CORRECTION TO OUR PAPER: PROJECTIVE SYSTEMS SUPPORTED ON THE COMPLEMENT OF TWO LINEAR SUBSPACES (BULL. KOREAN MATH. SOC. 37 (2000), 493–505)

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ABSTRACT. In our previous paper (Bull. Korean Math. Soc. 37 (2000), 493–505), we claimed a theorem on a certain subset of a projective space over a finite field (Theorem 3.1). Recently, however, Professor Kato pointed out that our proof does not work if the field consists of two elements. Here we give an alternative proof of the theorem for the exceptional case.

In our previous paper, we studied the union of two linear subspaces in general position of $\mathbb{P}^n$ over a finite field $\mathbb{F}_q$ from a coding theoretical viewpoint, and presented the following theorem [1, Theorem 3.1].

THEOREM. *Let S be a subset of the projective n-space $\mathbb{P}^n$ over the field $\mathbb{F}_q$ of q elements such that $\#S = N(s) + N(t)$ for nonnegative integers s and t with $s \geq t$, $s + t = n - 1$. Here $N(s)$ denotes the number of points of $\mathbb{P}^s$ over $\mathbb{F}_q$, that is, $N(s) = (q^{s+1} - 1)/(q - 1)$. Furthermore, we suppose the following two conditions on S:*

(1) *for any hyperplane H in $\mathbb{P}^n$, $\#(H \cap S)$ is either $N(s) + N(t-1)$ or $N(s-1) + N(t)$ or $N(s-1) + N(t-1)$;*

(2) *the number of hyperplanes H with $\#(H \cap S) = N(s) + N(t-1)$ is $N(t)$ and that of hyperplanes H with $\#(H \cap S) = N(s-1) + N(t)$ is $N(s)$.*

(When $s = t$, we understand the condition (2) to be saying that the number of hyperplanes H with $\#(H \cap S) = N(s) + N(s-1)$ is $2N(s)$.) Then we can conclude that S is the union of two linear subspaces of

dimension s and t respectively, and these linear subspaces do not meet each other.

But there was a gap in our proof. We first showed that if a line contains at least 3 points of S, then the line is contained in S [1, Claim 1 in the proof of Theorem 3.1], and next that if H_0 is a hyperplane with $\#(H_0 \cap S) = N(s-1) + N(t)$, then every line containing two points of $S \setminus H_0$ meets H_0 at a point of S [1, Claim 2 in the proof of Theorem 3.1]. From those facts, we deduced the linear span of $S \setminus H_0$ was contained in S. The reasoning works well if $q \geq 3$ because of the following characterization of a linear subvariety, or a subvariety of degree 1, in $\mathbb{A}^n$:

> *Let k be a field with more than two elements, and A a subset of an affine n-space $\mathbb{A}^n$ over k. If any (affine) line joining two points of A is contained in A, then A is a linear subvariety of $\mathbb{A}^n$.*

Obviously, the statement is not correct if $k = \mathbb{F}_2$ because any subset of $\mathbb{A}^n$ over $\mathbb{F}_2$ satisfies the assumption; therefore the proof presented in [1] is valid for the case $q > 2$. So we give an alternative proof of Theorem for $q = 2$. We divide the proof into two cases, which is either $s > t$ or $s = t$. The proof for the case $s > t$ works well for any q, but the proof for $s = t$ is effective only for $q = 2$.

1. Preliminary

In both of two cases, a fundamental idea was already appeared in [1].

LEMMA 1.1. *Under the situation stated in Theorem, if H is a hyperplane with $\#(H \cap S) > N(s-1) + N(t-1)$, then every line containing two points of $S \setminus H$ meets H at a point of S.*

Proof. The proof of Claim 2 in [1, Theorem 3.1] can be understood as a proof of this lemma. □

In our new proof, we need the following lemma.

LEMMA 1.2. *Let $H_1, \ldots, H_r$ be hyperplanes of $\mathbb{P}^n$ satisfying that $\dim \cap_{i=1}^r H_i = k$. Then $r \leq N(n-k-1)$.*

Proof. Since $\dim \cap_{i=1}^r H_i = k$, the number of hyperplanes containing the linear subspace is at most $N(n-k-1)$. □

COROLLARY 1.3. *For a positive integer α, we have*

$$\#\left(\bigcap_{i=1}^{N(\alpha)} H_i\right) \leq N(n - \alpha - 1),$$

where $H_1, \ldots, H_{N(\alpha)}$ are hyperplanes.

Proof. Let $k = \dim \cap_{i=1}^{N(\alpha)} H_i$. Then $N(\alpha) \leq N(n - k - 1)$ by the above lemma. Hence $\alpha \leq n - k - 1$ because $N(\alpha)$ is an increasing function on α, and we have

$$\#\left(\bigcap_{i=1}^{N(\alpha)} H_i\right) = N(k) \leq N(n - \alpha - 1),$$

as desired. $\square$

2. The case $s > t$

In this section, we assume that $s > t$. A hyperplane H is said to be of type $\mathrm{I_a}$ or $\mathrm{I_b}$ or II according as $\#(H \cap S)$ is equal to $N(s) + N(t - 1)$ or $N(s - 1) + N(t)$ or $N(s - 1) + N(t - 1)$ respectively.

LEMMA 2.1. *Let H_1 and H_2 be two hyperplanes, each of which is of type $\mathrm{I_a}$ or $\mathrm{I_b}$. If $H_1 \cup H_2 \not\supseteq S$, then both of the hyperplanes are the same type.*

Proof. Let $P_0 \in S \setminus (H_1 \cup H_2)$. From Lemma 1.1, we can define a map

$$(H_1 \setminus H_1 \cap H_2) \cap S \longrightarrow (H_2 \setminus H_1 \cap H_2) \cap S$$

by $Q \mapsto \ell(P_0, Q) \cap H_2$, where $\ell(P_0, Q)$ denotes the line joining P_0 and Q. Since we can define the inverse of the map in similar way, $\#(H_1 \cap S) = \#(H_2 \cap S)$. $\square$

Proof of Theorem for the case $s > t$. Let $\{H_i\}_{i=1}^{N(t)}$ be the set of hyperplanes of type $\mathrm{I_a}$, and $\{K_j\}_{j=1}^{N(s)}$ that of type $\mathrm{I_b}$. Since $S \subseteq H_i \cup K_j$ by Lemma 2.1, $S \subseteq (\cap_{i=1}^{N(t)} H_i) \cup (\cap_{j=1}^{N(s)} K_j)$. Hence

$$(2.1) \qquad S = \left(\left(\bigcap_{i=1}^{N(t)} H_i\right) \cap S\right) \bigcup \left(\left(\bigcap_{j=1}^{N(s)} K_j\right) \cap S\right).$$

From Corollary 1.3 and the assumption $s + t = n - 1$, we have two formulas

$$(2.2) \quad \# \left(\left(\bigcap_{i=1}^{N(t)} H_i \right) \cap S \right) \leq \# \left(\bigcap_{i=1}^{N(t)} H_i \right) \leq N(n - t - 1) = N(s)$$

and

$$(2.3) \quad \# \left(\left(\bigcap_{j=1}^{N(s)} K_j \right) \cap S \right) \leq \# \left(\bigcap_{j=1}^{N(s)} K_j \right) \leq N(n - s - 1) = N(t).$$

Adding (2.2) and (2.3), and considering (2.1), we have

$$
\begin{aligned}
\#S \quad \leq \quad & \# \left(\left(\bigcap_{i=1}^{N(t)} H_i \right) \cap S \right) + \left(\left(\bigcap_{j=1}^{N(s)} K_j \right) \cap S \right) \\
\leq \quad & \# \left(\bigcap_{i=1}^{N(t)} H_i \right) + \# \left(\bigcap_{j=1}^{N(s)} K_j \right) \\
\leq \quad & N(s) + N(t).
\end{aligned}
$$

(2.4)

Since $\#S = N(s) + N(t)$, equality holds in each step in (2.2), (2.3) and (2.4). Therefore $S = (\cap_{i=1}^{N(t)} H_i) \cup (\cap_{j=1}^{N(s)} K_j)$ and $(\cap_{i=1}^{N(t)} H_i) \cap (\cap_{j=1}^{N(s)} K_j) = \emptyset$. This completes the proof. $\qquad \square$

3. The case $s = t$ and $q = 2$

In this section, we give a proof of Theorem for the case $s = t$ and $q = 2$. In this case,

$$n = 2s + 1$$

and

$$\#S = 2N(s) = 2^{s+2} - 2.$$

Moreover, if H is a hyperplane of $\mathbb{P}^n$, then $\#(H \cap S)$ is equal to either $2^{s+1} + 2^s - 2$ or $2^{s+1} - 2$; a hyperplane H is said to be of type I or II according as $\#(H \cap S) = 2^{s+1} + 2^s - 2$ or $2^{s+1} - 2$ respectively. By our assumption, the number of hyperplanes of type I is $2^{s+2} - 2$. We denote by $\mathcal{H}$ the set of hyperplanes of type I. The statement is obviously true if $s = 0$; so we may assume $s > 0$.

For two hyperplanes $H_1, H_2 \in \mathcal{H}$, we denote by $H_1 \sim H_2$ if $H_1 \cup H_2 \not\supseteq S$.

Step 1. The relation $\sim$ is an equivalence relation on $\mathcal{H}$.

Proof. The relation is obviously reflexive and symmetric. We check its transitivity. Let H_1, H_2 and H_3 be hyperplanes in $\mathcal{H}$ such that $H_1 \sim H_2$ and $H_2 \sim H_3$. By definition, we can choose points $P \in S \setminus H_1 \cup H_2$ and $Q \in S \setminus H_2 \cup H_3$. If either $P \notin H_3$ or $Q \notin H_1$, then we have $H_1 \cup H_3 \not\supseteq S$. So we may assume that $P \in H_3$ and $Q \in H_1$. Note that $P \neq Q$ because $P \in H_3$ but $Q \notin H_3$. Let R be the remaining point of the line $\ell(P, Q)$, that is, $\ell(P, Q) = \{P, Q, R\}$. Since $P \in H_3$ but $Q \notin H_3$, R does not lie on H_3, and since $Q \in H_1$ but $P \notin H_1$, R does not lie on H_1, either. Now we show that $R \in S$. Since neither P nor Q lies on H_2, $\ell(P, Q) \cap H_2 = \{R\}$. Since H_2 is of type I, R is a point of S by Lemma 1.1. Therefore $H_1 \sim H_3$. $\qquad\square$

Now we introduce new notation. Let H_1 and H_2 be distinct hyperplanes of our $\mathbb{P}^n$. Since our base field is $\mathbb{F}_2$, there are precisely three hyperplanes that contain $H_1 \cap H_2$. We denote by $H_1 + H_2$ the third hyperplane. Note that

$$(3.1) \qquad (H_1 + H_2) \cap H_1 = (H_1 + H_2) \cap H_2 = H_1 \cap H_2$$

and

$$(3.2) \qquad H_1 \cup H_2 \cup (H_1 + H_2) = \mathbb{P}^n.$$

Step 2. If $H_1, H_2 \in \mathcal{H}$ such that $H_1 \sim H_2$ but $H_1 \neq H_2$, then $H_1 + H_2 \in \mathcal{H}$ and $H_1 + H_2 \sim H_1 \sim H_2$.

Proof. Since $S \not\subseteq H_1 \cup H_2$, we have

$$\begin{aligned} 2^{s+2} - 2 = \#S \quad > \quad & \#(H_1 \cap S) + \#(H_2 \cap S) - \#(H_1 \cap H_2 \cap S) \\ = \quad & 2(2^{s+1} + 2^s - 2) - \#(H_1 \cap H_2 \cap S). \end{aligned}$$

Hence $\#(H_1 \cap H_2 \cap S) > 2^{s+1} - 2$. Since $H_1 + H_2 \supseteq H_1 \cap H_2$, we have

$$\#((H_1 + H_2) \cap S) > 2^{s+1} - 2,$$

which means that $H_1 + H_2$ is also of type I.

We denote $H_1 \cap H_2$ by D. Then

$$\begin{aligned} & 2^{s+2} - 2 \\ = \quad & \#S \\ = \quad & \#(H_1 \cap S) + \#(H_2 \cap S) + \#((H_1 + H_2) \cap S) - 2\#(D \cap S) \\ = \quad & 3(2^{s+1} + 2^s - 2) - 2\#(D \cap S) \\ & \text{(because } H_1 + H_2 \text{ is also of type I)} \end{aligned}$$

by (3.1) and (3.2). Hence $\#(D \cap S) = 2^{s+1} + 2^{s-1} - 2$, and hence $\#((H_2 \setminus D) \cap S) = 2^{s-1} > 0$. Therefore $(H_1 + H_2) \cup H_1 \not\supseteq S$, which means $H_1 + H_2 \sim H_1$. $\square$

Step 3. There are precisely two equivalence classes for the relation $\sim$.

Proof. By Step 2, the set of hyperplanes in any particular equivalence class forms a linear subspace of the dual projective space of our $\mathbb{P}^n$. Hence the number of hyperplanes in an equivalence class is odd. But $\#\mathcal{H}$ is even. So $\mathcal{H}$ is divided into at least two equivalence classes.

Let $H_1, H_2 \in \mathcal{H}$ such that $H_1 \not\sim H_2$, that is, $H_1 \cup H_2 \supseteq S$. We want to show that any $H_3 \in \mathcal{H}$ is equivalent to either H_1 or H_2. Suppose that $H_3 \in \mathcal{H}$ is equivalent to neither H_1 nor H_2, i.e., $H_1 \cup H_3 \supseteq S$ and $H_2 \cup H_3 \supseteq S$. Since

$$(H_1 \cup H_2) \cap (H_2 \cup H_3) \cap (H_3 \cup H_1) \supseteq S,$$

we have

(3.3) $$(H_1 \cap H_2) \cup (H_2 \cap H_3) \cup (H_3 \cap H_1) \supseteq S.$$

Let $L := H_1 \cap H_2 \cap H_3$, $x_{ij} = \#((H_i \cap H_j \setminus L) \cap S)$ and $y = \#(L \cap S)$. From (3.3), we have

(3.4) $$x_{12} + x_{23} + x_{31} + y = 2^{s+2} - 2.$$

On the other hand, for any permutation (i, j, k) of $(1, 2, 3)$, we have

(3.5) $$x_{ij} + x_{jk} + y = 2^{s+1} + 2^s - 2,$$

because the left hand side of (3.5) is just the number $\#(H_j \cap S)$. From (3.4) and (3.5), we have

$$y = 2^s - 2 \quad \text{and} \quad x_{12} = x_{23} = x_{31} = 2^s.$$

Fix a point $P_0 \in (H_2 \cap H_3 \setminus L) \cap S$. For any point $P \in (H_2 \cap H_3 \setminus L) \cap S \setminus \{P_0\}$, the line $\ell(P_0, P)$ meets H_1 at a point of S by Lemma 1.1. We denote by $\varphi_{P_0}(P)$ the point $\ell(P_0, P) \cap H_1$. On the other hand, since $P_0, P \in H_2 \cap H_3$, $\varphi_{P_0}(P)$ is also in $H_2 \cap H_3$. So φ_{P_0} is a map

$$(H_2 \cap H_3 \setminus L) \cap S \setminus \{P_0\} \to L \cap S.$$

Since $\ell(P_0, P)$ consists of three points, the map φ_{P_0} is injective; however, it is impossible because $x_{23} = 2^s$ and $y = 2^s - 2$. $\square$

Final Step. From Step 3, $\mathcal{H}$ consists of two equivalence classes, say $\mathcal{H}_a$ and $\mathcal{H}_b$. Recall that $S \subseteq H \cup K$ if $H \in \mathcal{H}_a$ and $K \in \mathcal{H}_b$. Hence

$$S \subseteq \bigcap_{(H,K)\in\mathcal{H}_a\times\mathcal{H}_b} (H \cup K) = \left(\bigcap_{H\in\mathcal{H}_a} H\right) \cup \left(\bigcap_{K\in\mathcal{H}_b} K\right).$$

On the other hand, since the set of hyperplanes of each equivalence class forms a projective space, there are two nonnegative integers s_1 and s_2 so that $\#\mathcal{H}_a = N(s_1)$ and $\#\mathcal{H}_b = N(s_2)$. Hence $2^{s_1+1} + 2^{s_2+1} = 2^{s+2}$ because $\#\mathcal{H}_a + \#\mathcal{H}_b = \#\mathcal{H}$, which implies $s_1 = s_2 = s$. Hence $\#\left(\bigcap_{H\in\mathcal{H}_a} H\right)$ and $\#\left(\bigcap_{K\in\mathcal{H}_b} K\right)$ are at most $N(s) = 2^{s+1}-1$ by Corollary 1.3. Therefore S can be decomposed into two linear subspaces as

$$S = \left(\bigcap_{H\in\mathcal{H}_a} H\right) \cup \left(\bigcap_{K\in\mathcal{H}_b} K\right) \quad \text{with} \quad \left(\bigcap_{H\in\mathcal{H}_a} H\right) \cap \left(\bigcap_{K\in\mathcal{H}_b} K\right) = \emptyset,$$

and each linear subspace is of dimension s because $\#S = 2N(s)$. This completes the proof. $\qquad\square$

References

[1] M. Homma, S. J. Kim and M. J. Yoo, *Projective systems supported on the complement of two linear subspaces*, Bull. Korean Math. Soc. **37** (2000), 493–505.

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