q^\star, u),$$

which yields the solution in (6.7). Using the continuity of $q^\star$ and the terminal condition imposed on it, we conclude that when sufficiently close to the terminal time T, $u^\star = \underline{u}\mathbf{1}$. □

Next, we demonstrate that the optimal controller (6.5) can exhibit multiple switches. Consider the network shown in Fig. 6.3, and let $d = [1, 1, 10, 1, 1]^T$ such that node 3 has a high cost on control. Also, let $p(0) = [0.1, 0.01, 0.9, 0.01, 0.01]^T$, where we assigned a high probability of infection to node 3. Let $\underline{u} = 0.1$, $\overline{u} = 1$, $T = 100$, and $c = \mathbf{1}$. Unity infection rates were assigned to all the nodes, i.e., $\beta_i = 1$ for all $i \in [6]$. The edge weights a_{ij} were generated randomly.

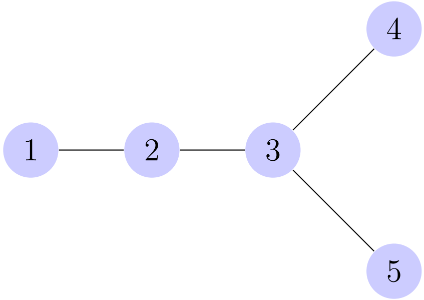


Figure 6.3: An infected graph with node 3 having high probability of infection and high cost on control.

Figure 6.4 shows the state of the network above after implementing the controller given in (6.5). Note that $u_3 = \underline{u}$ throughout $[0, T]$, because controlling this node is expensive. Nevertheless, although the neighboring nodes have low initial probability of infections, the optimal controller intelligently increases the curing rates of these nodes, which enjoy low control cost, in order to help cure node 3. It is interesting to note that all the controllers, except u_3, exhibit multiple switches between $\underline{u}$ and $\overline{u}$.

Remark 6.1. *It is important to note that the designer was able to cure the entire network without needing to apply the maximum vaccination level to node* 3*. This demonstrates that curing the network does not require applying high vaccination levels to the entire network.* •

6.5 Static Approaches

Note from (6.5)-(6.7) that the state, costate, and optimal control are interrelated and cannot be solved in closed form. Hence, besides simulations, it is not apparent how one can analytically study the properties of the optimal controller, such as the number of switches between the bounds $\underline{u}$ and $\overline{u}$.

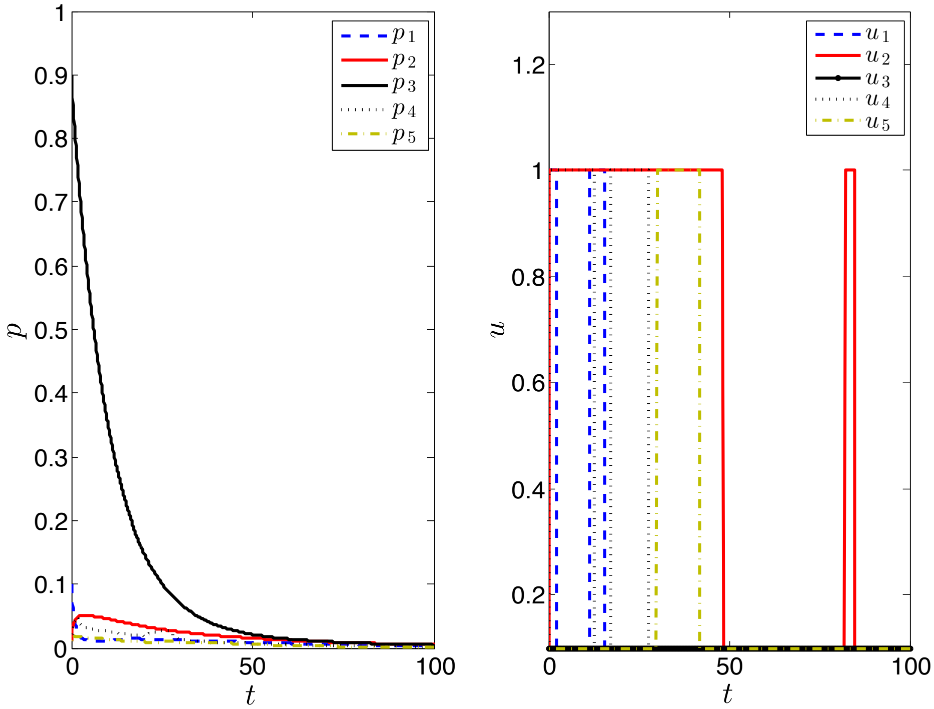


Figure 6.4: State and optimal controller of a network with a highly infected node whose control cost is high.

We have shown above that the optimal controller (6.5) can exhibit multiple switches between the vaccination levels, which may not be practical for certain scenarios. For scenarios where a *static* controller, i.e., a controller that does not exhibit any switching, is more appropriate, it is instructive to compare the performances of static controllers with that of the optimal controller in (6.5). In this section, we will propose two approaches to obtain an efficient static controller.

6.5.1 Optimal Static Controller

Instead of allowing the control input u to change its value dynamically, we would like to choose the control input at $t = 0$ and fix it for the remaining portion of the problem's horizon. We will denote the static controller by u_i^{c}, $i \in \mathcal{V}$. When the controller is fixed, we can readily obtain the solution of the linearized dynamics in (6.4) as

$$p = e^{(AB - U^{\mathrm{c}})t} p_0, \tag{6.8}$$

where $U^{\mathrm{c}} = \mathrm{diag}(u^{\mathrm{c}})$. The objective function of the designer in this case becomes

$$J(u^{\mathrm{c}}) = \int_0^T c^T e^{(AB-U^{\mathrm{c}})t} p_0 dt + T d^T u^{\mathrm{c}},$$

and his optimization problem becomes

$$\min_{u^{\mathrm{c}} \in W} \quad J(u^{\mathrm{c}}).$$

Since $J(u^{\mathrm{c}})$ is continuous in u^{c}, and W is closed and bounded, it follows from Weierstrass's Extreme Value Theorem that a globally optimum solution exists. Hence, although the objective function is not convex for all parameter values, we can still obtain the global minimum using standard search algorithms. However, in general, it is not possible to obtain the optimal static controller, $u^{\mathrm{c}\star}$, in closed form.

6.5.2 Sub-Optimal Static Controller

When the horizon of the problem is small enough, we can obtain a static controller in closed form. To this end, consider the first order Taylor expansion of (6.8)

$$e^{(AB-U^{\mathrm{c}})t} p_0 = p_0 + t(AB - U^{\mathrm{c}})p_0 + \mathcal{O}\left(t^2\right).$$

The objective function, up to second order, can then be written as

$$\hat{J}(u^{\mathrm{c}}) = Tc^T p_0 + \frac{T^2}{2} c^T (AB - U^{\mathrm{c}}))p_0 + T d^T u^{\mathrm{c}}.$$

Using this objective function, an alternative optimization problem for the designer is

$$\min_{u^{\mathrm{c}} \in W} \quad \hat{J}(u^{\mathrm{c}}).$$

The solution of this problem can be readily obtained, and it is given by

$$\hat{u}_i^{\mathrm{c}\star} \quad = \quad \begin{cases} \overline{u}, & d_i < \frac{T}{2} c_i p_i(0) \\ \underline{u}, & d_i > \frac{T}{2} c_i p_i(0) \\ \{\underline{u}, \overline{u}\}, & \text{otherwise} \end{cases} .$$

6.5.3 Performance Comparison

We will now compare the performances of the optimal dynamic controller $u^\star$, the optimal static controller $u^{c\star}$, and the sub-optimal static controller $\hat{u}^{c\star}$ for different graphs. Let $p_0 = 0.5\mathbf{1}$, $T = 1$, $c = 5\mathbf{1}$, and $d = \mathbf{1}$. Also, let $B = I$, $a_{ij} = 1$ for all $(i, j) \in \mathcal{E}$, and $\underline{u} = 0.1$. Depending on the graph, the value of $\overline{u}$ is chosen to ensure that $\mathcal{R}_o$ is satisfied, and hence that the all-healthy state is GAS as per Proposition 5.2.

Figure 6.5 compares the cost incurred by the three control laws for a path graph with a varying number of nodes. The maximum vaccination level was fixed at $\overline{u} = 2$. As expected, the dynamic controller achieves the best performance, while the static controllers incur the same cost. A similar scenario arises over a cycle graph as shown in Fig. 6.6, while the performances of all three control laws are almost identical over a star graph as shown in Fig. 6.7.

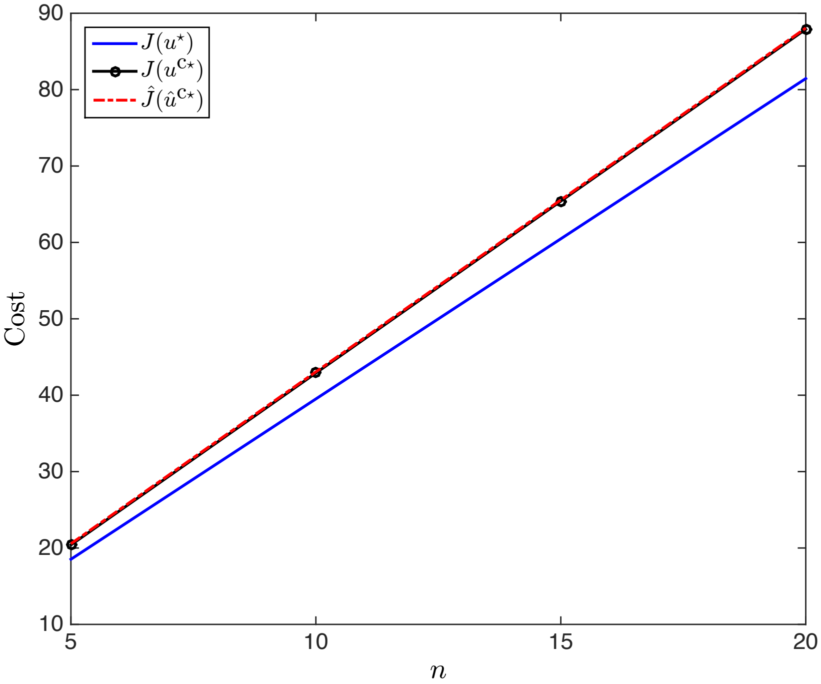


Figure 6.5: Costs incurred by dynamic and static control laws over a path graph.

Figure 6.8 illustrates the performances of each of the three control laws over a complete graph with a varying number of nodes. The value of $\overline{u}$ was chosen to be 23. Unlike the above experiments, the performance gap

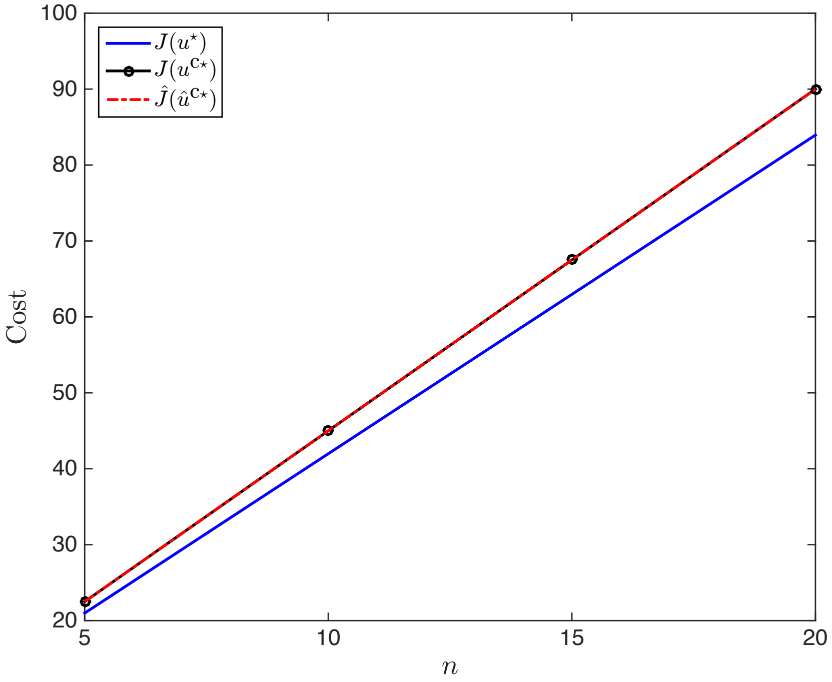


Figure 6.6: Costs incurred by dynamic and static control laws over a cycle graph.

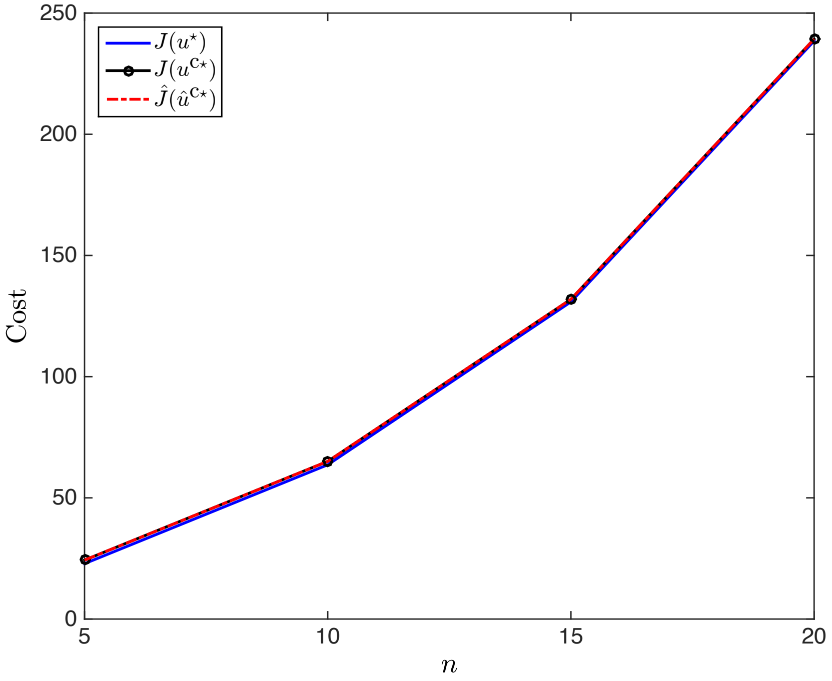


Figure 6.7: Costs incurred by dynamic and static control laws over a star graph.

between the dynamic controller and the static ones is quite large. Further, the performance gap between the optimal and the sub-optimal static controllers is also large.

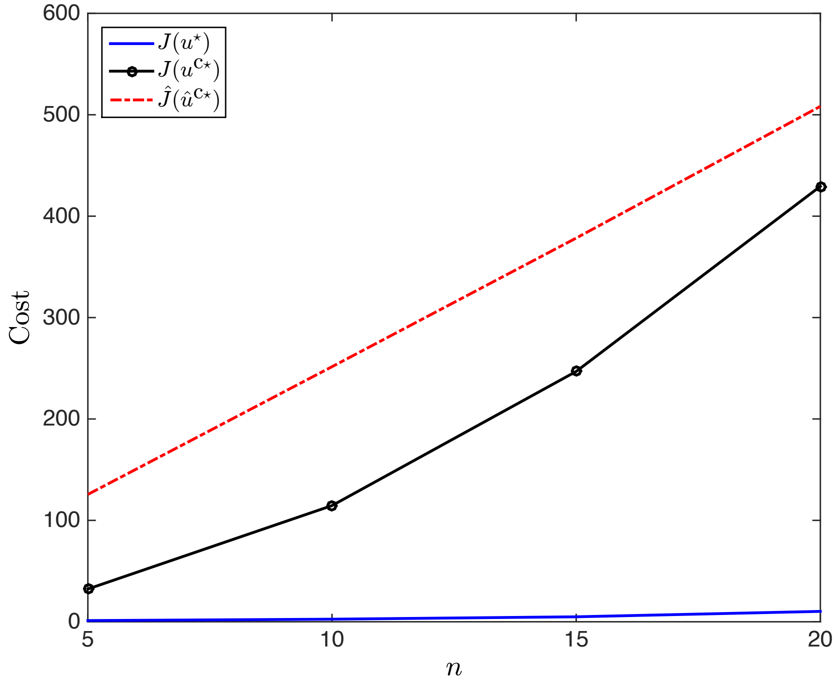


Figure 6.8: Costs incurred by dynamic and static control laws over a complete graph.

Remark 6.2. *The performances of the dynamic and static control laws is comparable over* sparse *networks such as paths and stars. However, the performance gap becomes significant in graphs with many connections as in the complete graph case. It is worth noting that the sub-optimal static controller* $\hat{u}^{c\star}$ *achieved a performance comparable to that of the optimal static controller* $u^{c\star}$*, although in the derivation of* $\hat{u}^{c\star}$ *we relied on second-order terms only. Future work will focus on characterizing these gaps analytically.* •

6.6 Linear Transformation

The fact that the system we are studying is affine in controls makes the canonical equations provided by the MP intractable. As a sub-optimal approach, consider instead the following linear-in-control dynamics:

$$\dot{p}_i = -\delta_i p_i + \sum_{j \neq i} a_{ij}\beta_j p_j - u_i,$$

or in matrix form

$$\dot{p} = (AB - D)p - u. \tag{6.9}$$

Note that the dynamics in (6.9) provide a lower bound to those in (5.1). Hence, using the comparison lemma, the cost associated with the state in (6.9) provides a lower bound for the cost associated with the state in (5.1). Moreover, similar transformations have been studied in the literature, where the control input u_i is interpreted as extra vaccination provided to node $i \in \mathcal{V}$ [39]. The problem we want to solve now is

$$\begin{aligned} \inf_{u \in \mathcal{U}} \quad & J(u) \\ \text{subject to} \quad & \dot{p} = (AB - D)p - u, \\ & 0 \leq p_i \leq 1, \quad \forall i \in \mathcal{V}, \\ & \underline{u} \leq u_i \leq \overline{u}, \quad \forall i \in \mathcal{V}, \end{aligned}$$

where we have added a constraint on the state p_i to ensure that it is a valid probability of infection, for all $i \in \mathcal{V}$, and all $t \in \mathbb{R}_{\geq 0}$. The following theorem provides the solution to this problem.

Theorem 6.1. *For all $i \in \mathcal{V}$, the optimal controller is given by*

$$u_i^\star \;=\; \begin{cases} \overline{u}, & d_i - q_i^\star < 0 \\ \underline{u}, & d_i - q_i^\star > 0 \\ \{\underline{u}, \overline{u}\}, & \textit{otherwise} \end{cases} . \tag{6.10}$$

For all $i \in \mathcal{V}$, the optimal controller $u_i^\star$ switches at most once. If $u_i^\star$ exhibits a switch at time $t^\star \in \mathbb{R}_{\geq 0}$, then $u_i^\star = \overline{u}$ for $t \leq t^\star$, and $u_i^\star = \underline{u}$ for $t > t^\star$.

Proof. The Hamiltonian in this case becomes

$$H(p, q, u) = (c - \lambda)^T p + (d - q)^T u + q^T (AB - D)p,$$

where λ is the Lagrange multiplier associated with the positivity constraint on p. From the Hamiltonian minimization condition provided by the MP, we conclude that the optimal controller is as claimed. In order to completely characterize the optimal controller, we must find λ. By complementary slackness, when $p_i > 0$, we have $\lambda_i = 0$. When, $p_i = 0$, we must have $\lambda_i > 0$. However, optimality dictates that $u_i = \underline{u}$ when $p_i = 0$; hence, we do not need to find the explicit value of λ_i in this case.

To determine the switching behavior, we study the costate equation. The costate equation is given by

$$\dot{q} = -(AB - D)^T q - c + \lambda, \quad q(T) = 0,$$

whose solution is

$$q = \int_t^T e^{-(AB-D)^T(t-\tau)}(\lambda(\tau) - c)d\tau.$$

Note that q is independent of d. When $d_i = 0$ for some $i \in \mathcal{V}$, i.e., there is no cost on control, optimality dictates that we must have $u_i^\star = \overline{u}$. This implies that $q_i^\star > 0$ for all $t \in [0, T)$. Using the terminal condition $q_i(T) = 0$, and since i was arbitrary, we conclude that q_i is nonnegative for all $t \in [0, T]$, for all $i \in \mathcal{V}$. Further, from the structure of q, we conclude that $q_i - d_i$ can become zero at most once. Also, from the terminal condition and the continuity of q, it follows that there is an $\epsilon > 0$ such that $q(t) = 0$ for all $t \in [T - \epsilon, T]$, and therefore (6.10) implies that $u_i(t) = \underline{u}$ for all $t \in [T - \epsilon, T]$. Using these facts, we conclude that if u_i exhibits a switch at time $t^\star \in \mathbb{R}_{\geq 0}$, then $u_i = \overline{u}$ for $t \leq t^\star$, and $u_i = \underline{u}$ for $t > t^\star$ as claimed. □

Remark 6.3. *The fact that the optimal controllers in* (6.5) *and* (6.10) *switch to* $\underline{u}$ *towards the end of the horizon demonstrates that applying high curing rates across the entire horizon of the problem is not required. Most of the current approaches in the literature require applying constant curing rates across the entire horizon of the problem; the optimal control framework we provide here (based on linear approximation) and the one provided in Section 6.4 prove that this is in fact wasteful, and one can switch to low curing rates once the nodes start approaching the all-healthy state.* •

In Fig. 6.9, we repeat the same experiment, but we implement the optimal controller provided in Theorem 6.1. We observe similar behavior as that shown in Fig. 6.4; however, each control input switches at most once as shown.

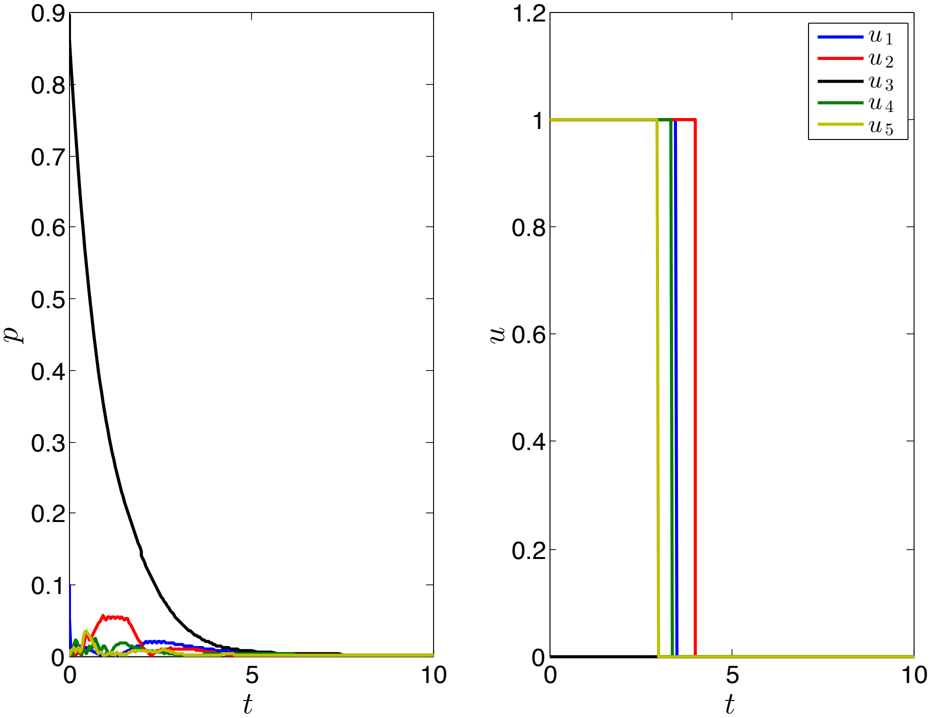


Figure 6.9: A demonstration of the optimal controller provided by Theorem 6.1.

6.7 Summary

We derived sufficient conditions for stabilizing the all-healthy state for a class of graphs using a limited number of controllers. We compared the performances of constant and nonlinear controllers. Further, we formulated the infection diffusion and control of curing in networks as an optimal control problem and studied the switching behavior of the optimal controller. We proposed two static control laws: one is based on optimizing the vaccination levels at time zero, and the other one is based on a second-order expansion of the objective function. We compared the performances of the dynamic and static controllers and identified graph classes over which the three control laws exhibit comparable performances. Finally, we studied an optimal control problem subject to a transformed version of the n-intertwined dynamics and showed that the optimal controller in this case exhibits at most one switch. The optimal controller was shown to switch to the lowest possible vaccination level when the nodes approach the all-healthy state, which demonstrates that high levels of vaccination are not required across the entire time horizon of the problem as previously assumed in the literature.

CHAPTER 7

OPEN PROBLEMS

The problems we have studied in the previous chapters point to many unexplored avenues in the area of control of spread of information. We identify several potential directions below.

Chapters 2 & 3

- We have assumed that the players know the state and the network topology completely. An interesting line of research is to derive the optimal strategies when the knowledge of the players about the state and the topology is restricted.
- When applying necessary conditions for optimality, e.g., the MP, to the min–max or the max–min problem, one must first prove the existence of optimal controllers. Such results can be viewed as existence results for equilibria in the general framework of Stackelberg games, which are not available in the literature.
- Other future directions include: removing Assumption 3.1 and showing that Zeno behavior can be ruled out in optimality, formulating the problems in discrete time, and deriving the optimal randomized strategies for both players.

Chapter 4

- We have assumed that the adversary has already acted on the network and introduced a modeling uncertainty, and as a result, the system matrix was time-invariant. The logical next step would be to allow the system to be time-varying, which would translate to the scenario where the adversary is continuously attacking the network. Modeling the adversary as a switching signal, and characterizing the worst-case switching behavior would add a robustness notion to this problem.

- Making Condition 4.2 less strict would strengthen the framework. The problem can also be generalized to tracking of a reference model with a reference input signal, instead of set-point tracking.

- In the distributed supervisory control framework, game-theoretic notions can be applied to this problem in order to design incentive schemes to ensure that the majority of the agents identify the underlying network.

Chapters 5 & 6

- We identified the number of controllers required to stabilize a class of undirected graphs. Finding the optimal set of nodes to control in order to stabilize the network is the next logical step. However, this problem might be NP-hard. An alternative approach in this case would be to construct a polynomial-time algorithm that provides near-optimal solutions.

- The effort towards stabilizing an infected network using a limited number of controllers that are *bounded* starts by quantifying the amount of control needed to stabilize the network. The first step required to solve this problem is to find conditions under which a single controller can make the closed-loop system stable, when arbitrarily small self-loops are implemented at the remaining nodes.

- When stabilizing the all-healthy state is not possible, finding control laws capable of minimizing the probability of infection at the endemic state is an important development.

- Studying the effect of malicious nodes on the evolution of probability of infections is an important future direction. In such scenarios, a game-theoretic approach would be appropriate, and different competitions between friendly and malicious nodes can be formulated.

- We have focused on the n-intertwined model in our study of infected networks. However, the proof methods we developed can be applied to other, more general, epidemiological models.

CHAPTER 8

CONCLUSION

In this thesis, we focused on designing optimal and stabilizing controllers for the purpose of controlling spread of information in networks. We considered two models to describe information spread: linear distributed averaging and the n-intertwined model. Designing controllers with practical constraints was the main feature of our designs for both dynamical models.

For distributed averaging networks, we considered two types of adversarial attacks. Both attacks have the common objective of slowing down the convergence of the computation at the nodes to the global average. We introduced a network designer whose objective is to assist the nodes in reaching consensus by countering the attacks of the adversary. ATTACK-I involves an adversary and a network designer who are capable of targeting links. We have formulated and solved two problems that capture the competition between the players in this attack.

We considered practical models for the players by constraining their actions along the problem horizon. The derived strategies were shown to exhibit a low worst-case complexity. We also proved that the optimal strategies admit a potential-theoretic analogy. Finally, we showed that when the link weights are sufficiently diverse, an SPE exists for the zero-sum game between the designer and the adversary.

ATTACK-II, on the other hand, involves an adversary and a network designer who are able to modify the values of the nodes by injecting signals of bounded power and energy. We utilized the maximum principle to completely characterize the optimal strategies of the players and showed that an SPE exists in this case.

When the adversary introduces a large modeling uncertainty in the system, we have proposed a distributed mechanism for the agents to stabilize the network. We extended the classical centralized supervisory control framework to a distributed setting, and we provided sufficient conditions for the nodes to achieve set-point tracking by relying on local information only, and without requiring the individual agents to have explicit knowledge of this set-point. This is particularly useful for distributed computation, distributed optimization, and synchronization problems, where agents use local information in order to compute quantities that are unknown to them a priori.

For infected networks, we borrowed tools from positive systems theory to characterize the stability properties of the n-intertwined Markov model over arbitrary networks. For strongly connected digraphs, we proved that when the basic reproduction number is less than or equal to 1, the all-healthy state is GAS. When the basic reproduction number is greater than 1, we proved that the endemic state is globally asymptotically stable, and that locally around this equilibrium, the convergence is exponentially fast. Furthermore, we studied the stability properties of weakly connected graphs, and we showed that a weak endemic state could emerge over such networks. By viewing weakly connected graphs as a cascade of nonlinear systems, and establishing input-to-state stability for those systems, we proved the global asymptotic stability of the equilibria that emerge over such graphs.

Moreover, we have proposed a dynamical model that describes the interaction among nodes in an infected network as a concave game and demonstrated that it subsumes the n-intertwined Markov model. This alternative description provides a new condition, which can be checked collectively by agents, for the stability of the origin. We have also formulated multiple control design questions over infected networks. In particular, we provided sufficient conditions for stabilizing various networks by controlling a limited number of nodes. Further, we have proposed an optimal control framework that allows a network designer to minimize the total infection in the network at minimal cost.

This thesis serves as a demonstration of control and game theoretic questions that arise in the area of control of spread of information over networks. We have studied various problems, developed solution methodologies, and highlighted that this new emerging area leads to interesting theoretical explorations. We have also identified open problems for research in the longer term.

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