G. On Rings Having G–Global Dimension Zero

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Let R be a ring with identity. There have been several interesting results which assert under suitable assumption that R has a ring decomposition $R = R_0 \oplus R_1$, where R_0 is always semisimple and R_1 has some properties depending to that of R. For example, if R is QF, then R has a ring decomposition $R = R_0 \oplus R_1$, where R_0 is semisimple and R_1 is a ring with essential singular ideal. If R is a ring with finitely generated left socle and has no nilpotent minimal left ideals, then R is the ring direct sum of a semisimple ring and a ring with zero left socle. However, it seems to us that these decomposition theorems can be treated more generally. In this paper we shall try to unify and to extend these decomposition theorems.

Let $I = (\mathrm{soc}({}_R R))^2$ and let

$$C = \{{}_R M \mid IM = M\}, \quad T = \{{}_R M \mid IM = 0\} \quad \text{and} \quad F = \{{}_R M \mid r_M(I) = 0\},$$

where $r_M(-)$ means the right annihilator in M.

In Section 1, we shall characterize rings R with the ring decomposition of the form $R = I \oplus r_R(I)$. Generalizing a previous result, we shall show that R has the decomposition of this form if and only if there exists an idempotent e in R such that Re is contained in I and $R(1-e)$ has no simple direct summand . Semiperfect left P–injective rings, semiperfect left mininjective rings, and rings for which every left exact preradical is a radical are examples of R having the decomposition of this form.

In Section 2, we shall characterize rings R with the ring decomposition of the form $R = I \oplus G(R)$, where G means the Goldie torsion functor. It is shown that R has the decomposition of this form if and only if R has G–global dimension zero.

We shall treat semilocal rings in Section 3. Every semilocal ring R has a decomposition $R = R_0 \oplus R_1$ of left ideals R_0 and R_1, where R_0 is semisimple and J, the Jacobson radical of R, is essential in R_1. This decomposition coincides with $R = I \oplus r_R(I)$ if and only if $J \in T$ and coincides with $R = I \oplus G(R)$ if and only if $J \in T(G)$. If R is a semilocal ring with essential left socle and $J \in T$, then (C, T, F) has length 2 and hence R has G–global dimension zero. Examples of such rings are QF–rings, dual rings, left GPF rings, left min–PF rings, and also left PF–rings.

Finally, in Section 4, we shall remark that, if R has G–global dimension zero, then some structures of R are determined by that of $G(R)$.

The detailed version of this paper will be submitted in publication elsewhere.