Optimization Problems in Elementary Geometry

A K Mallik

Optimization, a principle of nature and engineering design, in real life problems is normally achieved by using numerical methods. In this article we concentrate on some optimization problems in elementary geometry and Newtonian mechanics. These include Heron's problem, Fermat's principle, Brachistochrone problems, Fagano's problem, geodesics on the surface of a parallelepiped, Fermat/Steiner problem, Kakeya problem and the isoperimetric problem. Some of these are very old and historically famous problems, a few of which are still unresolved. Close connection between Euclidean geometry and Newtonian mechanics is revealed by the methods used to solve some of these problems. Examples are included to show how some problems of analysis or algebra can be solved by using the results of these geometrical optimization problems.

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Introduction

Optimization is a way of life. We always try to minimize the effort (cost) and/or maximize the benefit (profit). Nature too, through its strategy of random mutation and survival of the fittest, is optimizing the biological functions. Of late, genetic algorithm (GA), is trying to mimic this natural evolutionary process to solve some optimization problems. Optimization has become an important tool for engineering design. But much earlier, optimization was thought to be the guiding principle of formulation of natural physical processes. Pierre Louis Moreau de Maupertius (1698 – 1759), in his speech in 1746 as the President of The Academy of Sciences in Berlin, alluded to this principle in a metaphysical way

Keywords
Brachistochrone problem, Fermat, Steiner problem, Kakeya problem, geodesic.

Box 1.

"The description of right lines and circles, upon which geometry is based, belongs to mechanics. Geometry does not teach us how to draw these lines, but requires them to be drawn.''

— *Isaac Newton*

It may be worthwhile to remember that when we draw a straight line using a straight edge, we merely copy an exact straight line and do not generate one.

– he talked of 'God's intention to regulate physical phenomena by a principle of perfection'. He was ridiculed by Voltaire, but finally in the hands of great geniuses, like Euler, Lagrange and Hamilton, 'The Principle of Least Action' was established. In the process, a new branch of Mathematics, known as Calculus of Variations was born. This powerful mathematical tool has been applied successfully in optics, electrodynamics, mechanics and other branches of physics. In this article we will restrict ourselves to optimization problems in elementary (Euclidean) geometry only. Due to close connections between geometry and Newtonian mechanics, occasionally we will digress to some problems in mechanics (see *Box* 1).

A large number of interesting and useful optimization problems have been posed and solved in the area of Euclidean geometry. If the existence of a unique solution is assumed, then a direct solution can often be found. But the risk of such a method, in case no such solution exists, is that by following perfectly valid mathematical arguments one can land in nonsense. The only mistake is the assumption of the existence of the solution. This was first pointed out by Weierstrass, one of the most rigorous mathematicians.

Let us start with a trivial problem. What is the maximum possible area of a triangle with two sides of given length, say a and b? By denoting the angle between the two sides as θ, we can write the area as $(1/2)ab\sin\theta$. Therefore, the answer is $(1/2)ab$ and the two given sides are at 90^o to each other. For any other angle, the area will be less than $(1/2)ab$.

Let us now consider some classical optimization problems in geometry and physics.

Heron's Problem: Extremum Properties of Light Rays – The Law of Reflection

Referring to *Figure* 1, say a ray of light starting from the point A has to reach the point B via a reflecting surface (the mirror M), then where should it meet M? According to Heron, it should meet the mirror at a point P such that the distance AP + PB is minimum, because light travels along the shortest path. The point P is obtained as follows: Join the point A with the mirror image of the point B, i.e., B′ and P is the point of intersection of AB′ with the mirror M. It is quite trivial to prove that AP + PB = AB′. For any other point P′ on the mirror, AP′ + P′B = AP′+ P′B′, which is the broken path from A to B′ and consequently greater than the straight path AB′. From *Figure* 1, it is simple to see that all the angles indicated by θ are equal and this proves the law of reflection which states that the angle of incidence is equal to the angle of reflection, both being defined by $((\pi/2) - \theta)$.

Snell's Law of Refraction and Fermat's Principle

When a light ray meets an interface separating two different media 1 and 2, it is seen that the ray bends (see *Figure* 2). Thus starting from the point A, the light ray reaches the point B along the broken path which is obviously not the shortest path from A to B. Fermat conjectured that the light travels along the path of least *time*.

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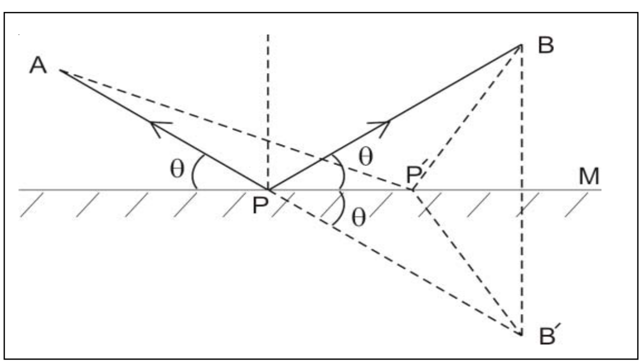
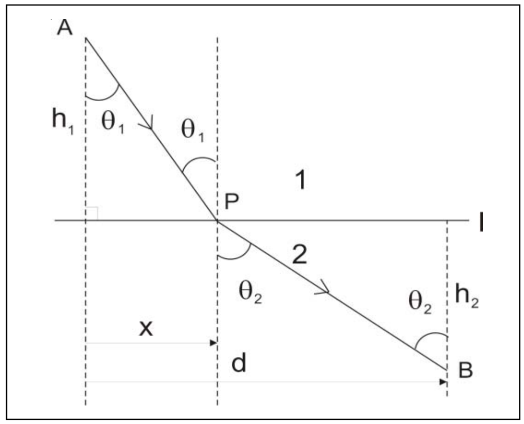


Figure 1.

Figure 2.



Let the ray from A meet the interface I at the point P in order to reach the point B in least time. The distances of the interface from the points A and B are indicated respectively, by h_1 and h_2. Let the speed of light in media 1 and 2 be v_1 and v_2, respectively. Referring to *Figure* 2, the total time of travel can be written as

$$T = \frac{\sqrt{h_1^2 + x^2}}{v_1} + \frac{\sqrt{h_2^2 + (d-x)^2}}{v_2}.$$

For T to be minimum, setting $\frac{\mathrm{d}T}{\mathrm{d}x} = 0$ one gets after simplification

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \tag{1}$$

or,

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} = n(\text{constant}). \tag{2}$$

Equation (2) is known as Snell's law (obtained experimentally). It may be pointed out that without any change of medium (during reflection), the shortest path and the path of least time are identical. It is now known that Fermat's principle of shortest time path should actually be modified as the stationary-time path. This means the path where a little change in it leaves the time of travel unaltered (implying both maximum and minimum).

Brachistochrone of (Johann) Bernoulli

In the last decade of the seventeenth century, Bernoulli posed the following problem. Consider two points A and B at different levels (see *Figure* 3). A smooth particle slides down a curve in the vertical plane containing A and B under the action of gravity starting from rest at A. Obtain the shape of the curve for which the time of descent is minimum. He solved the problem just using Fermat's principle discussed above (see *Box* 2). At a depth y below the point, the speed of the particle is $v = \sqrt{2gy}$, where g is the acceleration due to gravity. Considering the depth from A to B as consisting of a large number of layers, where the speed of the particle is different in each layer, to minimize the time of travel this bending should follow (1). In other words,

$$\frac{\sin\theta}{v} = \text{constant}$$

$$\text{or} \qquad \frac{\cos\phi}{v} = \text{constant, say } c_1, \tag{3}$$

where ϕ is the angle made by the velocity vector (tangent to the path) with the x–axis. So, the differential equation of the required curve is easily obtained, using (3) and substituting for v, as

$$\frac{dy}{dx} = \tan\phi = \sqrt{\frac{c-y}{y}}, \tag{4}$$

where $c = 1/(2c_1^2\, g)$.

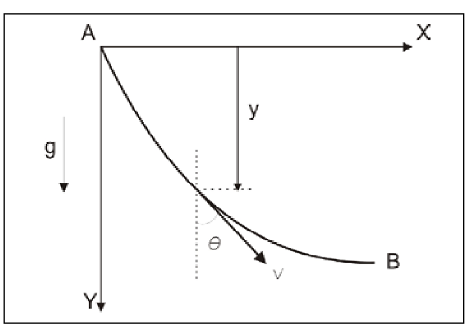
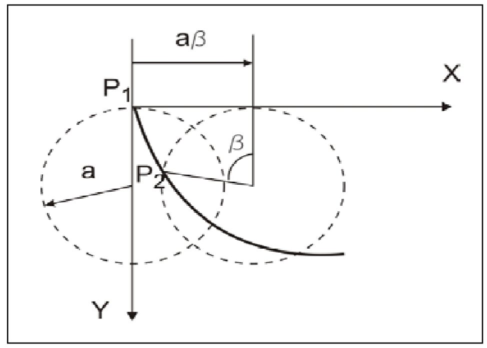


Figure 3.

Box 2.

Bernoulli was so proud of solving the brachisto (minimum) chrone (time) problem that he challenged Newton to solve it within six months. Newton said "I do not want to be dunned and teased by foreigners about mathematical things'' and solved the problem in a few days. He published the result anonymously. Bernoulli, a follower and admirer of Leibniz, did not like Newton. For once, even Bernoulli paid his tribute to Newton by saying "Lion is known by his paws.''

Figure 4.



The nonlinear differential equation can be solved by substituting

$$y = c\sin^2\alpha, \tag{5}$$

when (4) yields

$$\mathrm{d}x = c(1 - \cos 2\alpha)\mathrm{d}\alpha. \tag{6}$$

For the point A, $x = 0$ and $y = 0$ (which implies $\alpha = 0$). Integrating (6) and using this fact, finally one gets

$$x = a(\beta - \sin\beta),$$

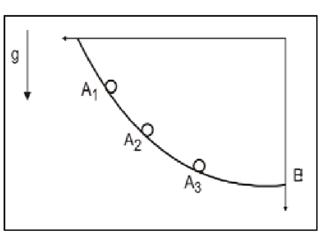
$$y = a(1 - \cos\beta),$$

$$a = c/2,\ \ \beta = 2\alpha. \tag{7}$$

Equation (7) represents a cycloid which is generated by a point on the rim of a circular disc of radius a rolling on a flat surface (see *Figure* 4). The name cycloid was coined by Galileo (see *Box* 3).

Huygens had earlier shown that a particle rolling down a cycloid (starting from rest) has another curious feature: the time of descent up to the lowest point (B in *Figure* 5) is independent of the starting point, i.e. the time of descent to B is the same from A_1 or A_2 or A_3 and so on. So he named it tauto(same)chrone. By noting that the involute of a cycloid is another cycloid, he designed a pendulum whose bob was constrained to move along a cycloid (by providing cycloidal cheeks – see *Figure* 6)

Figure 5.



Box 3.

Galileo, in his last book, reported the quadrant of a circle passing through A and B as the path having the shortest time of descent (see *Figure* A) among all the series of chords along the circle passing through A and B. Towards this end, he first showed that a particle, starting from rest, takes the same time (T) to reach the point B along the inclined planes AB and A′B for all values of θ. This can be easily established as the length $A'B = 2R \sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$ and the acceleration along A′B is $g\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$. Therefore, $T = 2\sqrt{R/g}$ is independent of θ. Then he showed that the time taken by the particle, starting from A, following two inclined planes AA′ and then A′B (i.e., the broken path AA′B) is less than T. Thereafter, with some fallacious arguments he reached the conclusion that if the path is continuously broken along the circle, the time of descent will be least. It can be shown that the time of descent along the quadrant of the circle comes out approximately $1.8541\sqrt{R/g}$, where R is the radius of the circle. If the points A and B are really joined by a cycloid (the correct brachistochrone of Bernoulli), then the time of descent turns out as $1.8257\sqrt{R/g}$. So, the quadrant of a circle as argued by Galileo has only 1.56% error!

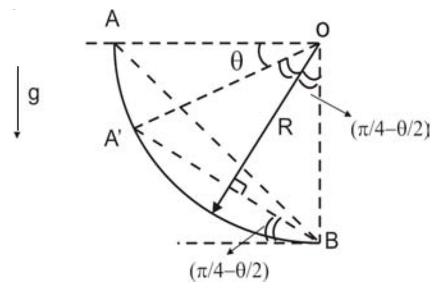
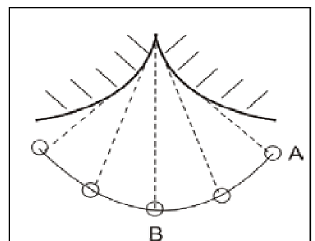


Figure A.

and consequently its time period was independent of the amplitude of oscillation (unlike in a simple pendulum, where the bob moves along a circle).



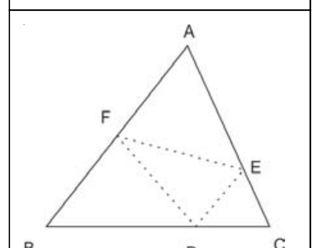


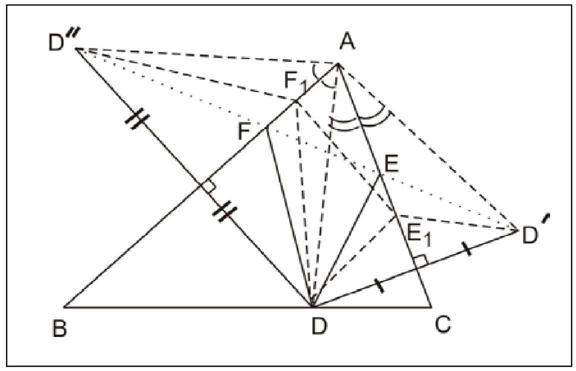
Figure 6. (top).
Figure 7. (bottom).

Fagano's Problem

To inscribe, in a given acute-angled triangle, the triangle of minimum perimeter. First we discuss the direct solution assuming its unique existence. The solution is in two steps.

First Step: Let ABC be the given acute-angled triangle (*Figure* 7). We consider a specified point D on the side BC. We would like to determine the points E (on AC) and F (on AB), such that for the given point D, the perimeter of the triangle DEF is minimum. Towards

Figure 8.



this end, we draw perpendiculars from the point D on to the sides AB and AC and extend these perpendiculars up to D″ and D′, respectively, so that AB and AC become bisectors of DD″ and DD′ (see *Figure* 8). Join D′D″ and let this line intersect AC at the required point E and AB at F. It is easy to see that the perimeter of the triangle DEF is equal to D′D″.

For any other choice, like E_1 (on AC) and F_1 (on AB), the perimeter of the triangle DE_1F_1 is given by the broken distance $D'E_1F_1$ D″ and is obviously more than that of the triangle DEF.

Second Step: In this step, the point D is so chosen that the distance D′D″ is minimized. Towards this end, we first note that $\angle D''AD' = 2\ \angle A$ = a given quantity, and $AD' = AD = AD''$. So for the base of the isosceles triangle AD′D″ (= D″D′) with a given vertex $\angle D''AD'$, to be minimum, the sides $AD''= AD'(= AD)$ should be minimum. Now, for AD to be minimum, D must be the foot of the altitude from the vertex A.

In the first step, we could have taken an arbitrary point E (on AC) or F (on AB) and in the second step, these would have turned out to be the feet of the altitudes drawn from B and C, respectively. If the solution is unique, then the inscribed triangle (DEF) of minimum perimeter is obviously obtained by joining the feet of the three altitudes.

Schwarz's Proof

Schwarz posed the problem as to prove that for an acute-angled triangle ABC, the altitude triangle PQR has the minimum perimeter amongst all the inscribed triangles (see *Figure* 9).

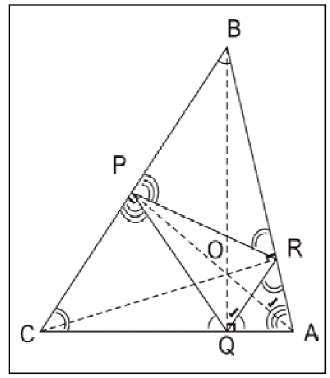


Figure 9.

It is easy to see that the quadrilaterals OQAR, OPCQ and ORBP are all cyclic quadrilaterals. Therefore,

$\angle RQA = 90^o - \angle OQR = 90^o - \angle OAR = \angle B$ and similarly $\angle PQC = \angle B$,

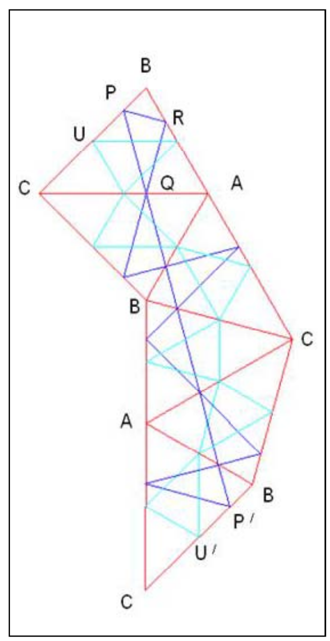
or, $\angle RQA = \angle PQC$.

Similarly, $\angle RPB = \angle QPC$ and $\angle QRA = \angle PRB$.

Thus a light ray starting from P returns to P after being reflected successively at Q and R with the sides of the triangle ABC acting as mirrors.

Consider two inscribed triangles one of which is the altitude triangle PQR (see *Figure* 10) and five successive reflections explained in the figure. After all the reflections the side BC becomes parallel to its original orientation. Therefore, the straight lines PP′ and UU′ are of equal length. It is readily seen that the distance PP′ is equal to twice the perimeter of the altitude triangle, whereas twice the perimeter of the other triangle is given by the zigzag length UU′. Hence the perimeter of the altitude triangle is minimum.

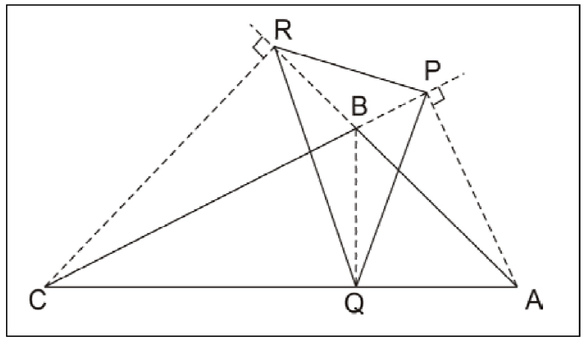
Figure 10.



It can be shown that the perimeter of the altitude triangle is less than twice the shortest altitude. One may ask what happens to this minimum property of the altitude triangle if the original triangle ABC is an obtuse-angled triangle (see *Figure* 11). In this case, the perimeter of the altitude triangle is more than twice of the shortest altitude (BQ in the figure).

It can be shown that, in this case, for the altitude trian-

Figure 11.



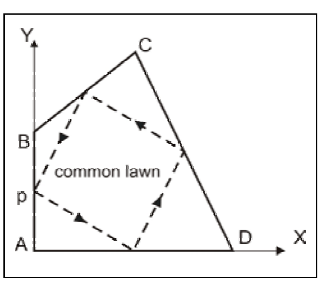
gle the expression $p + q - r$ attains a stationary value, where p, q and r represent the sides of the triangle PQR (P, Q and R are three points on the sides (if necessary extended)) with r opposite to the obtuse angle.

Minimum Path within a Quadrilateral

Consider a quadrilateral ABCD (*Figure* 12) and a given point p on one of the sides AB. We have to determine the shortest path starting from p and returning to p, touching all the sides of the quadrilateral once. In words, one can pose the problem as if there is a common lawn ABCD and a house at p. Where should the other three houses q, r and s on the other three sides be located so that the brick path joining the houses will be of minimum length?

The solution can be obtained by reflecting the quadrilateral successively on sides AD, DC and CB as explained in *Figure* 13. Join p with its final reflected position (p_3) and obtain q, r and s by noting the points of intersection of pp_3 with the different (reflected) sides.

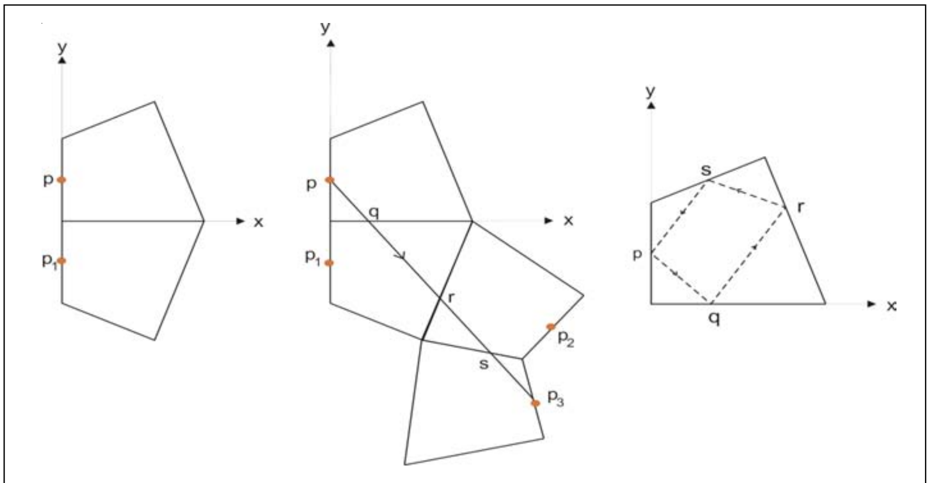
Figure 12.



Depending on the shape of the original quadrilateral, it may happen that the line pp_3 may not intersect the line BC (see *Figure* 14). In that case the path is via the corner C as shown in the figure.

Geodesics on the Surface of a Parallelepiped

The shortest path between two points on a surface (along the surface) is known as geodesic. Obviously, for a plane



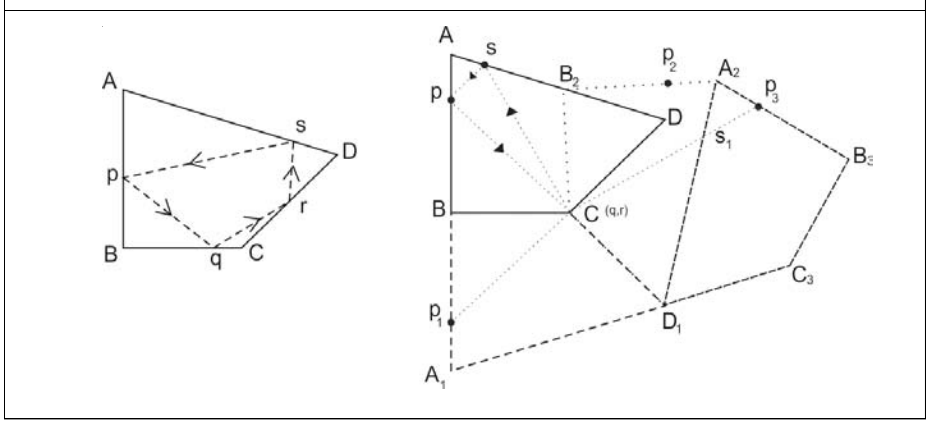
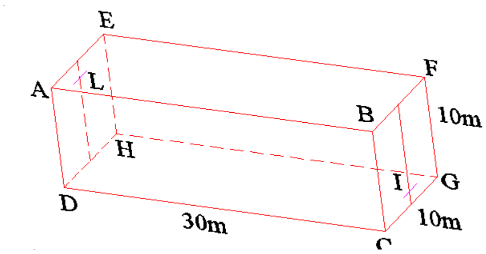
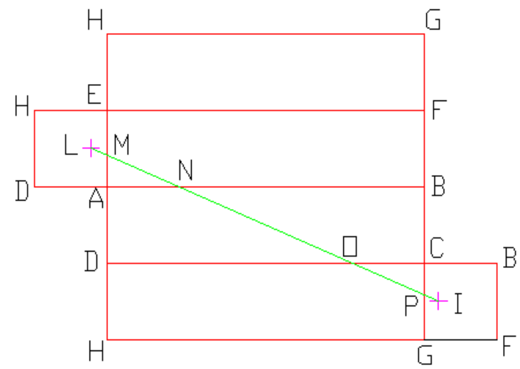


Figure 13 (top).
Figure 14 (bottom).

surface the geodesic is the line joining the two points. A rectangular parallelepiped consists of six different plane surfaces. Here the geodesic between two points lying on two different surfaces can be counter-intuitive. Geodesics, of course, will be a combination of a number of straight lines lying on different surfaces. Here we discuss three popular problems.

Problem 1: Referring to *Figure* 15, we consider a room of dimensions shown. On the centerlines of the two opposite vertical faces L indicates a lizard and I indicates an insect. The lizard is 1m below the ceiling and the insect is 1m above the floor. What is the length of the minimum path that the lizard has to cover to reach the insect? The lizard can travel on all faces.





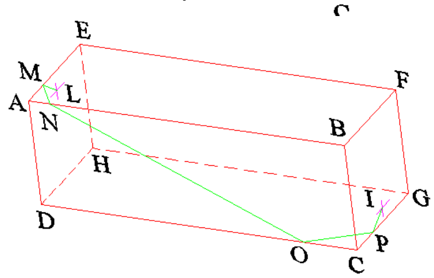


Figure 15 (top-left).
Figure 16 (top-right).
Figure 17 (bottom-left).

The most common (and wrong) answer is 40m. To get the correct answer we have to unfold different surfaces along edges to obtain a flat picture. This unfolding (in the parlance of engineering drawing – development) can be done in various ways. The geodesic is obtained by the unfolding shown in *Figure* 16. The path of the lizard is obtained by folding as shown in *Figure* 17. One should note that the lizard has to use five of the six faces. Obviously there is another symmetric path using the back vertical face instead of the front one. The length of the geodesic is easily seen to be $\sqrt{32^2 + 20^2} = 37.736$ m.

By using four surfaces the lizard can find a path which is longer than 37.736 m but shorter than 40 m (using only three surfaces). Here we leave two similar problems for the readers to try.

Problem 2: Note that the geodesic is defined for two given end points on the surfaces of the room. Suppose one of these is given as the corner D (*Figure* 15). The length of the geodesic then depends on the choice of the other point. Determine this other point so that the length of the geodesic is maximum. What is this maximum value? (The other end is not the point F and the maximum length is more than $\sqrt{1300}$m.)

Problem 3: If one is free to choose both the end points, then what is the maximum possible length of the geodesic? (It is more than 40 m.)

Fermat/Steiner Problem

Within an acute-angled triangle ABC[1], determine the point P such that PA+PB+PC is minimum. In the seventeenth century Fermat posed this problem to Torricelli, who solved it in more than one ways. The same problem was again discussed by Steiner in the nineteenth century.

[1] In fact, it is enough if each angle of the triangle is less than 120°.

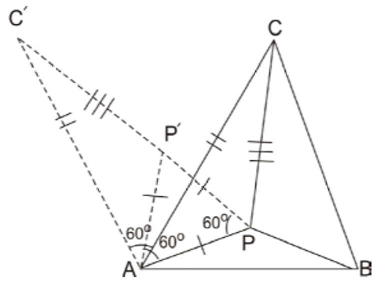
First we assume that a unique solution exists. Referring to *Figure* 18, we consider an inside point P. Now rotate the triangle APC about the point A as a rigid body through 60° in the counter-clockwise direction as shown, when the point C goes to C′ and P goes to P′. The zigzag path from B to C′ is

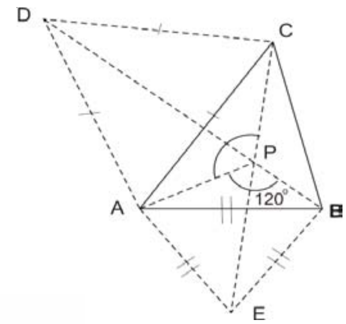
$$\mathrm{BP} + \mathrm{PP}' + \mathrm{P}'\mathrm{C}' = \mathrm{BP} + \mathrm{PA} + \mathrm{PC}$$

(since the triangle APP′ is equilateral).

Thus the required sum is minimum when the zigzag path becomes straight. Note that the location of the point C′ is independent of the choice of the point P. For the path BPP′C to be straight, the angle $\angle \mathrm{APB} = 120^\circ$ (since $\angle \mathrm{APP}' = 60^\circ$). Instead of rotating the triangle APC about A, we could have rotated the triangle BPA about B or the triangle

CPB about C. Then in the same way, one would reach the conclusion that, at the required point P the sides BC and AC also subtend an angle 120°. Thus, the point P is located so that all the sides of the triangle ABC subtend an angle 120° at P. Though we started with an acute-angled triangle, the method explained above clearly tells that the solution is valid so long as all the angles of the triangle are less than 120°. If the angle at A is more than 120°, then the point C′ comes below the line AB.





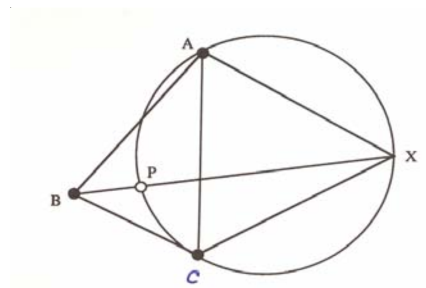


Figure 18 (top-left).
Figure 19 (top-right).
Figure 20 (bottom-left).

If any of the angle of the triangle is equal to or more than 120°, then the point P is identical with the vertex containing this obtuse angle. *Figures* 19 and 20 show two easy ways of locating the point P geometrically. In *Figure* 19, ABD and ACE are equilateral triangles and P is at the intersection of BE and CD. In *Figure* 21, ACX is an equilateral triangle and P is at the intersection of BX and the circumcircle of ACX.

An alternative proof and solution to the above problem is based on Viviani's theorem, which states that in an equilateral triangle, the sum of the distances of an inside point from the three sides is the same for all the points. This theorem can be proved as follows. Referring to *Figure* 22, for the equilateral triangle PQR of side s, its area Δ can be written as

$$\begin{aligned}\Delta &= \Delta\mathrm{QOR} + \Delta\mathrm{ROP} + \Delta\mathrm{POQ} \\ &= \left(\frac{1}{2}\right) s\,(\mathrm{OA} + \mathrm{OB} + \mathrm{OC})\,.\end{aligned}$$

Therefore, OA + OB + OC = $(2\Delta/s)$, a constant independent of the location of O.

Now for the Fermat/Steiner problem, consider the acute-angled triangle ABC (*Figure* 21). The point O, for OA + OB + OC to be minimum, should be chosen such that perpendiculars drawn to OA, OB and OC constitutes an equilateral triangle (PQR). This implies that the sides AB, AC and BC subtend 120° at O. The last statement is obvious if one notices the cyclic quadrilaterals OAQC, OBPC and OARB. The minimum property of the point O can be proved as follows.

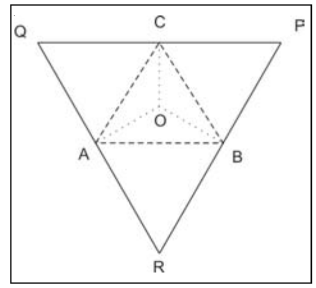


Figure 21.

If O_1 be another point inside the triangle ABC, then let $O_1 A_1$, $O_1 B_1$ and $O_1 C_1$ be the three perpendiculars on the sides of the equilateral triangle from O_1, then

$$A_1O_1 \leq AO_1, B_1O_1 \leq BO_1, C_1O_1 \leq CO_1$$

$$\text{or} \quad A_1O_1 + B_1O_1 + C_1O_1 < AO_1 + BO_1 + CO_1$$

since all equalities cannot be valid simultaneously. Now, from Viviani's theorem

$$OA + OB + OC = O_1A_1 + O_1B_1 + O_1C_1 <$$

$$O_1A + O_1B + O_1C.$$

Solution with a Mechanics Flavour (Leibniz)

Place the triangle ABC on a horizontal table (see *Figure* 22). Drill holes on the table at the three vertices A, B

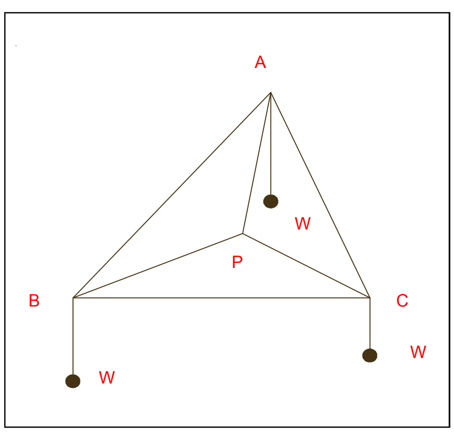


Figure 22.

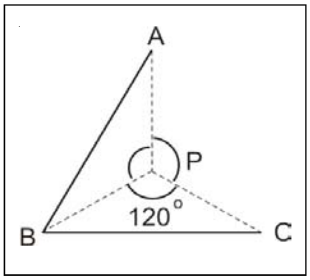


Figure 23.

and C. Through these holes hang three equal weights (W) by tying to strings. The free ends of the strings are tied in a knot. Place the knot on the table. When the system comes to equilibrium, the knot occupies the desired location P. The gravitational potential energy (PE) of the system (with the table as the datum) is

$$\mathrm{PE} = W(\mathrm{PA} + \mathrm{PB} + \mathrm{PC} - L_1 - L_2 - L_3)$$

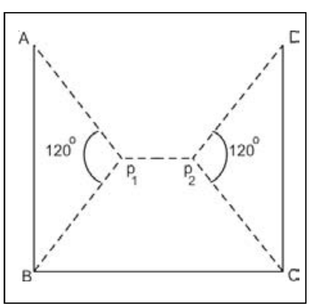
where L_1, L_2 and L_3 are the lengths of the strings passing, respectively, through A, B and C. Since at equilibrium the potential energy is minimum, (PA + PB + PC) will be minimum. Now as three equal forces (W), acting at the point P, are in equilibrium, they must be at 120° to one another. Thus all the sides of the triangle subtend 120° at the point P.

Steiner Points for Three Locations

We have just seen that for three locations (forming a triangle with no angle greater than 120°) to be interconnected by road or cable network, maximum saving can be achieved by creating a virtual hub at the point P (see *Figure* 23). Such virtual hub locations where three lines meet at lines 120° are called Steiner points. If the three locations are interconnected directly (e.g., AB + BC in *Figure* 24), then the total minimum of all such direct connections is called the spanning length. The total length of interconnections when minimized using Steiner points (e.g., PA + PB + PC in *Figure* 23) is called the Steiner length. For three locations at the vertices of an equilateral triangle, it is easily seen that the Steiner ratio, defined as,

$$\frac{\text{Steiner length}}{\text{Spanning length}} = \frac{\sqrt{3}}{2} \approx 0.866 \text{ (see } \mathit{Box}\ 4) \qquad (8)$$

Figure 24.



Steiner Points, Soap Films and Steiner Conjecture

Figure 24 shows that for four locations at the vertices of a square, one needs two Steiner points and the Steiner ratio comes out as $\frac{1+\sqrt{3}}{3}$. Note that at all Steiner points, three lines meet at 120^o. In this case, the shortening of the Steiner length from the spanning length is by approximately 8.9%.

For six locations at the vertices of two adjacent squares (see *Figure* 25a), one may think that the Steiner solution can be obtained by the combination of a square and a triangle. But that is not correct. The correct solution is shown in *Figure* 25b. It may be mentioned that for n given locations, there will be at most n-2 Steiner points.

These solutions can also be obtained by simple experiments. One can connect two transparent (perspex) sheets (of about 3 mm thickness) with a gap (say 3 mm) by a number of metal pins (representing the site locations to be interconnected). This assembly is dipped in a soap (home detergent) solution and taken out. By draining out the extra solution, a soap film (a little glycerin may be added to the soap solution for enhancing the stability of the film) connecting all the pins is formed. To minimize the surface energy, the surface area and hence the total length (since the other dimension of the film is equal to the gap) is minimized. Formation of Steiner points where three film surfaces meet at 120^o can be readily seen for three and four locations (like *Figures* 23 and 24). However, *Figure* 25b can be obtained, only by taking extreme care that all the pins are connected by a single film. Otherwise, *Figure* 25a or some other figure is obtained. *Figure* 25a is obtained when two films are formed, one connecting the vertices of a square and the other, the three vertices of a triangle. Even *Figure* 24 can be repeated side by side when each of the two films connects the vertices of one square each. (See *Box* 5).

Box 4.

Delta Airlines in USA had their hubs in New York, Chicago and Atlanta. They asked Bell Telephone Company to interconnect these three hubs by telephone cables. Bell Telephone charged on the basis of the length of the cables connecting New York – Chicago – Atlanta (spanning length). Someone in Delta Airlines knew that Steiner length will be shorter (cheaper), if a virtual hub is created at the required Steiner point. The three hubs formed approximately an equilateral triangle (for which the saving would have been approximately 13.4% as given by eqn. (8)) and Bell Telephone Company returned 15% of the fees.

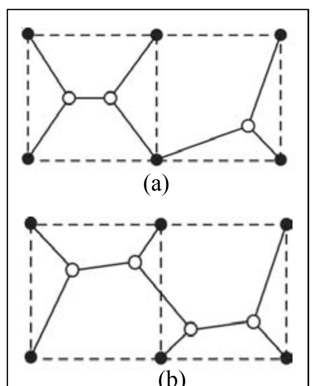


Figure 25.

Box 5.

In the nineteenth century, Belgian physicist Plateau conducted extensive experiments with soap bubbles and came up with two laws of their formation. One of which states that three soap films can meet along an edge with the surfaces at an angle 120° to one another. This is seen in the experiment described here. The second law says that six films can meet at a point with four edges at an angle $\cos^{-1}(-1/3)$ to one another. One can see this by dipping a wire frame in the form of a cube in the soap solution. This second law was proved mathematically in 1976. Incidentally, Plateau blinded himself by looking at the Sun while doing his doctoral thesis on physiological optics! The soap bubble experiments were conducted with the help of his assistants.

In 1960's mathematicians[2] conjectured that for any number of locations placed arbitrarily in a plane, the Steiner ratio is $\geq \sqrt{3}/2$ (≈ 0.866) (see (8)). This came to be known as Steiner ratio conjecture. In 1985, two mathematicians obtained the limit as 0.824. But their method was so cumbersome, that they themselves declared that their result can be improved and announced a prize for one who could prove this conjecture. In 1991, two mathematicians proved it using the concept of game theory.

[2] Obviously not Steiner who lived during the nineteenth century – it has been said that Steiner's only contribution to this problem is his name!

Turning a Transparent Square Opaque

Now let us consider a square of unit side with transparent sides. So light entering (along the plane of the square) through any side at any angle will pass through. The problem is to find the minimum length of black-body obstacle that needs to be put inside the square so that the square becomes opaque. In other words, any light entering the square will not come out of it. One possible solution is shown in *Figure* 26a, where black-body obstacles are put along the diagonals of total length $2\sqrt{2}(\approx 2.828)$. But this is not the minimum. *Figure* 26b shows a solution with the two Steiner points of a square, and the total length of the obstacle is $1+\sqrt{3}(\approx 2.732)$. No one knows what the minimum solution is. The best achieved so far is shown in *Figure* 26c, where three lines meet at 120° (the Steiner point of a triangle) and a fourth one covers half of a diagonal. Here the total length is approximately 2.639.

Kakeya Problem

What is the least area within which a line segment of unit length can be rotated through 360°? The line segment can be translated and rotated.

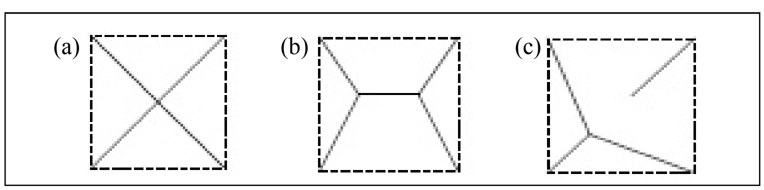
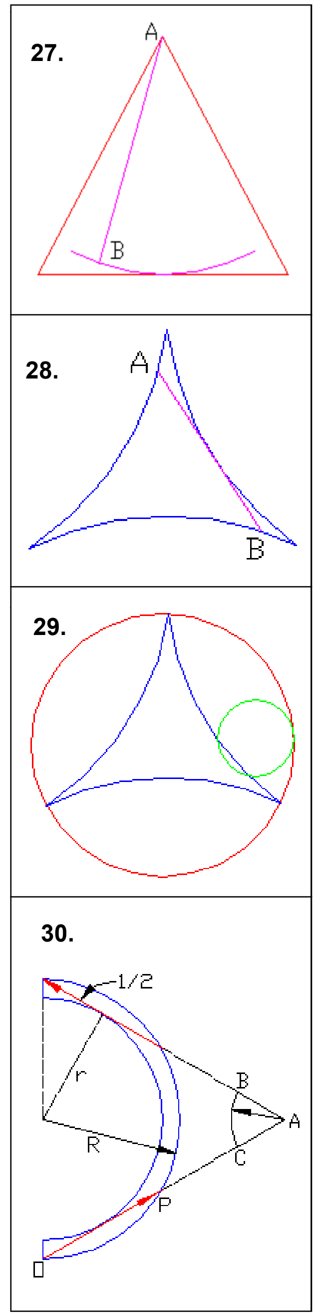


Figure 26.

An obvious area (not minimum) within which this can be accomplished is a circle of unit diameter with area of $\pi/4(\approx 0.7854)$ units. One can do it in a smaller area by considering an equilateral triangle of unit altitude (*Figure* 27) with an area of $1/\sqrt{3}(\approx 0.577)$ units. The line segment AB is rotated about the point A until it coincides with one of the sides of the triangle. Then it is translated so that the end B coincides with another vertex of the triangle. Then it is rotated through 60° about the end B. This process continues until the rod is rotated through 360°, while always remaining within the triangle. It has been proved that this is the minimum area of a *convex* figure within which the task can be accomplished. If a concave figure is permitted, then one can further reduce the area by considering a deltoid big enough to permit turning of a unit line (*Figure* 28). The deltoid is generated by a point on the rim of a wheel rotating inside a pipe of diameter three times that of the wheel (*Figure* 29). The area of this deltoid is $\pi/8(\approx 0.3927)$ units.



Figures 27–30.

Another figure of same area is shown in *Figure* 30. This figure is bounded by two semicircles, where the unit line segment OP can rotate in the counterclockwise direction through some angle. At the end of the turning, the line is translated along BA and then rotated to coincide with AC. Thereafter the rod is translated along AC to be placed within the annulus again and in the process the rod rotates through 180° (watch the arrowhead). The process is repeated once more to make a complete rotation. The area needed is that of the semicircular annulus and that of the sector ABC. The area of the sector ABC can be made negligibly small by making R as large as we want. It is easy to see that $R^2 = r^2 + (1/4)$ and the area of the annulus is $(\pi/2)(R^2 - r^2) = (\pi/8)$.

Since both *Figures* 29 and 30 have the same area, for a long time it was thought that the area cannot be reduced any further. But extending the idea of *Figure*

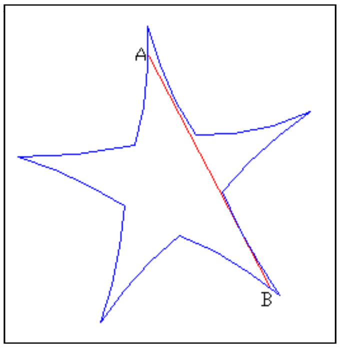


Figure 31.

31, Besicovitch proved that the area can be reduced to zero(!), if multiply-connected region is allowed. So far as a simply-connected region is concerned, the current minimum is obtained by a five-pointed star (*Figure* 31) having an area ≈ 0.2945 units. No one knows what the minimum area is and what the figure is. It is known that the greatest lower bound of area of a simply-connected region is less than or equal to $((5-2\sqrt{2})/24)\pi(\approx 0.2843)$.

Isoperimetric Problems in Two- and Three-Dimensions

It is a commonly known fact that, of all closed figures with the same perimeter, the circle encloses the maximum area. Similarly, of all closed surfaces with the same surface area, the sphere encompasses the maximum volume. The converses of these statements are also true, i.e., of all closed curves of equal area circle has the minimum perimeter and of all solids of same volume sphere has the minimum surface area. These results were known from the time of Greek geometers and some non-formal proofs were also available. Steiner, with a set of clever arguments, improved upon these proofs. But, for two dimensions, a mathematically formal proof was first given by Weierstrass around 1880 and published in 1927. In 1884, Schwarz first gave the formal proof for three dimensions.

The statements given above are used to define the following isoperimetric quotient (IQ) for closed figures and surfaces:

$$\text{For 2-D} \qquad \mathrm{IQ} = \frac{4\pi A}{P^2} \leq 1, \tag{9}$$

where A is the area and P is the perimeter with the equality sign valid only for a circle.

$$\text{For 3-D} \qquad \mathrm{IQ} = \frac{36\pi V^2}{A^3} \leq 1, \tag{10}$$

where V is the volume and A is the surface area with the equality sign valid only for a sphere. The isoperimetric quotient has been generalized for n dimensions as

$$\text{For } n\text{-D} \qquad \text{IQ} = \frac{2\pi^{n/2} n^{n-1} V^{n-1}}{A^n \Gamma(n/2)} \leq 1, \qquad (11)$$

where Γ represents the 'Gamma' function.

Equations (9) and (10) can be used to solve some analytical (non-geometrical) problems. Two such problems are discussed below.

Problem 1: Prove that

$$\int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\, \mathrm{d}t \geq \sqrt{4\pi \left[\pi ab + (a-b)^2\right]}.$$

Solution: We know that the parametric equation of an ellipse of semi-major axis a and semi-minor axis b (*Figure* 32) is given by $x = a\cos t,\ y = b\sin t$

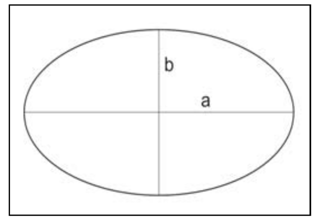


Figure 32.

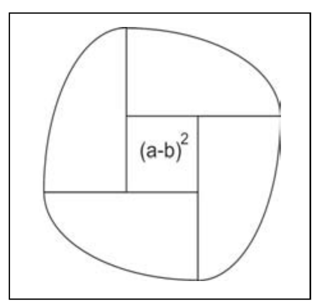
The perimeter of this ellipse P is given by

$$P = \oint \mathrm{d}s = \oint \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\, \mathrm{d}t.$$

Now cut the ellipse into four equal pieces and rearrange the pieces to obtain the closed figure shown in *Figure* 33. This non-circular figure has the same perimeter as the ellipse and the enclosed area A is given by

$$A = \pi ab + (a-b)^2.$$

Figure 33.



Substituting the expressions for A and P in (9) proves the desired result. The equality sign holds only for $a = b$.

Problem 2: Prove that for n real numbers $x_i, i = 1, \ldots, n$,

$$\left(\sum_{i=1}^{n} x_i^2\right)^3 \geq \left(\sum_{i=1}^{n} x_i^3\right)^2.$$

Solution: First let us assume all x_i's to be positive. Now consider n spheres whose radii are given by $x_i, i = 1, \ldots, n$. Think of the (non-spherical) solid obtained by gluing these spheres one after another. The total surface area A of this solid is given by

$$A = 4\pi \sum_{i=1}^{n} x_i^2,$$

and the volume V of this solid is given by

$$V = \left(\frac{4}{3}\right) \pi \sum_{i=1}^{n} x_i^3.$$

Substituting for A and V in (10), the desired result is proved. If some of the x_i's are negative, then the left-hand side is unaltered and the right-hand side diminishes, so the inequality becomes even stronger. The equality sign holds only for the trivial case when all x_i's are equal.

So we have seen that in geometry, different scalar measures like length, area, etc., can be optimized. Sometimes the optimal shape of a figure or a body may be of interest. Close connections between Euclidean geometry and Newtonian mechanics is revealed by some of these optimization problems. Finally, difficult to prove analytical results can be easily obtained using the concepts of optimization in geometrical problems. It may be mentioned that solutions can be obtained deductively if the verbal optimization statements and restrictions are translated into the language of mathematics (e.g., that of variational calculus) to define the objective functions and constraints.

Acknowledgement

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