S. On Rings Having G–Global Dimension Zero

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Let R be a ring with identity. There have been several interesting results which assert under suitable assumption that R is a ring direct sum $R = R_0 + R_1$ where R_0 is semisimple and R_1 has, for example, essential singular ideal. However, it seems to us that these decomposition theorems can be treated in a mass. In this paper we shall try to unify these theorems.

Let $I = (\mathrm{soc}({}_RR))^2$ and let

$$C = \{{}_RM \mid IM = M\}, T = \{{}_RM \mid IM = 0\} \text{ and } F = \{{}_RM \mid r_M(I) = 0\},$$

where $r_M(-)$ means the right annihilator in M.

First, we shall characterize rings R with the ring decomposition of the form $R = I + r_R(I)$. Generalizing a previous result, we shall show that, over a ring R with a complete orthogonal set of primitive idempotents, R has the decomposition of this form if and only if the Jacobson radical J of R is in T. Semiperfect left P–injective rings and semiperfect left mininjective rings are examples of R having the decomposition of this form.

Next, we shall characterize rings R with the ring decomposition of the form $R = I + G(R)$. It is shown that R has the decomposition of this form if and only if R has G–global dimension zero.

Finally, we shall treat semilocal rings. Every semilocal ring R has a decomposition $R = R_0 + R_1$ of left ideals R_0 and R_1, where R_0 is semisimple and J is essential in R_1. It is shown that this decomposition coincides with $R = I + r_R(I)$ if and only if $J \in T$ and it coincides with $R = I + G(R)$ if and only if $J \in T(G)$. If R is a semilocal ring with essential left socle and $J \in T$, then (C, T, F) has length 2 and hence R has G–global dimension zero. Examples of such rings are semisimple rings, QF–rings, dual rings, left GPF rings, left min–PF rings, and also left PF–rings.

The detailed version of this paper will be submitted in publication elsewhere.