論説

Optimality under Demographic Shocks

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1 Introduction

Competitive equilibrium in the overlapping generations (OLG) model might not achieve an optimal allocation, even when markets operate perfectly, as in the Arrow-Debreu abstraction. It is understood that this sort of inefficiency is caused by the lack of a transversality condition at infinity. Therefore, lots of studies have tried to characterize equilibrium optimality of the OLG model in a various level of generality : Peled (1984), Aiyagari and Peled (1991), Manuelli (1990), Chattopadhyay and Gottardi (1999), Chattopadhyay (2001, 2006), and Bloise and Calciano (2008) are such examples.

These studies, however, considered models without demographic shocks. In order to design active policies such as social security which remedy inefficiency, it is important to consider the presence of demographic shocks. A triplet of studies per Demange and Laroque (1999, 2000, 2001) addressed this issue on optimality under demographic shocks. They showed in several OLG frameworks that optimality of intergenerational transfer is characterized by the condition on the dominant root of the matrix of marginal rates of substitution adjusted by population growth. More precisely, the dominant root must be equal to one.

Here, we should note that Demange and Laroque considered a model with time running from $-\infty$ to ∞, which implies the fact that their model has no *initial old*, which is the oldest generation in the initial period. In order to design social security systems, however, the policymaker might also consider olds generations in the period wherein the policy begins. Therefore, this article aims to reexamine optimality under demographic shocks in the model with the initial period and the initial old.

In order to shed light to the presence of demographic shocks and the initial old, we consider a very simple, but rather canonical, pure-endowment model of overlapping generations. Time runs from the initial period to infinity. The stochastic environment is described by a time-homogeneous Markov chain. At each date-event, there is a single perishable good and a new generation, consisting of homogeneous agents living for two periods, is born. The demographic shocks enter into the economy through the growth rates of population. In this stochastic OLG model, we first find that the Demange and Laroque (DL) criterion is equivalent to conditional golden rule optimality (CGRO), which is an optimality criterion completely ignoring welfare of the initial old.

Instead of CGRO, we also consider conditional Pareto optimality (CPO), which copes with welfare of initial old, as a criterion of optimality.[1] According to CGRO and CPO, agents' welfare is evaluated by conditioning their utility on the state at the date of their birth. Agents are therefore distinguished not only by their type and date of birth but also by the state at that date, and an agent's preference is defined over a set of contingent consumption streams available in the two periods of that agent's lifetime. CPO is then characterized by the condition on the dominant root of the matrix of marginal rates of substitution adjusted by population growth. More precisely, the dominant root is allowed to be less than one, whereas CGRO requires the dominant root begin equal to one.

This article also applies characterizations of CPO and CGRO to examine equilibrium welfare. It has been known that a stationary equilibrium with circulating money achieves CPO. By applying our results to welfare on stationary monetary equilibrium, we can conclude that a stationary monetary equilibrium achieves not only CPO but also CGRO. This result may be interpreted as the first welfare theorem in a financial economy.

The organization of this paper is as follows : Section 2 presents details of the model. Section 3 introduces the Demange and Laroque criterion for optimality and shows that it is equivalent to ignoring the welfare of the initial olds. Section 4 define CPO and provides its characterization. Section 5 applies previous results to equilibrium allocations. Proofs of results are provided in Section 6. Section 7 provides some concluding remarks.

2 The Model

This article considers a stationary, one-good, finite-state, pure-endowment stochastic overlapping generations model with demographic shocks, wherein agents live for two periods.

Time and Stochastic Structures. Time is discretely runs from 1 to ∞. The stochastic environment is modeled by a time-homogenous Markov process with its state space S, where S is a nonempty finite set and satisfies that $0 \notin S$. The state $s_0 \in S$ in (implicitly defined) period 0 is treated as given. The date-event tree, Γ, is then defined as follows : (i) the root of the tree is s_0 ; (ii) the set of nodes at date t is denoted by Σ_t where we set $\Sigma_1 := \{s_0\} \times S$ and, iteratively, $\Sigma_t := \Sigma_{t-1} \times S$ for $t \geq 2$; and (iii) $\Sigma := \cup_{t \geq 1} \Sigma_t$ and $\Gamma := \{s_0\} \cup \Sigma$.[2] Given any node $\sigma \in \Gamma$, its *predecessor node* is uniquely defined and denoted by σ^-. Moreover, its Markov (terminal) state is denoted by $s(\sigma)$.

Demographic Structure. Demographic shocks are assumed to enter the economy through the

1 CPO was first proposed by Muench (1977).

2 This is a standard de finition of the date-event tree. See Chattopadhyay (2001) for example.

growth rates of population. At each node $\sigma \in \Sigma$, a new generation consisting of homogenous agents living for two periods is born. The population of the generation born at each $\sigma \in \Sigma$ is denoted by $N(\sigma)$ and assumed to satisfy that $N(\sigma) = n_{s(\sigma)} N(\sigma^-)$, where $n_s > 0$ is the growth rate of population at state $s \in S$ and $N(s_0)$ is a given positive number.

Endowments and Preference Structures. We assume that the economy is stationary, i.e.: the endowments and preference structures of each agent depend only on the realizations of the Markov state during his/her lifetime, not on time or on past realizations. Therefore, (i) the endowment stream of each agent born at state $s \in S$ is denoted by $\omega_s = (\omega_s^1, (\omega_{ss'}^2)_{s' \in S}) \in \Re_{++} \times \Re_{++}^S$ and (ii) his/her lifetime utility function is denoted by $U^s : \Re_+ \times \Re_+^S \to \Re$, where ω_s^1 and $(\omega_{ss'}^2)_{s' \in S}$ describe the endowments at birth and all states in the following period. It is assumed that $\omega_s \gg 0$ and U^s is strictly monotone increasing, strictly quasi-concave, and continuously differentiable on the interior of its domain.[3]

Initial Olds. In addition, a one-period lived generation, the members of which are called *initial old agents* or simply *initial olds*, is born after the realization of state $s_1 \in S$ in period 1. The population of the initial olds is given by $N(s_0)$ as defined above. Each initial old born at state s_1 in the initial period is assumed to be endowed with $\omega_{0s_1}^2 := \omega_{s_0 s_1}^2$ units of the consumption good in his/her lifetime and his/her consumption streams $c_{0s_1}^2 \in \Re_+$ is ranked according to a utility function $u_0(c_{0s_1}^2) := c_{0s_1}^2$.

Stationary Feasible Allocations. We denote by $c_\sigma(\sigma')$ the amount of the consumption good at node σ' consumed by an agent born at node σ (by an initial old if $\sigma = s_0$). Then, the resource constraints of this economy can be given by

$$(\forall \sigma \in \Sigma_1) \qquad N(\sigma)c_\sigma(\sigma) + N(s_0)c_0^2 = N(\sigma)\omega_{s(\sigma)}^1 + N(s_0)\omega_{0s}^2(\sigma)$$
$$(\forall t \geq 2)(\forall \sigma \in \Sigma_t) \qquad N(\sigma)c_\sigma(\sigma) + N(\sigma^-)\, c_{\sigma^-}(\sigma) = N(\sigma)\omega_{s(\sigma)}^1 + N(\sigma^-)\omega_{s(\sigma^-)s(\sigma)}^2$$

or equivalently

$$(\forall \sigma \in \Sigma_1) \qquad n_{s(\sigma)}c_\sigma(\sigma) + c_0^2 = n_{s(\sigma)}\omega_{s(\sigma)}^1 + \omega_{0s(\sigma)}^2$$
$$(\forall t \geq 2)(\forall \sigma \in \Sigma_t) \qquad n_{s(\sigma)}c_\sigma(\sigma) + c_{\sigma^-}(\sigma) = n_{s(\sigma)}\omega_{s(\sigma)}^1 + \omega_{s(\sigma^-)s(\sigma)}^2.$$

In order to bring out the sharp contrast between our results and those of Demange and Laroque (1999), we concentrate our attention on "stationary" feasible allocations, not on "all" feasible allocations, throughout the rest of this paper. Let $S_0 := \{0\} \cup S$ and $\bar{\omega}_{ss'} := n_{s'}\omega_{s'}^1 + \omega_{ss'}^2$ for each $(s, s') \in S_0 \times S$, which is the total endowment when the current and preceding states are s' and s, respectively.[4] A *stationary feasible allocation* of this economy is a pair $c = (c^1, c^2)$ of functions $c^1 : S \to \Re_+$ and $c^2 : S_0 \times S \to \Re_+$ such that

$$(\forall (s, s') \in S_0 \times S) \; n_{s'}c_{s'}^1 + c_{ss'}^2 = \bar{\omega}_{ss'},$$

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3 In the rest of this study, we denote by $U_1^s(c^1, c^2)$ and $U_{s'}^s(c^1, c^2)$ the partial derivatives $\partial U^s(c^1, c^2)/\partial c^1$ and $\partial U^s(c^1, c^2)/\partial c_{s'}^2$ for all $s, s' \in S$ and all $(c^1, c^2) \in \Re_+ \times \Re_+^S$, respectively.

4 We introduce S0 to distinguish initial olds' consumption from consumption in the second period of newly born agents.

where $c^2_{0s_1} \in \Re_+$ is the consumption of initial olds born at period 1 state s_1, and $c_s = (c^1_s, (c^2_{ss'})_{s' \in S}) \in \Re_+ \times \Re^S_+$ is the consumption stream of the agent born at the Markov state s. Note that the stationary feasiblity of an allocation c does not necessarily imply that $c^2_{0s'} = c^2_{ss'}$ for each $s, s' \in S$, whereas it holds that $\bar{\omega}_{0s'} - c^2_{0s'} = \bar{\omega}_{ss'} - c^2_{ss'}$. Let A be the set of all stationary feasible allocations with its generic element c. Note that A is nonempty, bounded, closed, and convex. A stationary feasible allocation c is *interior* if $c^1_s > 0$ and $c^2_{ss'} > 0$ for all $s, s' \in S$.

3 Conditional Golden Rule Optimality

Optimality under demographic shocks in a stochastic overlapping generations model is already studied per Demange and Laroque (1999, 2000, 2001). However, they considered a model *without* the initial period and therefore there exists no initial old in their model. Therefore, this section aims to reexamine their characterization of optimality in a model *with* the initial period (and the initial olds).[5]

In order to present Demange and Laroque's results, we introduce several notations. Given an interior stationary feasible allocation $c = (c_s)_{s \in S}$, let $m_{ss'}(c) = U^s_{s'}(c_s) / U^s_1(c_s)$ and let $M(c) = [n_s m_{ss'}(c)]_{s,s' \in S}$, which is the matrix of the marginal rates of substitution adjusted by population growth. The current restrictions on preferences imply that $M(c)$ is a positive square matrix. By the Perron-Frobenius theorem,[6] any positive square matrix M has a unique dominant root. This paper denotes by $\lambda^f(M)$ the dominant root of a positive square matrix M. Given these notations, we can present the Demange and Laroque (DL) criterion of optimality as follows :

DL Criterion. An interior stationary feasible allocation c satisfies the *DL criterion* if it satisfies that $\lambda^f(M(c)) = 1$.

The DL criterion says that the dominant root of the matrix of marginal rates of substitution, adjusted by population growth, is equal to one. In order to examine this criterion, we also introduce a criterion of optimality, conditional golden rule optimality (CGRO). For any two stationary feasible allocations $b, c \in A$, we say that b *CGRO-dominates* c if

$$(\forall s \in S) \; U^s(b_s) \geq U^s(c_s)$$

with strict inequality somewhere. CGRO is then defined as follows :

Conditional Golden Rule Optimality. A stationary feasible allocation c is *conditionally golden rule optimal* if there exists no stationary feasible allocation b that CGRO-dominates c.

Note in this definition of CGRO that agents' welfare is evaluated by conditioning their utility on the state at the date of their birth. Agents are thus distinguished not only by their type and date of birth

5 Precisely, Demange and Laroque (1999, 2001) considers productive labour supply and storage technology. However, their argument can be easily tailored to our pure-endowment environment.

6 See, for example, Debreu and Herstein (1953) and Takayama (1974) for more details on the Perron-Frobenius theorem.

but also by the state at that date, and an agent's preference is defined over a set of contingent consumption streams available in the two periods of that agent's lifetime. Obviously, CGRO does not cope with welfare of the initial olds.

Our first finding is the equivalency between the DL criterion and CGRO.

Theorem 1 *An interior stationary feasible allocation satisfies the DL criterion if and only if it is conditionally golden rule optimal.*

In other words, we can find that the DL criterion of optimality does not consider welfare of the initial olds at all. Although there exists no doubt that the DL criterion is one of important criteria of optimality, there seems no reason to allow to ignore welfare of the initial olds. Therefore, we will incorporate welfare of the initial old into the concept of optimality and provide its characterization in the next section.

4 Conditional Pareto Optimality

This section introduces "conditional Pareto optimality" (CPO), which is a criterion of optimality of stationary feasible allocations considering welfare of the initial old, and provide its characterization.[7] For any two stationary feasible allocations b and c, we say that b *CPO-dominates* c if

$$(\forall s \in S) \qquad \begin{aligned} b_{0s}^2 &\geq c_{0s}^2, \\ U^s(b_s) &\geq U^s(c_s) \end{aligned}$$

with strict inequality somewhere. CPO is then defined as follows:

Conditional Pareto Optimality. A stationary feasible allocation c is said to be *conditionally Pareto optimal* if there exists no other stationary feasible allocation b that CPO-dominates c.

Differently from CGRO, CPO considers the welfare of the initial olds. This section now characterizes CPO.

Theorem 2 *An interior stationary feasible allocation c is conditionally Pareto optimal if and only if $\lambda^f(M(c))$ is less than or equal to unity.*

This theorem says that the dominant root of the matrix of marginal rates of substitution, adjusted by population growth, is less than or equal to one. By Theorems 1 and 2, we now know that a stationary feasible allocation c satisfying the DL criterion implies that $\lambda^f(M(c)) \leq 1$ and therefore it is CPO. Remark that this characterization of CPO extends Aiyagari and Peled (1991, Theorem 1) by characterizing the CPO of not an interior stationary "equilibrium" allocation but of an interior stationary "feasible" allocation under demographic shocks.

7 More precisely, conditional Pareto optimality can be applied to not only "stationary" feasible allocations but also all feasible allocations. See for example Chattopadhyay and Gottardi (1999).

This section characterized optimality criteria of stationary feasible allocations by the dominant root of a matrix related to them in an economy with strictly convex preferences. While the dominant root of a matrix related to a CGRO allocation must be equal to one, whereas that related to a CPO allocation is allowed to be less than one. By their characterizations, we might say that the CGRO is a stronger criterion of optimality than CPO.

5 Optimality of Stationary Equilibrium Allocations

The previous two sections characterized CPO and CGRO of stationary “feasible” allocations. The results also correspond to welfare analysis of stationary equilibrium. This section examines the relationship between optimality criteria and stationary “equilibrium” allocations.

5.1 Complete Market

We define a stationary equilibrium with complete market, i.e.: a stationary equilibrium at which agents can buy and sell all contingent commodities in a centralized market.

Denition 1 A pair (Π^*, c^*) of a positive price matrix $\Pi^* = [\pi^*_{ss'}]_{s, s' \in S}$ of contingent commodities and a stationary feasible allocation $c^* = (c^*_s)_{s \in S}$ is called a *stationary equilibrium* if

- for all $s \in S$, c^*_s belongs to the set
$$\underset{(c^y_s,\, c^o_s) \in \Re_+ \times \Re^S_+}{\arg\max} \left\{ U^s(c_s) : c^1_s + \sum_{s' \in S} c^2_{ss'} \pi^*_{ss'} \leq \omega^1_s + \sum_{s' \in S} \omega^2_{ss'} \pi^*_{ss'} \right\}$$
given π^*_s; and
- for all $s, s' \in S$, $n_{s'} c^{*1}_{s'} + c^{*2}_{ss'} = \bar{\omega}_{ss'}$.

In this definition, the former condition is the optimization problem of each agent $s \in S$ subject to a lifetime budget constraint, and the latter is the market clearing conditions.

Let (Π, c) be a stationary equilibrium with $c_s \gg 0$ for all $s \in S$, if any. Since, for all $s \in S$, c_s must be a solution of the optimization problem of agent born at state s, it follows from the Kuhn-Tucker theorem that there exists some $\lambda^s \geq 0$ such that

$$U^s_1(c_s) = \frac{\partial U^s}{\partial c^1_s}(c_s) = \lambda^s,$$

$$(\forall s' \in S) \quad U^s_{s'}(c_s) = \frac{\partial U^s}{\partial c^2_{s'}}(c_s) = \lambda^s \pi_{ss'},$$

where λ^s is the Lagrange multiplier of the lifetime budget constraint. Note that $\lambda^s > 0$ because U^s is strictly monotone increasing. Thus, we can observe that

$$(\forall_{s, s'} \in S) \; \pi_{ss'} = \frac{U^s_{s'}(c_s)}{U^s_1(c_s)} = m_{ss'}(c),$$

Therefore, the stationary equilibrium contingent claim price matrix Π can be always represented by the matrix of marginal rates of substitution at the stationary equilibrium allocation c.

The next two propositions follow immediately from the previous observation and Theorems 1 and 2, respectively.

Proposition 1 *For every stationary equilibrium* (Π, c) *with* $c_s \gg 0$ *for all* $s \in S$*, c is conditionally Pareto optimal if and only if* $\lambda^f([n_s \pi_{ss'}]_{s,s' \in S}) \leq 1$.

Proposition 2 *For every stationary equilibrium* (Π, c) *with* $c_s^h \gg 0$ *for all* $s \in S$*, c is conditionally golden rule optimal if and only if* $\lambda^f([n_s \pi_{ss'}]_{s,s' \in S}) = 1$.

These propositions characterize optimality of stationary equilibrium allocations. While the CPO of stationary equilibrium allocations is characterized by the dominant root of the contingent claim price matrix, adjusted by population growth, being less than or equal to one, their CGRO has the dominant root exactly equal to one. These are natural extensions of Propositions 1 and 2 of Ohtaki (2013) to an environment with demographic shocks. Remark in these propositions that both the equilibrium price matrix, Π, and the growth rates of population, n, are observable variables. This indicates that we can examine equilibrium welfare by observing these variables and do not necessarily require information about preferences and the initial endowments. This fact is often called the *testability* or *observability* of optimality (Barbie et al., 2007).

5.2 Sequentially Complete Markets with Money

As shown in the previous proposition, a stationary equilibrium itself might not be optimal even when markets operate perfectly. However, we can construct a market mechanism which generates an optimal allocation by introducing an infinitely-lived outside asset, which yields no dividend, money. Suppose in the this subsection that there exists one unit of money. We denote by $p(\sigma)$ and $q(\sigma)$ the real money price and the per-capita real money balance at node $\sigma \in \Sigma$, respectively. Obviously, these have a one-to-one relation to each other, i.e.: $p(\sigma) = q(\sigma)N(\sigma)$ at each node σ. Also suppose that spot markets of one-period contingent claims exist and are complete.

Denition 2 A triplet (q^*, Π^*, c^*) of a positive per-capita real money balance vector $q^* \in \Re^S_{++}$, a positive price matrix $\Pi^* = [\pi^*_{ss'}]_{s,s' \in S}$ of contingent claims, and a stationary feasible allocation $c^* = (c^*_s)_{s \in S}$ is called a *stationary equilibrium with circulating money* if there exists some money holding process $m^*: \Sigma \to \Re$ and some contingent claim portfolio process $\theta^*: \Sigma \to \Re^S$ such that

- at each node $\sigma \in \Sigma$, $(c^*_{s(\sigma)}, m^*_\sigma, \theta^*_\sigma)$ belongs to the set

$$\underset{(c^y_s, c^o_s, m, \theta)}{\arg\max} \left\{ U^s(c_s) \ : \begin{array}{l} c^y_s = \omega^y_s - q^*_s N(\sigma)\, m - \Sigma_{s' \in S}\, \theta_\sigma(s')\, \pi_{s(\sigma)s'} \\ (\forall s' \in S)\ c^o_{s(\sigma)s'} = \omega^o_{s(\sigma)s'} + q^*_{s'} N(\sigma, s')\, m_s + \theta_\sigma(s') \end{array} \right\}$$

  given q^* and p^*_s; and
- at each node $\sigma \in \Sigma$, $m^*_\sigma = 1$ and $\theta^*_\sigma = 0$.

In this definition, the former condition is the optimization problem of each agent $s \in S$ subject to sequential budget constraints, and the latter is the asset market clearing conditions. One can easily verify that the good market equilibrium condition also holds at a stationary equilibrium with circulating money. We can then find that an introduction of money may generate a CGRO allocation:

Proposition 3 *An interior stationary feasible allocation of a stationary equilibrium with circulating money, if any, is always conditionally golden rule optimal.*

In other words, when a stationary equilibrium with circulating money exists, it always generates a CGRO allocation. This financial intermediate role of money for remedying inefficiency in the OLG model is a well-known result in the literature and the last theorem showed that the result still holds even in the presence of demographic shocks.

6 Proofs

Proof of Theorem 1. We first claim that $c_{0s}^2 > 0$ for each interior stationary feasible allocation c and each $s \in S$. In order to verify this claim, let c be an interior stationary feasible allocation and $s \in S$. Because c is a stationary feasible allocation, we can obtain that

$$n_s c_s^1 + c_{s0s}^2 = \omega_{s0s} = \omega_{0s} = n_s c_s^1 + c_{0s}^2,$$

which implies that

$$0 < c_{s0s}^2 = c_{0s}^2,$$

where the first inequality follows from the fact that c is interior.

Let c be an interior stationary feasible allocation. It is easy to verify that c is a CGRO allocation if and only if there exist Pareto weights $\gamma: S \to \Re_{++}$ such that

$$c \in \arg\max_{b \in A} \sum_{s \in S} \gamma^s U^s(b_s).$$

Define the Lagrangian L by

$$L = \sum_{s \in S} \gamma^s U^s(c_s) - \sum_{(s,s') \in S_0 \times S} \lambda_{ss'} \left[\bar{\omega}_{ss'} - (n_{s'} c_{s'}^1 + c_{ss'}^2) \right],$$

where λ is the Lagrange multipliers for the resource constraint. Note that the objective function is strictly quasi-concave. Therefore, by Arrow and Enthoven (1961), the CGRO of c can be completely characterized by the existence of Pareto weights $\gamma: S \to \Re_{++}$and Lagrange multipliers $\lambda : S_0 \times S \to \Re_+$ which satisfy that

$$(\forall s \in S)\ \gamma^s U_1^s(c_s) = \sum_{s' \in S} \lambda_{s's}\, n_s + \lambda_{0s}\, n_s, \tag{1}$$
$$(\forall s \in S)(\forall s' \in S)\ \gamma^s U_{s'}^s(c_s) = \lambda_{ss'}, \tag{2}$$
$$(\forall s \in S) - \lambda_{0s} \leq 0 \text{ with equality if } c_{0s}^{h2} > 0. \tag{3}$$

Note that, as observed above, we can treat λ_{0s} as zero for each $s \in S$, because $c_{0s}^2 > 0$ for each $s \in S$. Therefore, we can ignore Eq.(3) and remove $\lambda_{0s} n_s$ from Eq.(1).

We should now claim the equivalence between the existence of γ and λ satisfying Eqs.(1) and (2) with $\lambda_{0s} = 0$ and $\lambda^f(M(c)) = 1$. Assume the existence of γ and λ satisfying Eqs.(1) and (2) with$\lambda_{0s} = 0$ to show $\lambda^f(M(c)) = 1$. Note that, by strict monotonicity of U^s, $m_{ss'}(c)$ is positive for all$s, s' \in S$.

We can then obtain from Eqs.(1) and (2) with $\lambda_{0s} = 0$ that

$$(\forall s, s' \in S)\ n_{s'} m_{ss'}(c) = \frac{\lambda_{ss'}}{\Sigma_{\tau \in S} \lambda_{\tau s'}}.$$

Then, it follows that

$$(\forall s, s' \in S)\ \lambda_{ss'} = \sum_{T \in S} \lambda_{\tau s} n_s m_{ss'}.$$

Summing this equation over $s \in S$, we have

$$(\forall s, s' \in S)\ \alpha = \alpha M(c),$$

where $\alpha_s := \Sigma_{\tau \in S}\lambda_{\tau s}$. Note that $M(c)$ is an $S \times S$ matrix with positive coefficients. Therefore, it follows from the Perron-Frobenius theorem that $\lambda^f(M(c)) = 1$.

Assume now that $\lambda^f(M(c)) = 1$. Because $M(c)$ is an $S \times S$ matrix with positive coefficients, we can pick up the row eigenvector $\alpha \gg 0$ of $M(c)$. Note that it satisfies that $\alpha \cdot (I - M) = 0$, where I is the $S \times S$ identity matrix. For all $s, s' \in S$, define γ_s and $\lambda_{ss'}$ by

$$\gamma^s := \frac{\alpha_s n_s}{U_1^s(c_s)},$$
$$\lambda_{ss'} := \gamma^s U_s^s(c_s).$$

By their definitions, we can obtain that

$$(\forall s, s' \in S)\ \lambda_{ss'} = \alpha_s n_s \frac{U_s^s(c_s)}{U_1^s(c_s)} = \alpha_s n_s m_{ss'},$$

so that $\alpha_{s'} = \Sigma_{s \in S}\lambda_{ss'}$ for all $s' \in S$. It is now easy to verify that γ and λ satisfies Eqs.(1) and (2) with $\lambda_{0s} = 0$. This completes the proof. Q.E.D.

Proof of Theorem 2. Let c be an interior stationary feasible allocation. It is easy to verify that c is CPO if and only if there exist Pareto weights $\gamma : S \rightarrow \Re_{++}$ and $\gamma_0 : S \rightarrow \Re_{+}$ such that

$$c \in \arg\max_{b \in A} \left(\sum_{s \in S} \gamma^s U^s(b_s) + \sum_{s \in S} \gamma_0^s b_{0s}^2 \right).$$

Define the Lagrangian L by

$$L = \sum_{s \in S} (\gamma^s U^s(c_s) + \gamma_0^s c_{0s}^2) - \sum_{(s,s') \in S_0 \times S} \lambda_{ss'} \left[\bar{\omega}_{ss'} - \sum_{s' \in S} (n_{s'} c_{s'}^1 + c_{ss'}^2) \right],$$

where λ is the Lagrange multipliers for the resource constraint. Note that the objective function is strictly quasi-concave. Therefore, by Arrow and Enthoven (1961), the CPO of c can be completely characterized by the existence of Pareto weights $\gamma : H \times S \rightarrow \Re_{++}$ and $\gamma_0 : H \times S \rightarrow \Re_{+}$ and Lagrange multipliers $\lambda : S_0 \times S \rightarrow \Re_{+}$, which satisfy that

$$(\forall s \in S)\ \gamma^s U_1^s(c_s) = \sum_{s' \in S} \lambda_{s's} n_s + \lambda_{0s} n_s, \tag{4}$$
$$(\forall s \in S)(\forall s' \in S)\ \gamma^s U_{s'}^s(c_s) = \lambda_{ss'}, \tag{5}$$
$$(\forall s \in S)\ \gamma_0^s - \lambda_{0s} \leq 0 \text{ with equality if } c_{0s}^{h2} > 0. \tag{6}$$

Therefore, we should claim equivalence between the existence of γ, γ^0, and λ, and λ^0 satisfying Eqs.(4)–(6) and $\lambda^f(M(c)) \leq 1$. However, we omit the proof of this claim because its proof strategy is nearly identical to that of Theorem 1 of Aiyagari and Peled (1991). Q.E.D.

Proof of Proposition 3. By the sequential budget constraints of an agent, we can obtain the agent's lifetime budget constraint such that: at each node $\sigma \in \Sigma$,

$$c^1_{s(\sigma)} + \sum_{s' \in S} c^2_{s(\sigma)s'} \pi_{s(\sigma)s'} \leq \omega^y_{s(\sigma)} + \sum_{s' \in S} \omega^o_{s(\sigma)s'} \pi_{s(\sigma)s'} + \left(\sum_{s' \in S} q_{s'} n_{s'} \pi_{ss'} - q_{s(\sigma)} \right) N(\sigma)\ m.$$

By this equation, we can obtain the no arbitrage condition when the per-capita real money balance is positive, i.e.: $q = \Pi \cdot q$ for any stationary equilibrium with circulating money, (q, Π, c), with $c_s \gg 0$ for each $s \in S$. In order to verify this, we should show that

$$(\forall s \in S)\ q_s = \sum_{s' \in S} q_{s'} n_{s'} \pi_{ss'}.$$

Suppose the contrary that $q_s \neq \Sigma_{s' \in S}\, q_{s'} n_{s'} \pi_{ss'}$ for some $s \in S$. If $q_s < \Sigma_{s' \in S}\, q_{s'} n_{s'} \pi_{ss'}$, then agent born at state s will choose ∞ as m and his/her optimization problem has no solution. On the other hand, if $q_s > \Sigma_{s' \in S}\, q_{s'} n_{s'} \pi_{ss'}$, then agent born at state s will choose $-\infty$ as m and his/her optimization problem has no solution.[8] In any cases, we obtain a contradiction, so that $q_s = \Sigma_{s' \in S}\, q_{s'} n_{s'} \pi_{ss'}$ for all $s \in S$.

Suppose now that there exists at least one stationary equilibrium with circulating money, (q, Π, c), satisfying that $c_s \gg 0$ for all $s \in S$. We have obtained that $[n_s \pi_{ss'}]_{s',s \in S} \cdot q = q$, at which the lifetime budget constraint coincides with that in the complete market. Because q_s is now positive for all $s \in S$, it follows from the Perron-Frobenius theorem that the $S \times S$ matrix $[n_s \pi_{ss'}]_{s',s \in S}$ with positive coefficients has the dominant root equal to unity. Now it follows from Proposition 2 that the equilibrium allocation c is CGRO. This completes the proof of Proposition 3. Q.E.D.

7 Concluding Remarks

This article examines optimality under demographic shocks in a stochastic overlapping generations model with the initial old. It has been shown that the well-known characterization of optimality in the previous studies is equivalent to conditional golden rule optimality (CGRO), which is a criterion ignoring welfare of initial old. Therefore, we have introduced conditional Pareto optimality (CPO) to our model and characterized it. We have shown that both these criteria are characterized by conditions on the dominant root of the agents' matrix of marginal rates of substitution adjusted by population growth. While CGRO requires that the dominant root is exactly equal to unity, CPO allows it to be less than unity, because CPO copes with an initial condition, whereas CGRO does not. Thus, on the basis of their characteristics, we might say that CGRO is stronger than CPO as a criterion of optimality.

It has been known that a stationary monetary equilibrium achieves CPO. By applying our results to welfare on stationary monetary equilibrium, we can conclude that a stationary monetary equilibrium achieves not only CPO but also CGRO. This result can be interpreted as the first welfare theorem.
Finally, we have concentrate our attention to the space of all "stationary" feasible allocations. Charac-

8 When one wished to impose the lower bound for possible m, $m \geq 0$ for example, the agent born state s chooses 0 as the amount of money holding. However, this contradicts the fact that m should be equal to 1 at a stationary equilibrium with circulating money.

terizations of optimality of general feasible allocations under demographic shocks are left to the future research.

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References

Aiyagari, S.R. and D. Peled (1991) "Dominant root characterization of Pareto optimality and the existence of optimal equilibria in stochastic overlapping generations models," *Journal of Economic Theory* **54**, 69–83.

Arrow, K.J. and Alain C. Enthoven (1961) "Quasi-concave programming," *Econometrica* **29**, 779–800.

Barbie, M., M. Hagedorn, and A. Kaul (2007) "On the interaction between risk sharing and capital accumulation in a stochastic OLG model with production," *Journal of Economic Theory* **137**, 568–579.

Bloise, G. and F.L. Calciano (2008) "A characterization of in efficiency in stochastic overlapping generations economies," *Journal of Economic Theory* **143**, 442–468.

Chattopadhyay, S. (2001) "The unit root property and optimality : a simple proof," *Journal of Mathematical Economics* **36**, 151–159.

Chattopadhyay, S. (2006) "Optimality in stochastic OLG models : Theory for tests," *Journal of Economic Theory* **131**, 282–294.

Chattopadhyay, S. and P. Gottardi (1999) "Stochastic OLG models, market structure, and optimality," *Journal of Economic Theory* **89**, 21–67.

Debreu, G. and I.N. Herstein (1953) "Nonnegative square matrix," *Econometrica* **21**, 597–607.

Demange, G. and G. Laroque (1999) "Social security and demographic shocks," *Econometrica* **67**, 527–542.

Demange, G. and G. Laroque (2000) "Social security, optimality, and equilibria in a stochastic overlapping generations economy," *Journal of Public Economic Theory* **2**, 1–23.

Demange, G. and G. Laroque (2001) "Social security with heterogenous populations subject to demographic shocks," *Geneva Papers on Risk and Insurance Theory* **26**, 5–24.

Manuelli, R. (1990) "Existence and optimality of currency equilibrium in stochastic overlapping generations models : The pure endowment case," *Journal of Economic Theory* **51**, 268–294.

Muench, T.J. (1977) "Optimality, the interaction of spot and futures markets, and the nonneutrality of money in the Lucas model," *Journal of Economic Theory* **15**, 325–344.

Ohtaki, E. (2013) "Golden rule optimality in stochastic OLG economies," *Mathematical Social Sciences* **65**, 60–66.

Peled, D. (1984) "Stationary Pareto optimality of stochastic asset equilibria with overlapping generations," *Journal of Economic Theory* **34**, 396–403.

Takayama, A. (1974) *Mathematical Economics*. The Dryden Press : Hinsdale. IL.