総合論文

クオンテイルと完備べき等左半環の表現定理

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Representation Theorems for Quantales and Complete Idempotent Left Semirings

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１．Introduction

This paper shows representation theorems for quantales and complete idempotent left semirings.

Quantales were introduced by Mulvey(1) in order to provide a constructive formulation of foundations of quantum mechanics. They are complete join semilattices together with a monoid structure satisfying the distributive laws. In the literature, they are also known as complete idempotent semirings or standard Kleene algebras(2).

There is a relational quantale whose elements are binary relations on a set, whose order is given by inclusion, and whose monoid structure is given by relational composition and the identity relation. Relational quantales play an important role in computer science. For example, they are models for the semantics of non-deterministic while-programs(3,4), they also provides a sound and complete class of models for linear intuitionistic logic(5), and so on.

In Stone's representation theorem for Boolean algebra or Priestley's representation theorem for bounded distributive lattices, a powerset is regarded as a standard Boolean algebra or a standard bounded distributive lattice(6,7). On the other hand, since a relational quantale has been regarded as a `standard quantale', some results called `relational representation theorem for quantales' have been shown in the literature.

However, relational quantales in their results are not equal to the standard relational quantales.

For example, Valentini(8) shows that a quantale Q is isomorphic to a sub-quantale of the quantale whose elements are binary relations on Q. However, the order of the quantale is not given by inclusion, but the opposite order of inclusion.

Brown and Gurr(9) show that a completely coprime algebraic quantale Q is isomorphic to a sub-quantale of the quantale whose elements are binary relations on Q and whose order is given by inclusion. However, the unit of the monoid structure of the quantale is not equal to the identity relation.

Palmigiano and Re(10) give a sufficient condition for a quantale to be isomorphic to a sub-quantale of the quantale whose elements are binary relations on a set and whose order is given by inclusion and whose monoid structure is given by relational composition and the identity relation. Indeed, this result means `relational representation theorem'. It is important point to embed a quantale Q in the relations not on Q but on the set of all atoms of Q.

However, the result given in the paper(10) is a relational representation theorem for `unital involutive quantales'. A quantale Q is involutive if it is endowed with a unary operation f. such that, for all $a, b \in Q$ and every $S \subseteq Q$,

1. $f(f(a)) = a$,
2. $f(a \cdot b) = f(b) \cdot f(a)$,
3. $f(\bigvee S) = \{f(a) \mid a \in S\}$.

This paper gives a relational representation theorem for quantales which are not in general involutive. Similarly to the papers(10,11), our representation theorem shows that a quantale Q satisfying some condition is isomorphic to a sub-quantale of the quantale whose elements are binary relations on the set of all atoms of Q (12,13).

The main theorem Theorem 4 of this paper says that for a quantale Q, the following are equivalent.

1. Q has a relational representation in our way and it is CCP-invertible.

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2. Q is isomorphic to $\wp(X)$ as complete join semilattice for some set X.
3. Q is atom-algebraic and it is a frame.
4. Q is atom-algebraic and its atoms are completely coprime.
5. Q is completely coprime algebraic and its completely coprime elements are atoms.
6. Q is is completely coprime algebraic and the order of its completely coprime elements is discrete.

This theorem asserts that a quantale has a relational representation, if it is isomorphic to a powerset as complete join semilattice. A powerset quantale has four other equivalent conditions. The notion of a CCP-invertible quantale is defined in this paper. When a quantale is CCP-invertible, it has a relational representation in our way if and only if it is isomorphic to a powerset as complete join semilattice.

Powerset quantales are examples of `completely coprime algebraic quantales' which is the sufficient condition given in the paper[9]. However, our result is not an application of the result in the paper[9] to powerset quantales, since our representation theorem embeds a quantale Q in the relations not on Q but on the set of all atoms of Q.

Similarly to quantales, we also show a representation theorem for powerset complete idempotent left semirings[14]. Complete idempotent left semirings are a relaxation of quantales by giving up strictness and distributivity of composition over arbitrary joins from the left. We show that powerset complete idempotent left semirings can be represented not only by relations but also by multirelations.

This paper is organized as follows. Section 2 defines completely coprime algebraic quantales and Section 3 defines powerset quantales. Section 4 shows that a quantale has a relational representation, if it is isomorphic to a powerset as complete join semilattice. Section 5 shows that a quantale satisfies the condition, if it has a relational representation in our way and it is CCP-invertible. Section 6 shows a multirelational representation theorem for powerset complete idempotent left semirings. Section 7 summarizes this work and discusses future work.

２．Completely Coprime Algebraic Quantales

In this section, we recall the notion of completely coprime algebraic quantales and show some examples. A quantale is called completely coprime algebraic depending only on its underlying complete join semilattice structure. The terminology in this section relies on the paper[9].

Definition 1 (complete join semilattice).

A complete join semilattice *is a tuple* $(K, \leq, \bigvee)$ *with the following properties:*

1. $(K, \leq)$ *is a partially ordered set.*
2. $\bigvee S$ *is the join (i.e., the least upper bound) for each subset* S *of* K.

A complete join semilattice must have the least element, which is the join of the empty subset. We write $\bot$ for it.

A complete join semilattice must be a complete lattice, since the meet of a subset S is the join of all lower bounds of S. We write $a \wedge b$ for the meet of $\{a, b\}$.

Definition 2.

An element x *of a complete join semilattice* Q *is called* completely coprime (or completely join-prime)*, if*

$$x \leq \bigvee S \Leftrightarrow \exists a \in S. x \leq a$$

for each subset S *of* Q.

We write $\mathbf{CCP}(Q)$ for the set of all completely coprime elements of Q.

Remark 1.

$\bot$ is not completely coprime, since $\bot = \bigvee \emptyset$.

Definition 3 (completely coprime algebraic).

A complete join semilattice Q *is called* completely coprime algebraic (or CCPA) *if for each* $a \in Q$*, the following equation holds.*

$$a = \bigvee \{ x \in \mathbf{CCP}(Q) \mid x \leq a \}$$

Definition 4 (quantale (or complete idempotent semiring)).

A quantale *is a tuple* $(K, \leq, \bigvee, \cdot, 1)$ *with the following properties:*

1. $(K, \cdot, 1)$ *is a monoid.*
2. $(K, \leq, \bigvee)$ *is a complete join semilattice.*
3. $(\bigvee S) \cdot a = \bigvee\{b \cdot a \mid b \in S\}$ *for each element* a *and each subset* S *of* K.
4. $a \cdot (\bigvee S) = \bigvee\{a \cdot b \mid b \in S\}$ *for each element* a *and each subset* S *of* K.

A quantale $(K, \leq, \bigvee, \cdot, 1)$ is called completely coprime algebraic if $(K, \leq, \bigvee)$ is completely coprime algebraic.

Example 1.

For a set A, the tuple $\mathbf{Rel}(A) = (K, \leq, \bigvee, \cdot, 1)$ forms a quantale where

- K is the set of all binary relations on A,
- $\leq$ is the inclusion $\subseteq$,
- $\bigvee$ is the union operator $\cup$,

- $R \cdot Q$ is the composition of R and Q, and
- 1 is the identity (diagonal) relation on A.

Here, the composition $R \cdot Q$ is defined as follows.

$$(a, b) \in R \cdot Q \Leftrightarrow \exists c \in A. (a, c) \in R \text{ and } (c, b) \in Q)$$

A binary relation on A is completely coprime in $\mathbf{Rel}(A)$ if and only if it is a singleton subset of $A \times A$. $\mathbf{Rel}(A)$ is completely coprime algebraic, since for $R \in \mathbf{Rel}(A)$,

$$\begin{aligned} &\bigcup \{ Q \in \mathbf{CCP}(\mathbf{Rel}(A)) \mid Q \subseteq R \} \\ &= \bigcup \{ \{(a, a')\} \mid \{(a, a')\} \subseteq R \} \\ &= \{ (a, a') \mid (a, a') \in R \} \\ &= R \end{aligned}$$

■

The powerset of a monoid forms a quantale. This paper gives only two examples for monoids.

Example 2.

For a set Σ, the tuple $\wp(\Sigma^*) = (K, \leq, \vee, \cdot, 1)$ forms a quantale where

- K is the powerset of Σ^* where Σ^* is the set of all finite sequences of elements of Σ,
- $\leq$ is the inclusion $\subseteq$,
- $\vee$ is the union operator $\cup$,
- $R \cdot Q$ is $\{ \sigma\pi \mid \sigma \in R, \pi \in Q \}$, and
- 1 is the singleton set of the empty sequence on A.

A subset of Σ^* is completely coprime if and only if it is a singleton subset. $\wp(\Sigma^*)$ is completely coprime algebraic.

Example 3.

The tuple $\wp(\mathbb{Z}_2) = (K, \leq, \vee, \cdot, 1)$ forms a quantale where

- K is the powerset of the group $\mathbb{Z}_2 = (\{0,1\}, +, 0, -)$,
- $\leq$ is the inclusion $\subseteq$,
- $\vee$ is the union operator $\cup$,
- $A \cdot B$ is $\{ a + b \mid a \in A, b \in B \}$, and
- 1 is the set $\{0\}$.

A subset of $\mathbb{Z}_2$ is completely coprime if and only if it is a singleton subset. This quantale is completely coprime algebraic.

Example 4.

The tuple $\mathbb{N} \cup \{\omega\} = (K, \leq, \vee, \cdot, 1)$ forms a quantale where

- K is the set of all natural numbers and the additional element ω,
- $a \leq b$ if and only if $a = \omega$, $a = b$, or a is a natural number greater than b,
- $\vee S$ is the minimum number of S except for $\vee\{\omega\} = \vee \emptyset = \omega$,
- $a \cdot b$ is $a + b$ except for $\omega \cdot a = a \cdot \omega = \omega$, and
- 1 is the zero number.

This quantale is completely coprime algebraic and $\mathbf{CCP}(\mathbb{N} \cup \{\omega\}) = \mathbb{N}$.

3. Powerset Quantales

In this section, we recall the notion of atom and define the notion of atom-algebraic. We compare a powerset semilattice with the four conditions based on completely coprime elements or atoms. Finally, we define the notion of powerset quantale.

Definition 5 (atom).

An atom *of a complete join semilattice Q is an element x with the following properties:*

1. $x \neq \perp$.
2. $a < x$ *implies* $a = \perp$.

We write $\mathbf{Atom}(Q)$ for the set of all atoms of Q.

Definition 6 (atom-algebraic).

A complete join semilattice Q is called atom-algebraic *if for each $a \in Q$, the following holds.*

$$a = \bigvee \{ x \in \mathbf{Atom}(Q) \mid x \leq a \}$$

We also recall the notion of frame.

Definition 7 (frame).

A complete join semilattice Q is called a frame *if*

$$a \wedge \bigvee S = \bigvee \{ a \wedge s \mid s \in S \}$$

for each element a and each subset S of Q.

Example 5.

In $\mathbf{Rel}(A)$ of Example 1, $\wp(\Sigma^*)$ of Example 2, and $\wp(\mathbb{Z}_2)$ of Example 3, an element is an atom if and only if it is a singleton subset. They are atom-algebraic and they are frames.

Theorem 1.

For a complete join semilattice Q, the following are equivalent.

1. *Q is isomorphic to $\wp(X)$ for some set X.*
2. *Q is atom-algebraic and it is a frame.*
3. *Q is atom-algebraic and its atoms are completely coprime.*
4. *Q is completely coprime algebraic and its completely coprime elements are atoms.*
5. *Q is completely coprime algebraic and the order of its completely coprime elements is discrete.*

Proof.

$(1 \Rightarrow 2)$

A subset Y of X is an atom in $\wp(X)$ if and only if Y is a singleton subset. $\wp(X)$ is an atom-algebraic frame.

(2 ⇒ 3)

Let us show that an atom x of a frame Q is completely coprime, that is,

$$x \leq \bigvee S \Leftrightarrow \exists s \in S. x \leq s$$

for each subset S of Q. RHS implies LHS, since $x \leq s \leq \bigvee S$. To show that LHS implies RHS, assume that $x \leq \bigvee S$. Since Q is a frame and $x \leq \bigvee S$, we have $x = x \wedge \bigvee S = \bigvee\{ x \wedge s \mid s \in S \}$. Since x is an atom, we have $\bigvee\{ x \wedge s \mid s \in S \} \neq \bot$. Therefore, there exists $s \in S$ satisfying $x \wedge s \neq \bot$. Since x is an atom and $\bot \neq x \wedge s \leq x$, we have $x \wedge s = x$. Therefore, we have $x \leq s$.

(3 ⇒ 4)

Let Q be atom-algebraic and assume that its atoms are completely coprime. Q is completely coprime algebraic, since the following holds.

$$a = \bigvee\{ x \in \mathbf{Atom}(Q) \mid x \leq a \}$$

$$\leq \bigvee\{ x \in \mathbf{CCP}(Q) \mid x \leq a \}$$

$$\leq a$$

Let a be completely coprime. The following holds.

$a \leq a$

$\Leftrightarrow a = \bigvee\{ x \in \mathbf{Atom}(Q) \mid x \leq a \}$

$\Leftrightarrow \exists x \in \mathbf{Atom}(Q). a \leq x \leq a$

$\Leftrightarrow a \in \mathbf{Atom}(Q)$

Therefore, a is an atom.

(4 ⇒ 5)

Let Q be completely coprime algebraic and assume that its completely coprime elements are atoms. Assume that there are completely coprime elements a, b satisfying $a \leq b$ and $a \neq b$. Since b is also an atom, $a = \bot$. But since a is also an atom, $a \neq \bot$. It is a contradiction. Therefore, the order of completely coprime elements is discrete.

(5 ⇒ 1)

Let Q be completely coprime algebraic and assume that the order of its completely coprime elements is discrete. Let f be a function $f : Q \to \wp(\mathbf{CCP}(Q))$ such that $f(a) = \{x \in \mathbf{CCP}(Q) \mid x \leq a\}$. Let g be a function $g : \wp(\mathbf{CCP}(Q)) \longrightarrow Q$ such that $g(S) = \bigvee S$. We have $a = g(f(a))$ for all $a \in Q$, since $g(f(a)) = \bigvee\{ x \in \mathbf{CCP}(Q) \mid x \leq a \}$ and Q is completely coprime algebraic. Since the order of $\mathbf{CCP}(Q)$ is discrete, an arbitrary subset of $\mathbf{CCP}(Q)$ is down-closed. Therefore, $Y \subseteq \mathbf{CCP}(Q)$ satisfies

$f(g(Y))$

$= \{x \in \mathbf{CCP}(Q) \mid x \leq \bigvee Y\}$

$= \{x \in \mathbf{CCP}(Q) \mid \exists y \in Y. x \leq y\}$

$= Y.$

■

When these conditions are satisfied by a quantale, we call it a *powerset quantale*. Every powerset quantale Q satisfies $\mathbf{CCP}(Q) = \mathbf{Atom}(Q)$.

Example 6.

$\mathbf{Rel}(A)$ in Example 1, $\wp(\Sigma^*)$ in Example 2, and $\wp(\mathbb{Z}_2)$ in Example 3 are powerset quantales.

We also give an example of completely coprime algebraic quantale which is not a powerset quantale.

Example 7.

$\mathbb{N} \cup \{\omega\}$ in Example 4 is completely coprime algebraic and it is a frame. However, it has no atoms. Therefore, it is not a powerset quantale.

４．Representation Theorem for Powerset Quantales

This section shows that a powerset quantale has a relational representation.

Theorem 2.

Let $(Q, \leq, \bigvee, \cdot, 1)$ be a quantale. If Q is a powerset quantale, then the following function $\eta : Q \to \mathbf{Rel}(\mathbf{CCP}(Q))$ is an injective homomorphism of quantales.

$$\eta(a) = \{ (x, y) \mid x \in \mathbf{CCP}(Q), y \in \mathbf{CCP}(Q), x \leq a \cdot y\}$$

Proof.

By Theorem 1, Q is completely coprime algebraic and the order of its completely coprime elements is discrete.

(η preserves joins)

$(x, y) \in \eta(\bigvee S)$

$\Leftrightarrow x \leq (\bigvee S) \cdot y$

$\Leftrightarrow x \leq \bigvee\{ a \cdot y \mid a \in S \}$

$\Leftrightarrow \exists a \in S. x \leq a \cdot y$ (by $x \in \mathbf{CCP}(Q)$)

$\Leftrightarrow \exists a \in S. (x, y) \in \eta(a)$

$$\Leftrightarrow (x, y) \in \bigcup\{ \eta(a) \mid a \in S \}.$$

(η preserves $\cdot$)

$(x, y) \in \eta(a \cdot a')$

$\Leftrightarrow x \leq a \cdot a' \cdot y$

$\Leftrightarrow x \leq a \cdot \bigvee\{ z \in \mathbf{CCP}(Q) \mid z \leq a' \cdot y \}$

$\Leftrightarrow x \leq \bigvee\{ a \cdot z \mid z \in \mathbf{CCP}(Q), z \leq a' \cdot y \}$

$\Leftrightarrow \exists z \in \mathbf{CCP}(Q). x \leq a \cdot z$ and $z \leq a' \cdot y$

$\Leftrightarrow \exists z \in \mathbf{CCP}(Q). (x,z) \in \eta(a)$ and $(z,y) \in \eta(a')$

(η preserves 1)

$(x,y) \in \eta(1)$

$\Leftrightarrow x \leq 1 \cdot y$

$\Leftrightarrow x \leq y$

$\Leftrightarrow x = y$ (since $\mathbf{CCP}(Q)$ is discrete)

(η is injective)

For all $a \in Q$, we have the following equation.

a

$= a \cdot 1$

$= \bigvee\{ x \in \mathbf{CCP}(Q) \mid x \leq a \cdot 1 \}$

$= \bigvee\{ x \in \mathbf{CCP}(Q) \mid x \leq a \cdot \bigvee\{y \in \mathbf{CCP}(Q) \mid y \leq 1\} \}$

$= \bigvee\{x \in \mathbf{CCP}(Q) \mid x \leq \bigvee\{a \cdot y \mid y \in \mathbf{CCP}(Q), y \leq 1\}\}$

$= \bigvee\{ x \in \mathbf{CCP}(Q) \mid \exists y \in \mathbf{CCP}(Q). x \leq a \cdot y, y \leq 1 \}$

$= \bigvee\{x \in \mathbf{CCP}(Q) \mid \exists y. (x,y) \in \eta(a), y \leq 1\}$

Here, if $a, a' \in Q$ satisfy $\eta(a) \subseteq \eta(a')$, then we have $a \leq a'$ as follows.

a

$= \bigvee\{x \in \mathbf{CCP}(Q) \mid \exists y. (x,y) \in \eta(a), y \leq 1\}$

$\leq \bigvee\{x \in \mathbf{CCP}(Q) \mid \exists y. (x,y) \in \eta(a'), y \leq 1\}$

$= a'$

■

Since the above function η is an injective homomorphism, the image of Q by η is isomorphic to Q and it is a sub-quantale of $\mathbf{Rel}(\mathbf{CCP}(Q))$.

We give some examples.

Example 8.

$\mathbf{Rel}(A)$ in Example 1 is a powerset quantale. Therefore, by Theorem 2, $\mathbf{Rel}(A)$ has a relational representation. The injective map η from $\mathbf{Rel}(A)$ to $\mathbf{Rel}(\mathbf{CCP}(\mathbf{Rel}(A)))$ is given as follows.

$$\eta(R) = \{ (P,Q) \mid P,Q \in \mathbf{CCP}(\mathbf{Rel}(A)), P \leq R \cdot Q \}$$

Since $\mathbf{CCP}(\mathbf{Rel}(A))$ is the set of singleton subsets of $A \times A$, η can be also given as follows.

$\eta(R)$

$= \{(\{(s,t)\},\{(u,v)\}) \mid s,t,u,v \in A, (s,t) \in R \cdot \{(u,v)\}\}$

$= \{(\{(s,t)\},\{(u,t)\}) \mid s,t,u \in A, (s,t) \in R \cdot \{(u,t)\}\}$

$= \{(\{(s,t)\},\{(u,t)\}) \mid t \in A, (s,u) \in R\}$

Example 9.

Similarly, $\wp(\Sigma^*)$ in Example 2 has a relational representation. The map η is injective from $\wp(\Sigma^*)$ to $\mathbf{Rel}(\mathbf{CCP}(\wp(\Sigma^*))) \cong \mathbf{Rel}(\Sigma^*)$. The map η is given as follows.

$$\eta(R) = \{ (P,Q) \mid P,Q \in \mathbf{CCP}(\wp(\Sigma^*)), P \leq R \cdot Q \}$$

Since $\mathbf{CCP}(\wp(\Sigma^*))$ is the set of singleton subsets of Σ^*, η can be also given as follows.

$\eta(R) = \{ (\{\sigma\},\{\pi\}) \mid \sigma,\pi \in \Sigma^*, \{\sigma\} \subseteq R \cdot \{\pi\} \}$

$= \{ (\{\sigma\},\{\pi\}) \mid \sigma,\pi \in \Sigma^*, \sigma \in R \cdot \{\pi\} \}$

$= \{ (\{\tau\pi\},\{\pi\}) \mid \pi \in \Sigma^*, \tau \in R \}$

Example 10.

Similarly, $\wp(\mathbb{Z}_2)$ in Example 3 has a relational representation. The injective map η from $\wp(\mathbb{Z}_2)$ to $\mathbf{Rel}(\mathbf{CCP}(\wp(\mathbb{Z}_2)))$ is given as follows.

$$\eta(A) = \{ (B,C) \mid B,C \in \mathbf{CCP}(\wp(\mathbb{Z}_2)), B \leq A \cdot C \}$$

Since $\mathbf{CCP}(\wp(\mathbb{Z}_2))$ is the set of singleton subsets of $\mathbb{Z}_2$, η can be also given as follows.

$\eta(A) = \{ (\{b\},\{c\}) \mid b,c \in \mathbb{Z}_2, \{b\} \subseteq A \cdot \{c\} \}$

$= \{ (\{b\},\{c\}) \mid b,c \in \mathbb{Z}_2, b \in A \cdot \{c\} \}$

$= \{ (\{a+c\},\{c\}) \mid c \in \mathbb{Z}_2, a \in A \}$

Example 11.

For a frame $(Q, \leq, \bigvee)$, the tuple $(Q, \leq, \bigvee, \wedge, \mathrm{T})$ is a quantale where T is the greatest element. If it is also atom-algebraic, then it has a relational representation by Theorem 2 and Theorem 1.

５．CCP-invertible Quantale

Section 4 shows that a powerset quantale has a relational representation. Conversely, if a quantale has a relational representation in the same way as Section 4, is it then a powerset quantale? The answer is 'Yes', if it is CCP-invertible.

Definition 8 (CCP-invertible).

A quantale Q is called CCP-invertible*, if for all $x, y \in \mathbf{CCP}(Q)$, for all $a \in Q$, the following holds.*

$$x \leq a \cdot y \Leftrightarrow \exists z \in \mathbf{CCP}(Q). x \leq z \cdot y \text{ and } z \leq a$$

Theorem 3.

Let $(Q, \leq, \bigvee, \cdot, 1)$ be a quantale. The following are equivalent.

1. The following function $\eta : Q \to \mathbf{Rel}(\mathbf{CCP}(Q))$
$\eta(a) = \{(x, y) \mid x, y \in \mathbf{CCP}(Q)), x \leq a \cdot y\}$
is an injective homomorphism of quantales and Q is CCP-invertible.

2. Q is a powerset quantale.

Proof.

$(1 \Longrightarrow 2)$

Let a be an element of Q. Since η is a homomorphism of quantales and Q is CCP-invertible, we have the following diagram.

$\eta(\bigvee\{z \in \mathbf{CCP}(Q) \mid z \leq a\})$

$= \bigcup\{\eta(z) \mid z \in \mathbf{CCP}(Q), z \leq a\}$

$= \{(x, y) \mid \exists z \in \mathbf{CCP}(Q), (x, y) \in \eta(z), z \leq a\}$

$= \{(x, y) \mid x, y \in \mathbf{CCP}(Q), \exists z \in \mathbf{CCP}(Q), x \leq z \cdot y, z \leq a\}$

$= \{(x, y) \mid x, y \in \mathbf{CCP}(Q), x \leq a \cdot y\}$

$= \eta(a)$

Moreover, since η is injective, we have

$$\mathrm{a} = \bigvee\{z \in \mathbf{CCP}(Q) \mid z \leq a\}$$

Therefore, Q is completely coprime algebraic.

Since η preserves 1, for $x, y \in \mathbf{CCP}(Q)$, $\mathrm{x} \leq \mathrm{y}$ if and only if $\mathrm{x} = \mathrm{y}$. Therefore, the order of completely coprime elements of Q is discrete.

$(2 \Longrightarrow 1)$

By Theorem 2, η is an injective homomorphism of quantales. Let a be an element of Q and x, y elements of $\mathbf{CCP}(Q)$. Since Q is completely coprime algebraic, we have

$x \leq a \cdot y$

$\Leftrightarrow x \leq \left(\bigvee\{z \in \mathbf{CCP}(Q) \mid z \leq a\}\right) \cdot y$

$\Leftrightarrow x \leq \bigvee\{z \cdot y \mid z \in \mathbf{CCP}(Q), z \leq a\}$

$\Leftrightarrow \exists z \in \mathbf{CCP}(Q). x \leq z \cdot y$ and $z \leq a$

Therefore, Q is CCP-invertible.

■

Example 12.

$\mathbf{Rel}(A)$ in Example 1 is CCP-invertible, since z in Definition 8 is given by $x \cdot y^{\circ}$ where y° is the opposite relation of y.

Example 13.

$\wp(\Sigma^*)$ in Example 2 is CCP-invertible, since z in Definition 8 is given by $\{\sigma\}$ where $x = \{\sigma\pi\}$ and $y = \{\pi\}$.

Example 14.

$\wp(\mathbb{Z}_2)$ in Example 3 is CCP-invertible, since z in Definition 8 is given by $z = \{s + t\} = \{s - t\}$ where $x = \{s\}$ and $y = \{t\}$.

Example 15.

$\mathbb{N} \cup \{\omega\}$ in Example 4 is CCP-invertible, since z in Definition 8 is given by $x - y$. However, the order of its completely coprime elements is not discrete. Therefore, by Theorem 3, η is not an injective homomorphism of quantales.

Example 16.

The ordered set of Fig 1 forms a quantale where $a \cdot b = a \wedge b$ except for $1 \cdot a = a \cdot 1 = a$. It is not CCP-invertible, since the set of its completely coprime elements is $\{s, t\}$ and $t \leq 1 \cdot t$ but $t \nleq s \cdot t$.

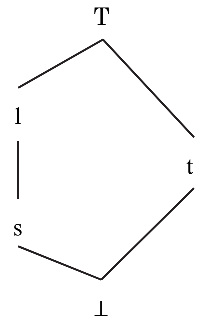

Fig.1. A quantale which is not CCP-invertible

Example 17.

The ordered set of Fig 2 forms a quantale where $a \cdot b = a \wedge b$ except for $1 \cdot a = a \cdot 1 = a$. This quantale is CCP-invertible. However, this quantale is not completely coprime algebraic, since the set of its completely coprime elements is $\{s, 1\}$, but $t \neq \bigvee\{s\}$. Therefore, by Theorem 3, η is not an injective homomorphism of quantales.

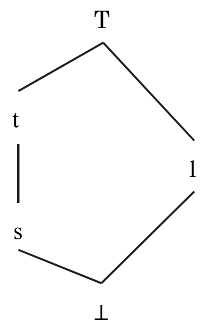

Fig. 2. A quantale which is CCP-invertible

Remark that there exist other relational representations of Example 17, for example, the following function $\eta' : Q \to \mathbf{Rel}(\{\alpha,\beta,\gamma\})$[9].

$\eta'(T) = \{(\alpha,\beta),(\alpha,\alpha),(\beta,\beta),(\gamma,\gamma)\}$
$\eta'(t) = \{(\alpha,\beta),(\alpha,\alpha),(\beta,\beta)\}$
$\eta'(s) = \{(\alpha,\beta),(\alpha,\alpha)\}$
$\eta'(1) = \{(\alpha,\alpha),(\beta,\beta),(\gamma,\gamma)\}$
$\eta'(\bot) = \emptyset$

Example 18.

$(Q,\leq,\vee,\wedge,T)$ in Example 11 is CCP-invertible, since z in Definition 8 is given by $z = x$.

We can summarize our results for quantales as follows. Palmigiano and Re[10] show `relational representation theorem' for `unital involutive quantales'. On the other hand, our relational representation theorem can be applied to quantales which are not in general involutive.

Theorem 4.

For a quantale Q, the following are equivalent.

1. *The following function $\eta : Q \to \mathbf{Rel}(\mathbf{CCP}(Q))$*
   $$\eta(a) = \{(x,y) \mid x,y \in \mathbf{CCP}(Q), x \leq a \cdot y\}$$
   is an injective homomorphism of quantales and Q is CCP-invertible.
2. *Q is isomorphic to $\wp(X)$ as complete join semilattice for some set X.*
3. *Q is atom-algebraic and it is a frame.*
4. *Q is atom-algebraic and its atoms are completely coprime.*
5. *Q is completely coprime algebraic and its completely coprime elements are atoms.*
6. *Q is completely coprime algebraic and the order of its completely coprime elements is discrete.*

Proof.

This theorem is implied by Theorem 1 and Theorem 3.

■

6. Representation Theorem for Powerset Complete Idempotent Left Semirings

This section shows a representation theorem for powerset complete idempotent left semirings.

It is known that the set of up-closed multirelations over a set forms a complete idempotent left semiring together with union, multirelational composition, the empty multirelation, and the membership relation. Similarly to quantales, this section shows the powerset condition is sufficient for a complete idempotent left semiring to be isomorphic to a complete idempotent left semiring consisting of up-closed multirelations, in which all joins, the least element, multiplication, and the unit element are respectively given by unions, empty multirelations, the multirelational composition, and the membership relation.

Definition 9 (complete idempotent left semiring).

A complete idempotent left semiring *is a tuple $(K,\leq,\vee,\cdot,1)$ with the following properties:*

1. *$(K,\cdot,1)$ is a monoid.*
2. *$(K,\leq,\vee)$ is a complete join semilattice.*
3. *$(\vee S)\cdot a = \vee\{b \cdot a \mid b \in S\}$ for each element a and each subset S of K.*

A complete idempotent left semiring $(K,\leq,\vee,\cdot,1)$ is called completely coprime algebraic if $(K,\leq,\vee)$ is completely coprime algebraic. A powerset complete idempotent left semiring is a complete idempotent left semiring which is isomorphic to $\wp(X)$ as complete join semilattice for some set X.

Definition 10 (multirelation).

A multirelation *on a set A is a subset of $A \times \wp(A)$. A multirelation R on a set A is called* up-closed *if $(a,X) \in R$ and $X \subseteq Y$ imply $(a,Y) \in R$.*

Example 19.

For a set A, the tuple $\mathbf{UMRel}(A) = (K,\leq,\vee,\cdot,1)$ forms a complete idempotent left semiring where

- K is the set of all multirelations on A,
- $\leq$ is the inclusion $\subseteq$,
- $\vee$ is the union operator $\cup$,
- $R \cdot Q$ is defined by
  $\{(a,X) \mid \exists Y.(a,Y) \in R, \forall y \in Y.(y,X) \in Q\}$,
- 1 is $\{(a,X) \mid a \in A, X \subseteq A\}$.

The next theorem means that a powerset complete idempotent left semiring has a multirelational representation. This theorem is proved in the paper[14].

Theorem 5.

Let $(Q,\leq,\vee,\cdot,1)$ be a complete idempotent left semiring. If Q is a powerset complete idempotent left semiring, then the following function $\eta : Q \to \mathbf{UMRel}(\mathbf{CCP}(Q))$ is an injective homomorphism of complete idempotent left semirings.

$$\eta(a) = \{(x,\delta(b)) \mid x \in \mathbf{CCP}(Q), b \in Q, x \leq a \cdot b\}$$

Here, we write $\delta(b)$ for the following set of completely coprime elements.

$$\delta(b) = \{y \in \mathbf{CCP}(Q) \mid y \leq b\}$$

Example 20.

The ordered set of Fig 3 forms a complete idempotent left semiring where $a \cdot b = a \wedge b$ except for $1 \cdot a = a \cdot 1 = a$

and $2 \cdot \top = \top$. It is a powerset complete idempotent left semiring but not a powerset quantale.

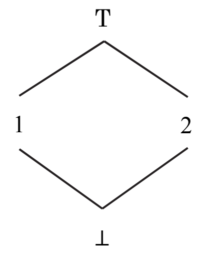

Fig. 3. A powerset complete idempotent left semiring

Our multirelational representation theorem for powerset complete idempotent left semirings can be also extended as follows. The proof is described in the paper(14).

Theorem 6.

Let $(Q, \leq, \vee, \cdot, 1)$ be a complete idempotent left semiring. The following are equivalent.

1. *The following function $\eta : Q \to \mathbf{UMRel}(\mathbf{CCP}(Q))$ is an injective homomorphism of complete idempotent left semirings, Q is CCP-invertible, and η reflects the order.*

$$\eta(a) = \{ (x, \delta(b)) \mid x \in \mathbf{CCP}(Q), b \in Q, x \leq a \cdot b \}$$

2. *Q is a powerset complete idempotent left semiring.*

７．Conclusion

Our main theorem is a relational representation theorem for powerset quantales. Conversely, if a quantale has a relational representation in the same way and it is CCP-invertible, then it is a powerset quantale. Similarly, a multirelational representation theorem for powerset complete idempotent left semirings is also proved.

It is future work to extend our representation theorem to a Stone-type duality(7).

As shown in Example 15, η for $\mathbb{N} \cup \{\omega\}$ is not an injective homomorphism. However, we do not know whether there exist other relational representations of $\mathbb{N} \cup \{\omega\}$ than η. It is also future work.

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References

(1) C.J.Mulvey, "&", In Second Topology Conference. Rendiconti del Circolo Matematico di Palermo, ser.2, supplement no. 12 (1986) pp.99--104.

(2) J. H. Conway, "Regular Algebra and Finite Machines", Chapman and Hall, London, (1971).

(3) He Jifeng, C. A. R. Hoare, "Weakest Prespecification", Information Processing Letters, vol. 24 (1987).

(4) S. Vickers, "Topology via Logic", Cambridge University Press (1989).

(5) D. N. Yetter, "Quantales and (Noncommutative) Linear Logic", Journal of Symbolic Logic vol. 55 (1990) pp.41--64.

(6) B. A. Davey and H. A. Priestley, "Introduction to Lattices and Order", 2nd edn. Cambridge University Press (2002).

(7) Peter T. Johnstone, "Stone Spaces", Cambridge University Press (1982).

(8) Silvio Valentini, "Representation Theorems for Quantales", Math. Log. Q. vol. 40 (1994) pp.182--190.

(9) Carolyn Brown and Doug Gurr, "A Representation Theorem for Quantales", Journal of Pure and Applied Algebra vol. 85 (1993) pp.27--42.

(10) B. Jo'nsson and A. Tarski, "Relational Representation of Groupoid Quantales", Order (2011) pp.1--19.

(11) A. Palmigiano and R. Re, "Boolean algebras with operators. Part II", Amer. J. Math. vol.74 (1952) pp.127--162.

(12) H. Furusawa, K. Nishizawa, "Relational and multirelational representation theorem for complete idempotent left semirings". LNCS, vol. 6663, Springer (2011) pp.148—163.

(13) K. Nishizawa, H. Furusawa, "Relational representation theorem for powerset quantales". LNCS vol. 7560, Springer(2012) pp.207–218.

(14) H. Furusawa, K. Nishizawa, "Multirelational representation theorem for complete idempotent left semirings". submitted.